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# An Asymptotic Behavior Property of High-Order Nonlinear Dynamic Equations on Time Scales

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**Abstract:** In this work, by using one dynamic Gronwall-Bihari type integral inequality on time scales, an interesting asymptotic behavior property of high-order nonlinear dynamic equations on time scales was obtained, which also generalized two classical results belong to Máté and Neval's and Agarwal and Bohner's respectively.

**Keywords:** Gronwall-Bihari type dynamic inequality; high-order dynamic equations; on time scales; asymptotic behavior

**MSC:** 2010: 34N05, 34E05, 34E10

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## 1. Introduction

Since Stenfan Hilger has introduced the theory of time scales which unify continuous and discrete analysis and extend the continuous and discrete theories to the case "in between", in the last few decades, the theories have gained considerable importance and attention due to their numerous applications to literally all branches of science such as statistics, biology, economics, finance, engineering, physics, and operations research have been given. The literature on such dynamic differential equations and their applications is vast; see the monographs of Martin Bohner and Alian Peterson [2, 3], Martin Bohner and Svetlin G. Georgiev [4] and the references given therein.

It is well known that Gronwall-type integral inequalities and their discrete analogues play a dominant role in the study of quantitative properties of solutions of differential, integral and difference equations. During the last few years, some Gronwall-type integral inequalities on time scales and their applications have been investigated by many authors. For example, we refer readers to [5–13]. In this paper, motivated by the paper [5,16], we using a Gronwall-Bihari type dynamic inequality to have established an interesting asymptotic behavior property of high-order dynamic equations on time scales. For all the detailed definitions, notation and theorems on time scales, we refer the readers to the excellent monographs [2,3] and references given therein. We also present some preliminary results that are needed in the remainder of this paper as useful lemmas for the discussion of our proof. In what follows, R denotes the set of real numbers,$R+ = [0, +\infty)$; $C(M, S)$ denotes the class of all continuous functions defined on set M with range in set S, T is an arbitrary time scale, and $C_{rd}$ denotes the set of rd-continuous functions. Throughout this paper, we always assume that $t_0 \in T, T_0 = [t_0, +\infty) \cap T$.

## 2. Some Lemmas and Main Result

**Lemma 2.1** ([11]). *Let* $\mathrm{T}$ *be an unbounded time scale* $t, t_0 \in \mathrm{T}$*; and* $u(t), a(t), b(t)$ *be nonnegative continuous functions defined for* $t \in \mathrm{T}$*. Assume that* $a(t)$ *is nondecreasing for* $t \in \mathrm{T}$ *and* $0 < r \leq 1$*. If for* $t \in \mathrm{T}$ *we have*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(s)u^r(s)\Delta s, \tag{2.1}$$

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*then*

$$u(t) \le \begin{cases} a(t)e_b(t,t_0), \ \ r=1, \\ \\ a(t)\left[1+(1-r)\int_{t_0}^{t} b(s)\Delta s\right]^{\frac{1}{1-r}}, \ \ 0<r<1, \end{cases} \tag{2.2}$$

*where* $e_b(t,t_0)=\exp\left\{\int_{t_0}^{t}\xi_{\mu(s)}(b(s))\Delta s\right\}$, *and the cylinder transformation* $\xi_h : C_h \to Z_h$ *defined by*

$$\xi_h(z)=\frac{1}{h}Log(1+zh),$$

*where Log is the principal logarithm function.*

**Lemma 2.2.** *For any rd-continuous nonnegative function* $b(t)$*, we have inequality*

$$e_b(t,t_0) \le \exp\int_{t_0}^{t} b(s)\Delta s. \tag{2.3}$$

**Proof.** By the representation [2, (2.15)], we have

$$e_b(t,t_0)=\exp\left\{\int_{t_0}^{t}\xi_{\mu(s)}(b(s))\Delta s\right\}.$$

If $\mu(s)=0$, it flows that

$$\xi_{\mu(s)}(b(s))=b(s);$$

If $\mu(s)>0$, we have

$$\begin{aligned}\xi_{\mu(s)}(b(s)) &= \frac{Log(1+\mu(s)b(s))}{\mu(s)} = \frac{\log(1+\mu(s)b(s))}{\mu(s)} \\ &= b(s)-\frac{\mu(s)b(s)-\log(1+\mu(s)b(s))}{\mu(s)}. \end{aligned} \tag{2.4}$$

Setting $f(x)=x-\log(1+x)$ for $x>-1$, from (2.4) we obtain that

$$\xi_{\mu(s)}(b(s))=b(s)-\frac{f(\mu(s)b(s))}{\mu(s)} \le b(s),$$

since $f(x)\ge 0$ for $x>-1$. □

**Theorem 2.3.** *Let* $I=[1,\infty), \mathrm{T}_1=\mathrm{T}\cap I, n\in \mathrm{N}^+$, *rd-continuous functions* $p_i:\mathrm{T}_1\to \mathrm{R}^+, 1\le i\le n-1$. *If for* $t\in \mathrm{T}_1$, $r_i(i=0,1,2,...,n-1)$ *are constants with* $0<r_i\le 1$, *a function* $y$ *is* $n$ *times differentiable on* $\mathrm{T}_1^{\check{}\mathrm{n}}$ *and assume that*

$$|y^{\Delta^n}(t)| \le \sum_{i=0}^{n-1} p_i(t)|y^{\Delta^i}(t)|^{r_i} \tag{2.5}$$

*and*

$$\int_1^{+\infty} s^{(n-i-1)r_i}p_i(s)\Delta s < +\infty. \tag{2.6}$$

*Then there exists* $\gamma>0$ *such that*

(*i*) $|y^{\Delta^k}(t)| = O(\gamma t^{n-k-1}), \ \ k=0,1,2,\cdots,n-1;$

*and*

*(ii)* $\lim\limits_{t\to+\infty} y^{\Delta^{n-1}}(t)$ *exists.*

**Proof.** From (2.5), for any $t \geq t_0 \geq 1$ we have

$$|y^{\Delta^{n-1}}(t) - y^{\Delta^{n-1}}(t_0)| \leq \sum_{i=0}^{n-1} \int_{t_0}^{t} p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s, \tag{2.7}$$

which follows that for $t_0 = 1$

$$|y^{\Delta^{n-1}}(t)| \leq |y^{\Delta^{n-1}}(1)| + \sum_{i=0}^{n-1} \int_{1}^{t} p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s$$

$$= \big[|y^{\Delta^{n-1}}(1)| + \sum_{i=0}^{n-2} \int_{1}^{t} p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s\big] + \int_{1}^{t} p_{n-1}(s)|y^{\Delta^{n-1}}(s)|^{r_{n-1}}\Delta s.$$

Without loss of generality, assume that $|y^{\Delta^{n-1}}(1)| \geq 1$, from the last inequality and by Lemma 2.1 and 2.2 we obtain that

$$|y^{\Delta^{n-1}}(t)| \leq \left[|y^{\Delta^{n-1}}(1)| + \sum_{i=0}^{n-2} \int_{1}^{t} p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s\right] G_1(t), \tag{2.8}$$

where

$$G_1(t) = \begin{cases} \exp \displaystyle\int_{1}^{t} p_{n-1}(s)\Delta s, \ \ r_{n-1} = 1, \\ \left[1 + (1 - r_{n-1}) \displaystyle\int_{1}^{t} p_{n-1}(s)\Delta s\right]^{\frac{1}{1-r_{n-1}}}, 0 < r_{n-1} < 1. \end{cases} \tag{2.9}$$

From (2.8), (2.9) and condition (2.6), we have

$$|y^{\Delta^{n-1}}(t)| \leq K_1 + \sum_{i=0}^{n-2} \int_{1}^{t} M_1 p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s, \tag{2.10}$$

where

$$K_1 = |y^{\Delta^{n-1}}(1)|M_1, \ M_1 = G_1(+\infty).$$

Integrating (2.10) from 1 to $t, t \geq 1$ and using the change of order integration formula [17, Lemma 2.1], we obtain that

$$|y^{\Delta^{n-2}}(t)| \leq |y^{\Delta^{n-2}}(1)| + K_1 t + t \sum_{i=0}^{n-2} \int_{1}^{t} M_1 p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s, \tag{2.11}$$

which follows that

$$\frac{|y^{\Delta^{n-2}}(t)|}{t} \leq (|y^{\Delta^{n-2}}(1)| + K_1) + \sum_{i=0}^{n-3} \int_{1}^{t} M_1 p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s +$$

$$+ \int_{1}^{t} M_1 s^{r_{n-2}} p_{n-2}(s) \left(\frac{|y^{\Delta^{n-2}}(s)|}{s}\right)^{r_{n-2}} \Delta s.$$

Using Lemma 2.1 and 2.2 to the last inequality again, we have

$$\frac{|y^{\Delta^{n-2}}(t)|}{t} \leq \left[|y^{\Delta^{n-2}}(1)| + K_1 + \sum_{i=0}^{n-3} \int_{1}^{t} M_1 p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s\right] G_2(t), \tag{2.12}$$

where

$$G_2(t)=\begin{cases}\exp\int_1^t M_1 s^{r_{n-2}}p_{n-2}(s)\Delta s, & r_{n-2}=1,\\ \left[1+(1-r_{n-2})\int_1^t M_1 s^{r_{n-2}}p_{n-2}(s)\Delta s\right]^{\frac{1}{1-r_{n-2}}}, & 0<r_{n-2}<1.\end{cases}\tag{2.13}$$

(2.12) implies that

$$|y^{\Delta^{n-2}}(t)|\le K_2 t+\sum_{i=0}^{n-3} t\int_1^t M_2 p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s,\tag{2.14}$$

where

$$K_2=(|y^{\Delta^{n-2}}(1)|+K_1)G_2(+\infty), M_2=M_1G_2(+\infty).$$

By mathematical induction, we will derive that

$$|y^{\Delta^{n-j}}(t)|\le K_j t^{j-1}+\sum_{i=0}^{n-j-1} t^{j-1}\int_1^t M_j p_i(s)|y^{\Delta^i}(s)|^{r_i}\Delta s,\tag{2.15}$$

where $K_j$ and $M_j$ are constants, $j=1,2,\cdots,n-1$. Especially, for $j=n-1$, we have

$$|y^{\Delta}(t)|\le K_{n-1}t^{n-2}+t^{n-2}\int_1^t M_{n-1}p_0(s)|y(s)|^{r_0}\Delta s,\tag{2.16}$$

Integrating (2.16) from 1 to $t, t\ge 1$ and using the change of order integration formula again, we can obtain that

$$|y(t)|\le K_0 t^{n-1}+t^{n-1}\int_1^t M_{n-1}p_0(s)|y(s)|^{r_0}\Delta s$$

where $K_0$ is a suitable constant. (2.16) can be re-written as

$$\frac{|y(t)|}{t^{n-1}}\le K_0+\int_1^t M_{n-1}p_0(s)s^{(n-1)r_0}\left(\frac{|y(s)|}{s^{n-1}}\right)^{r_0}\Delta s$$

Using Lemma 2.1 and 2.2 to the last inequality, we have

$$|y(t)|\le M_0 t^{n-1},\tag{2.17}$$

where

$$M_0=\begin{cases}K_0\exp\int_1^{+\infty} M_1 s^{(n-1)r_0}p_0(s)\Delta s, & r_0=1,\\ K_0\left[1+(1-r_0)\int_1^{+\infty} M_1 s^{(n-1)r_0}p_0(s)\Delta s\right]^{\frac{1}{1-r_0}}, & 0<r_0<1.\end{cases}\tag{2.18}$$

From (2.15)-(2.18) we can derive that

$$|y^{\Delta^k}(t)|\le a_k t^{n-k-1}, k=0,1,2,\cdots,n-1.\tag{2.19}$$

where $a_k$, $(k=0,1,2,\cdots,n-1)$ are some constants.

Set $\gamma=\max\limits_{0\le k\le n-1}\{a_k\}$, from (2.19) we have proved (i);

By condition (2.6), combing with (2.7) and (2.19) we obtain that

$$\lim_{t,t_0\to+\infty}|y^{\Delta^{n-1}}(t)-y^{\Delta^{n-1}}(t_0)|=0.$$

From the Cauchy criterion [4], it follows that $\lim_{t\to+\infty} y^{\Delta^{n-1}}(t)$ exist. □

By Theorem 2.3, we can get the following corollary easily.

**Corollary 2.4.** *Consider initial value problem*

$$y^{\Delta^n} = f(t, y, y^{\Delta}, \cdots, y^{\Delta^{n-1}}),\ \ y^{\Delta^i}(1) = K_i \ for\ 0 \le i \le n-1 \tag{2.20}$$

*where* $f : \mathrm{T} \times \mathrm{R}^n \to \mathrm{R}$ *is supposed to satisfy*

$$|f(t, u_0, \cdots, u_{n-1})| \le \sum_{i=0}^{n-1} p_i(t)|u_i|^{r_i} \tag{2,21}$$

*for all* $t \in \mathrm{T}_1$, $\{u_i : 0 \le i \le n-1\} \subset \mathrm{R}$; $p_i(t)(i = 0, 1, \cdots, n-1)$ *are defined as in Theorem 2.3 and satisfy condition (2.6). Then there exists* $\gamma > 0$ *such that for every solution* $y$ *of (2.20) satisfies*
*(i)*

$$|y^{\Delta^k}(t)| = O(\gamma t^{n-k-1}),\ \ k = 0, 1, 2, \cdots, n-1; \tag{2.22}$$

*and (ii)*

$$\lim_{t\to+\infty} y^{\Delta^{n-1}}(t)\ \ exists.$$

**Remark 1.** *When* $\mathrm{T} = \mathrm{R}$, $r_i = 1, i = 0, 1, \cdots, n-1$, *from Theorem 2.3, we can get a main result of Máté and Neval's ([16], Lemma 2); when* $\mathrm{T} = \mathrm{Z}$, $r_i = 1, i = 0, 1, \cdots, n-1$ *with some suitable conditions we can get one another main result of Máté and Neval's ([16], Lemma 6).*

**Remark 2.** *When* $r_i = 1, i = 0, 1, \cdots, n-1$, *from (2.22), we can get the similar result of Agarwal and Bohner's ([17],Theorem 7) under some simpler conditions on* $p_i(t), (i = 0, 1, \cdots, n-1)$.

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