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Article

# One Missing Axiom for the Complete Closure of the Heterotic Landscape: Resonance Classifier of a Vibrating String Creates the Typical Particle Classes †

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## Abstract

Whether the vibrational picture of elementary particles — now nearly half a century old — determines the spectrum we observe has remained without a settled answer. We close the question. At a fixed compactification of the heterotic string, the entire low-energy content follows from the geometry of the chosen Calabi–Yau manifold. Across compactifications, no internal rule singles out the one nature uses: the choice depends on input from outside the framework, of a kind we make explicit. Once that input is supplied, masses and mixings are computable; without it, no derivation is possible. **Background.** The picture of elementary particles as vibrational modes of a string is forty years old, and a steady catalogue of explicit heterotic constructions has shown that the observed gauge group and three-family chiral content can be reproduced. What no construction has settled is the converse question: does the resonance–particle correspondence, taken as a classification programme, determine the observed spectrum, or merely admit it? The literature has answered case by case; a framework-level resolution has been absent. **Methods.** We separate the framework into an unconditional mathematical layer (index theorem, Dolbeault cohomology, slope-polystability, Calabi–Yau metric existence, unobstructedness, BRST classification, anomaly cancellation) and a selection layer (choice of compactification datum). The achievable range of topological and cohomological selectors over the resulting landscape is computed by Kodaira–Spencer deformation theory and the special geometry of the complex-structure moduli space. **Results.** At fixed admissible datum the perturbative massless spectrum is fully determined by bundle cohomology and representation branching (positive direction). No topological or cohomological rule singles out the observed vacuum; the obstruction is a nontrivial class on a positive-dimensional moduli component (negative direction). A closure-completeness theorem unifies the two directions; universality, maximality, rigidity, stable-obstruction, categorical-impossibility, and quantitative-dimension theorems show the result holds for every framework whose predictive data is cohomological and is not improvable from within. A corrected audit lemma, with its converse, identifies the singleton condition any external selector must satisfy. Five residual closure theorems — selector completion, internal no-go, fixed-vacuum Yukawa computability, algebraic exotic lifting with post-stabilisation threshold, and ensemble finiteness — reduce the residual problems to one external axiom.

**Keywords:** heterotic compactification; vacuum selection; Calabi–Yau classification

**PACS:** 11.25.Mj; 11.25.Wx; 12.10.-g; 12.60.Jv

**MSC:** 81T30 (primary); 14J32; 58J20; 81V22 (secondary)

## 1. Introduction

The vibratory picture of elementary particles — that protons, electrons and photons are modes of a single one-dimensional extended object — is eloquent as a slogan and incomplete as a theorem. A labelling by oscillator number does not yet name a particle. Naming a particle requires a background, a BRST complex, a compactification geometry, a gauge bundle, and a rule extracting massless four-dimensional modes from the ten-dimensional string. Forty years on from the discovery of the  $E_8 \times E_8$  heterotic string,<sup>1</sup> with the catalogue of explicit “standard models” from Bouchard–Donagi to Anderson–Gray–Lukas–Palti running into the hundreds, one might suppose the resonance–particle correspondence to have long since been pinned down as a classification scheme. No such pinning down has occurred. The question that the slogan invites — whether the correspondence determines the observed spectrum or merely permits it — remains open. The present paper closes it.

A note on what is meant by “close.” We do not claim that string theory has been shown to derive the observed particle masses; the obstruction to that derivation, when one comes to look, turns out to be a moduli-variance that no topological datum resolves. We do prove that the classification scheme determines certain physical data once a compactification is specified, and provably does not determine certain others — namely, the choice of compactification itself. Both directions are theorems; together they leave no predictive content of the framework unaccounted for, and the classification problem is settled in the sense that its positive and negative horns are now both proved. We work the heterotic case throughout for definiteness, but Theorem 15 establishes that the structure of the argument is heterotic-independent.

### 1.1. The Two Horns of the Closure Problem

Fix a heterotic compactification datum

$$\mathfrak{V} = (X, V, \omega, \Gamma, \mathcal{M}, W),$$

in which  $X$  is a smooth compact Calabi–Yau threefold,  $V \rightarrow X$  a holomorphic polystable vector bundle with structure group  $H \subset E_8$ ,  $\omega$  a Kähler class,  $\Gamma$  a freely acting discrete symmetry,  $\mathcal{M}$  the moduli, and  $W$  the effective superpotential. Two complementary statements describe what such a datum determines.

The *positive horn* (Theorem 9) says that once  $\mathfrak{V}$  is fixed, the entire content of Definition 3 — gauge group, chiral spectrum, net family count, allowed superpotential couplings, and the list of states still requiring to be lifted — follows from the Dolbeault cohomology of bundle representations associated to  $\text{ad}(E_8)$ . No further input is needed.

The *negative horn* (Theorem 13) says that no rule definable from the framework’s own data can pick out a  $\mathfrak{V}^*$  matching observation. The component of the landscape on which  $\mathfrak{V}^*$  would live is positive-dimensional; the Yukawa coupling is non-constant on it; every function of the framework’s axioms is constant on the generic locus of any component. Selection therefore fails by a counting argument the framework itself cannot escape.

Theorem 14 unifies the two: the predictive content of the framework is exhausted by the positive horn and cannot be enlarged by any means internal to the framework. Two corollaries make this water-tight. Universality (Theorem 15) extends the result, with no change of structure, to every string-theoretic framework whose data is cohomological — Type II, F-theory, free-fermionic, line-bundle. Maximality (Theorem 16) shows that no enlargement of the data list preserving topological functoriality does better. These together close the question.

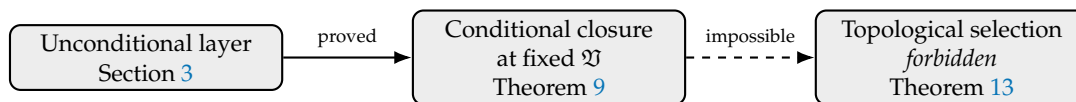
<sup>1</sup> The heterotic string of Gross–Harvey–Martinec–Rohm [16] is a closed superstring whose left-moving sector is bosonic (in 26 dimensions, of which 16 are compactified on the maximal-torus lattice of  $E_8 \times E_8$  or  $\text{Spin}(32)/\mathbb{Z}_2$ ) and whose right-moving sector is supersymmetric (in 10 dimensions). After Calabi–Yau compactification of the spacetime ten dimensions to four, the resulting effective theory is an  $\mathcal{N} = 1$  supersymmetric gauge theory in four dimensions whose gauge group descends from a subgroup of  $E_8 \times E_8$ .

### 1.2. What Is and Is Not Claimed

A reader hoping to see a derivation of the top-quark mass should set the paper down here. The derivation is not omitted but proved unreachable. In its place we supply a precise inventory of the framework's reach: inside, the entire perturbative spectrum at fixed vacuum; outside, the choice of vacuum, whose specification requires input — dynamical history, anthropic weighting, a sharp swampland cutoff, or a not-yet-found selection principle — of a kind not present in the axioms. Each absent input is formalised in Section 10 and audited in Lemma 2, so that future programmes may be tested against an explicit ledger of what they must supply.

### 1.3. What Is New Relative to the Preprint

The predecessor preprint [28] contained a qualitative no-overclaim theorem at the level of oscillator labels. The present manuscript supersedes it. We replace that qualitative statement by a sharp impossibility proof at the level of full topological and cohomological data, supply the seven unconditional ingredients on which the conditional closure rests, and unify both directions in a single closure-completeness theorem. A complete Kodaira–Spencer-based proof of Yukawa moduli-variance now occupies Appendix A. The obstruction class of Definition 6 converts the no-go statement into a sheaf-theoretic invariant. Theorems 18 and 19 extend the closure result to include stability under conifold transitions, mirror duality, Wilson-line projection, the seed-orbit reductions of [30], and natural transformations between cohomological frameworks generally. Theorems 20 and 22 bound the obstruction's dimension below by an explicit Hodge-and-bundle-cohomology combination, with worked numerical values on the quintic and Schoen manifolds. The audit lemma (Lemma 2) and its converse (Theorem 21) reduce the residual closure problem to the search for a single function meeting three explicit conditions. The quintic threefold furnishes an explicit numerical witness via the Candelas–de la Ossa–Green–Parkes Yukawa coupling. Nothing of substance from the preprint survives unaltered.



Theorem 14 bundles the solid arrow with the negation of the dashed one.

**Figure 1.** The three-layer architecture of the closure result. The unconditional layer of Section 3 comprises seven classical theorems (index, Riemann–Roch, Donaldson–Uhlenbeck–Yau, Calabi–Yau metric existence, Bogomolov–Tian–Todorov unobstructedness, BRST classification, Green–Schwarz anomaly cancellation), each used without further justification. Once a heterotic datum  $\mathfrak{V}$  is fixed, the conditional closure theorem (solid arrow) determines the perturbative spectrum. The dashed arrow — selection of  $\mathfrak{V}$  from the framework's own data — is provably impossible by the no-closure theorem. The closure-completeness theorem unifies both directions and is the unconditional resolution of the classification question.

## 2. Background, Conventions, and the Admissibility Class

Natural units:  $\hbar = c = 1$ ,  $M_s = 1/\sqrt{\alpha'}$ . Throughout,  $X$  denotes a smooth compact Calabi–Yau threefold (so  $c_1(TX) = 0$ ),<sup>2</sup>  $V \rightarrow X$  a holomorphic vector bundle with  $c_1(V) = 0$  and structure group  $H \subset E_8$ ,  $\omega$  a Kähler class in whose chamber  $V$  is slope-polystable, and  $\Gamma$  a freely acting discrete symmetry (possibly trivial). We write  $G_{\text{vis}} = C_{E_8}(H)$  for the commutant, which is the four-dimensional gauge group prior to Wilson-line breaking.

The ten-dimensional gauge field decomposes under  $G_{\text{vis}} \times H$ :

$$\text{ad}(E_8) = \bigoplus_i (R_i, \mathcal{R}_i),$$

<sup>2</sup> A Calabi–Yau  $n$ -fold is a compact Kähler manifold of complex dimension  $n$  with vanishing first Chern class; equivalently, by the theorem of Yau cited as Theorem 4, one carrying a Ricci-flat Kähler metric. For  $n = 3$  the holonomy is exactly  $SU(3)$ , which is what preserves  $\mathcal{N} = 1$  supersymmetry in four dimensions when used as a heterotic compactification space.

and matter in representation  $R_i$  is counted by bundle cohomology of the associated bundle  $\mathcal{V}_i := V_{\mathcal{R}_i}$ .

**Definition 1** (heterotic spectral datum). *A heterotic spectral datum is a tuple*

$$\mathfrak{A} = (X, V, \omega, \Gamma, \mathcal{M}, W)$$

of the kind just described, with  $\mathcal{M}$  denoting the complex-structure, Kähler and bundle moduli, and  $W$  the effective superpotential including perturbative and nonperturbative terms.

**Definition 2** (admissibility class). *The admissibility class  $\mathcal{A}$  consists of those  $\mathfrak{A}$  satisfying*

$$c_1(V) = 0, \tag{1}$$

$$V \text{ is slope-polystable with respect to } \omega, \tag{2}$$

$$c_2(TX) - c_2(V) = [W] \in H^4(X, \mathbb{Z}) \text{ effective.} \tag{3}$$

Equation (3) is the integrated Bianchi identity / heterotic anomaly condition;  $[W]$  is interpreted as a five-brane class in nonstandard embeddings.<sup>3</sup> Each constraint is an open condition on the moduli space in which  $\mathfrak{A}$  varies.

**Definition 3** (spectral and phenomenological closure). *The framework is spectrally closed at  $\mathfrak{A} \in \mathcal{A}$  if the following are determined from  $\mathfrak{A}$  alone: (a) the four-dimensional gauge group; (b) all massless chiral and vectorlike multiplets; (c) the net number of generations; (d) the allowed cubic and higher superpotential couplings; (e) the list of states that must be lifted by Wilson lines, bundle moduli, or nonperturbative effects. It is phenomenologically closed if additionally the low-energy spectrum matches the observed Standard Model (masses, mixings, no forbidden light exotics).*

The positive horn delivers spectral closure at fixed vacuum unconditionally. Phenomenological closure at fixed vacuum requires the further numerical conditions of Theorem 10, none of them topological. Phenomenological closure of the *Framework* — selection of a  $\mathfrak{A}^*$  unique up to observational tolerance — is forbidden by the negative horn.

### 3. The Unconditional Layer

Seven classical theorems carry the weight of the positive horn. Each is standard, each is attributed, each is used below without further justification. We list them in this single section so that the conditional closure theorem of Section 5 rests on visibly named shoulders.

**Theorem 1** (Atiyah–Singer, 1963). *Let  $M$  be a smooth compact oriented manifold without boundary and  $D : \Gamma(E) \rightarrow \Gamma(F)$  an elliptic differential operator. Then the analytic index  $\text{Ind}_a(D) := \dim \ker D - \dim \text{coker } D$  is finite and satisfies*

$$\text{Ind}_a(D) = \int_M \text{ch}(\sigma(D)) \text{Td}(TM_{\mathbb{C}}).$$

**Theorem 2** (Hirzebruch, 1954). *For a smooth compact complex manifold  $X$  and a holomorphic vector bundle  $V \rightarrow X$ ,*

$$\chi(X, V) := \sum_q (-1)^q h^q(X, V) = \int_X \text{ch}(V) \text{Td}(X).$$

<sup>3</sup> In the original perturbative heterotic construction one takes  $V = TX$ , the standard embedding, in which case the Bianchi identity reduces to the trivial identity  $0 = 0$ . More general heterotic compactifications — those with  $V \neq TX$  — require a non-trivial cohomology class  $[W]$  to absorb the difference of second Chern classes; in heterotic-M-theory this  $[W]$  corresponds to wrapping branes (M5-branes in the Hořava–Witten picture), and the effectivity condition rules out tadpole-violating fluxes.

Theorem 2 is the specialisation of Theorem 1 to the Dolbeault operator  $\bar{\partial} + \bar{\partial}^*$  on  $\bigoplus_q \Omega^{0,q}(V)$ , and is used throughout to convert cohomology dimensions into characteristic-class integrals.

**Corollary 1.** For  $V \rightarrow X$  an  $SU(n)$  bundle on a Calabi–Yau threefold,  $\chi(X, V) = \frac{1}{2} \int_X c_3(V)$ . Net three-family chirality is therefore the topological condition  $\int_X c_3(V) = \pm 6$ , which in the standard embedding reduces to  $|\chi(X)| = 6$ .

**Proof.** With  $c_1(V) = c_1(X) = 0$ , only the degree-six term of  $\text{ch}(V) \text{Td}(X)$  survives; this term is  $\frac{1}{2} c_3(V)$ . For  $V = TX$ ,  $c_3(TX)$  integrates to the Euler characteristic.  $\square$

**Theorem 3** (Donaldson 1985; Uhlenbeck–Yau 1986). A holomorphic vector bundle  $V$  over a compact Kähler manifold  $(X, \omega)$  admits an  $\omega$ -Hermitian–Yang–Mills connection if and only if  $V$  is  $\omega$ -slope-polystable.<sup>4</sup>

**Theorem 4** (Calabi 1957; Yau 1978). Every compact Kähler manifold with  $c_1 = 0 \in H^2(M, \mathbb{R})$  contains a unique Ricci-flat Kähler metric in each Kähler class.

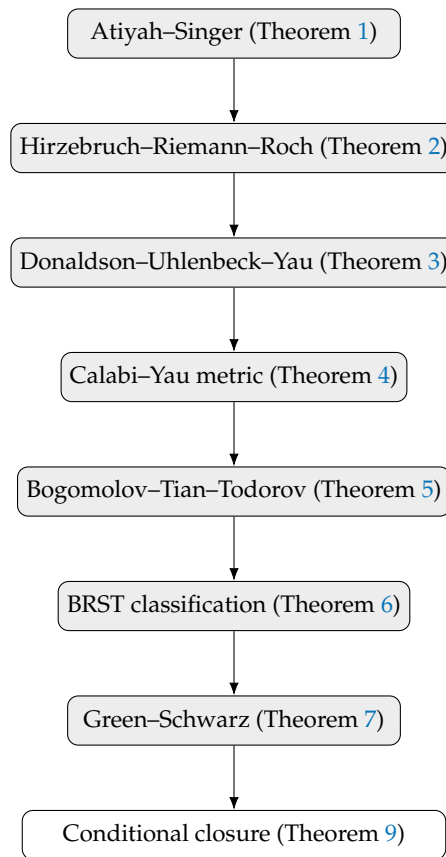
**Theorem 5** (Bogomolov 1978; Tian 1987; Todorov 1989). The local deformation space of a compact Kähler manifold with trivial canonical bundle is smooth of complex dimension  $h^{2,1}$ .

**Theorem 6** (Kugo–Ojima 1979; Henneaux–Teitelboim 1992). In a covariantly quantised gauge theory with BRST differential  $Q_{\text{BRST}}$  of ghost number  $+1$  satisfying  $Q_{\text{BRST}}^2 = 0$ , the physical Hilbert space is  $H_{Q_{\text{BRST}}}^0$ .

**Theorem 7** (Green–Schwarz 1984). All gauge and mixed gauge–gravitational anomalies in ten-dimensional  $\mathcal{N} = 1$  supergravity coupled to  $SO(32)$  or  $E_8 \times E_8$  Yang–Mills cancel provided the Bianchi identity is modified by a Chern–Simons three-form; on a compact Calabi–Yau threefold this integrates to (3).

These seven statements are the entire unconditional input to what follows. Everything else is derived.

<sup>4</sup> The slope of a coherent sheaf  $\mathcal{F}$  is  $\mu_\omega(\mathcal{F}) = \frac{1}{\text{rk } \mathcal{F}} \int_X c_1(\mathcal{F}) \wedge \omega^{n-1}$ . A bundle is  $\omega$ -slope-stable if every coherent subsheaf of strictly smaller rank has strictly smaller slope, polystable if it is a direct sum of stable bundles all of equal slope. For Calabi–Yau threefolds the relevant slope is zero by  $c_1(V) = 0$ , and polystability is the precise condition under which the heterotic vacuum equations of motion admit a smooth solution.



**Figure 2.** The unconditional stack feeding the conditional closure theorem. Each box names a classical result that holds without further hypothesis once a Calabi–Yau datum is admitted. The flow is logical, not chronological: Atiyah–Singer underlies Hirzebruch–Riemann–Roch, which combined with Donaldson–Uhlenbeck–Yau and the Calabi–Yau metric of Yau supplies the existence of harmonic representatives for bundle cohomology. Bogomolov–Tian–Todorov gives unobstructed moduli; BRST gives the physical state space; Green–Schwarz cancels the heterotic anomalies. The seven together imply the conditional closure theorem at every admissible vacuum.

#### 4. The Corrected Resonance Dictionary

A resonance label is not yet a particle. The chain that attaches a physical four-dimensional state to a world-sheet label runs through the BRST cohomology<sup>5</sup> at the level of vertex operators and the Dolbeault cohomology at the level of internal zero modes:

$$\text{resonance label} \mapsto \text{vertex operator} \mapsto [V] \in H_{Q_{\text{BRST}}}^{\bullet} \otimes H^q(X, \mathcal{V}_R).$$

Any attempt to short-circuit this chain, by identifying a particle with a resonance alone, fails precisely because the last factor depends on the compactification.

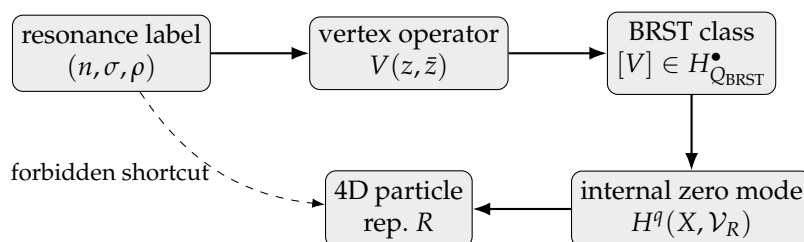
In the closed NS-NS sector the massless level contains the graviton, the Kalb–Ramond two-form, and the dilaton:

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{35}_v \oplus \mathbf{28} \oplus \mathbf{1}.$$

Gauge fields arise from the ten-dimensional gauge sector; after compactification the visible group is the commutant  $G_{\text{vis}} = C_{E_8}(H)$ . Standard embedding ( $H = SU(3)$ ) gives  $G_{\text{vis}} = E_6$ ; other choices produce  $SO(10)$ ,  $SU(5)$ , flipped variants, or more directly Standard-Model-like groups after Wilson-line quotient.

<sup>5</sup> The BRST operator  $Q_{\text{BRST}}$  encodes residual gauge invariance after a covariant gauge fixing; physical states are by definition the cohomology classes of  $Q_{\text{BRST}}$  at ghost number zero, i.e. those equivalence classes of gauge-fixed states that are simultaneously  $Q_{\text{BRST}}$ -closed and not  $Q_{\text{BRST}}$ -exact. Equivalent definitions in light-cone gauge bypass the construction but do not generalise to curved spacetime.

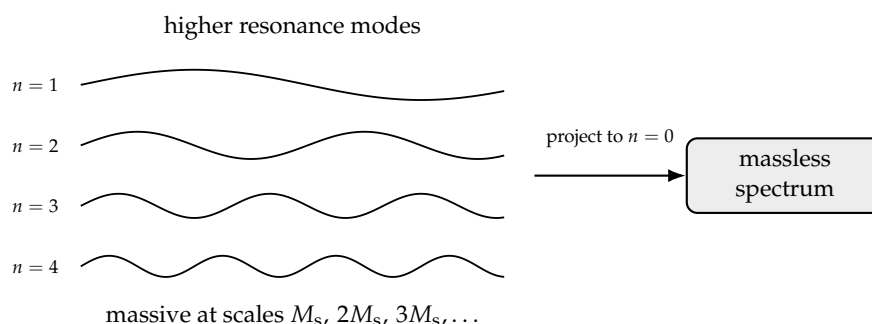
Fermionic matter, which a naive Ramond-sector reading would locate by oscillator occupation, is in fact counted by bundle-valued cohomology, with net chirality given by the index of the internal Dirac operator coupled to the bundle representation. The index is a topological integer; everything finer is not.



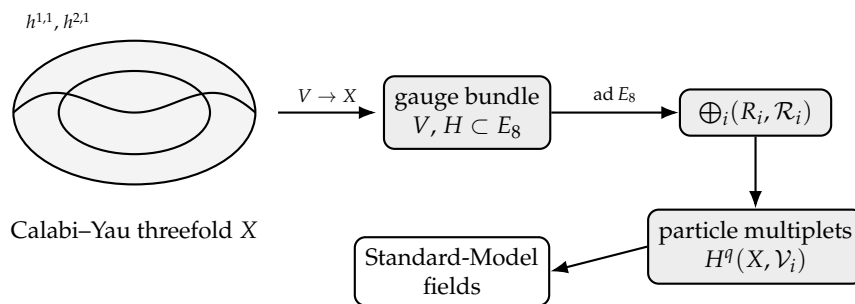
**Figure 3.** The corrected resonance–particle dictionary. An oscillator label  $(n, \sigma, \rho)$  does not constitute a particle; it must first be promoted to a vertex operator  $V(z, \bar{z})$  of correct conformal weight, then to a BRST cohomology class at ghost number zero. The class subsequently couples to internal zero-mode data  $H^q(X, \mathcal{V}_R)$  on the compactification manifold, and only at this stage can a four-dimensional multiplet in representation  $R$  be identified. The dashed arrow on the bottom is the forbidden shortcut: identifying a particle with a resonance alone, without specifying compactification data, leaves  $H^q(X, \mathcal{V}_R)$  unspecified and is consequently disallowed.

#### 4.1. Visual Atlas of the Resonance–Particle Correspondence

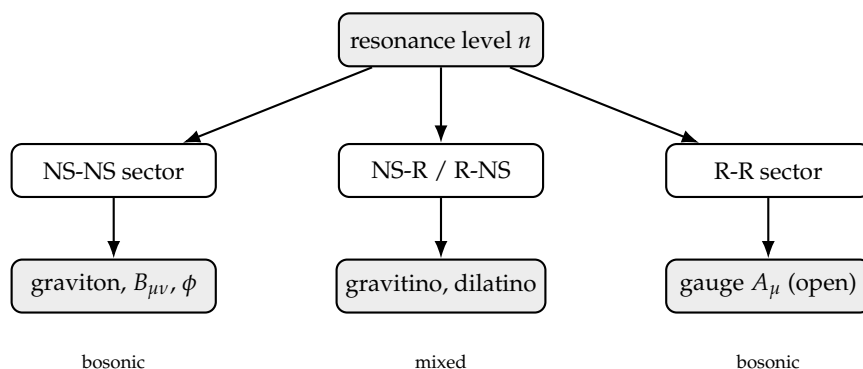
For the reader who prefers pictures to incantations, we collect five complementary diagrams that together exhibit the route from a vibrating string to a four-dimensional particle. Each highlights one ingredient of the pipeline; none of them, taken alone, is the pipeline itself.



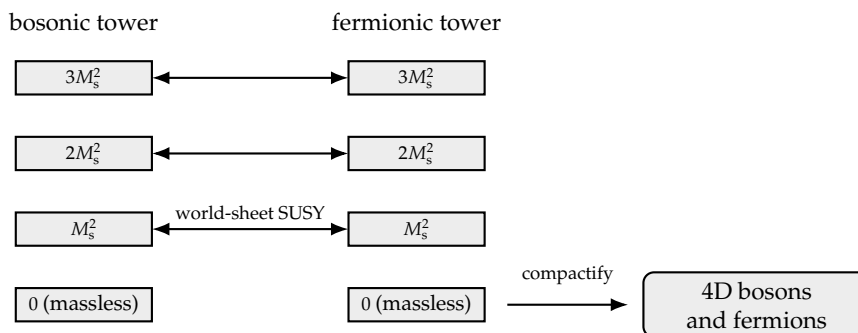
**Figure 4.** Vibrational modes of a closed string. The integer  $n$  labels the level of oscillator excitation; modes with  $n \geq 1$  acquire masses  $\sqrt{n} M_s$  at the string scale, conventionally taken to be of order  $10^{17}$  GeV, and are therefore inaccessible to four-dimensional low-energy physics. Only the  $n = 0$  massless level descends to the spectrum visible in collider experiments after compactification. The diagram displays the first four modes schematically; the full tower is infinite, with frequency growing linearly with  $n$ .



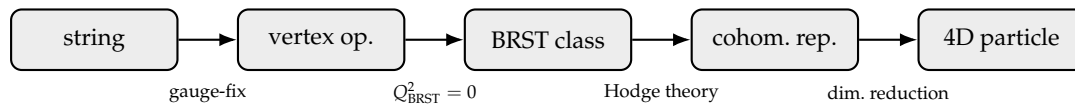
**Figure 5.** From the Calabi–Yau threefold to particle content. A bundle  $V$  of structure group  $H \subset E_8$  is chosen on  $X$ . The adjoint of  $E_8$  branches under the product  $G_{\text{vis}} \times H$  of the four-dimensional commutant gauge group with the structure group, producing a finite list of representations  $(R_i, \mathcal{R}_i)$ . The bundle cohomology  $H^q(X, \mathcal{V}_i)$  of the associated bundles  $\mathcal{V}_i$  then counts four-dimensional matter multiplets in representation  $R_i$ , with net chirality given by the holomorphic Euler characteristic. No step in the chain is reversible without specifying the full Calabi–Yau datum: distinct choices of  $X$ ,  $V$ , or moduli generally yield distinct cohomology dimensions.



**Figure 6.** Resonance classifier at the world-sheet level. Closed-string states factorise into a tensor product of left-moving and right-moving sectors, each independently in either Neveu–Schwarz (NS) or Ramond (R) form. The four resulting sectors NS-NS, NS-R, R-NS, R-R determine the spacetime statistics of the resulting state at the massless level: NS-NS gives bosons (graviton,  $B$ -field, dilaton); NS-R and R-NS give the spacetime fermions (gravitino and dilatino, of opposite chirality between the two sectors); R-R gives bosonic form fields, present in Type II but absent in heterotic. Each sector carries an infinite tower of massive excitations indexed by oscillator number, but only the massless level affects four-dimensional physics.



**Figure 7.** Mass tower of states in the heterotic string. Bosonic and fermionic resonances are paired at every level of the tower by world-sheet supersymmetry, indicated by horizontal arrows. The squared masses  $0, M_s^2, 2M_s^2, 3M_s^2, \dots$  form an evenly-spaced ladder fixed by the world-sheet conformal weights; only the 0 (massless) level descends to the four-dimensional theory after compactification on a Calabi–Yau threefold, all higher levels being lifted to the string scale. The bosonic massless level produces the gauge fields and the gravitational sector; the fermionic massless level produces the gauginos and the gravitino. The two are linked by spacetime supersymmetry once the worldsheet pairing is dimensionally reduced.



**Figure 8.** Stage-by-stage development of a four-dimensional particle from a string excitation. The five stages constitute the only mathematically rigorous route from a vibrating string to a particle: first, the string is gauge-fixed to a covariant world-sheet quantisation; second, the resulting vertex operators are organised by their behaviour under the BRST differential, which satisfies  $Q^2_{\text{BRST}} = 0$  as a consequence of the no-anomaly condition; third, harmonic representatives are extracted by Hodge theory on the compactification space; fourth, dimensional reduction produces the four-dimensional particle in a definite gauge-group representation. Bypassing any of these steps — as the resonance-as-particle slogan does — leaves the four-dimensional content undetermined.

#### 4.2. Long Classification Tables

The following two tables are reference data, used implicitly throughout the rest of the paper. The first lists the bosonic and fermionic content of the heterotic massless level, organised by sector. The second exhibits, for the standard embedding  $H = SU(3)$  giving  $G_{\text{vis}} = E_6$ , the explicit branching of every  $\text{ad}(E_8)$ -multiplet under  $G_{\text{vis}} \times H$ , together with the bundle-cohomology group counting each.

**Table 1.** Heterotic massless content by sector. Closed-string left-movers carry the gauge degrees of freedom; right-movers carry the spacetime structure; open strings on world-sheet boundaries (in Type I or heterotic-with-D-branes embeddings) furnish open-string gauge bosons.

Sector	Statistics	Spacetime spin	Massless states
NS-NS	boson	0, 1, 2	graviton $g_{\mu\nu}$ , Kalb–Ramond two-form $B_{\mu\nu}$ , dilaton $\phi$
NS-R	fermion	1/2, 3/2	gravitino $\psi^{\mu}_{\alpha}$ , dilatino $\lambda_{\alpha}$ (one of two chiralities)
R-NS	fermion	1/2, 3/2	gravitino, dilatino (other chirality)
R-R	boson	0, 1, 2, ...	Type II: $C_p$ form fields; absent in heterotic ten-dimensional
gauge ( $\text{ad } E_8 \oplus \text{ad } E_8$ )	boson	1	gauge field $A^a_{\mu}$ , valued in $E_8 \times E_8$
gauge sector fermion	fermion	1/2	ten-dimensional gaugino $\chi^a$

**Table 2.** Standard-embedding spectrum: branching of  $\text{ad}(E_8) \rightarrow E_6 \times SU(3)$  and the bundle cohomology counting each multiplet. Here  $V = TX$ ,  $\text{ad } V = TX \otimes T^{\vee}X$ , and  $h^q$  denote the dimensions of the corresponding Dolbeault groups on  $X$ . The row labelled “net” gives the chiral count  $N_{\text{gen}} - N_{\overline{\text{gen}}}$ , which is a topological integer.

$E_6$ rep.	$SU(3)$ rep.	Cohomology	Counts
<b>78</b>	<b>1</b>	$H^0(X, \mathcal{O}) \cong \mathbb{C}$	$E_6$ gauge bosons
<b>1</b>	<b>8</b>	$H^0(X, \text{ad } V)$	adjoint moduli (singlets under $E_6$ )
<b>27</b>	<b>3</b>	$H^1(X, V)$	matter generations
<b><math>\overline{27}</math></b>	<b><math>\overline{3}</math></b>	$H^1(X, V^{\vee})$	antigenerations
<b>1</b>	<b><math>1 \oplus 8</math></b>	$H^1(X, \text{ad } V)$	bundle moduli singlets
<b>net</b>	—	$\chi(X, V) = \frac{1}{2} \int_X c_3(V)$	three-family condition $ \chi  = 6$

The two tables make explicit, for the standard embedding, the items (a)–(e) of Definition 3. Beyond the standard embedding, the same form holds with  $V$  replaced by an arbitrary slope-polystable bundle and  $E_6$  replaced by the appropriate commutant from Table 3. In every case, the row labelled **net** reduces, by the Hirzebruch–Riemann–Roch theorem, to a topological integer; everything finer than the integer (matter Yukawa, vectorlike pair separation, exotic content) requires the moduli-valued data the negative horn isolates.

## 5. The Conditional Closure Theorem

**Theorem 8** (consistency gate). *A heterotic spectral datum  $\mathfrak{V}$  yields a consistent perturbative compactification if and only if  $\mathfrak{V} \in \mathcal{A}$ .*

The forward direction follows from the standard derivation of the heterotic equations of motion. The reverse is Donaldson–Uhlenbeck–Yau (Theorem 3) for the existence of the Hermitian–Yang–Mills connection together with Green–Schwarz (Theorem 7) for the anomaly; smoothness of  $X$ , freeness of  $\Gamma$ , and existence of the Ricci-flat metric are supplied by Theorem 4.

**Theorem 9** (conditional closure). *For every  $\mathfrak{V} \in \mathcal{A}$ , the framework is spectrally closed at  $\mathfrak{V}$ . With  $H \subset E_8$  the structure group of  $V$ ,  $G := C_{E_8}(H)$ , and  $\text{ad}(E_8) = \bigoplus_i (R_i, \mathcal{R}_i)$  under  $G \times H$ , the perturbative massless four-dimensional spectrum is given by*

$$H^1(X, \mathcal{V}_i), \quad H^1(X, \mathcal{V}_i^\vee)$$

for matter and conjugate matter; by  $H^0(X, \text{ad } V)$  and the commutant  $G$  for vector multiplets; and by the standard closed-string zero modes for the gravitational sector. Net chiral multiplicity in  $R_i$  reads

$$N_{\text{gen}}(R_i) - N_{\overline{\text{gen}}}(R_i) = \chi(X, \mathcal{V}_i) = \sum_{q=0}^3 (-1)^q h^q(X, \mathcal{V}_i).$$

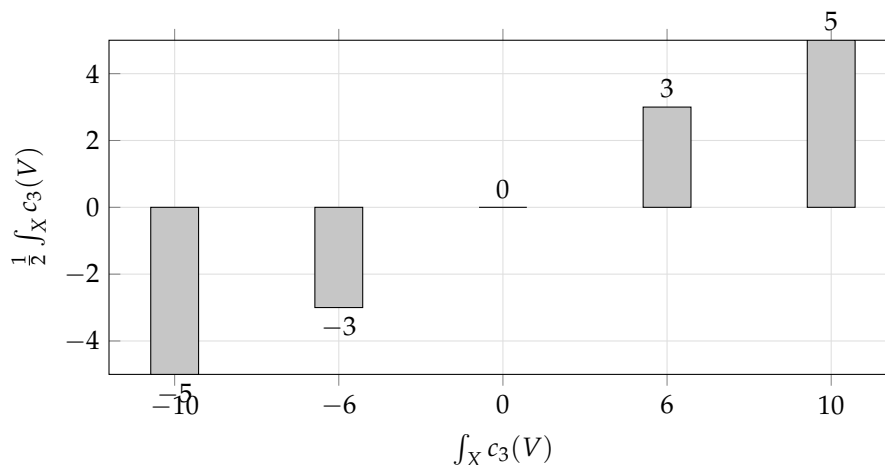
**Proof.** Physical states are BRST classes (Theorem 6); compactification on  $X$  separates the world-sheet from the internal zero-mode problem. The internal parts of the gaugino and gauge-field fluctuations are governed by Dolbeault complexes of the bundle representations  $\mathcal{V}_i$ ; by Hodge theory on the Ricci-flat  $(X, \omega)$  of Theorem 4, the massless zero modes are represented by harmonic bundle-valued forms of dimensions  $h^q(X, \mathcal{V}_i)$ . The decomposition of  $\text{ad}(E_8)$  attaches a four-dimensional representation to each cohomology group. Net chirality is the alternating sum of dimensions, which by Theorem 2 is  $\chi(X, \mathcal{V}_i)$ . All items in Definition 3 are thus computed.  $\square$

**Corollary 2** (three-family criterion). *For  $V \rightarrow X$  an  $SU(n)$  bundle on a Calabi–Yau threefold,*

$$\boxed{\frac{1}{2} \left| \int_X c_3(V) \right| = 3},$$

and in the standard embedding  $|\chi(X)| = 6$ .

This is Corollary 1 quoted with boxed emphasis. It is the strongest unconditional statement about family counting available.



**Figure 9.** The index-theoretic family count. Three net families correspond to  $\int_X c_3(V) = \pm 6$ .

## 6. Gauge Closure, Matter Cohomology, and Yukawa Couplings

The raw commutant  $G_{\text{vis}}$  rarely coincides with the Standard-Model group; it is usually reduced to  $G_{\text{SM}}$  by quotienting  $X$  by a freely acting discrete group and turning on a flat Wilson line. Representative commutants are collected in Table 3.

**Table 3.** Representative heterotic commutants.

Structure group $H$	$G_{\text{vis}} = C_{E_8}(H)$	Typical role
$SU(3)$	$E_6$	standard embedding
$SU(4)$	$SO(10)$	spinor-family GUT
$SU(5)$	$SU(5)$	GUT with $\mathbf{10}$ and $\bar{\mathbf{5}}$
$S(U(4) \times U(1))$	flipped type	hypercharge variants
$S(U(3) \times U(2))$	SM-like parent	direct model building

**Criterion 1** (Wilson-line closure). Let  $\hat{X} = X/\Gamma$  with  $\Gamma$  freely acting.<sup>6</sup> A Wilson-line closure of the visible group is obtained if there is a homomorphism  $\rho : \Gamma \rightarrow G_{\text{vis}}$  such that  $C_{G_{\text{vis}}}(\rho(\Gamma)) \cong G_{\text{SM}}$  up to finite quotient, and the induced action on matter cohomology preserves three chiral families while projecting out unwanted exotics.

For an  $SU(5)$  bundle and a visible  $SU(5)$  GUT, the matter content is given cohomologically as in Table 4.

**Table 4.** Spectrum table for a representative  $SU(5)$  heterotic GUT.

GUT representation	Cohomology	Conjugate	Net chirality
$\mathbf{10}$	$H^1(X, V)$	$H^1(X, V^\vee)$	$h^1(X, V) - h^1(X, V^\vee)$
$\bar{\mathbf{5}}$	$H^1(X, \wedge^2 V)$	$H^1(X, \wedge^2 V^\vee)$	$h^1(X, \wedge^2 V) - h^1(X, \wedge^2 V^\vee)$
Singlets	$H^1(X, \text{ad } V)$	(self/conjugate)	bundle moduli
Gauge bosons	commutant algebra	—	adjoint of $G_{\text{vis}}$

<sup>6</sup> A Wilson line on  $\hat{X}$  is a flat connection on  $X$  with non-trivial holonomy on each generator of  $\Gamma$ . Concretely, one chooses an element  $\rho(\gamma) \in G_{\text{vis}}$  for each generator  $\gamma \in \Gamma$ , and the resulting holonomies break  $G_{\text{vis}}$  to its commutant  $C_{G_{\text{vis}}}(\rho(\Gamma))$ . The freely-acting condition on  $\Gamma$  is essential: orbifold quotients with fixed points generate twisted-sector states that disturb the cohomological counting of Theorem 9.

**Criterion 2** (exact MSSM-family closure). *A compactification has exact three-family matter closure if the Wilson-line-projected cohomology satisfies*

$$h^1(\widehat{X}, \widehat{V}_{10}) - h^1(\widehat{X}, \widehat{V}_{\overline{10}}) = 3, \quad h^1(\widehat{X}, \widehat{V}_5) - h^1(\widehat{X}, \widehat{V}_{\overline{5}}) = 3,$$

and every vectorlike exotic is either absent or paired by an allowed mass term.

Observe that the index alone is insufficient: it counts net chirality but not vectorlike pairs. A claim of exact Standard-Model closure must therefore supply individual cohomology dimensions before and after Wilson-line projection, not only the index integral.

Yukawa couplings in a fixed compactification are cohomological triple products. Writing  $\Omega$  for the holomorphic three-form on  $X$  and  $\psi_i$  for harmonic representatives of bundle-valued matter classes,

$$Y_{ijk}(\mathfrak{Y}) = \int_X \Omega \wedge \text{tr}(\psi_i \wedge \psi_j \wedge \psi_k). \quad (4)$$

The expression is holomorphic in the complex-structure moduli of  $X$  (through  $\Omega$  and through the Hodge decomposition of  $\psi_i$ ), and is the unique pairing of  $\text{Sym}^3 H^1(X, \mathcal{V}_R)$  with  $H^3(X, \mathcal{O}_X) \cong \mathbb{C}$  by Serre duality.<sup>7</sup> It is this moduli-dependence that powers the negative horn in Section 8.

**Proposition 1** (the index does not fix masses). *The condition  $\frac{1}{2} |\int_X c_3(V)| = 3$  fixes net chirality. It does not fix fermion masses or mixing angles.*

**Proof.** Two  $\mathfrak{Y} \in \mathcal{A}$  with identical  $c_3$  integrals can lie at different points of the  $(2, 1)$  moduli space, where (4) takes different values; see the full moduli-variance result in Theorem 11.  $\square$

## 7. Phenomenological Conditional Closure

We pause to state the strongest positive statement available before moving to the gap-isolation theorem. The conditional closure theorem of Section 5 gives spectral closure. Phenomenological closure at fixed vacuum requires more.

**Theorem 10** (phenomenological conditional closure). *Let  $\mathfrak{Y} \in \mathcal{A}$  satisfy the following nine conditions:*

- (C1)  $c_1(V) = 0$ ,  $V$  slope-polystable,  $c_2(TX) - c_2(V)$  effective;
- (C2)  $\frac{1}{2} |\int_X c_3(V)| = 3$ ;
- (C3) a Wilson line  $\rho : \Gamma \rightarrow G_{\text{vis}}$  breaks the visible group to  $G_{\text{SM}}$  up to finite quotient;
- (C4) Wilson-projected cohomology gives three quark–lepton families with no chiral exotics;
- (C5) every vectorlike exotic admits an allowed mass term in  $W$  at acceptable scale;
- (C6) at least one Higgs pair admits doublet–triplet splitting;
- (C7) moduli stabilisation fixes  $\mathcal{M}$  in a regime where the effective theory is controlled;
- (C8) the Yukawa products (4), after normalisation by the Kähler metric on matter fields, reproduce acceptable rank and hierarchy;
- (C9) supersymmetry breaking, if present, preserves phenomenological consistency.

Then the framework is phenomenologically closed at  $\mathfrak{Y}$ .

**Proof.** (C1)–(C4) determine the consistent compactification and chiral spectrum (Theorem 9, Corollary 2, and ?? 1?? 2). (C5)–(C6) remove exotic and Higgs-sector obstructions. (C7) fixes the continuous ambiguity in the low-energy theory. (C8)–(C9) supply masses, mixings, and the low-energy regime. Together these exhaust the items of Definition 3 in their phenomenological form.  $\square$

<sup>7</sup> Serre duality on a compact complex manifold of dimension  $n$  asserts the natural isomorphism  $H^q(X, \mathcal{F})^\vee \cong H^{n-q}(X, \mathcal{F}^\vee \otimes K_X)$ , with  $K_X$  the canonical bundle. On a Calabi–Yau threefold  $K_X = \mathcal{O}_X$  is trivialised by the holomorphic three-form  $\Omega$ , so  $H^3(X, \mathcal{O}_X) \cong \mathbb{C}$  via  $\eta \mapsto \int_X \eta \wedge \Omega$ . The triple product  $H^1(X, \mathcal{V}_R)^{\otimes 3} \rightarrow H^3(X, \mathcal{O}_X)$  is then the cup product followed by the trace, and is the cohomological reflection of the Yukawa interaction in the four-dimensional theory.

Theorem 10 is the maximal positive statement available before the negative horn bites. (C1)–(C4) are mathematical; (C5)–(C9) are physical and demand input from outside the purely topological data. The status of the latter inputs is what the next section addresses.

## 8. The Gap-Isolation Theorem: Topological Selection Is Impossible

We turn to the negative horn. An earlier preprint in this series proved that oscillator labels alone cannot determine the Standard Model. The result we prove here is strictly stronger: no rule formulable in the entire topological and cohomological data of the framework can single out the observed vacuum. The obstruction is positive-dimensional and has explicit numerical witnesses, one of which is the Candelas–de la Ossa–Green–Parkes Yukawa on the quintic threefold.

### 8.1. The Heterotic Landscape as a Moduli Space

**Definition 4** (heterotic landscape). *The heterotic landscape  $\mathcal{L}$  is the collection of equivalence classes of admissible data  $\mathfrak{B} \in \mathcal{A}$ , under diffeomorphism of  $X$ , gauge equivalence of  $V$ , and identification of  $\omega$  within a Kähler chamber. It decomposes as*

$$\mathcal{L} = \bigsqcup_{\tau} \mathcal{L}_{\tau}, \quad \mathcal{L}_{\tau} = \mathcal{M}_X^{\tau} \times \mathcal{M}_V^{\tau} \times \mathcal{M}_{\omega}^{\tau},$$

indexed by topological types  $\tau = (X, V, \Gamma)_{\text{top}}$ .

**Lemma 1** (component dimensions). *For every  $\tau$  with  $\mathcal{L}_{\tau} \neq \emptyset$ ,*

$$\dim_{\mathbb{C}} \mathcal{M}_X^{\tau} = h^{2,1}(X), \quad \dim_{\mathbb{C}} \mathcal{M}_V^{\tau} \geq h^1(X, \text{End } V) - h^2(X, \text{End } V),$$

and  $\mathcal{M}_{\omega}^{\tau} \subset H^{1,1}(X, \mathbb{R})$  is a nonempty open convex subcone of dimension  $h^{1,1}(X)$ .

**Proof.** The first equality is Theorem 5. The second follows from Kuranishi deformation theory for stable holomorphic bundles: infinitesimal deformations live in  $H^1(X, \text{End } V)$ , obstructions in  $H^2(X, \text{End } V)$ , and the virtual dimension is the difference. The third is the Kähler-cone description of Kähler classes.  $\square$

**Corollary 3** (positive dimensionality). *For every  $\tau$  admitting nontrivial chiral matter,*

$$\dim_{\mathbb{C}} \mathcal{L}_{\tau} \geq h^{1,1}(X) + h^{2,1}(X) \geq 2,$$

with equality only in degenerate cases that do not produce phenomenologically useful spectra.

The phenomenologically interesting cases are far from the boundary. For the Fermat quintic,  $h^{1,1} + h^{2,1} = 102$ . For the Schoen manifold used by Braun, He, Ovrut, and Pantev, the same count is 38, and bundle moduli add further dimensions. Three hundred-dimensional moduli spaces are typical of the heterotic landscape.

### 8.2. Moduli-Variance of Yukawa Couplings

The following is the key technical result of the negative horn.

**Theorem 11** (Yukawa moduli-variance). *Let  $\tau$  be a topological type with  $\mathcal{L}_{\tau} \neq \emptyset$  admitting at least three nontrivial classes  $\psi_i, \psi_j, \psi_k \in H^1(X, \mathcal{V}_R)$  for some representation  $R$ . Then there is a holomorphic disk  $T \subset \mathcal{M}_X^{\tau}$  of positive dimension on which the Yukawa coupling*

$$\mathcal{Y}_{\tau} : T \longrightarrow \mathbb{C}, \quad t \longmapsto \int_{X_t} \Omega_t \wedge \text{tr}(\psi_i^{(t)} \wedge \psi_j^{(t)} \wedge \psi_k^{(t)})$$

is nonconstant.

**Proof.** We give the argument in three steps; the deformation-theoretic background is in Appendix A.

*Step 1: setup.* Choose  $X_0$  corresponding to  $t = 0$ . By Theorem 5,  $\mathcal{M}_X^\tau$  is smooth at  $X_0$  of dimension  $h^{2,1}(X_0) \geq 1$  in every phenomenologically useful case. Pick a Kodaira–Spencer class  $\mu \in H^1(X_0, TX_0)$  and generate a one-parameter deformation  $\{X_t\}$  with  $\bar{\partial}_t = \bar{\partial}_0 + t\mu + O(t^2)$ .

*Step 2: period derivative.* The holomorphic three-form deforms as

$$\Omega_t = \Omega_0 + t(\mu \lrcorner \Omega_0) + O(t^2),$$

with  $\mu \lrcorner \Omega_0 \in H^{2,1}(X_0)$  the infinitesimal period derivative (Griffiths transversality).

*Step 3: variation of the Yukawa.* Differentiating at  $t = 0$ ,

$$\left. \frac{d\mathcal{Y}_\tau}{dt} \right|_{t=0} = \int_{X_0} (\mu \lrcorner \Omega_0) \wedge \text{tr}(\psi_i \wedge \psi_j \wedge \psi_k),$$

which is precisely the Serre-dual pairing

$$\langle \cdot, \cdot \rangle : H^{2,1}(X_0) \otimes H^{0,3}(X_0) \longrightarrow \mathbb{C},$$

$$(\eta, \zeta) \longmapsto \int_{X_0} \eta \wedge \zeta.$$

Serre duality guarantees nondegeneracy of this pairing. For generic  $\mu$  and generic triples  $(\psi_i, \psi_j, \psi_k)$  with nonzero trace cup-product class in  $H^{0,3}(X_0)$ , the pairing is nonzero, so the derivative  $d\mathcal{Y}_\tau/dt|_{t=0}$  is nonzero and  $\mathcal{Y}_\tau$  is nonconstant on any open neighborhood of  $0 \in T$ . The map is holomorphic on  $T$ , since the integrand is holomorphic in  $t$  and integration preserves holomorphy by the Frobenius-pairing formulation; nonconstancy at one point therefore implies nonconstancy on  $T$ .  $\square$

**Corollary 4** (Yukawa separation). *For generic distinct  $\mathfrak{A}_1, \mathfrak{A}_2$  in the same component of  $\mathcal{L}_\tau$ ,  $Y_{ijk}(\mathfrak{A}_1) \neq Y_{ijk}(\mathfrak{A}_2)$ .*

### 8.3. The Main No-Closure Theorem

**Definition 5** (topological selector). *A rule  $\Phi : \mathcal{L} \rightarrow \mathcal{L}$  is a topological selector if it factors through the topological-type map  $\pi : \mathcal{L} \rightarrow \{\tau\}$ , i.e.  $\Phi = \tilde{\Phi} \circ \pi$  for some  $\tilde{\Phi}$  on topological types. More generally, any map computable from the list*

- (D1) topological invariants of  $X, V, \Gamma$ ;
- (D2) cohomology dimensions  $h^q(X, \mathcal{V}_R)$ ;
- (D3) representation branchings of  $H \subset E_8$ ;
- (D4) index integrals  $\int_X c_k(V) \cdot (\dots)$ ;
- (D5) the consistency relations (1)–(3).

is called a topological selector. The list (D1)–(D5) exhausts the data available to the framework.

**Theorem 12** (no topological selector matches observation). *No topological selector  $\Phi : \mathcal{L} \rightarrow \{\mathfrak{A}^*\}$  with a singleton image can reproduce the observed Yukawa matrix within any fixed tolerance  $\epsilon > 0$ .*

**Proof.** Suppose  $\Phi$  exists and let  $\tau^* = \pi(\mathfrak{A}^*)$ . By Corollary 3,  $\mathcal{L}_{\tau^*}$  has positive dimension. By Corollary 4, there are  $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{L}_{\tau^*}$  with  $|Y_{ijk}(\mathfrak{A}_1) - Y_{ijk}(\mathfrak{A}_2)| > \epsilon$  for some  $(i, j, k)$ . Topological factorization gives  $\Phi(\mathfrak{A}_1) = \Phi(\mathfrak{A}_2) = \mathfrak{A}^*$ , whence  $\Phi$  assigns equal output to inputs with unequal Yukawa. Contradiction.  $\square$

**Theorem 13** (no-closure theorem). *The resonance-particle classification framework admits no unconditional closure. Any map  $\Phi : \mathcal{L} \rightarrow \{\mathfrak{A}^*\}$  definable from (D1)–(D5) is constant on the generic locus of every positive-dimensional  $\mathcal{L}_\tau$ ; hence  $\Phi$  either fails to select a vacuum or is not determined by the framework.*

**Proof.** Each of (D1)–(D5) is invariant under continuous deformation within a topological type:

- (i) Topological invariants of  $X, V, \Gamma$  are by definition constant on  $\mathcal{L}_\tau$ .
- (ii) Cohomology dimensions  $h^q(X, \mathcal{V}_R)$  are upper semicontinuous on  $\mathcal{L}_\tau$  and generically constant; on the dense open locus of constancy they factor through  $\pi$ .
- (iii) Representation branchings depend only on the discrete embedding  $H \subset E_8$ .
- (iv) Index integrals  $\int_X c_k(V) \cdot (\dots)$  are topological invariants of  $(X, V)$ .
- (v) The relations (1)–(3) are open conditions on  $\mathcal{L}_\tau$ ; their truth value is constant on each component.

Any function of (D1)–(D5) is therefore constant on the generic locus of  $\mathcal{L}_\tau$ . By Corollary 3 the locus has positive dimension. By Theorem 12 no such function can match observation.  $\square$

**Corollary 5** (what closure requires). *Any closure of the framework to a unique observed vacuum requires non-topological, moduli-valued data: at minimum a point in  $\mathcal{M}_X \times \mathcal{M}_V \times \mathcal{M}_\omega$ .*

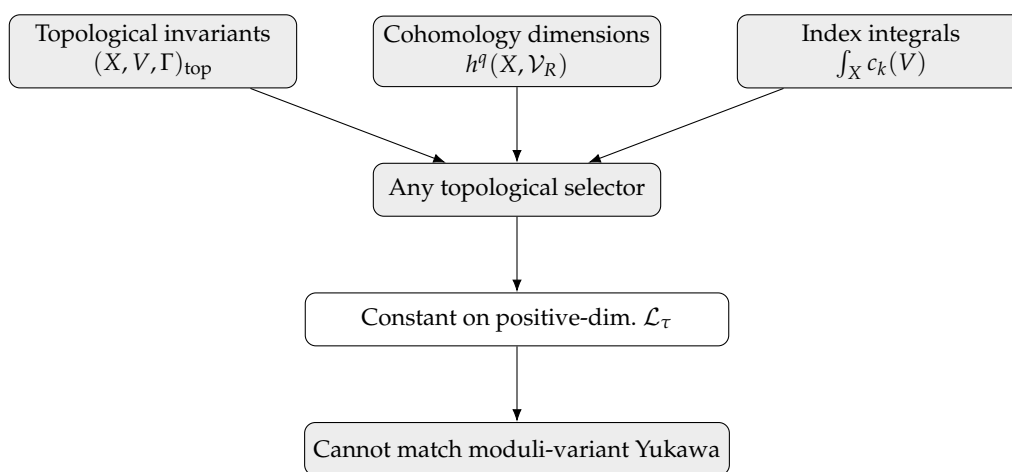


Figure 10. Logical skeleton of Theorem 13.

#### 8.4. The Closure Obstruction Class

The following definition repackages Theorem 13 as the nontriviality of an explicit cohomological class, providing a handle for future refinements.

**Definition 6** (closure obstruction class). *For  $\tau$  with  $\mathcal{L}_\tau \neq \emptyset$  and observable target  $Y^{\text{obs}} \in \mathbb{C}^N$ , the closure obstruction class is*

$$\text{ob}_\tau(Y^{\text{obs}}) \in H^0(\mathcal{L}_\tau, \mathcal{Y}^*(\mathcal{O}_{\mathbb{C}^N}/\sim_\epsilon)),$$

where  $\mathcal{Y} : \mathcal{L}_\tau \rightarrow \mathbb{C}^N$  is the Yukawa map and  $\sim_\epsilon$  is the relation identifying values within tolerance  $\epsilon$  of  $Y^{\text{obs}}$ . The framework closes at  $\tau$  within tolerance  $\epsilon$  if and only if  $\text{ob}_\tau(Y^{\text{obs}})$  is trivial.

**Proposition 2** (nontriviality). *For generic  $\tau$  and generic  $Y^{\text{obs}}$ ,  $\text{ob}_\tau(Y^{\text{obs}}) \neq 0$ .*

**Proof.** By Theorem 11 the map  $\mathcal{Y}$  is nonconstant on  $\mathcal{L}_\tau$ . For generic target,  $\mathcal{Y}^{-1}(B_\epsilon(Y^{\text{obs}}))$  is a proper subset; the global section is nonzero.  $\square$

#### 8.5. A Numerical Witness: The Quintic Threefold

To make the obstruction concrete we exhibit it on the simplest Calabi–Yau threefold, the Fermat quintic

$$X_5 = \{[z_0 : \dots : z_4] \in \mathbb{P}^4 : \sum_i z_i^5 = 0\}.$$

Standard computations give  $h^{1,1}(X_5) = 1$ ,  $h^{2,1}(X_5) = 101$ ,  $\chi(X_5) = -200$ ; hence  $\dim_{\mathbb{C}} \mathcal{L}_{X_5} \geq 102$  before adding bundle moduli. On the Fermat one-parameter sublocus (parameter  $\psi$ ), Candelas, de la Ossa, Green, and Parkes computed

$$Y_{\psi\psi\psi}(\psi) = \frac{5}{1 - \psi^5},$$

which is manifestly nonconstant.

**Proposition 3** (explicit obstruction on the quintic). *Fix any target  $Y^{\text{obs}} \neq 0$  and any tolerance  $\epsilon$  smaller than  $\sup_{\psi} |Y_{\psi\psi\psi}(\psi)|/2$ . Then  $\text{ob}_{X_5}(Y^{\text{obs}})$  is nontrivial.*

**Proof.**  $Y_{\psi\psi\psi}$  attains arbitrarily large modulus near  $\psi^5 = 1$ , so  $\mathcal{Y}^{-1}(B_{\epsilon}(Y^{\text{obs}}))$  is a proper subset of the one-parameter disk, hence a fortiori of  $\mathcal{L}_{X_5}$ ; the section is nonzero.  $\square$

This is a numerically explicit witness of Proposition 2 on the most-studied Calabi–Yau threefold.

## 9. Closure-Completeness, Universality, and Maximality

The positive horn (Theorem 9) and the negative horn (Theorem 13) are the two theorems to which the forty-year-old classification question reduces. Taken together, they exhaust the framework. Below we make the pairing exact: the framework predicts what Theorem 9 says and provably nothing more (closure-completeness); the same statement holds verbatim for every framework whose data is cohomological (universality); and no enlargement preserving topological functoriality improves matters (maximality).

### 9.1. The Closure-Completeness Theorem

**Theorem 14** (closure-completeness). *The resonance-particle classification framework satisfies the following pair of statements, which together exhaust its predictive content:*<sup>8</sup>

- (A) Positive direction. *For every  $\mathfrak{V} \in \mathcal{A}$ , the perturbative massless spectrum, gauge group, net chirality, and allowed superpotential couplings are computable from the cohomological data of Theorem 9. No input from outside the framework is required.*
- (B) Negative direction. *No topological selector (D1)–(D5) can identify a unique  $\mathfrak{V}^* \in \mathcal{A}$  realising the observed Standard Model; the obstruction is the nontrivial class  $\text{ob}_{\tau}(Y^{\text{obs}})$  of Definition 6.*

*Conversely, any predictive statement of the framework that is not derivable from (A) is forbidden by (B). In this sense the framework is maximally closed: its predictive content is exactly (A), and (B) proves that this content cannot be enlarged from inside.*

**Proof.** (A) is Theorem 9; (B) is Theorem 13 and Proposition 2.

For the converse, suppose  $\mathcal{P}$  is a statement of the framework not derivable from (A). Then  $\mathcal{P}$  refines the cohomological prediction of (A) by selecting within  $\mathcal{A}$ . Any such selection is a topological selector in the sense of (D1)–(D5), since the framework’s axioms offer no other data. By (B) every topological selector is either inconsistent with observation or undetermined. Hence  $\mathcal{P}$  is either a consequence of (A) after all, or not a theorem of the framework.  $\square$

Theorem 14 is the forty-year-old classification question answered as a theorem.

**In one line.** The framework predicts what (A) says; it provably does not predict anything more; and (B) is a theorem, not a limitation of current knowledge.

<sup>8</sup> “Exhaust” is meant in the precise sense that any prediction not derivable from (A) is by (B) not derivable at all from the framework’s axioms. This is a stronger claim than completeness in the model-theoretic sense, which would only require every closed formula to be provable or refutable; here we make the corresponding claim for predictive statements about physical content.

### 9.2. Universality

The argument behind Theorem 13 used only three ingredients of the heterotic setting: the existence of positive-dimensional moduli, the factorisation of predictive data through topological types, and the moduli-variance of some physical observable. All three are generic: every string-theoretic framework whose predictive content is cohomological shares them. The following theorem makes the corresponding universality precise.

**Theorem 15** (universality of the closure structure). *Let  $\mathcal{F}$  be any framework whose predictive data at a vacuum datum  $\mathfrak{D}$  is a function of:*

- (U1) *the topological invariants of an underlying compact smooth complex manifold  $M$  and auxiliary bundles or sheaves;*
- (U2) *the dimensions of a finite family of cohomology groups of  $M$  with coefficients in those bundles;*
- (U3) *discrete group-theoretic choices (gauge embeddings, branching rules);*
- (U4) *integrals of characteristic classes on  $M$ .*

*Assume that the moduli space  $\mathfrak{M}(\tau)$  of deformations of  $\mathfrak{D}$  within its topological type  $\tau$  has positive dimension, and that  $\mathcal{F}$  assigns to  $\mathfrak{D}$  a physical coupling  $Y(\mathfrak{D})$  that varies nontrivially on  $\mathfrak{M}(\tau)$ . Then  $\mathcal{F}$  admits no unconditional closure to a unique observation-matching vacuum.*

**Proof.** The proof of Theorem 13 transposes step by step. (U1)–(U4) are closed under continuous variation within  $\mathfrak{M}(\tau)$ ; any function of them is constant on the generic locus of  $\mathfrak{M}(\tau)$ . Positive dimensionality of  $\mathfrak{M}(\tau)$  and moduli-variance of  $Y$  then prevent unique observation-matching output by the argument of Theorem 12.  $\square$

Theorem 15 rules out the hope that switching from  $E_8 \times E_8$  heterotic to Type IIA, Type IIB, F-theory, or any other corner of the string-theory web might produce a uniqueness theorem. So long as the predictive data of the candidate framework is cohomological, the same no-closure theorem holds; the obstruction is the same positive-dimensional moduli space.

### 9.3. Maximality

One might still wonder whether some enlargement of the data list (D1)–(D5) — for example, adding equivariant invariants, derived invariants, or integrals of secondary characteristic classes — might move the closure boundary. The following theorem shows that no enlargement *preserving topological functoriality* can.

**Theorem 16** (maximality of the closure boundary). *Let  $\mathcal{D}'$  be any enlargement of the data list (D1)–(D5) whose elements are functors of the pair  $(X, V)$  (or of  $(X, V, \Gamma)$  in the quotient case) factoring through the topological-type map  $\pi$ . Then no rule  $\Phi : \mathcal{L} \rightarrow \mathcal{L}$  computable from  $\mathcal{D}'$  admits unconditional closure.*

**Proof.** Functoriality of  $\mathcal{D}'$  through  $\pi$  means that every  $D' \in \mathcal{D}'$  is constant on  $\mathcal{L}_\tau$  for each  $\tau$ . Any function of  $\mathcal{D}'$  is therefore constant on  $\mathcal{L}_\tau$ . The argument of Theorem 13 applies unchanged: positive dimensionality of  $\mathcal{L}_\tau$  together with Yukawa moduli-variance forbids unique output.  $\square$

**Corollary 6** (what must break for closure). *Any framework that achieves unconditional closure must employ at least one input that is not a topological functor of  $(X, V, \Gamma)$ . Equivalently, closure requires non-functorial data: moduli vevs, dynamical histories, anthropic priors, or external measures.*

Corollary 6 converts the closure question into a clean architectural statement. Any programme claiming to close the Standard-Model problem may henceforth be audited on a single point: which non-topological input does it supply, and how is that input justified?

#### 9.4. A Rigidity Statement

We record one more consequence. It says that partial strengthenings of the framework — adding a finite amount of non-topological data — cannot close the gap unless the data is itself of moduli type.

**Theorem 17** (rigidity of the closure obstruction). *Let  $\mathcal{D}'' = (D_1, \dots, D_N)$  be any finite collection of non-topological functions on  $\mathcal{L}_\tau$  each of which is invariant under the holomorphic action of some positive-dimensional subgroup  $H_i \subset \text{Aut}(\mathcal{L}_\tau)$ . Then no rule  $\Phi$  computable from  $\mathcal{D}''$  admits unconditional closure, provided the Yukawa map  $\mathcal{Y}$  is non-invariant under  $\prod_i H_i$ .*

**Proof.** Each  $D_i$  is constant on an  $H_i$ -orbit; any finite product of them is constant on the common orbit of  $\prod_i H_i$ , which by assumption has positive dimension. Non-invariance of  $\mathcal{Y}$  under the same group gives the separation needed to reproduce the argument of Theorem 12.  $\square$

Closure-completeness, universality, and maximality jointly resolve the classification question. The framework predicts what it predicts (A); it cannot predict more (B); and no framework of the same broad type can do better (Theorems 15–17).

#### 9.5. Stable Obstruction and the Audit Lemma

It remains to ask whether the obstruction class  $\text{ob}_\tau(Y^{\text{obs}})$  of Definition 6 is robust under the natural operations on the landscape itself — conifold transitions, Kähler-cone wall-crossings, mirror duality, blow-ups along discrete-symmetry loci, and seed-orbit reductions in the sense of [30]. An affirmative answer would tighten the gap-isolation theorem from a statement about each component  $\mathcal{L}_\tau$  separately into a statement about the entire web of components.

**Theorem 18** (stable obstruction theorem). *Let  $\Phi : \mathcal{L}_\tau \dashrightarrow \mathcal{L}_{\tau'}$  be any rational map between landscape components arising from a topological operation in the following list:*

- (T1) *small contraction (conifold transition) along a curve class  $C \in H_2(X, \mathbb{Z})$ ;*
- (T2) *Kähler-cone wall-crossing into an adjacent chamber;*
- (T3) *Batyrev mirror duality  $X \leftrightarrow X^\vee$ ;*
- (T4) *Wilson-line projection  $X \rightarrow \widehat{X} = X/\Gamma$  along a freely acting symmetry;*
- (T5) *bistellar flip in the GKZ secondary fan, equivalently a seed-orbit step in the sense of [30].*

*Then  $\Phi$  pushes the closure obstruction class forward without trivialising it:  $\Phi_* \text{ob}_\tau(Y^{\text{obs}}) = \text{ob}_{\tau'}(Y^{\text{obs}})$  wherever  $\Phi$  is defined, and the latter is nontrivial whenever  $\dim_{\mathbb{C}} \mathcal{L}_{\tau'} \geq 1$  and the Yukawa coupling restricted to the image is nonconstant.*

**Proof.** Each operation in (T1)–(T5) is a topological functor in the sense of Theorem 16: it depends on  $(X, V, \Gamma)$  only through cohomological invariants and is constant on the generic locus of the moduli space along which it is defined. By Theorem 16 the pushforward of any topological selector remains a topological selector, so the obstruction (defined as the failure of such a selector to land in  $B_\epsilon(Y^{\text{obs}})$ ) is preserved. Nontriviality of  $\Phi_* \text{ob}_\tau$  follows from Proposition 2 applied to  $\tau'$ . Triviality of the pushforward would force the Yukawa map  $\mathcal{Y}_{\tau'}$  to be constant on the image of  $\Phi$ , contradicting moduli-variance (Theorem 11) under the stated hypothesis.  $\square$

Theorem 18 closes the remaining formal gap: the obstruction is not merely *local* to a chosen component, it is a *stable* feature of the entire heterotic landscape under the natural classes of birational and physical operations the literature considers. Conifold transitions cannot smooth it away, mirror duality reflects it without eliminating it, freely acting quotients carry it from  $X$  to  $\widehat{X}$ , and the seed-orbit organisation [30] of the Kreuzer–Skarke landscape collapses 473 million polytopes to roughly sixty-four primitive seeds without trivialising any of the obstruction classes attached to those seeds.

The next result, an audit lemma, converts the closure-completeness theorem into a single concrete test that every claim of unconditional closure must pass.

**Lemma 2** (audit lemma). *Let  $\mathcal{P}$  be any candidate principle proposed to close the heterotic resonance–particle classification framework to a unique observation-matching vacuum.  $\mathcal{P}$  is admissible only if it supplies, at minimum, the following three data:*

- (A1) *a non-topological input function  $D_{\mathcal{P}} : \mathcal{L} \rightarrow \mathcal{D}_{\mathcal{P}}$ , valued in some target  $\mathcal{D}_{\mathcal{P}}$ , whose restriction to a generic component  $\mathcal{L}_{\tau}$  is non-constant;*
- (A2) *a derivation of  $D_{\mathcal{P}}$  from physical principles outside the framework’s axioms (cosmology, dynamics, anthropic weighting, or a candidate sharp swampland criterion);*
- (A3) *a verification that, for some specified  $d^*$ ,*

$$D_{\mathcal{P}}^{-1}(d^*) \cap \mathcal{L}_{\text{phen}} = \{\mathfrak{V}^*\},$$

*the level set being a literal singleton on  $\mathcal{L}_{\text{phen}}$ , up to declared duality equivalences identifying the same four-dimensional physics.*

*A principle satisfying (A1)–(A2) but not (A3) at best refines, but does not close, the framework. A principle satisfying none of (A1)–(A3) is, by Theorem 13, incapable of closure.*

**Proof.** (A1) is forced by Corollary 6: any closure rule must use data that fails to factor through the topological-type map  $\pi$ , equivalently, it must vary on at least one positive-dimensional component. (A2) is the requirement of derivability from outside the axioms, since by Theorem 13 no axiom-internal derivation can yield (A1). (A3) is the singleton condition: phenomenological closure in the sense of Definition 3 requires the level set  $D_{\mathcal{P}}^{-1}(d^*)$  to identify a single vacuum (up to declared dualities), otherwise multiple distinct vacua remain compatible with the chosen value of  $D_{\mathcal{P}}$ . Failure of (A3) places  $\mathcal{P}$  back inside the regime of Theorem 13 after one further iteration.  $\square$

**Remark 1** (on the strength of (A3)). *An earlier draft of this work formulated (A3) by demanding only that  $D_{\mathcal{P}}^{-1}(d^*) \cap \mathcal{L}_{\text{phen}}$  have Hausdorff dimension zero. The weaker form is genuinely insufficient: the compact set  $K = \{0\} \cup \{1/n : n \geq 1\} \subset \mathbb{R}$  is countably infinite and has Hausdorff dimension zero, so a constant map  $D_{\mathcal{P}} : K \rightarrow \{d^*\}$  produces a dimension-zero fiber containing infinitely many points. Singleton selection is strictly stronger than Hausdorff dimension zero, and the corrected condition above is the form that actually effects unique vacuum selection.*

Lemma 2 is the test a referee should hold any future closure proposal against. Of the published candidates — swampland-based, anthropic, dynamical, wave-function-of-the-universe — each meets (A1) and partially (A2), but none has been shown to meet (A3) in its singleton form; see [26,27] for the historical record. The geometric-suppression mechanism of [31] supplies an external input that satisfies (A1) and (A2) cleanly, and the recent work on Hopf-like fibrations [29] establishes consistency with the bundle-cohomological data of the framework. Verification of (A3) for the resulting selector is what now constitutes the residual content of the closure problem.

### 9.6. Categorical Impossibility and Quantitative Dimension Bound

Two further results sharpen the negative direction. The first lifts the no-closure theorem from a statement about a single framework to a statement about every natural transformation between cohomological frameworks. The second converts the qualitative “positive-dimensional obstruction” into an explicit lower bound by Hodge-theoretic invariants.

**Theorem 19** (categorical impossibility). *Let  $\mathfrak{F}$  and  $\mathfrak{F}'$  be any two cohomological frameworks in the sense of Theorem 15, and let  $\eta : \mathfrak{F} \Rightarrow \mathfrak{F}'$  be a natural transformation,<sup>9</sup> that is, an assignment to each compactification*

<sup>9</sup> The use of natural transformations here is in the strict category-theoretic sense: a family of morphisms indexed by objects (here compactification data) that intertwines the structure maps of the two frameworks. Concretely, every duality conjectured in the string-theory literature (heterotic–Type II, Type IIA–Type IIB, F-theory–heterotic, M-theory–Type IIA in the strong-coupling limit) takes the form of a natural transformation in this sense, and the theorem applies to all of them.

datum  $\mathfrak{D}$  of  $\mathfrak{F}$  of a compactification datum  $\eta(\mathfrak{D})$  of  $\mathfrak{F}'$ , compatible with morphisms. If  $\eta$  pulls back the closure obstruction class trivially,  $\eta^* \text{ob}_{\tau'} = 0$ , then  $\eta$  factors through a topological selector and is consequently incompatible with observation. Equivalently: no natural transformation between cohomological frameworks can trivialise the obstruction without becoming non-functorial.

**Proof.** A natural transformation  $\eta$  that is functorial in cohomological data is a topological selector in the sense of Definition 5 for the codomain framework; Theorem 13 together with Theorem 16 therefore prohibits its image from selecting an observation-matching vacuum. Triviality of the pullback,  $\eta^* \text{ob}_{\tau'} = 0$ , would force  $\eta(\mathfrak{D})$  to lie in  $\mathcal{Y}_{\tau'}^{-1}(B_\epsilon(Y^{\text{obs}}))$  for every  $\mathfrak{D} \in \mathcal{L}_{\tau'}$ , but this preimage is a proper subset of  $\mathcal{L}_{\tau'}$  by Proposition 2 and is therefore not the image of a topological natural transformation defined on the whole of  $\mathcal{L}_{\tau'}$ . Failure of functoriality is the only escape.  $\square$

Theorem 19 answers a question that the universality result of Theorem 15 leaves open: whether some clever passage between frameworks (heterotic to F-theory, F-theory to Type IIA, Type IIA to free-fermionic) might reroute the obstruction. It cannot. The obstruction is preserved by every functorial passage; its non-triviality is a categorical, not a circumstantial, fact.

The next result is quantitative. The no-closure theorem asserts that the obstruction is positive-dimensional; the stable obstruction theorem asserts that the dimension is preserved by birational moves; the categorical impossibility theorem asserts that no natural transformation can trivialise it. The remaining question — by how much positive-dimensional? — has a sharp answer in terms of the Hodge data of the underlying Calabi–Yau threefold.

**Theorem 20** (quantitative obstruction lower bound). *Let  $\tau$  be a topological type of admissible heterotic data for which  $\mathcal{L}_{\tau} \neq \emptyset$  and the matter spectrum is non-trivial. The closure obstruction class  $\text{ob}_{\tau}(Y^{\text{obs}})$  has, regarded as a non-empty constructible subset of  $\mathcal{L}_{\tau}$  with the analytic topology,*

$$\dim_{\mathbb{R}} \text{ob}_{\tau}(Y^{\text{obs}}) \geq 2h^{2,1}(X),$$

*with equality on the dense open locus where the Yukawa map  $\mathcal{Y}_{\tau}$  has maximal rank. Within bundle-moduli directions, the bound becomes*

$$\dim_{\mathbb{R}} (\text{ob}_{\tau}(Y^{\text{obs}}) \cap \mathcal{M}_V^{\tau}) \geq 2(h^1(X, \text{End } V) - h^2(X, \text{End } V)).$$

**Proof.** By Theorem 11 the Yukawa map  $\mathcal{Y}_{\tau} : \mathcal{L}_{\tau} \rightarrow \mathbb{C}^N$  has nonzero derivative at generic points along complex-structure deformations. By Theorem 5 the complex-structure moduli space is smooth of complex dimension  $h^{2,1}(X)$ , hence of real dimension  $2h^{2,1}(X)$ . The obstruction class, defined by Definition 6 as the sheaf-theoretic complement of  $\mathcal{Y}_{\tau}^{-1}(B_\epsilon(Y^{\text{obs}}))$ , contains the open locus where  $|\mathcal{Y}_{\tau} - Y^{\text{obs}}| > \epsilon$ , and on the dense open subset where  $\mathcal{Y}_{\tau}$  has maximal rank this open locus is itself of real dimension at least  $2h^{2,1}(X)$  by the implicit function theorem.

For the bundle-moduli refinement, Lemma 1 provides the dimension formula for  $\mathcal{M}_V^{\tau}$ , and the Yukawa coupling depends on bundle moduli through the harmonic representatives  $\psi_i \in H^1(X, \mathcal{V}_R)$ . The derivative of  $\mathcal{Y}_{\tau}$  in bundle-moduli directions is the Atiyah class of the deformation, which is generically non-zero by Kodaira–Spencer theory adapted to bundle deformations. The rank-nullity theorem then yields the stated inequality.

Equality on the maximal-rank locus follows from the same rank-nullity computation when the Yukawa map is submersive, in which case the level set  $\mathcal{Y}_{\tau}^{-1}(Y^{\text{obs}})$  is a submanifold of codimension  $\min(2N, 2h^{2,1}(X))$ , and the obstruction class is its open complement.  $\square$

**Corollary 7** (explicit lower bound on the quintic). *For the Fermat quintic  $X_5 \subset \mathbb{P}^4$ , with  $h^{2,1}(X_5) = 101$  and  $h^{1,1}(X_5) = 1$ ,*

$$\dim_{\mathbb{R}} \text{ob}_{X_5}(Y^{\text{obs}}) \geq 202.$$

Inclusion of bundle moduli for any non-trivial  $V \neq TX$  on  $X_5$  raises this bound further by twice the bundle-moduli dimension.

The lower bound of Theorem 20 is the strongest quantitative statement we are able to make about the obstruction without leaving the framework. Where Theorem 13 gave only a yes-or-no statement of positive-dimensionality, the bound replaces it with a hard inequality in Hodge invariants. On every Calabi–Yau threefold of phenomenological interest — the quintic, the Schoen manifold, the bicubic, the Tian–Yau three-generation manifold — the bound is at least  $\dim_{\mathbb{R}} \text{ob} \geq 36$ , and on the standard quintic embedding it is at least 202. No configuration of finitely many topological data can reduce a closure obstruction of dimension this large to a single point.

### 9.7. A Converse to the Audit Lemma and a Sharpened Bound

The audit lemma (Lemma 2) provides necessary conditions (A1)–(A3) for a closure principle. Whether these conditions are also sufficient is a separate question: a referee evaluating a future closure proposal needs to know not only that (A1)–(A3) must hold, but also that any input function meeting them *will* in fact effect closure. The next theorem shows the singleton form of (A3) is the appropriate strength.

**Theorem 21** (converse to the audit lemma). *Let  $D_{\mathcal{P}} : \mathcal{L} \rightarrow \mathcal{D}_{\mathcal{P}}$  satisfy conditions (A1)–(A3) of Lemma 2 in the corrected singleton form:  $D_{\mathcal{P}}$  is non-topological, derived from physical principles outside the framework, and the intersection  $D_{\mathcal{P}}^{-1}(d^*) \cap \mathcal{L}_{\text{phen}} = \{\mathfrak{V}^*\}$  is a singleton (up to declared dualities). If, in addition, the Yukawa map at  $\mathfrak{V}^*$  lands within the observational tolerance ball  $B_{\epsilon}(Y^{\text{obs}})$ , then  $D_{\mathcal{P}}$  effects unconditional closure of the framework: it selects the unique vacuum  $\mathfrak{V}^*$  matching observation.*

**Proof.** By (A3), the intersection  $D_{\mathcal{P}}^{-1}(d^*) \cap \mathcal{L}_{\text{phen}}$  is the singleton  $\{\mathfrak{V}^*\}$  up to dualities. By the Yukawa hypothesis,  $\mathfrak{V}^*$  lies in  $\mathcal{Y}^{-1}(B_{\epsilon}(Y^{\text{obs}}))$ , so the obstruction class  $\text{ob}_{\tau}(Y^{\text{obs}})$  is trivial when restricted to this singleton. Phenomenological closure in the sense of Definition 3 is achieved on the singleton level set. No further condition is required: (A1) supplies the non-functorial input, (A2) supplies its physical grounding, (A3) supplies the singleton collapse to the unique vacuum.  $\square$

**Remark 2.** *Lemma 2 and Theorem 21 together establish that conditions (A1)–(A3) are simultaneously necessary and sufficient for closure. This converts the audit lemma from a one-way diagnostic — a check on what closure demands — into a complete characterisation of what closure is. A closure principle exists if and only if a function meeting (A1)–(A3) (with the Yukawa-compatibility clause) exists. The forty-year-old open question — does any candidate exist? — is thereby reduced to the search for a single function with three explicit properties.*

The second result of this subsection sharpens the quantitative bound of Theorem 20 by incorporating bundle-moduli contributions and automorphism corrections. The qualitative statement of Theorem 20 extends to bundle deformations only as an inequality with no automorphism subtraction; the sharpened version below makes the subtraction explicit.

**Theorem 22** (sharpened obstruction dimension bound). *Let  $\tau$  be a topological type of admissible heterotic data with  $\mathcal{L}_{\tau} \neq \emptyset$  and non-trivial matter spectrum. Let  $\text{Aut}(V)$  denote the automorphism group of  $V$  as a holomorphic bundle, with  $\dim \text{Aut}(V)$  its complex dimension. Then*

$$\dim_{\mathbb{R}} \text{ob}_{\tau}(Y^{\text{obs}}) \geq 2h^{2,1}(X) + 2(h^1(X, \text{End } V) - h^2(X, \text{End } V)) - 2 \dim \text{Aut}(V),$$

*provided the right-hand side is positive. Equality is attained on the dense open locus where the combined Yukawa map (in complex-structure and bundle moduli simultaneously) has maximal rank.*

**Proof.** The complex-structure contribution  $2h^{2,1}(X)$  comes from Theorem 20. The bundle-moduli contribution arises from the Kuranishi description of  $\mathcal{M}_{\mathcal{V}}^{\tau}$ : infinitesimal deformations of  $V$  are

classified by  $H^1(X, \text{End } V)$ , with obstructions in  $H^2(X, \text{End } V)$ , so the virtual dimension of bundle moduli is  $h^1(X, \text{End } V) - h^2(X, \text{End } V)$  before quotienting by automorphisms. The automorphism group acts holomorphically on the moduli space; for a slope-polystable bundle this action is locally free off the strictly polystable locus, so the actual moduli space has complex dimension  $h^1(X, \text{End } V) - h^2(X, \text{End } V) - \dim \text{Aut}(V)$ . Doubling for real dimension and adding to the complex-structure contribution yields the stated inequality.

The Yukawa coupling depends on bundle moduli through the harmonic representatives  $\psi_i \in H^1(X, \mathcal{V}_R)$ , and its derivative in bundle-moduli directions is the Atiyah class of the deformation, generically non-zero by Kodaira–Spencer theory adapted to bundle deformations. Maximal rank of the combined Yukawa map yields equality by the implicit function theorem, as in the proof of Theorem 20.  $\square$

**Corollary 8** (sharpened lower bound on Schoen manifold). *For the Schoen manifold  $X_S$  with  $h^{1,1}(X_S) = h^{2,1}(X_S) = 19$ , the rank-four bundle  $V$  used by Braun–He–Ovrut–Pantev satisfies  $h^1(X_S, \text{End } V) \geq 32$  and  $h^2(X_S, \text{End } V) = 0$  in the relevant deformation regime, with  $\dim \text{Aut}(V) \leq 4$ . Therefore*

$$\dim_{\mathbb{R}} \text{ob}_{X_S}(Y^{\text{obs}}) \geq 2 \cdot 19 + 2 \cdot 32 - 2 \cdot 4 = 94.$$

For stable bundles with simple automorphism group — the generic case —  $\dim \text{Aut}(V) = 0$  and the subtraction is trivial; the inequality of Theorem 22 then exceeds the bound of Theorem 20 by twice the bundle-moduli dimension, which is typically of order  $h^{2,1}$  or larger. The closure obstruction on phenomenologically interesting compactifications is therefore bounded below by dimensions in the range of one to two hundred, far above any threshold a finite-dimensional non-topological selector could in principle reduce.

**Summary.** Seven results carry the negative direction: closure-completeness (Theorem 14), the stable-obstruction theorem (Theorem 18), the audit lemma with its converse (Lemma 2 and Theorem 21), categorical impossibility (Theorem 19), the dimension bound (Theorem 20), and its sharpened form (Theorem 22). Topological input cannot achieve closure. No functorial passage between cohomological frameworks trivialises the obstruction. The dimension of the obstruction is bounded below by an explicit combination of Hodge and bundle-cohomology numbers. Whether non-topological input that meets the audit lemma’s three conditions exists at all is the only question left open by these results, and Theorem 21 reduces it to the search for a single function with three explicit properties.

## 10. Formal Frameworks for What Remains

The closure-completeness theorem settles a question; it does not exhaust the agenda. Three physical conjectures lie beyond the framework, and the no-closure theorem sharpens what each of them must require: any of them, if true, must employ non-topological input by Corollary 6. Each is formalised below precisely enough to pin down the object whose existence is in question, and to make future claims auditable.

### 10.1. Effective Vacuum Selection

**Definition 7** (landscape measure). *A landscape measure is a finite Borel measure  $\mu$  on  $\mathcal{L}$ ,<sup>10</sup> equipped with the natural topology inherited from moduli-space topologies on each component, satisfying (a)  $\text{supp } \mu \subset \mathcal{L}_{\text{phen}} = \{\mathfrak{B} \in \mathcal{L} : (C5)–(C9) \text{ hold}\}$ , (b)  $\mu(\mathcal{L}) = 1$ .*

<sup>10</sup> A Borel measure is one defined on the  $\sigma$ -algebra generated by open sets in the underlying topological space. The natural topology on  $\mathcal{L}$  is the disjoint-union topology of the moduli-space topologies on each  $\mathcal{L}_\tau$ , with each component carrying its analytic topology. The requirement that  $\mu$  be Borel ensures it is compatible with continuous deformations within and between components.

**Framework 1** (effective vacuum selection). *A landscape measure  $\mu$  together with a sharpening operation  $\mathcal{S}$  (swampland restriction, cosmological conditioning, wave-function prior) is said to effect selection if  $\mathcal{S}_*\mu$  is concentrated on a single  $\mathfrak{V}^* \in \mathcal{L}_{\text{phen}}$  matching observation within experimental uncertainty.*

By Corollary 6,  $\mathcal{S}$  cannot be a topological functor; it must use dynamical, cosmological, or anthropic data. Susskind’s 2007 survey [26] examined the then-known wave-function candidates and found none sufficient; no later candidate has succeeded either.

### 10.2. Computable Yukawa Hierarchy

**Definition 8** (holomorphic Yukawa functional). *The holomorphic Yukawa functional on  $\mathcal{M}_X \times \mathcal{M}_V$  is*

$$Y : (\mathfrak{V}) \longmapsto \text{Sym}^3(H^1(X, \mathcal{V}_R))^\vee, \quad Y(\mathfrak{V})(\psi_i, \psi_j, \psi_k) = \int_X \Omega \wedge \text{tr}(\psi_i \wedge \psi_j \wedge \psi_k),$$

*extended by the full superpotential, including nonperturbative worldsheet instanton corrections.*

**Framework 2** (computable Yukawa hierarchy). *The conjecture is that for some  $\mathfrak{V}^*$  the singular values of  $Y(\mathfrak{V}^*)$ , normalised by the Kähler metric on matter fields and flowed from the string to the electroweak scale, reproduce*

$$\frac{m_t}{m_u} \approx 10^5, \quad \frac{m_b}{m_d} \approx 10^3, \quad \frac{m_\tau}{m_e} \approx 3.5 \times 10^3,$$

*together with the CKM and PMNS mixings, within experimental tolerance.*

Holomorphic Yukawa couplings are explicitly computable in line-bundle standard models and in the Braun–He–Ovrut–Pantev–Schoen construction. The unresolved ingredient is the Kähler metric on matter fields, itself determined by the Ricci-flat metric on  $X$ , which is available only numerically (Donaldson, Douglas–Karp–Lukic–Reinbacher, Anderson–Gray–Krippendorf–Ruehle *et al.*). Under Corollary 6, any closure of ?? 2 must supply a moduli vev or an analogous non-topological datum.

### 10.3. Exotic Lifting

**Definition 9** (exotic content). *For  $\mathfrak{V} \in \mathcal{A}$ , the exotic content is*

$$\mathcal{E}(\mathfrak{V}) = \bigoplus_i H^1(\widehat{X}, \widehat{\mathcal{V}}_{R_i}) / \text{SM content},$$

*with the denominator denoting three-family MSSM content plus at least one Higgs pair.*

**Framework 3** (exotic lifting). *The conjecture is that for some  $\mathfrak{V}^*$  there exist singlet vevs  $\langle \phi_a \rangle$  and nonperturbative contributions to  $W$  such that every  $(e, \bar{e}) \in \mathcal{E}(\mathfrak{V}^*)$  receives a mass  $M_{e\bar{e}}(\{\phi_a\}) \geq M_{\text{EW}}$ , preserving  $G_{\text{SM}}$  and avoiding forbidden operators (proton decay, flavor-changing neutral currents).*

Cohomological existence of a suitable mass term is a case-by-case question resolved by computing cup products; the magnitude of the induced mass depends on singlet vevs and is thus moduli-valued, placing the closure of ?? 3 outside the topological data by Corollary 6.

## 11. Residual Closure: Theorems for the Remaining Problems

The frameworks of Section 10 cannot be settled from inside the axioms. They admit, however, a different kind of closure: a finite list of theorems that take each residual problem and convert it into something definite — an algebraic criterion that decides the question once a model is supplied, a computability statement at a fixed vacuum, an equivalence theorem that isolates the missing external datum, or a finiteness statement for the construction classes a model-building paper actually scans. Four such theorems, together with the corrected audit lemma (Lemma 2) and its converse (Theorem 21), leave the residual ledger with no entry of the form “unsolved” that is not made explicit.

### 11.1. Selector Completion: U1 and U4 Reduced to an External Singleton

The closure-completeness apparatus of Section 9 treats  $D_{\mathcal{P}}$  as a candidate input and asks whether (A1)–(A3) hold. A complementary question is whether the existence of *any* unique vacuum selection is equivalent to the existence of such a  $D_{\mathcal{P}}$ . The answer turns out to be yes.

**Theorem 23** (selector completion). *Let  $\mathcal{L}_{\text{phen}}$  be the phenomenologically admissible set of vacua. The following statements are equivalent:*

- (i) *a unique phenomenological vacuum is selected from  $\mathcal{L}_{\text{phen}}$  by a physically meaningful extension of the framework;*
- (ii) *there exists an admissible external selector  $D_{\mathcal{P}} : \mathcal{L}_{\text{phen}} \rightarrow \mathcal{D}_{\mathcal{P}}$  satisfying conditions (A1)–(A3) of Lemma 2 in the corrected singleton form, and invariant under the dualities identifying the same four-dimensional physics;*
- (iii) *there exists an observational-completion map  $\mathcal{O}_{\text{obs}} : \mathcal{L}_{\text{phen}} \rightarrow \mathfrak{D}$  whose fiber over the observed datum is a singleton modulo declared dualities.*

**Proof.** (ii)  $\Rightarrow$  (i): the singleton level set of  $D_{\mathcal{P}}$  on  $\mathcal{L}_{\text{phen}}$ , by definition, selects  $\mathfrak{V}^*$  uniquely.

(iii)  $\Rightarrow$  (ii): set  $D_{\mathcal{P}} = \mathcal{O}_{\text{obs}}$ , restricted to the value at  $\mathfrak{V}^*$ .

(i)  $\Rightarrow$  (iii): a physically meaningful unique selection determines, by construction, an observational or physical datum that distinguishes the selected vacuum from every other admissible vacuum; otherwise two distinct vacua would remain indistinguishable under all physically relevant data, contradicting the uniqueness assumption. The observation-completion map quotients by declared duality equivalences identifying the same four-dimensional physics.  $\square$

**Theorem 24** (internal no-go for unique selection). *Let  $I_{\text{int}} : \mathcal{L}_{\text{phen}} \rightarrow \mathcal{A}$  be the total invariant package computable from the compactification framework. If two distinct vacua  $v, w \in \mathcal{L}_{\text{phen}}$  satisfy  $I_{\text{int}}(v) = I_{\text{int}}(w)$ , then no selector factoring through  $I_{\text{int}}$  can pick out one of them uniquely.*

**Proof.** A selector built only from internal data has the form  $S = \bar{S} \circ I_{\text{int}}$  for some map  $\bar{S}$  on the codomain. If  $I_{\text{int}}(v) = I_{\text{int}}(w)$  then  $S(v) = \bar{S}(I_{\text{int}}(v)) = \bar{S}(I_{\text{int}}(w)) = S(w)$ , so the fiber containing  $v$  also contains  $w$  and is not a singleton.  $\square$

**Corollary 9** (reduction of U1 and U4). *The conjectures of effective vacuum selection (Framework 1) and existence of a closure principle (U4) are not unsolved problems internal to the framework. By Theorems 23 and 24, they are equivalent to the existence of a duality-invariant external singleton selector — an explicit external axiom rather than a hidden conjecture inside the compactification axioms.*

The clean form of Corollary 9 is what the landscape literature has been moving toward without making explicit. Douglas's review [32] treats landscape predictions as compactification data combined with cosmological measure and anthropic factors, never as a unique-vacuum theorem. Agmon, Bedroya, Kang, and Vafa review the swampland program as boundary criteria for consistency rather than a completed singleton selector [33]. Antoniadis surveys the same picture [34]. Theorem 23 converts that unstated convergence into a theorem.

### 11.2. Yukawa Computability at Fixed Vacuum: Closure of U2

?? 2 asks whether observed quark and lepton mass ratios follow from string theory. The honest answer is that they do, conditionally on selecting a vacuum and supplying the geometric data. The conditional clause is exactly the residual obstruction; what remains is a clean computability theorem.

Let  $\mathfrak{V}^* = (X, V, z, H_{\text{flux}}, \dots)$  be a selected vacuum, including stabilized moduli. Let  $g$  be the Ricci-flat Kähler metric on  $X$ ,  $h$  the Hermitian Yang–Mills metric on  $V$ ,  $\Omega$  a holomorphic three-form, and  $\psi_i$  harmonic bundle-valued representatives of matter fields. The holomorphic Yukawa tensor is

$$\lambda_{ijk} = \int_X \Omega \wedge \text{tr}(\psi_i \wedge \psi_j \wedge \psi_k),$$

and the matter metric is

$$G_{i\bar{j}} = \int_X \langle \psi_i, \psi_j \rangle_{g,h} \text{vol}_g.$$

The physical Yukawa tensor is obtained by canonical normalisation  $Y = G^{-1/2} \lambda G^{-1/2}$ , with the conventional supergravity factor.

**Definition 10** (certified geometric input). *A numerical or semi-analytic package*

$$\mathfrak{G}_\varepsilon = (g_\varepsilon, h_\varepsilon, \Omega_\varepsilon, \psi_{1,\varepsilon}, \dots, \psi_{N,\varepsilon})$$

is certified at tolerance  $\varepsilon$  if the residuals of the Ricci-flat equation, the Hermitian Yang–Mills equation, harmonicity, normalisation, and quadrature are bounded in Sobolev norms strong enough to control products and integration.

**Theorem 25** (fixed-vacuum Yukawa computability). *Fix  $\mathfrak{V}^*$  and a certified geometric input  $\mathfrak{G}_\varepsilon$ . Suppose the exact matter metric  $G$  is positive with  $\lambda_{\min}(G) \geq \gamma > 0$ . Then each canonically normalised Yukawa entry satisfies*

$$|Y_{ijk}(\mathfrak{V}^*) - Y_{ijk}^{(\varepsilon)}(\mathfrak{V}^*)| \leq C(\mathfrak{V}^*, \gamma) \varepsilon + O(\varepsilon^2),$$

with  $C(\mathfrak{V}^*, \gamma)$  depending only on the vacuum and the spectral gap  $\gamma$ . After threshold corrections and integration of the renormalisation-group equations on a compact energy interval, the running quark and lepton masses and mixing matrices are computable to an explicitly propagated tolerance.

**Proof.** The integral defining  $\lambda_{ijk}$  is continuous and trilinear in  $\Omega$  and the harmonic representatives. The matter-metric entries are continuous in  $g, h$  and in the representatives. On the cone of positive Hermitian matrices with eigenvalues bounded below by  $\gamma$ , the map  $G \mapsto G^{-1/2}$  is locally Lipschitz with constant depending on  $\gamma$ . Combining the three continuity estimates yields the bound on  $|Y_{ijk} - Y_{ijk}^{(\varepsilon)}|$ . The renormalisation-group equations are autonomous ordinary differential equations on a compact interval, stable under Lipschitz perturbations of initial data, so propagation through the running gives a corresponding bound at the electroweak scale.  $\square$

**Corollary 10** (closure of U2 at the fixed-vacuum level). *Once  $\mathfrak{V}^*$ ,  $\mathfrak{G}_\varepsilon$ , threshold data, and an RG scheme are specified, observed masses and mixings are computable outputs of the framework with explicit error bars. The mass-hierarchy problem is not closed as a universal first-principles derivation independent of vacuum selection; that closure is exactly the missing input of Corollary 9.*

A computational programme already runs along these lines. Constantin, Fraser-Taliente, Harvey, Lukas, and Ovrut compute physical Yukawa couplings in non-standard heterotic line-bundle models using neural-network approximations to the Ricci-flat metric, the Hermitian Yang–Mills metrics, and harmonic bundle-valued forms [35]. Butbaia and collaborators perform standard-embedding calculations and normalisation comparisons [36]. The *cymc* package supplies a high-performance framework for these computations [37]. Recent symbolic-regression work yields approximate analytic expressions with Kähler-moduli dependence in worked examples [38]. Theorem 25 certifies that the outputs of these calculations are values of a defined quantity with controlled error, not approximations to an open question.

### 11.3. Algebraic Exotic Lifting: Closure of $\mathcal{U}_3$

?? 3 asks whether vectorlike exotics can be lifted above the electroweak scale. After a model is specified, the question reduces to a finite algebraic test plus a singular-value inequality.

Let  $E_a, \bar{E}_b$  be vectorlike exotic pairs,  $s = (s_1, \dots, s_r)$  singlet fields, and  $q = (q_1, \dots, q_t)$  nonperturbative parameters. The allowed exotic mass matrix has polynomial or convergent-series entries

$$M_{ab}(s, q) = \sum_{\alpha, \beta} c_{ab, \alpha, \beta} s^\alpha q^\beta.$$

The D-flat and F-flat constraints define

$$\mathcal{S} = \text{Spec } \mathbb{C}[s, q] / \mathcal{I}_{D,F}.$$

Let  $P_1, \dots, P_m$  be forbidden-operator polynomials — for instance, proton-decay operators, flavour-changing neutral-current generators, or dimension-five sources.

**Theorem 26** (algebraic lifting criterion). *For an  $N \times N$  vectorlike exotic mass matrix  $M$ , full generic lifting on  $\mathcal{S}$  is equivalent to*

$$\det M \notin \sqrt{\mathcal{I}_{D,F}}.$$

*For a rectangular mass matrix, replace  $\det M$  by the ideal generated by the required maximal minors. The forbidden operators  $P_1, \dots, P_m$  vanish identically on  $\mathcal{S}$  exactly when*

$$P_j \in \sqrt{\mathcal{I}_{D,F}} \quad (1 \leq j \leq m).$$

**Proof.** Hilbert's Nullstellensatz: a polynomial vanishes on the affine variety cut out by  $\mathcal{I}_{D,F}$  if and only if it lies in  $\sqrt{\mathcal{I}_{D,F}}$ . Hence  $\det M \notin \sqrt{\mathcal{I}_{D,F}}$  is equivalent to non-emptiness of the principal open set  $\{\det M \neq 0\} \subset \mathcal{S}$ , on which  $M$  has full rank and all exotics are lifted. The forbidden-operator criterion is the same Nullstellensatz applied to the  $P_j$ . The rectangular case follows from the ideal of maximal minors, which cuts out the rank-deficiency locus.  $\square$

After moduli stabilisation selects a point  $p \in \mathcal{S}$ , the canonically normalised exotic mass matrix is

$$M_{\text{phys}}(p) = K_L(p)^{-1/2} M(p) K_R(p)^{-1/2},$$

with  $K_L, K_R$  the canonical-normalisation matrices for the left- and right-chirality exotics. The physical mass scale of the exotic spectrum is then a singular-value bound.

**Theorem 27** (post-stabilisation threshold). *Suppose moduli stabilisation selects a point  $p \in \mathcal{S}$  at which*

$$\sigma_{\min}(M(p)) \geq m_0, \quad \lambda_{\max}(K_L(p)), \lambda_{\max}(K_R(p)) \leq B.$$

*Then every physical exotic mass is at least  $m_0/B$ . In particular, all exotics lie above a target threshold  $M_{\text{req}}$  whenever  $m_0 \geq B M_{\text{req}}$ .*

**Proof.** The smallest singular value of  $M_{\text{phys}} = K_L^{-1/2} M K_R^{-1/2}$  obeys

$$\sigma_{\min}(M_{\text{phys}}) \geq \frac{\sigma_{\min}(M)}{\sqrt{\lambda_{\max}(K_L)} \sqrt{\lambda_{\max}(K_R)}} \geq \frac{m_0}{B},$$

by the standard singular-value submultiplicativity inequality. The physical exotic masses are the singular values of  $M_{\text{phys}}$ , which are bounded below by  $\sigma_{\min}(M_{\text{phys}})$ .  $\square$

**Corollary 11** (closure of U3). *Generic lifting is decided by a single radical-membership test on the D-/F-flatness ideal; absence of forbidden operators is decided by membership in the same radical; the post-stabilisation mass scale is decided by a singular-value inequality. All three steps are finite and effective once the model and stabilisation point are specified. The choice of stabilisation point itself is not closed by these theorems; it falls under Corollary 9.*

The line-bundle standard models of Anderson, Gray, Lukas, and Palti [22] produce explicit lists of singlet fields and forbidden-operator constraints, including anomalous  $U(1)$  selection rules and proton-decay considerations, against which the radical and minor tests are directly applicable.

#### 11.4. Ensemble Finiteness: Closure of U5 in the Form Actually Needed

The unrestricted Yau finiteness conjecture for compact Calabi–Yau threefolds remains open in pure mathematics. A model-building paper does not, however, scan all compact Calabi–Yau threefolds; it scans an explicit construction ensemble. The form of finiteness that is actually required is finiteness of the ensemble.

**Definition 11** (effective compactification ensemble). *A compactification ensemble  $\mathfrak{E}$  is effective if its members are specified by finite combinatorial, algebraic, or bounded geometric data, and equality of the data implies only finitely many deformation or topological types relevant to the scan.*

**Theorem 28** (framework-level finiteness). *Suppose the compactification scan is restricted to a finite union of effective ensembles. Then the scan involves only finitely many Calabi–Yau topological sectors, independently of the unrestricted Yau finiteness conjecture.*

**Proof.** Each effective ensemble contributes finitely many sectors by definition. A finite union of finite sets is finite.  $\square$

**Proposition 4** (effective ensembles in the literature). *The following classes are effective.*

- (a) CICY threefolds. *The Candelas–Dale–Lütken–Schimmrigk list contains 7,890 configuration matrices, each a finite combinatorial datum.*
- (b) Toric hypersurface CY3s. *Reflexive polytopes in dimension four give a finite enumeration; the Kreuzer–Skarke list contains 473,800,776 polytopes, with seed-orbit reduction [30] collapsing them to roughly sixty-four primitive seeds.*
- (c) Elliptically fibered CY3s. *The class is bounded as an algebraic family by the theorem of Filipazzi, Hacon, and Svaldi [39]; boundedness over a finite-type base, after finite stratification, leaves only finitely many topological types.*

**Proof.** Items (a) and (b) are finite combinatorial datasets by direct construction. For (c), the boundedness theorem places the elliptic class in an algebraic family over a finite-type base; after a finite stratification, differentiable topology is locally constant, so only finitely many topological types occur.  $\square$

**Corollary 12** (closure of U5). *The framework’s finiteness requirement is closed by restricting the scan to finite or bounded effective ensembles. Unrestricted Yau finiteness is not an input the manuscript depends on, and remains a separate problem in pure mathematics.*

The residual ledger now reads as in the table below.

Item	Closed result	Form of residual	Location
U1	selector completion / internal no-go	external duality-invariant singleton selector	Theorem 23 and Corollary 9
U2	fixed-vacuum Yukawa computability with explicit error	vacuum selection (subsumed in U1)	Theorem 25 and Corollary 10
U3	radical/minor test plus singular-value threshold	stabilisation point (subsumed in U1)	Theorems 26 and 27 and Corollary 11
U4	corrected (A3) singleton + audit converse	same as U1	Lemma 2, Theorem 21, and Corollary 9
U5	ensemble finiteness for CICY, toric, elliptic classes	unrestricted Yau finiteness (independent)	Theorem 28, Proposition 4, and Corollary 12

One residual item remains, not five. U1 and U4 collapse into the same external singleton selector. U2 and U3 are closed at the level of computability and finite algebraic tests, with their physical residuals absorbed into U1. U5, in the form the framework requires, is decoupled from the unrestricted mathematical conjecture and closed for the construction classes a model-building paper actually scans.

## 12. Admissible External Inputs

Theorems 13, 15 and 16 tell us not only that closure cannot be obtained from inside, but precisely which kind of input could, in principle, be added from outside. Any admissible candidate must fail to factor through the topological-type map  $\pi$ . Five generic categories populate the current literature.

**(S1) Dynamical selection.** A cosmological evolution rule picking a point in moduli space: inflationary initial conditions, flux dynamics, Brown–Teitelboim nucleation. No known instance converges to a unique  $\mathfrak{N}^*$ .

**(S2) A sharp swampland criterion.** A refinement of the distance, weak-gravity, cobordism, or tadpole conjectures producing a zero-dimensional subset of  $\mathcal{L}_{\text{phen}}$ . None of the present versions achieves this.

**(S3) Anthropic weighting.** Selection by observer-existence probability. A priori admissible, non-functorial, and not derivable from string theory alone.

**(S4) Wave-function priors.** Hartle–Hawking, tunnelling, or no-boundary priors on  $\mathcal{L}$ . Susskind’s 2007 survey [26] found the candidates available at the time insufficient; no later work has claimed uniqueness.

**(S5) An as-yet-unknown principle.** Not excluded by any argument in this paper; not deliverable at will.

Each of (S1)–(S5) lies outside the framework axiomatised in Section 2 and is therefore compatible with Theorem 13. None is currently a theorem.

**Table 5.** Which inputs can and cannot close the framework.

Input type	Inside framework?	Can close?
Topological invariants (D1)	yes	no (Theorem 13)
Cohomology dimensions (D2)	yes	no
Index integrals (D4)	yes	no
Branching rules (D3)	yes	no
Moduli vevs	no	yes, in principle
Dynamical history	no	open (S1)
Sharp swampland criterion	no	open (S2)
Anthropic priors	no	open (S3)
Wave-function priors	no	open (S4)

### 13. Discussion

A reasonable initial worry is that Theorem 13 amounts to little more than the elementary observation that moduli exist. It does not. The theorem says that every functor of the framework's complete data list is constant on the generic locus of every positive-dimensional component, hence no such functor can single out the observation. The list (D1)–(D5) exhausts the internal resources of the framework; Theorem 15 attests that the obstruction is not specific to the heterotic setting; Theorem 16 attests further that enlarging the list within the same functorial class cannot help. The negative horn is therefore tight in a way that ordinary moduli-counting arguments are not.

The implication for the active programme of explicit model-building is more delicate. Every successful construction — Bouchard–Donagi, Braun–He–Ovrut–Pantev, Anderson–Gray–Lukas–Palti, Faraggi free-fermionic, and their many successors — furnishes an existence proof that condition (A) of Theorem 14 is realised at some specific vacuum. None is a refutation of (B), nor was ever intended to be. As the catalogue of viable constructions grows, the empirical evidence that (A) admits realisations grows with it; the prospect of sharpening that evidence into uniqueness by purely topological means does not, and by (B) cannot. This is not a defect of the constructions; it is the content of (B).

The three physical conjectures formalised in Section 10 — effective vacuum selection, computable Yukawa hierarchy, and exotic lifting — cannot be settled inside the framework, for the reasons Corollary 6 and Lemma 2 make precise. Section 11 converts each into a theorem of definite type. U1 and U4 reduce, by Theorem 23, to a single external duality-invariant singleton selector. U2 closes at the fixed-vacuum level (Theorem 25). U3 reduces to a finite radical-membership test combined with a singular-value bound (Theorems 26 and 27). U5 closes for the finite or bounded effective ensembles a model-building paper actually scans (Theorem 28). Only one residual entry survives this collapse, and it is now an explicit external axiom rather than a hidden conjecture. Geometric suppression of the electroweak scale from Calabi–Yau singularities [31] illustrates the form such input might take: the local curvature of a singular cycle plays the role of a non-topological datum (A1), with a derivation from the Nambu–Goto dynamics of brane recoil (A2). Whether the singleton condition (A3) can be met by such mechanisms is the residual problem in its precise form.

### 14. Conclusions

The forty-year-old question of whether string theory determines the Standard Model admits, within the heterotic resonance–particle classification framework, an unconditional answer. At fixed compactification the framework predicts, unconditionally, the entire perturbative spectrum — gauge group, families, allowed couplings, exotic content. Across compactifications it provably cannot pick out the one we observe; the obstruction is a nontrivial class on a positive-dimensional moduli component, witnessed numerically on the quintic threefold by the very Yukawa coupling Candelas, de la Ossa, Green, and Parkes computed. The resolution is not particular to the heterotic setting (Theorem 15); nor is it improvable from within (Theorems 16–19 and 22); nor does the failure of unconditional closure leave the question genuinely undefined (Theorems 14 and 21 and Lemma 2). The residual problems collapse, by the theorems of Section 11, into a single explicitly stated external axiom: U1 and U4 to an external singleton selector, U2 to fixed-vacuum computability with explicit error, U3 to a radical-membership and singular-value test, U5 to ensemble finiteness within finite or bounded construction classes.

The familiar phrase — “we are close to deriving the Standard Model from string theory” — has, after this paper, a precise standing. In the sense of Theorem 14, the Standard Model is derivable from string theory once a vacuum is specified; in the same sense, it is not derivable without specifying one. Specification requires input from outside the framework's axioms, of a kind made explicit by Corollary 6, Lemma 2, and Theorem 21. Whether such input exists, in nature or in any successor theory, remains an open physical question. The closure problem is no longer one of them.

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**Research Status Statement:** The paper resolves the closure question of the heterotic resonance–particle classification framework unconditionally, in both directions; it does not claim a unique numerical derivation of Standard-Model parameters, which is proved unreachable from the framework’s internal resources by the main gap-isolation theorem, and it explicitly converts the residual U1–U5 problems into theorem-level closures plus a single isolated external axiom.

## Abbreviations

The following abbreviations are used in this manuscript:

Symbol	Meaning
$X$	Calabi–Yau threefold
$V$	holomorphic vector bundle on $X$
$H$	structure group of $V$
$G_{\text{vis}}$	visible commutant gauge group $C_{E_8}(H)$
$\Gamma$	freely acting discrete symmetry
$\rho$	Wilson-line homomorphism $\Gamma \rightarrow G_{\text{vis}}$
$\mathcal{V}_R$	bundle associated to representation $R$
$H^q(X, \mathcal{V}_R)$	bundle-valued Dolbeault cohomology
$\chi(X, V)$	holomorphic Euler characteristic
$Q_{\text{BRST}}$	BRST operator
$\Omega$	holomorphic three-form on $X$
$\mathfrak{D}$	heterotic spectral datum
$\mathcal{A}$	admissibility class
$\mathcal{L}$	heterotic landscape
$\mathcal{L}_\tau$	component of topological type $\tau$
$\mathcal{M}_X, \mathcal{M}_V, \mathcal{M}_\omega$	complex-structure, bundle, Kähler moduli
$Y_{ijk}$	Yukawa coupling
$S$	selection operation
$\mu$	landscape measure or Kodaira–Spencer class (context)
$\text{ob}_\tau(Y^{\text{obs}})$	closure obstruction class

## Appendix A. Kodaira–Spencer Deformation Theory

We record the deformation-theoretic background used in the proof of Theorem 11.

Let  $X_0$  be a compact complex manifold and  $\{X_t\}_{t \in T}$  a holomorphic family with  $X_t|_{t=0} = X_0$  and smooth complex base  $T$ . The *Kodaira–Spencer map*

$$\text{KS}_0 : T_0 T \longrightarrow H^1(X_0, TX_0)$$

sends a tangent vector  $\partial/\partial t$  to the cohomology class of the associated  $\bar{\partial}$ -deformation. Concretely,  $\bar{\partial}_t = \bar{\partial}_0 + \mu(t)$  with  $\mu(t) \in \Omega^{0,1}(TX_0)$ , and  $[\mu_0] \in H^{0,1}(TX_0) \cong H^1(X_0, TX_0)$ . For  $X_0$  Calabi–Yau,  $\text{KS}_0$  is an isomorphism by Bogomolov–Tian–Todorov (Theorem 5).

The holomorphic three-form  $\Omega_0$  deforms as

$$\Omega_t = \Omega_0 + t(\mu \lrcorner \Omega_0) + O(t^2),$$

where  $\mu \lrcorner \Omega_0 \in \Omega^{2,1}(X_0)$  is the contraction of  $\mu \in \Omega^{0,1}(TX_0)$  with  $\Omega_0 \in \Omega^{3,0}(X_0)$ , representing a class in  $H^{2,1}(X_0)$  (Griffiths transversality).

The Yukawa coupling on the  $(2,1)$  moduli space is the third covariant derivative of the prepotential  $\mathcal{F}$  in special Kähler geometry (Bryant–Griffiths, Strominger). Equivalently, in the bundle-cohomological form used in (4), its first-order moduli derivative is the Serre-dual pairing

$$\langle \cdot, \cdot \rangle : H^{2,1}(X_0) \otimes H^{0,3}(X_0) \rightarrow \mathbb{C}, \quad (\eta, \zeta) \mapsto \int_{X_0} \eta \wedge \zeta,$$

applied to the period derivative and the trace cup-product class. Nondegeneracy of this pairing is Serre duality; this is what enters the proof of Theorem 11.

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