Article

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The Collatz Conjecture "The Non-Existence of Divergent Orbits Through an Analysis of the Codification of Linear Diophantine Dynamical Systems"

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Abstract

We define the Collatz function $\mathrm{Col} : \mathbb{N} \to \mathbb{N}$ as $\mathrm{Col}(n) = 3n + 1$ if n is odd, and $\mathrm{Col}(n) = \frac{n}{2}$ if n is even. The Collatz conjecture postulates that the orbit of any positive integer will eventually reach 1, or equivalently, fall into the periodic cycle $\{4, 2, 1\}$. Two conditions could invalidate this conjecture: the existence of a divergent orbit or the presence of a non-trivial cycle. We study the dynamics of these orbits through the density of even terms within them. It is established that if the asymptotic parity density of an orbit exceeds the critical threshold of $\dfrac{\ln(3)}{\ln(2)}$, the orbit is bounded. The main result of this work is to prove that there are no natural numbers for which the accumulation points of this parity density are strictly less than $\dfrac{\ln(3)}{\ln(2)}$. Consequently, we demonstrate that there are no divergent orbits.

Keywords: Collatz conjecture; dynamical system

Contents

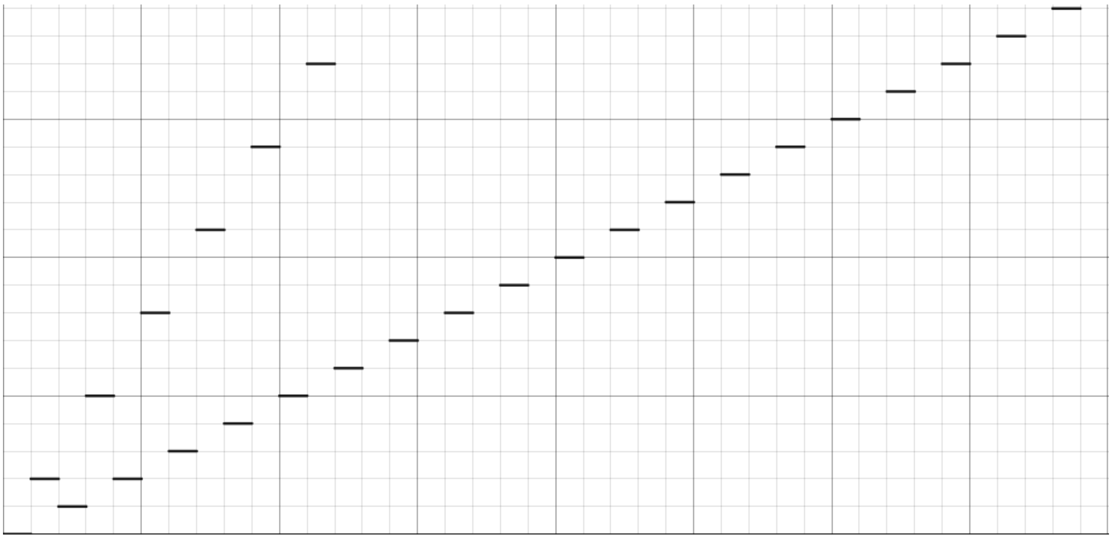
1. Collatz's Conjecture

The Collatz conjecture, also known as the $3n + 1$ conjecture, is an unsolved problem in number theory proposed by the German mathematician Lothar Collatz in 1937. Despite its seemingly simple formulation, it has challenged mathematicians for decades due to the extreme difficulty of proving its validity or finding a counterexample.

The formal formulation of the conjecture is as follows:

Let Col : $\mathbb{N} \to \mathbb{N}$ be defined by:

$$\mathrm{Col}(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even,} \end{cases}$$



Then, for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that:

$$\mathrm{Col}^k(n) = 1.$$

An equivalent formulation of the conjecture argues that starting from any positive integer, the sequence will eventually reach the cycle $\{4,2,1\}$. Two scenarios could invalidate the conjecture: the existence of a cycle strictly different from $\{4,2,1\}$, or the existence of a divergent orbit (an orbit that grows toward infinity). To date, no evidence has been found for either of these exceptions; however, no rigorous proof has completely ruled them out. In 2019, Terence Tao [16] presented a major breakthrough demonstrating that almost all orbits attain almost bounded values (falling very close to the $\{4,2,1\}$ cycle).

Example 1.

$$\begin{aligned} 27 &\to 82 \to 41 \to 124 \to 62 \to 31 \to 94 \to 47 \to 142 \to 71 \\ &\to 214 \to 107 \to 322 \to 161 \to 484 \to 242 \to 121 \to 364 \to 182 \\ &\to 91 \to 274 \to 137 \to 412 \to 206 \to 103 \to 310 \to 155 \to 466 \\ &\to 233 \to 700 \to 350 \to 175 \to 526 \to 263 \to 790 \to 395 \to 1186 \\ &\to 593 \to 1780 \to 890 \to 445 \to 1336 \to 668 \to 334 \to 167 \to 502 \\ &\to 251 \to 754 \to 377 \to 1132 \to 566 \to 283 \to 850 \to 425 \to 1276 \\ &\to 638 \to 319 \to 958 \to 479 \to 1438 \to 719 \to 2158 \to 1079 \to 3238 \\ &\to 1619 \to 4858 \to 2429 \to 7288 \to 3644 \to 1822 \to 911 \to 2734 \\ &\to 1367 \to 4102 \to 2051 \to 6154 \to 3077 \to 9232 \to 4616 \to 2308 \\ &\to 1154 \to 577 \to 1732 \to 866 \to 433 \to 1300 \to 650 \to 325 \to 976 \\ &\to 488 \to 244 \to 122 \to 61 \to 184 \to 92 \to 46 \to 23 \to 70 \\ &\to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \\ &\to 16 \to 8 \to 4 \to 2 \to 1 \end{aligned}$$

Example 2.

$$\begin{aligned}
871 &\to 2614 \to 1307 \to 3922 \to 1961 \to 5884 \to 2942 \to 1471 \to 4414 \to 2207 \\
&\to 6622 \to 3311 \to 9934 \to 4967 \to 14902 \to 7451 \to 22354 \to 11177 \to 33532 \\
&\to 16766 \to 8383 \to 25150 \to 12575 \to 37726 \to 18863 \to 56590 \to 28295 \to 84886 \\
&\to 42443 \to 127330 \to 63665 \to 190996 \to 95498 \to 47749 \to 143248 \to 71624 \\
&\to 35812 \to 17906 \to 8953 \to 26860 \to 13430 \to 6715 \to 20146 \to 10073 \to 30220 \\
&\to 15110 \to 7555 \to 22666 \to 11333 \to 34000 \to 17000 \to 8500 \to 4250 \to 2125 \\
&\to 6376 \to 3188 \to 1594 \to 797 \to 2392 \to 1196 \to 598 \to 299 \to 898 \to 449 \to 1348 \\
&\to 674 \to 337 \to 1012 \to 506 \to 253 \to 760 \to 380 \to 190 \to 95 \to 286 \\
&\to 143 \to 430 \to 215 \to 646 \to 323 \to 970 \to 485 \to 1456 \to 728 \to 364 \\
&\to 182 \to 91 \to 274 \to 137 \to 412 \to 206 \to 103 \to 310 \to 155 \to 466 \\
&\to 233 \to 700 \to 350 \to 175 \to 526 \to 263 \to 790 \to 395 \to 1186 \to 593 \\
&\to 1780 \to 890 \to 445 \to 1336 \to 668 \to 334 \to 167 \to 502 \to 251 \to 754 \\
&\to 377 \to 1132 \to 566 \to 283 \to 850 \to 425 \to 1276 \to 638 \to 319 \to 958 \\
&\to 479 \to 1438 \to 719 \to 2158 \to 1079 \to 3238 \to 1619 \to 4858 \to 2429 \to 7288 \\
&\to 3644 \to 1822 \to 911 \to 2734 \to 1367 \to 4102 \to 2051 \to 6154 \to 3077 \to 9232 \\
&\to 4616 \to 2308 \to 1154 \to 577 \to 1732 \to 866 \to 433 \to 1300 \to 650 \to 325 \\
&\to 976 \to 488 \to 244 \to 122 \to 61 \to 184 \to 92 \to 46 \to 23 \to 70 \\
&\to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \\
&\to 4 \to 2 \to 1
\end{aligned}$$

1.1. Main Idea of This Work

The primary contribution of this work is not simply a proof concerning the Collatz Conjecture, but the development of a comprehensive mathematical framework: the **Theory of Infinite Systems of Linear Diophantine Equations**. For decades, the Collatz problem has resisted traditional approaches because its discrete dynamics are highly chaotic. To conquer this, our central premise establishes that the existence of divergent orbits must be translated into a rigid algebraic structure. Specifically, a divergent orbit can only exist if there is a simultaneous natural solution to an endlessly growing, infinitely constrained system of linear Diophantine equations.

To navigate and eventually break this infinite system, the majority of this paper is dedicated to building the necessary theoretical machinery from the ground up. We do this in three fundamental steps:

1. Construction of the Diophantine System: We associate to each natural number a unique binary sequence ξ that represents the parity of its iterations. Every finite truncation of this sequence generates a specific affine linear transformation, which in turn defines a linear Diophantine equation. A divergent orbit would require a single natural number to perfectly satisfy this endless sequence of equations.

2. The Sigma Function (σ_a) as the Analytical Engine: A critical challenge in this infinite system is tracking the ever-shifting constants generated by the affine transformations. To solve this, we introduce the **Sigma function** (σ_a). Far from being a mere auxiliary calculation, the Sigma function is the core algebraic tool of our theory. It exactly parametrizes and tracks the minimal non-negative solutions (ρ_0) of these Diophantine equations across infinite iterations, allowing us to measure the precise "friction" between multiplications and divisions.

3. Topological Classification and the Critical Threshold: To evaluate whether a simultaneous solution can exist, the sequence is encoded by assigning 0 if the iteration is even, and 10 if it is odd. We define the density function as the ratio $\dfrac{a_{k+1}}{k}$, where a_k is the total number of 0's up to the k-th occurrence of the digit 1.

The critical threshold $\frac{\ln(3)}{\ln(2)}$ arises naturally from the geometric growth of the Collatz operations. An odd step multiplies the value by roughly 3, while an even step divides it by 2. After k odd steps and a_k even steps, the global magnitude of the orbit is scaled by a factor of approximately $\frac{3^k}{2^{a_k}}$. Setting this overall growth factor to 1 to find the state of equilibrium yields $3^k = 2^{a_k}$. By taking the natural logarithm, we obtain the precise ratio $\frac{a_k}{k} = \frac{\ln(3)}{\ln(2)}$. Thus, this constant represents the exact arithmetic break-even point between the expansion and contraction of the orbit.

Using this threshold, our theory classifies all infinite sequences into three distinct sets:

- G_0**:** The set of sequences ξ where $\liminf_{k\to\infty} \frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)}$. Here, divisions by 2 strictly dominate. We prove that any integer whose coding falls into G_0 must inevitably have a bounded orbit.
- G_∞**:** The set of sequences ξ where $\limsup_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)}$. Here, multiplications by 3 dominate. We prove that no natural number can exist in G_∞, because the "real-valued" limits of such sequences strictly map to negative integers or rationals.
- G_1**:** The exact boundary where $\lim_{l\to\infty} \frac{a_{k_l+1}}{k_l} = \frac{\ln(3)}{\ln(2)}$. This is the critical equilibrium threshold. By bridging our Diophantine theory with bounds from transcendental number theory, we demonstrate that this delicate balance is *positively unstable*. It cannot contain the coding of any natural number without the orbit exploding in a mathematically contradictory way.

Once this theoretical framework is fully established, the resolution of the Collatz divergence problem emerges as a natural, unavoidable consequence. Formally, we establish the following foundational results:

1. There is no $n \in \mathbb{N}$ whose coding satisfies $\limsup_{n\to\infty} \frac{a_{n+1}}{n} \le \frac{\ln(3)}{\ln(2)}$.
2. The coding of $n \in \mathbb{N}$ belongs to G_0 if and only if its orbit is bounded.

We leverage this machinery to demonstrate the main theorem of this work:

***Theorem* 12:** *There are no divergent orbits for the Collatz function on the natural numbers.*

The core idea of the final proof is by contradiction: Suppose there exists an $n \in \mathbb{N}$ whose orbit is divergent. Then, necessarily, its coding cannot belong to G_0. This means there must exist a subsequence of its density function whose limit is less than or equal to $\frac{\ln(3)}{\ln(2)}$. Moreover, the density function cannot have accumulation points strictly greater than $\frac{\ln(3)}{\ln(2)}$, because otherwise, there would exist a sub-orbit of n that is bounded (a consequence of Corollary 2).

Trapped by these restrictions, all accumulation points of the density function must be strictly less than or equal to $\frac{\ln(3)}{\ln(2)}$. However, our established theory proves that G_∞ contains no natural numbers, and the equilibrium in G_1 is positively unstable. Hence, the encoding of a divergent natural number has absolutely no valid mathematical space to exist within the topology of $\mathbb{Z}_2$, making divergent orbits an impossibility.

1.2. Notations and Conventions

In this work, we denote the set of positive integers as $\mathbb{N}$, the set of non-negative integers as $\mathbb{N}_0$, the greatest common divisor of a and b as (a, b), and the least common multiple of a and b as $[a, b]$. We use the following symbology to refer to an arbitrary composition of functions:

$$\bigodot_{i=1}^{n} f_i(x) = f_n \circ f_{n-1} \circ \ldots \circ f_2 \circ f_1.$$

1.3. Organization of the Paper

Due to the multidisciplinary nature of the mathematical tools employed, this work is structurally divided into three distinct parts. This division guides the reader from discrete arithmetic to continuous topological analysis, culminating in the resolution of the conjecture's main problems.

Part I: Elementary Theory of Infinite Systems of Diophantine Equations (Chapters 1–5)

This first part establishes the discrete and algebraic framework of the problem, transforming dynamic orbits into coded sequences and systems of linear equations.

Main Definitions:

- **Binary Coding of the Orbits (ξ):** The rule that translates the trajectory of a number into a symbolic sequence, assigning 0 for even steps and 10 for odd steps.
- **Local Affine Transformations (S_k):** The representation of each finite truncation of the coded sequence as a linear affine map of the form $S(x) = \dfrac{3^b x + N}{2^a}$.
- **Stability of Integer Sets:** The discrete criteria defining how the set of valid integer solutions evolves and is restricted as more iterations are added to the system.

Main Results:

- **Diophantine Equivalence:** It is demonstrated that the existence of any Collatz orbit is strictly equivalent to the existence of a simultaneous integer solution for an infinite system of Diophantine equations of the form $2^a y - 3^b x = N$.
- We prove that by composing the affine functions, the sets of integer solutions are strictly nested within each other ($\mathbb{E}(S \circ H) \subset \mathbb{E}(H)$).

Chapters Overview:

- **Chapter 1 & 2 (Collatz's Conjecture & Background):** Introduce the $3n + 1$ problem, the scenarios that would invalidate the conjecture, and the fundamental mathematical tools (metric spaces, p-adic analysis, Diophantine equations).
- **Chapter 3 (Set Generated by θ and ψ^q):** Models the Collatz operations as affine transformations and formalizes the integer sets generated by their truncations.
- **Chapter 4 (Stability and Instability of Integer Sets):** Establishes the algebraic rules governing the bounds of the integer solution sets.
- **Chapter 5 (Coding of the Orbits):** Details the symbolic binary assignment and explores the extension of the Collatz function over the odd rationals ($\mathbb{Q}_{odd}$).

Part II: Analytic Theory of Infinite Systems of Diophantine Equations (Chapters 6–11)

Here, the research takes a qualitative leap: the discrete problem is translated into the calculus of limits, 2-adic topology, and linear forms in logarithms.

Main Definitions:

- **Parity Density Function:** The asymptotic ratio $\frac{a_{k+1}}{k}$, which measures the proportion between divisions by 2 and multiplications by 3.
- **The Topological Sets (G_0, G_∞, G_1):** The partition of all possible codings based on the critical threshold $\frac{\ln(3)}{\ln(2)}$. G_0 represents division-dominated orbits, G_∞ multiplication-dominated ones, and G_1 the perfect equilibrium.
- **The 2-adic Space ($\mathbb{Z}_2$) and π Functions:** The extension of the domain to treat infinite sequences as convergent real series.
- **The Sigma Function (σ_a):** A fundamental auxiliary function introduced to trace the evolution of the constants in the Diophantine system.

Main Results:

- **Exact Parameterization (Theorem 6):** It is proved that the iterations of the Sigma Function exactly construct the minimal non-negative solution (ρ_0) of any generated Diophantine equation.
- **Invariance of G_∞:** We demonstrate topologically that any sequence dominated by multiplications by 3 corresponds strictly to negative integers or rational numbers, never to a natural number.
- **The Positive Instability of G_1:** By invoking the **Baker-Wüstholz Theorem** (bounds for linear forms in logarithms), we prove that the delicate equilibrium at the threshold $\frac{\ln(3)}{\ln(2)}$ is positively unstable. The transcendental bound forces an infinite accumulation of binary carries, destroying the possibility of an integer solution.

Chapters Overview:

- **Chapter 6 (The G_0, G_∞ and G_1 Sets):** Divides the total space of coding sequences based on their asymptotic density limits.
- **Chapter 7 & 8 ($\mathbb{Z}_2$ Extension & Real Functions π^1, π^2):** Translates the system's domain to the 2-adic integers and constructs weight functions to map topological evolution to real values.
- **Chapter 9 (The Sigma Function):** Introduces the main algebraic tool used to find minimal solutions for the affine mappings.
- **Chapter 10 (Coding of Set G_∞):** Topologically proves that natural numbers cannot exhibit multiplication-dominated behavior.
- **Chapter 11 (The Set G_1 is Unstable):** The climax of the analytic theory, where transcendental number theory is applied to break the critical threshold.

Part III: The Proof of the Non-Existence of Divergent Orbits (Chapter 12)

The resolution section of the paper. All the analytic machinery developed in Parts I and II is directly applied to prove that divergent trajectories are mathematically impossible.

Main Definitions:

- **Divergent Orbits:** Trajectories generated by the Collatz function over the natural numbers that escape to infinity, i.e., $\lim_{k\to\infty} \mathrm{Col}^k(n) = \infty$.

Main Results:

- **Non-Existence of Divergent Orbits (Main Theorem):** We formally conclude that, since the codings of natural numbers cannot belong to G_∞ or G_1 (as proven by the topological and transcendental bounds of Part II), every natural number strictly belongs to the set G_0. This mathematically guarantees that the density of divisions by 2 eventually strictly dominates the multiplications by 3, implying that **every orbit is bounded** and no divergent orbits can exist.

Chapters Overview:

- **Chapter 12 (The Problem of Divergence):** Consolidates the theorems from the analytic theory to definitively prove that no natural number can have a divergent trajectory, successfully ruling out this major scenario for the Collatz Conjecture.

2. Background

2.1. Metric Space

A **metric space** is a set X equipped with a function $d : X \times X \to \mathbb{R}$, called a *metric*, that satisfies the following properties for all $x, y, z \in X$:

- **Non-negativity:** $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
- **Symmetry:** $d(x, y) = d(y, x)$.
- **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$.

The function $d(x, y)$ measures the "distance" between any two points x and y in the set X. The pair (X, d) is called a metric space.

Examples of Metrics:

- **Euclidean Metric (on $\mathbb{R}^n$):**
$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$
This metric defines the usual distance between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in Euclidean space $\mathbb{R}^n$.
- **Discrete Metric:**
$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$
In this metric, the distance between two distinct points is always 1, and the distance from any point to itself is 0.
- **Taxicab Metric (or Manhattan Metric, on $\mathbb{R}^n$):**
$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$
This metric measures the distance between two points x and y as the sum of the absolute differences of their coordinates. It corresponds to the distance a taxi would drive on a grid of city streets.
- **p-adic Metric (on $\mathbb{Q}$):**
$$d_p(x, y) = p^{-ord_p(x-y)}$$
Here, p is a fixed prime number, and $ord_p(x - y)$ denotes the p-adic valuation of $x - y$, which
$$ord_p(x) = \begin{cases} \text{the highest power } p \text{ which divides } x & \text{if} \quad x \in \mathbb{Z} \\ ord_p(a) - ord_p(b) & \text{if} \quad x = \dfrac{a}{b} \in \mathbb{Q} \end{cases}$$
This metric measures the distance between two rational numbers based on their divisibility by p. The p-adic metric induces a non-Archimedean topology, meaning that the "triangle inequality" is strengthened to $d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\}$.

A **complete metric space** is a metric space in which every *Cauchy sequence* converges to a point within the space. Formally, a metric space (X, d) is called **complete** if, for every sequence $\{x_n\} \subset X$ that is Cauchy (i.e., for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$), there exists a point $x \in X$ such that:
$$\lim_{n \to \infty} x_n = x.$$
In other words, all Cauchy sequences in X must have a limit in X.

Examples:

- **The Real Numbers $\mathbb{R}$ with the Euclidean Metric:** The set of real numbers $\mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$ is a complete metric space. This is because every Cauchy sequence of real numbers converges to a real number.
- **The Rational Numbers $\mathbb{Q}$ with the Euclidean Metric:** The set of rational numbers $\mathbb{Q}$ with the Euclidean metric is not complete. For example, the sequence defined by $x_0 = 1$ and $x_{n+1} = \dfrac{x_n + \dfrac{2}{x_n}}{2}$ that approximate $\sqrt{2}$ is Cauchy in $\mathbb{Q}$ but does not converge to a rational number (since $\sqrt{2} \notin \mathbb{Q}$).

- **The p-adic Numbers $\mathbb{Q}_p$:** The set of p-adic numbers $\mathbb{Q}_p$, equipped with the p-adic metric $d_p(x,y) = \|x-y\|_2 = p^{-ord_p(x-y)}$, is a complete metric space. Every Cauchy sequence in $\mathbb{Q}_p$ converges to a p-adic number within $\mathbb{Q}_p$.

2.2. Limit Superior ($\limsup$) *and Limit Inferior* ($\liminf$) *of a Sequence:*

Given a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers and $X_n = \{x_n, x_{n+1}, \ldots\}$. Let us consider the following subsequences of $\{x_n\}_{n\in\mathbb{N}}$ given by

$$a_n = \inf X_n$$
$$b_n = \sup X_n$$

we have that the sequence $\{a_n\}$ is monotonically increasing and $\{b_n\}$ is monotonically decreasing, that is:

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1$$

Therefore there are limits

$$a = \lim_{n\to\infty} a_n = \sup_n a_n = \sup_n(\inf X_n)$$
$$b = \lim_{n\to\infty} b_n = \inf_n b_n = \inf_n(\sup X_n)$$

We will write $a = \liminf_{n\to\infty} x_n$ and $b = \limsup_{n\to\infty} x_n$ and we will call *Limit Inferior* and *Limit Superior* respectively.

Limit Superior ($\limsup$) **and Limit Inferior** ($\liminf$) **of a Sequence of Sets:** Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets in a space X. The limit superior and limit inferior of the sequence of sets are defined as follows:

1. **Set Sequence limit superior (**$\limsup_{n\to\infty} A_n$**):**

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

2. **Set Sequence limit inferior (**$\liminf_{n\to\infty} A_n$**):**

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

3. **Limit of a Sequence of Sets** If $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, then the sequence $\{A_n\}$ converges, and its limit is denoted as:

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

Particular Case: Monotone Sequences of Sets Non-Decreasing Sequence ($A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$**):** If A_n is a non-decreasing sequence (i.e., $A_n \subseteq A_{n+1}$ for all n), then:

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

In this case, the limit of the sequence is simply the union of all the sets in the sequence. **Non-Increasing Sequence (**$A_1 \supset A_2 \supset A_3 \supset \cdots$**):** If A_n is a non-increasing sequence (i.e., $A_n \supset A_{n+1}$ for all n), then:

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

In this case, the limit of the sequence is simply the intersection of all the sets in the sequence.

2.3. Number Theory

A **linear Diophantine equation** is an equation of the form

$$ax + by = c,$$

where $a, b, c \in \mathbb{Z}$. Such an equation has integer solutions if and only if the greatest common divisor $d = (a, b) := \gcd(a, b)$ divides c. When solutions exist, they form an infinite family given by:

$$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \quad \text{for all } t \in \mathbb{Z},$$

where (x_0, y_0) is a particular solution.

The **Carmichael function** $\lambda(n)$ is defined for each integer $n \geq 1$ as the smallest positive integer m such that

$$a^m = 1 \pmod{n}$$

for every integer a with $\gcd(a, n) = 1$. That is, $\lambda(n)$ is the exponent of the multiplicative group of units modulo n.

Some properties of $\lambda(n)$ include:

- If n is a power of an odd prime, say $n = p^k$ with p odd and $k \geq 1$, then $\lambda(n) = (p-1)p^{k-1}$.
- If $n = 2^k$, then:

$$\lambda(2^k) = \begin{cases} 1 & \text{if } k = 1, \\ 2 & \text{if } k = 2, \\ 2^{k-2} & \text{if } k \geq 3. \end{cases}$$

- If n has the prime power decomposition $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then:

$$\lambda(n) = \left[\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots, \lambda(p_r^{k_r})\right] := \operatorname{lcm}\left(\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots, \lambda(p_r^{k_r})\right).$$

2.4. The Baker–Wüstholz Theorem on Linear Forms in Logarithms (Rational Case)

In this subsection we present a simplified and completely elementary version of the main theorem of Baker and Wüstholz (1993), restricted exclusively to the case where the numbers α_i are **rational**, distinct from zero and one. In this way, any reference to algebraic number fields of degree greater than 1 is avoided.

Let α be a nonzero rational number, written in irreducible form as $\alpha = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{N}$ and $\gcd(|p|, q) = 1$. The **logarithmic height** of α is defined by

$$h(\alpha) = \ln \max(|p|, q).$$

The associated **modified height** is

$$h'(\alpha) = \max\big(|\ln \alpha|,\ 1\big),$$

Theorem 1 (Baker–Wüstholz, 1993 – rational version)**.** *Let $\alpha_1, \dots, \alpha_n$ be rational numbers, none zero or one. Consider the linear form*

$$L(z_1, \dots, z_n) = b_1 z_1 + \cdots + b_n z_n$$

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with integer coefficients $b_i \in \mathbb{Z}$, not all zero. If

$$A = L(\log \alpha_1, \ldots, \log \alpha_n) \neq 0$$

and $|b_j| > e$ then

$$\log |A| > -C(n,1)\, h'(\alpha_1) \cdots h'(\alpha_n)\, \ln(B),$$

where

$$C(n,1) = 18(n+1)!\, n^{n+1}\, 32^{n+2}\, \log(2n)$$

and

$$B = \max_j |b_j|.$$

Particular case: $A = a \ln 2 - b \ln 3$ with $a, b \in \mathbb{Z}$ not both zero and $|a|, |b| > e$. Take $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 3$ and the linear form $L(z_1, z_2) = az_1 - bz_2$. Let $B = \max(|a|, |b|)$.

Then:

1. $C(2,1) \approx 1.25594 \times 10^9$
2. $h'(2) = 1$ (since $\ln 2 \approx 0.693 < 1$)
3. $h'(3) = \ln 3 \approx 1.0986$

Then

$$\ln |a \ln 2 - b \ln 3| > -1.3798 \times 10^9 \cdot \log B,$$

or equivalently

$$|a \ln 2 - b \ln 3| > B^{-1.3798 \times 10^9}.$$

2.5. *The Ring of 2-Adic Integers $\mathbb{Z}_2$*

To establish a rigorous algebraic foundation for the subsequent analysis, we must introduce the ring of 2-adic integers and its topological properties. This provides the necessary framework to study sequences and limits within a context where the standard Euclidean metric is replaced by one based on divisibility by 2.

Definition 1 (2-adic Valuation and Norm)**.** *For any non-zero rational number $x \in \mathbb{Q}$, we can uniquely write it in the form $x = 2^v \dfrac{p}{q}$, where p, q, and v are integers, and both p and q are odd. We define the **2-adic valuation** as $Ord_2(x) = v$. Conventionally, we set $Ord_2(0) = \infty$.*
*The **2-adic norm** is defined as:*

$$||x||_2 = \begin{cases} 2^{-Ord_2(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This norm induces an ultrametric given by $d_2(x, y) = ||x - y||_2$.

Definition 2 (2-adic Integers)**.** *The ring of 2-adic integers, denoted as $\mathbb{Z}_2$, is the metric completion of the set of integers $\mathbb{Z}$ with respect to the norm $|| \cdot ||_2$. Every element $\alpha \in \mathbb{Z}_2$ admits a unique canonical representation as an infinite power series:*

$$\alpha = \sum_{i=0}^{\infty} a_i 2^i \quad \text{where } a_i \in \{0, 1\}$$

The sequence of coefficients $\{a_i\}_{i=0}^{\infty}$ is known as the 2-adic expansion of α.

Lemma 1. *A rational number $r = \dfrac{p}{q}$ (in simplest form) belongs to $\mathbb{Z}_2$ if and only if its denominator q is odd.*

Proof. Let $r = \frac{p}{q} \in \mathbb{Q}$ with $\gcd(p, q) = 1$. The rational number r belongs to $\mathbb{Z}_2$ if and only if $||r||_2 \leq 1$, which is equivalent to $Ord_2(r) \geq 0$. By definition, $Ord_2(r) = Ord_2(p) - Ord_2(q)$. Since p and q are coprime, they cannot both be even. If q is even, then $Ord_2(q) > 0$ and $Ord_2(p) = 0$, leading to $Ord_2(r) < 0$, meaning $r \notin \mathbb{Z}_2$. If q is odd, then $Ord_2(q) = 0$, so $Ord_2(r) = Ord_2(p) \geq 0$, and thus $r \in \mathbb{Z}_2$. □

The following theorem is fundamental for characterizing rational numbers within the 2-adic framework.

Theorem 2 (Eventually Periodic Expansion of Rationals). *Let $\alpha \in \mathbb{Z}_2$. The 2-adic expansion of α is eventually periodic if and only if α is a rational number.*

Proof. Suppose the 2-adic expansion of α is eventually periodic. Then, we can write $\alpha = A + 2^k B$, where $A \in \mathbb{Z}$ is the pre-periodic integer part (of length k) and B is the strictly periodic part with period T. We can express B as an infinite geometric series:

$$B = P \sum_{j=0}^{\infty} (2^T)^j = \frac{P}{1 - 2^T}$$

where P is the integer formed by the periodic block of bits. Since P and $1 - 2^T$ are integers, B is a rational number. Because A and 2^k are also integers, we conclude that $\alpha = A + 2^k B \in \mathbb{Q}$.

Let $\alpha = \dfrac{p}{q} \in \mathbb{Q} \cap \mathbb{Z}_2$. By Lemma 1, we know q must be odd. Without loss of generality, suppose $q > 0$. By Euler's Theorem, since $\gcd(q, 2) = 1$, there exists an integer $T \geq 1$ such that $2^T \equiv 1 \pmod{q}$. Therefore, $2^T - 1 = q \cdot m$ for some integer m. Thus, we can rewrite the fraction as:

$$\alpha = \frac{p}{q} = \frac{p \cdot m}{q \cdot m} = \frac{-pm}{1 - 2^T}$$

By applying classical long division, the term $-pm$ generates a sequence of remainders that must necessarily repeat because the divisor $1 - 2^T$ is constant. This induces a sequence of coefficients in the power series expansion of 2 that is eventually periodic, with a period dividing T. □

An important geometric consequence arises when considering the sign of rational numbers within their 2-adic expansion.

Lemma 2 (2-adic Expansion of Negative Numbers). *Let $\alpha \in \mathbb{Q} \cap \mathbb{Z}_2$. If $\alpha < 0$ (in the sense of the standard metric in $\mathbb{R}$), then the 2-adic expansion of α contains an infinite number of non-zero coefficients (infinite '1' bits). In particular, if α is a negative integer, its expansion possesses an infinite, uninterrupted tail of ones.*

Proof. For the case of a negative integer, consider $\alpha = -1$. In $\mathbb{Z}_2$, we can evaluate the sum of the infinite geometric series:

$$S = \sum_{i=0}^{\infty} 2^i = 1 + 2 + 4 + 8 + \ldots$$

Multiplying by 2, we obtain $2S = \sum_{i=1}^{\infty} 2^i = S - 1$. Solving for S under 2-adic arithmetic yields $S = -1$. Hence, the canonical representation of -1 is $\ldots 11111_2$. Any other negative integer $-n$ can be expressed using two's complement arithmetic, which guarantees that the most significant digits will all be '1' from a certain point onward.

For the general case of a negative rational fraction $\alpha = -p/q$ (with $p, q > 0$ and q odd), we know from Theorem 2 that its expansion is eventually periodic. Assume for the sake of contradiction that the expansion contains only a finite number of ones. This would imply that after some index N, all coefficients are 0. However, if a 2-adic expansion ends in an infinite tail of zeros, the infinite series $\sum_{i=0}^{\infty} a_i 2^i$ is structurally just a finite sum of positive powers of 2. In $\mathbb{R}$, a finite sum of non-negative terms must be a non-negative integer. This strictly defines a positive integer or zero, which contradicts the initial premise that $\alpha < 0$. Therefore, the periodic part of the expansion of any negative rational must contain at least one '1' bit, thereby ensuring that the overall expansion possesses infinitely many '1' bits. ☐

Lemma 3 (Purely Periodic Expansion of Rationals)**.** *Let $\alpha \in \mathbb{Q} \cap \mathbb{Z}_2$. The 2-adic expansion of α is purely periodic (meaning it possesses no pre-periodic block) if and only if $-1 \leq \alpha \leq 0$. In particular, it is non-trivially purely periodic if and only if $\alpha \in (-1, 0)$.*

Proof. Suppose the 2-adic expansion of α is purely periodic with a period of length $T \geq 1$. We can express α as a direct infinite geometric series:

$$\alpha = P \sum_{j=0}^{\infty} (2^T)^j = \frac{P}{1 - 2^T}$$

where P is the integer value formed by the periodic block of T bits. The maximum possible value for a T-bit integer occurs when all its bits are '1', yielding $P_{\max} = \sum_{i=0}^{T-1} 2^i = 2^T - 1$. Since $0 \leq P \leq 2^T - 1$, we can bound α as follows:

$$-1 = \frac{2^T - 1}{1 - 2^T} \leq \frac{P}{1 - 2^T} \leq \frac{0}{1 - 2^T} = 0$$

Excluding the trivial constant sequences where $P = 0$ (yielding $\alpha = 0$) and $P = 2^T - 1$ (yielding $\alpha = -1$), any strictly non-trivial purely periodic sequence maps to a rational number $\alpha \in (-1, 0)$.

Conversely, let $\alpha \in (-1, 0)$ be a rational number in $\mathbb{Z}_2$. By Lemma 1, $\alpha = -\frac{p}{q}$ with $0 < p < q$ and q odd. By Euler's Theorem, since $\gcd(q, 2) = 1$, there exists an integer $T \geq 1$ such that q divides $2^T - 1$, meaning $2^T - 1 = q \cdot m$ for some integer $m > 0$. We can rewrite α as:

$$\alpha = \frac{-p \cdot m}{q \cdot m} = \frac{-pm}{2^T - 1} = \frac{pm}{1 - 2^T}$$

Since $0 < p < q$, multiplying the inequality by m yields $0 < pm < qm = 2^T - 1$. Let $P = pm$. Because $0 < P < 2^T - 1$, P can be exactly represented as a positive T-bit binary integer. Expanding the fraction back into a geometric series yields:

$$\alpha = \frac{P}{1 - 2^T} = P \sum_{j=0}^{\infty} (2^T)^j = P + P \cdot 2^T + P \cdot 2^{2T} + \dots$$

This explicitly constructs a purely periodic 2-adic expansion for α without any pre-periodic terms, completing the proof. ☐

2.6. Dynamical System

A **discrete dynamical system** is a model of the evolution of a state over discrete time steps. Formally, it consists of a set X (called the *state space*) and a function $f : X \to X$ that describes how the state evolves from one time step to the next. The system is described by the equation:

$$x_{n+1} = f(x_n)$$

where $x_n \in X$ represents the state of the system at the n-th time step. The evolution of the system is typically studied by iterating the function, f starting from an initial state x_0. The sequence $\{x_n\}$, where $x_{n+1} = f(x_n)$, is called the *orbit* or *trajectory* of the initial state x_0.

Topologically Conjugate Dynamical Systems: Two discrete dynamical systems (X, f) and (Y, g) are said to be **topologically conjugate** if there exists a homeomorphism $h : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, "h"] \\
Y \arrow[r, "g"] & Y
\end{tikzcd}$$

In other words, the systems (X, f) and (Y, g) are topologically conjugate if there is a bijective function $h : X \to Y$ such that:

- h is a **homeomorphism**, meaning h is continuous, bijective, and its inverse h^{-1} is also continuous.
- The following relation holds:
$$h \circ f = g \circ h.$$

This means that the dynamics of f on X and g on Y are the same up to a change of coordinates given by h. The systems (X, f) and (Y, g) have the same qualitative behavior, such as the structure of orbits and periodic points, despite potentially differing in their specific representations.

Properties of Topological Conjugation with Respect to Orbits and Periodic Points:

- **Preservation of Orbits:** If $f : X \to X$ and $g : Y \to Y$ are two topologically conjugate dynamical systems, with a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, then h preserves the orbits of points. Specifically, for any point $x \in X$, the orbit of x under f is mapped to the orbit of $h(x)$ under g by h. Mathematically, this means:
$$h(f^n(x)) = g^n(h(x)) \text{ for all } n \geq 0.$$
- **Preservation of Periodic Points:** If $x \in X$ is a periodic point of f with period p, then $h(x) \in Y$ is a periodic point of g with the same period p. Specifically, if $f^p(x) = x$, then:
$$g^p(h(x)) = h(f^p(x)) = h(x).$$
Conversely, if $y \in Y$ is a periodic point of g with period p, then $h^{-1}(y) \in X$ is a periodic point of f with the same period p.

Part I: Elementary Theory of Infinite Systems of Diophantine Equations

3. Set Generate by θ and ψ^q

In this section, we delve into functions generated by the composition of two real linear functions, θ and ψ^q, focusing on their properties over integers. We define the set $\langle\theta, \psi^q\rangle$, representing compositions of these functions, and examine their orbits and associated sets of integers. Before delving into their properties, we introduce the crucial concept of the integer set of a function. Denoted as $\mathbb{E}(S)$, this set represents the integers generated by the orbit of the function S. We emphasize the one-to-one correspondence between functions of the same length and the partition of integers into sets based on this length. These results provide a solid foundation for a detailed understanding of the properties of these functions and their application in the study of iterative functions over rational numbers.

3.1. Summary of Propositions in the Section

1. **Definition 3:** Introduces the set $\langle\theta, \psi^q\rangle$, generated by two real linear functions θ and ψ^q.
2. **Definition 4 :** Defines the Integer set of a function.
3. **Lemma 4** Monotonicity of Integer Set Lemma.
4. **Proposition 1 :** Establishes a relation of Monotonicity in the entire sets concerning the composition of functions.
5. **Lemma 5** Establishes a characterization of the integer sets.

6. **Proposition 2 :** Establishes a one-to-one correspondence between functions of the same length and integer sets of the same length, and Affirms that the integer sets of functions of the same length are disjoint.
7. **Theorem 3:** Ensures that the integer sets are the disjoint union of the integer sets of functions in $\langle\theta,\psi^q\rangle$ with the same length.
8. **Proposition 3:** Guarantees the existence of a unique sequence of elements for a function $S\in\langle\theta,\psi^q\rangle$.

3.2. Set Generate by θ and ψ^q

The Generated Spaces, denoted as $\langle\theta,\psi^q\rangle$. These spaces arise from the iterative composition of functions, where the individual contributions of θ and ψ^q combine to form an enriched dynamic structure.

Definition 3 (Set Generated by θ and ψ^q). *Let $q\in\mathbb{Z}$ and $\theta,\psi^q:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $\theta(x)=\dfrac{x}{2}$ and $\psi^q(x)=\dfrac{3x+q}{2}$, then we define the set $\langle\theta,\psi^q\rangle$ as:*

$$S:\mathbb{R}\longrightarrow\mathbb{R};\quad S(x)=\bigodot_{i=1}^{n}s_i(x),s_i(x)=\theta(x),\psi^q(x)\textit{ for } i\geq 1,s_0=id$$

We will call the number n length of S.

Let define The integer sets of a function, denoted by $\mathbb{E}(S)$, represent the integer values that a specific function takes on its domain. Examining $\mathbb{E}(S)$ allows for the identification of patterns and regularities in the interaction of the function with integers, which is essential for understanding the structure of spaces generated by such functions.

Definition 4 (Integer Set of a Function). *Let $S:\mathbb{R}\longrightarrow\mathbb{R}$ a function, we called integer set of f or the integers of f the set:*

$$\mathbb{E}(S)=\{n\in\mathbb{Z};S(n)\in\mathbb{Z}\}.$$

and we called the $k-$integer set of f the set

$$\mathbb{E}^k(S)=\{n\in\mathbb{Z};S(n),S^2(n),\dots,S^k(n)\in\mathbb{Z}\}.$$

In the following Proposition we are going to see that integer sets have a monotonic behavior concerning the composition of linear functions, this property will be fundamental to studying $\mathbb{E}^k(S)$.

Lemma 4 (Monotonicity of Integer Set Lemma). *Let $A,B,a,b\in\mathbb{Z}$ such that $(A,2)=(B,2)=1$ and $\alpha,\beta\in\mathbb{N}$. Let $S(x)=\dfrac{Ax+a}{2^\alpha}$ and $H(x)=\dfrac{Bx+b}{2^\beta}$, then $\mathbb{E}(H\circ S)\subset\mathbb{E}(S)$.*

Proof. In fact, we have $H\circ S(x)=\dfrac{ABx+Ba+2^\alpha b}{2^{\alpha+\beta}}$ Since $(AB,2)=1$ there are solutions. Let $x_0\in\mathbb{E}(H\circ S)$ then by definition $\dfrac{ABx_0+Ba+2^\alpha b}{2^{\alpha+\beta}}\in\mathbb{Z}$, then we have

$$\frac{ABx_0+Ba+2^\alpha b}{2^{\alpha+\beta}}=\frac{B}{2^\beta}\left(\frac{Ax_0+a}{2^\alpha}\right)+\frac{b}{2^\beta}\in\mathbb{Z}\Rightarrow B\left(\frac{Ax_0+a}{2^\alpha}\right)+b\in\mathbb{Z}$$

as $(B,2)=1$ then we have

$$\frac{Ax_0+a}{2^\alpha}\in\mathbb{Z}$$

□

Proposition 1 (Monotonicity of Integer Set)**.** *Let* $\bigodot_{i=1}^{n} s_i(x) \in \langle \theta, \psi^q \rangle$ *with* $s_i \in \{\theta, \psi^q\}$ *and* $L, M \in \mathbb{N}$. *Then if* $n \geq L \geq M$ *we have:*

$$\mathbb{E}\left(\bigodot_{i=1}^{L} s_i(x)\right) \subset \mathbb{E}\left(\bigodot_{i=1}^{M} s_i(x)\right)$$

Proof. As the functions generated by $\langle \theta, \psi^q \rangle$ are linear of the form $\dfrac{3^b x + N}{2^a}$ with $a, b.N \in \mathbb{N}$. The result follows inductively from Lemma 4. □

Example 3. *Let* $S_n(x) = \theta^{2n}\psi(x) \in \langle \theta, \psi \rangle$. *We will calculate the integer set of* S_n

$$2^{2n}y - 3x = 1$$

we have that $y = 1$ *and* $\dfrac{2^n - 1}{3} \in \mathbb{N}$ *are solutions of the Diophantine Equation, then the integer set is:*

$$\mathbb{E}(\theta^{2n}\psi) = \frac{2^n - 1}{3} + 2^{2n}\mathbb{Z}$$

Let $m \geq n$ *then*

$$\mathbb{E}(\theta^{2m}\psi) \subset \mathbb{E}(\theta^{2n}\psi)$$

indeed

$$\frac{3\left(\dfrac{2^m - 1}{3}\right) + 1}{2^n} = 2^{m-n} \in \mathbb{N}$$

The following lemma states that $u \in \mathbb{E}^k(S)$ if and only if $u \in \mathbb{E}(S^k)$.

Lemma 5 (Containment in Integer Sets)**.** *Let* $\{s_i\}_1^j \in \{\theta, \psi^q\}$, *then* $u \in \mathbb{E}\left(\bigodot_{i=1}^{j} s_i(x)\right)$ *if and only if* $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{E}\left(\bigodot_{i=r+1}^{j} s_i(x)\right)$ *with* $r \leq j$. *In particular if* $S \in \{\theta, \psi^q\}$, *then* $\mathbb{E}^k(S) = \mathbb{E}(S^k)$.

Proof. Let $u \in \mathbb{E}\left(\bigodot_{i=1}^{j} s_i(x)\right)$ then by proposition 1 we have $u \in \mathbb{E}\left(\bigodot_{i=1}^{r} s_i(x)\right)$ with $r < j$, then $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{Z}$. On the other hand, we have $\bigodot_{i=r+1}^{j} s_i\left(\bigodot_{i=1}^{r} s_i(u)\right) \in \mathbb{Z}$ then $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{E}\left(\bigodot_{i=r+1}^{j} s_i(x)\right)$. If $\bigodot_{i=1}^{r} s_i(x) \in \mathbb{E}\left(\bigodot_{i=r+1}^{j} s_i(x)\right)$ with $r < j$ then $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{Z}$ so

$$\bigodot_{i=r+1}^{j} s_i\left(\bigodot_{i=1}^{r} s_i(u)\right) = \bigodot_{i=1}^{j} s_i(u) \in \mathbb{Z}$$

then $u \in \mathbb{E}\left(\bigodot_{i=1}^{j} s_i(x)\right)$. ☐

In the following proposition, We will demonstrate that the integer sets associated with functions of the same length are disjoint. That is, if two integer sets share at least one element, then the functions must be the same.

Proposition 2 (One-to-One Correspondence and Disjointedness)**.** *Let $S, H \in \langle \theta, \psi^q \rangle$ of length k with q odd number, if $H \neq S$ if and only if $\mathbb{E}(H) \cap \mathbb{E}(S) = \emptyset$.*

Proof. Let $S(x) = \bigodot_{i=1}^{j} s_i(x)$ and $H(x) = \bigodot_{i=1}^{j} h_i(x)$ with $s_i, h_i \in \{\theta(x), \psi(x)\}$ and let k_0 be the largest index such that $s_i(x) = h_i(x)$ for all $i < k_0$
If $k_0 = 1$ This means that they have different first terms. Then $\mathbb{E}(S) \subset \mathbb{E}(\theta) = 2\mathbb{Z}$ and $\mathbb{E}(H) \subset \mathbb{E}(\psi^q) = 2\mathbb{Z} + 1$ or, $\mathbb{E}(H) \subset \mathbb{E}(\theta) = 2\mathbb{Z}$ and $\mathbb{E}(S) \subset \mathbb{E}(\psi^q) = 2\mathbb{Z} + 1$ in either case we have $\mathbb{E}(H) \cap \mathbb{E}(S) = \emptyset$.
Suppose that exist $u \in \mathbb{E}(H) \cap \mathbb{E}(S)$ by proposition 1 we have

$$u \in \mathbb{E}\left(\bigodot_{i=1}^{r} s_i(x)\right) \cap \mathbb{E}\left(\bigodot_{i=1}^{r} h_i(x)\right) \text{ with } r \leq j.$$

Taken $r = k_0$

$$u \in \mathbb{E}\left(\bigodot_{i=1}^{k_0} s_i(x)\right) \cap \mathbb{E}\left(\bigodot_{i=1}^{k_0} h_i(x)\right).$$

and by lemma 5

$$\bigodot_{i=1}^{k_0-1} s_i(u) = \bigodot_{i=1}^{k_0-1} h_i(u) \in \mathbb{E}\big(s_{k_0}(x)\big) \cap \mathbb{E}\big(s_{k_0}(x)\big) = 2\mathbb{Z} \cap (2\mathbb{Z} + 1) = \emptyset$$

which is a contradiction, On the other hand if $\mathbb{E}(H) \cap \mathbb{E}(S) = \emptyset$ then $S \neq H$ otherwise we would have

$$\mathbb{E}(S) = \mathbb{E}(H) = \emptyset$$

however, neither set can be empty ☐

As a consequence of the above proposition we have

Theorem 3 (Partition of Integers)**.** *Let $\{S_j\}_{j \in \mathbb{N}} \subset \langle \theta, \psi^q \rangle$ with length k and q odd number, then*

$$\mathbb{Z} = \bigsqcup_{S \text{ with length } k} \mathbb{E}(S)$$

i.e., the sets of integers are equal to the disjoint union of the integer sets of functions S of length k

Proof. it is evident that

$$\bigsqcup_{S \text{ with length } k} \mathbb{E}(S) \subset \mathbb{Z}$$

To prove the other contention we consider the $q-$Collatz function defined by $Col_q : \mathbb{Z} \to \mathbb{Z}$ given by

$$Col_q(u) = \begin{cases} \dfrac{3u+q}{2} & \text{if } u \text{ odd} \\ \dfrac{u}{2} & \text{if } u \text{ even,} \end{cases}$$

let's take an integer u and calculate its k-th orbit, this orbit can be written as compositions of functions in $\langle \theta, \psi^q \rangle$, let's call the resulting function S since all the values of the orbit are integers, we have by the lemma 5 we can conclude that u is in the entire set of the function S.

$$u \in \mathbb{E}(S) \subset \bigsqcup_{S \text{ with length } k} \mathbb{E}(S).$$

□

As a consequence of the Theorem, we have that each element generated by the functions θ and ψ^q can be generated by a single combination.

Proposition 3 (Uniqueness of Basis Representation). *Let $S \in \langle \theta, \psi^q \rangle$ with q odd number. Then there exists a unique sequence of k elements $\{s_j\}_{j=1}^{k}$ with $s_j \in \{\theta, \psi^q\}$ such that*

$$S(x) = \bigodot_{j=1}^{k} s_j(x)$$

Proof. Since $S \in \langle \theta, \psi^q \rangle$, then there exists a sequence of k_0 elements $\{s_j\}_{j=1}^{k_0}$ with $s_j \in \{\theta, \psi^q\}$ such that $S(x) = \bigodot_{j=1}^{k_0} s_j(x)$. Suppose for absurdity, that there is another sequence but of l_0 elements $\{h_j\}_{j=1}^{l_0}$ such that $S(x) = \bigodot_{j=1}^{l_0} h_j(x)$. Let $u \in \mathbb{E}(S)$, we have the following cases

1. If $k_0 > l_0$. We have

$$S(x) = \bigodot_{j=1}^{k_0} s_j(x) = \bigodot_{j=1}^{l_0} h_j(x)$$

$$u \in \mathbb{E}\left(\bigodot_{j=1}^{k_0} s_j(x) \right) = \mathbb{E}\left(\bigodot_{j=1}^{l_0} h_j(x) \right)$$

by Proposition 1 we have $u \in \mathbb{E}(s_1) \cap \mathbb{E}(h_1)$, since $s_1, h_1 \in \{\theta, \psi^q\}$ and $\mathbb{E}(\theta) \cap \mathbb{E}(\psi^q) = \emptyset$ then $s_1 = h_1$. Since s_1 and h_1 are invertible functions, we have

$$\bigodot_{j=2}^{k_0} s_j(x) = \bigodot_{j=2}^{l_0} h_j(x).$$

Following the same idea up to k_0, we have

$$\bigodot_{j=k_0}^{l_0} s_j(x) = x$$

The latter is impossible since the slope of the resulting line is of the form $\frac{3^b}{2^a}$ with $a, b \in \mathbb{N}$. The case $k_0 < l_0$ is completely analogous, therefore the case where k_0 and l_0 are different is not possible.

2. If $k_0 = l_0$. Since the sequences are different, there must exist some $k < k_0$ such that

$$\bigodot_{j=1}^{k} s_j(x) \neq \bigodot_{j=1}^{k} h_j(x)$$

then by Proposition 2 we have

$$\mathbb{E}\left(\bigodot_{j=1}^{k} s_j(x)\right) \cap \mathbb{E}\left(\bigodot_{j=1}^{k} h_j(x)\right) = \emptyset$$

However, this is a contradiction to the Proposition 1, because $u \in \mathbb{E}\left(\bigodot_{j=1}^{k} s_j(x)\right) \cap \mathbb{E}\left(\bigodot_{j=1}^{k} h_j(x)\right)$ for all $k \leq k_0$. Then both sequences must be identical.

□

4. Stability and Instability of Integer Set

In this section, we delve into the stability and instability of sequences associated with integer sets. We begin by defining functions ρ_0 and ρ_1 that map real functions to integers. We introduce the concepts of positive and negative stability for sequences $\{S_j(x)\}_{j=1}^{\infty}$. The Monotonicity of ρ_0 and ρ_1 is established through Proposition 1, demonstrating the non-decreasing of ρ_0 and the non-increasing of ρ_1 for a given sequence $S_j(x)$. Further, the Proposition formally defines positive and negative stability, incorporating limits and intersections of sets. The ensuing Stability Limit Theorem (4) establishes the asymptotic behavior of the integer set of an iterative sequence.

4.1. Summary of Propositions in the Section

1. **Definition 5 :** Definition of functions ρ_0 and ρ_1.
2. **Proposition 4:** Monotonicity of the functions ρ_0 and ρ_1.
3. **Definition 6:** Definition of positively (negatively) stable (unstable) sequences.
4. **Theorem 4:** Establishes the asymptotic behavior of the integer set when we have a positively (negatively) stable (unstable) sequence.

4.2. Stability and Instability of Integer Set

We initiate this section by introducing functions that associate each integer set with its minimum positive integer value and maximum negative integer value. These values are determined by the solutions closest to zero for the variable x in the Diophantine equation $2^a y - 3^b x = N$. This equation is representative of the Diophantine equation linked to an element within the space generated by ψ and θ.

Definition 5 (ρ_0 and ρ_1 functions.)**.** *Define the function* $\rho_0, \rho_1 : \{f : \mathbb{R} \longrightarrow \mathbb{R} : f \text{ function }\} \to \mathbb{Z}$ *by*

$$\rho_0(f) = \min\{\mathbb{E}(f) \cap \mathbb{N}\}$$

and

$$\rho_1(f) = \max\{\mathbb{E}(f) \cap -\mathbb{N}\}$$

As a consequence of the Proposition 1. We have that the functions ρ_0 and ρ_1 are monotone.

Proposition 4 (Monotonicity of ρ_0 and ρ_1)**.** *Let* $\{S_j(x)\}_{j\in\mathbb{N}} \subset \langle\theta,\psi^q\rangle$ *given by* $S_j(x) = \bigodot_{i=1}^{j} s_i(x)$ *then* $\rho_0(S_j(x))$ *is a non-decreasing function, and* $\rho_1(S_j(x))$ *it is a non-increasing function.*

Proof. By the proposition 1 we have that

$$\rho_0(S_{j+1}), \rho_1(S_{j+1}) \in \mathbb{E}(S_{j+1}) \subset \mathbb{E}(S_j)$$

then $\rho_0(S_j) \leq \rho_0(S_{j+1})$ and $\rho_1(S_j) \geq \rho_1(S_{j+1})$. ☐

From the result of the proposition above, we are going to make a classification of the sequences $S_j = \bigodot_{i=1}^{j} s_i(x)$ according to the behavior of the functions ρ_0 and ρ_1.

Definition 6 (Stability of Sequences)**.** *Let* $\{S_j(x)\}_{j=1}^{\infty}$ *sequence on* $\langle\theta,\psi^q\rangle$ *given by* $S_j = \bigodot_{i=1}^{j} s_i(x)$. *We will say that* $\{S_j(x)\}_{j=1}^{\infty}$ *is positively stable if* $\lim_{j\to\infty} \rho_0\left(\bigodot_{i=1}^{j} S_k(j)\right) < \infty$ *otherwise we will say that it is positively unstable. On the other hand, we will say that* $\{S_j(x)\}_{j=1}^{\infty}$ *is negatively stable if* $\lim_{j\to\infty} \rho_1\left(\bigodot_{i=1}^{j} S_k(j)\right) > -\infty$, *otherwise, we will say that it is negatively unstable.*

Now we will give the central theorem of this section, which establishes the asymptotic behavior of the integer sets of S_j from the stability or stability of this.

Theorem 4 (Stability Limit Theorem)**.** *Let* $\{S_j(x)\}_{j=1}^{\infty} \subset \langle\theta,\psi^q\rangle$ *and*

$$\mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) \cap \mathbb{N}$$

and

$$\mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) \cap -\mathbb{N}.$$

We have:

1. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is positively stable then*

$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \left\{\lim_{k\to\infty} \rho_0\left(\bigodot_{j=1}^{k} S_j(x)\right)\right\}$$

2. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is positively unstable, then*

$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \emptyset$$

analogously

1. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is negatively stable then*

$$\lim_{k\to\infty}\mathbb{E}^{-}\left(\bigodot_{j=1}^{k}S_j(x)\right)=\bigcap_{k=1}^{\infty}\mathbb{E}^{-}\left(\bigodot_{j=1}^{k}S_j(x)\right)=\left\{\lim_{k\to\infty}\rho_1\left(\bigodot_{j=1}^{k}S_j(x)\right)\right\}$$

2. *if* $\{S_j(x)\}_{j=1}^{\infty}$ *is positively unstable, then*

$$\lim_{k\to\infty}\mathbb{E}^{-}\left(\bigodot_{j=1}^{k}S_j(x)\right)=\bigcap_{k=1}^{\infty}\mathbb{E}^{-}\left(\bigodot_{j=1}^{k}S_j(x)\right)=\emptyset$$

Proof. : Let $\rho_0^k=\rho_0\left(\bigodot_{j=1}^{k}S_j(x)\right)$ and $\bigodot_{j=1}^{k}S_j(x)=\dfrac{3^{B_k}x+N}{2^{A_k}}$ with A_k =numbers from θ to $\bigodot_{j=1}^{k}S_j(x)$ and B_k =numbers from ψ^q to $\bigodot_{j=1}^{k}S_j(x)$. We have

$$\mathbb{E}\left(\bigodot_{j=1}^{k}S_j(x)\right)=\rho_0^k+2^{A_k}\mathbb{N}.$$

supposed that $\{S_j(x)\}_{j=1}^{\infty}$ is stable, we will first prove that $\lim\limits_{k\to\infty}\mathbb{E}^{+}\left(\bigodot_{j=1}^{k}S_j(x)\right)$ is non-empty. By Proposition 1 the sequence of sets $\mathbb{E}^{+}\left(\bigodot_{j=1}^{k}S_j(x)\right)$ is a decreasing sequence of sets i.e. that the next set is a subset of the previous one, then the limit set corresponds to the intersection of all the sets of the sequence.

$$\lim_{k\to\infty}\mathbb{E}^{+}\left(\bigodot_{j=1}^{k}S_j(x)\right)=\bigcap_{k\in\mathbb{N}}\mathbb{E}^{+}\left(\bigodot_{j=1}^{k}S_j(x)\right)$$

Now since ρ_0^k is a function of the natural ones in the natural ones and is convergent, it implies that this function reaches its limit in a finite amount of steps

$$\rho_0^k=cts \text{ for all } k>K$$

this implies

$$\lim_{k\to\infty}\rho_0^k\in\mathbb{E}^{+}\left(\bigodot_{j=1}^{k}S_j(x)\right)\text{ for all } k\in\mathbb{N}$$

then the limit set is non-empty.

Now we will prove the limit set contains a single element. Suppose there exists another element u_0 that is contained in all positive integer sets, then there exists a non-negative integer t such that

$$u_0=\rho_0^k+2^{A_k}t$$

without loss of generality, we can assume that ρ_0^k is constant. Solving the equation in terms of t, we have

$$t=\frac{u_0-\rho_0^k}{2^{A_k}}$$

This solution is a fraction less than 1 for k large enough., which contradicts the fact that t is an integer.

Now let us take the unstable case. Suppose there exists an element u_0 in the limiting set i.e. an element that is contained in all non-negative integer sets, then there exists t a non-negative integer such that

$$u_0 = \rho_0^k + 2^{A_k}t \geq \rho_0^k$$

as ρ_0^k diverges and u_0 constant, then there exists a K such that ρ_0^k is greater than u_0, then u_0 cannot belong to any integer set with $k > K$, which is a contradiction. Analogously for the other case. $\square$

Example 4. $S_n(x) = \theta^{2n}\psi(x)$ *is positively and negatively unstable. Indeed, by example 3, we have* $\mathbb{E}(\theta^{2n}\psi(x)) = \frac{2^n - 1}{3} + 2^{2n}\mathbb{Z}$ *where* $\rho_0(\theta^{2n}\psi(x)) = \frac{2^n - 1}{3} \to \infty$ *as* $n \to \infty$ *then* $\lim\limits_{n\to\infty} \mathbb{E}^+(\theta^{2n}\psi(x)) = \emptyset$*. On the other hand* $\rho_1(\theta^{2n}\psi(x)) = \frac{2^n - 1}{3} - 2^{2n} \to -\infty$.

Example 5. $S_n(x) = (\theta\psi(x))^n$ *is positively unstable and negatively stable. Let's calculate the integer set* $\mathbb{E}((\theta\psi(x))^n)$*, let's observe that*

$$\theta\psi(-1) = \frac{3(-1) + 1}{2} = -1$$

Then $\mathbb{E}(\theta\psi(x))^n = -1 + 2^n\mathbb{Z}$ *then we have* $\rho_1((\theta\psi(x))^n) = -1$ *and* $\rho_0((\theta\psi(x))^n) = 2^n - 1 \to \infty$ *as* $n \to \infty$

5. Coding of the Orbits

In this section, we will delve into the study of the coding of the orbits of the Collatz function. The main results of this section are the invariance of the coding between Cod_q and Cod on the fractions with denominator q and the one-to-one identification of each element of $\mathbb{Q}_{odd}$ with its coding.

5.1. Summary of Propositions in the Section

1. **Definition 7:** Coding maps and Σ_2^* the space of sequences 0 and 10.
2. **Proposition 5:** General form of the elements generated by ψ and θ.
3. **Proposition 6:** Fist Cod invariance: $Cod(S) = Cod_q(S_q)$.
4. **Definition 10:** Definition of $Col_q : \mathbb{Z} \to \mathbb{Z}$.
5. **Definition 8:** Extension of Collatz function on $\mathbb{Q} = \mathbb{Q}_{odd} \cup \mathbb{Q}_{even}$.
6. **Proposition 7**: Extension of Collatz function on $\mathbb{Q}$ is well-defined.
7. **Proposition 8:** $Col(\mathbb{Q}_{odd}) \subset \mathbb{Q}_{odd}$.
8. **Definition 9:** Extension of the Collatz function on $\mathbb{Q}_{odd}$.
9. **Proposition 9:** $Col - Col_q$ equivalence : if $\frac{p}{q} \in \mathbb{Q}_{odd}$ then $Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$.
10. **Proposition 10:** Second Cod invariance: $Cod^k\left(\frac{p}{q}\right) = Cod_q^k(p)$.
11. **Proposition 11:** $p \in \mathbb{E}(S)$ if and only if $Cod_q^k(p) = Cod_q(S)$.
12. **Proposition 12** $p \in \mathbb{E}(S)$ if and only if $Cod^k\left(\frac{p}{q}\right) = Cod_q(S)$.
13. **Proposition 13:** $Cod^k\left(\frac{p}{q}\right) = Cod^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right)$.
14. **Definition 11:** The Coding set $Cod^k(\xi)$.
15. **Proposition 14:** Monotonicity of the Coding set $Cod^{k+1}(\xi) \subset Cod^k(\xi)$.
16. **Proposition 15:** Generating property: if $\frac{p}{q}, \frac{t}{r} \in Cod^k(\xi)$ then $\frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr}$.
17. **Theorem 5:** Uniqueness of the full coding $\mathbb{Q}_{odd}$.

5.2. Coding of the Orbits

It is a common practice in dynamical systems to encode orbits based on specific criteria. In our case, we encode the orbits of the Collatz function according to the parity of their elements, assigning the value 1 when they are odd and 0 when they are even. Since our primary focus is on the Collatz function over $\mathbb{Q}_{\text{odd}}$, we modify this initial coding by assigning 10 when an element is odd, instead of just 1. We denote the space where these codings reside by Σ_2^*, as it is a subset of the sequence space consisting of 0s and 1s, usually denoted in dynamics by Σ_2. Formally, we express this as

Definition 7 (Coding of the Orbits)**.** *Let $S \in \langle \theta, \psi^q \rangle$ with length k. We define the following application $Cod_q : \langle \theta, \psi^q \rangle \to \{0, 10\}^k$ defined by*

$$Cod_q\left(\bigodot_{j=1}^{k} s_j(x) \right) = \{\xi_j\}_{j=1}^{k} \text{ with } s_j \in \{\theta, \psi^q\}$$

with

$$\xi_j = \begin{cases} 0 & \text{if} \quad s_j(x) \text{ is } \theta(x) \\ 10 & \text{if} \quad s_j(x) \text{ is } \psi^q(x) \end{cases}$$

To rigorously examine the properties of the coding, it is essential to establish a precise form for the elements generated by ψ and θ.

Proposition 5 (General form of S)**.** *Let $S \in \langle \theta, \psi^q \rangle$ and Let $\{\theta_j\}_{j=1}^{k+1} \in \mathbb{N}_0 \times \mathbb{N} \times \ldots \times \mathbb{N}$ and $Cod_q(S) = 0^{\theta_1}10^{\theta_2} \ldots 0^{\theta_k}10^{\theta_{k+1}}$ and let $a_j = \sum_{i=1}^{j} \theta_i$ and $N : \Sigma_2^* \to \mathbb{N}$ defined as*

$$N\big(Cod_q(S)\big) = \begin{cases} 3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k} & \text{if} \quad k > 0 \\ 0 & \text{if} \quad k = 0 \end{cases}$$

then

$$S(x) = \frac{3^k x + qN(Cod_q(S))}{2^{a_{k+1}}}$$

Proof. We will prove by induction on k. For $a_{k+1} = 1$ we have

1. $Cod_q(\psi^q(x)) = 10$, then, $k = 1, a_1 = 0, a_2 = 1$ and $N = 3^0 2^0 = 1$ then $S(x) = \dfrac{3x+q}{2} = \psi^q(x)$.
2. $Cod_q(\theta(x)) = 0$, then, $k = 0, a_1 = 1$ and $N = 0$ then $S(x) = \dfrac{3^0 x + 0q}{2} = \dfrac{x}{2} = \theta(x)$.

Suppose the statement is true up to k, let $S \circ H \in \langle \theta, \psi^q \rangle$ of length a_{k+1} with H of length a_k.

Claim 1: $Cod_q(\psi^q \circ H) = Cod_q(H)10$. We have:

1. $a_{k+2}(Cod_q(H)10) = a_{k+1} + 1$ and $a_j(Cod_q(H)10) = a_j(Cod_q(H))$ for $j \leq k+1$.
2. $N(Cod_q(H)10) = 3^{k+1}2^{a_1} + 3^k 2^{a_2} + \ldots + 3^1 2^{a_k} + 2^{a_{k+1}}$

On the other hand, we have:

$$\begin{aligned}\psi^q\left(\frac{3^k x + qN(Cod_q(H))}{2^{a_{k+1}}}\right) &= \frac{3^{k+1}x + q3N(Cod_q(H)) + q2^{a_{k+1}}}{2^{a_{k+1}+1}} \\ &= \frac{3^{k+1}x + q3(3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k}) + q2^{a_{k+1}}}{2^{a_{k+1}+1}} \\ &= \frac{3^{k+1}x + q(3^k 2^{a_1} + 3^{k-1}2^{a_2} + \ldots + 3^1 2^{a_k} + 2^{a_{k+1}})}{2^{a_{k+1}+1}} \\ &= \frac{3^{k+1}x + qN(Cod_q(H)10)}{2^{a_{k+2}}}\end{aligned}$$

where we observe that the values coincide with those calculated.

Claim 2: $Cod_q(\theta \circ H) = Cod_q(H)0$. We have

1. $a_{k+2}(Cod_q(H)0) = a_k(Cod_q(H)) + 1$.
2. $N(Cod_q(H)0) = 3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k} = N(Cod_q(H))$.

On the other hand, we have

$$\theta\left(\frac{3^k x + qN(Cod_q(H))}{2^{a_{k+1}}}\right) = \frac{3^k x + qN(Cod_q(H)0)}{2^{a_{k+2}}}$$

where we observe that the values coincide with those calculated, then the statement is true. □

Now we will see the first property of the coding

Proposition 6 (First Cod invariance). *Let* $S \in \langle \theta, \psi \rangle$ *given by* $S(x) = \dfrac{3^b x + N}{2^a}$ *and* q *odd number, we defined* $S_q \in \langle \theta, \psi^q \rangle$ *given by* $S_q(x) = \dfrac{3^b x + qN}{2^a}$. *Then* $Cod(S) = Cod_q(S_q)$.

Proof. Let $q \in \mathbb{Z}$, since q is odd number then does not change the parity of the argument. To prove that they have the same coding, we have to prove that they have the same decomposition in principle, except that where there is ψ we have a ψ^q. let us observe that q has commutative properties with θ and ψ.

1. $q\theta(x) = \dfrac{qx}{2} = \theta(qx)$.
2. $q\psi(x) = q\left(\dfrac{3x+1}{2}\right) = \dfrac{3(qx)+q}{2} = \psi^q(qx)$

As $S \in \langle \theta, \psi \rangle$ then there exists $s_j \in \{\theta, \psi\}$ such that

$$S(x) = \bigodot_{j=1}^{a} s_j(x).$$

For convenience we will denote $s_j^q \in \{\theta, \psi^q\}$. Then we have

$$S_q(x) = \frac{3^b x + qN}{2^a} = q\left(\frac{3^b\left(\frac{x}{q}\right) + N}{2^a}\right) = q\bigodot_{j=1}^{a} s_j\left(\frac{x}{q}\right) = \bigodot_{j=1}^{a} s_j^q\left(q\frac{x}{q}\right) = \bigodot_{j=1}^{a} s_j^q(x)$$

We have that if s_j is θ then s_j^q is still θ and if s_j is ψ then s_j^q corresponds to ψ^q. By Proposition 3 we have that the coding of S_q has to be the same as that of S. □

Let us contemplate a generalization of the Collatz function applied to integers. In this variant, rather than adding 1, the function adds $q \in \mathbb{Z}$, where q is an odd integer. Subsequently, we will establish the compatibility of this generalization with the extension of the Collatz function to $\mathbb{Q}_{odd}$.

5.3. Extension of the Collatz function on $\mathbb{Q}$

As the concept of parity is a concept defined for integers, the Collatz function can be naturally extended to the set of integers. This concept is not trivially extended to the set of rational numbers, as there is no unique representation, We are going to consider a modification of extension on the rationals proposed by Lagaria in [10], Lagaria defined the Collatz function for fractions $\frac{p}{q}$ such that $(q,2) = (q,3) = 1$. We will distinguish two subsets of $\mathbb{Q}$. The set of rationals with odd denominators, denoted by $\mathbb{Q}_{odd}$, and the set of rationals with even denominators such that the numerator and denominator are co-prime, denoted by $\mathbb{Q}_{even}$. We will say that a rational number in $\mathbb{Q}_{\text{odd}}$ is odd if the numerator is odd, and it is even if its numerator is even. In the case of $\mathbb{Q}_{\text{even}}$, since the denominator is already even and due to coprimality, all elements are odd. We are going to consider the following extension of the Collatz function.

Definition 8 (Extension of the Collatz function)**.** *Let's consider the following sets*

$$\mathbb{Q}_{odd} = \left\{ \frac{p}{q} \in \mathbb{Q}, \text{ such that } q \text{ is odd } \right\}$$

and

$$\mathbb{Q}_{even} = \left\{ \frac{p}{q} \in \mathbb{Q}, \text{ such that } q \text{ is even and } (p,q) = 1 \right\}.$$

We defined the Collatz's function by $Col : \mathbb{Q}_{odd} \cup \mathbb{Q}_{even} \to \mathbb{Q}_{odd} \cup \mathbb{Q}_{even}$ *by*

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

We are going to show that the extension of the Collatz function that we defined is well-defined on $\mathbb{Q}_{odd}$.

Proposition 7 (Well-Defined)**.** *The Collatz's function on* $\mathbb{Q}_{odd}$ *is well-defined.*

Proof. We are going to show that Col is well-defined over $\mathbb{Q}_{odd}$. Let $p, q \in \mathbb{Z}$ with $q \neq 0$ and $(q,2) = (p,q) = 1$. Let λ an odd number, then

$$Col\left(\frac{\lambda p}{\lambda q}\right) = \begin{cases} \dfrac{3\lambda p + \lambda q}{\lambda q} & \text{if } \lambda p \text{ odd} \\ \dfrac{\lambda p}{2\lambda q} & \text{if } \lambda p \text{ even,} \end{cases} = \begin{cases} \dfrac{3p+q}{q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases} = Col\left(\frac{p}{q}\right)$$

□

Let's observe that when we apply the Collatz function to $\frac{p}{q} \in \mathbb{Q}_{\text{odd}}$ with odd number, we always obtain a fraction with an even numerator, and when applied to $\mathbb{Q}_{\text{even}}$, we always obtain an odd number. This will be very important since in section 5, we are going to define how to coding the orbits, assigning 1 if it is odd and 0 if it is even, in the case of $\mathbb{Q}_{\text{even}}$, we will have that all its elements have the same encoding which is $1111\ldots$ unlike $\mathbb{Q}_{\text{odd}}$, where the codings will be generated by 10 and

0. For this reason we are going to work mainly on $\mathbb{Q}_{\text{odd}}$, let's simplify the Collatz function a bit, as $Col : \mathbb{Q}_{\text{odd}} \to \mathbb{Q}_{\text{odd}} \cup \mathbb{Q}_{\text{even}}$ given by

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{2q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

Proposition 8 (Invariance of $\mathbb{Q}_{odd}$)**.** *The Collatz function defined above satisfies that* $Col(\mathbb{Q}_{odd}) \subset \mathbb{Q}_{odd}$

Proof. We will show that Col does not change the parity of the numerator.

1. if p is odd, we have $3p + q = 2r$ with $r \in \mathbb{Z}$, then

$$\frac{3p+q}{2q} = \frac{2r}{2q} = \frac{r}{q}$$

Since q is odd, we have independent of the simplification $\frac{r}{q} \in \mathbb{Q}_{odd}$.

2. if $p = 2r$ with $r \in \mathbb{Z}$, we have

$$\frac{p}{2q} = \frac{2r}{2q} = \frac{r}{q}$$

Since q is odd, we have independent of the simplification, we have $\frac{r}{q} \in \mathbb{Q}_{odd}$.

□

Considering the proposition above, we are going to define the Collatz function on $\mathbb{Q}_{odd}$ as

Definition 9 (Extension of the Collatz function on $\mathbb{Q}_{odd}$)**.** *We define the Collatz Function on* $\mathbb{Q}_{odd}$ *by* $Col : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{2q} & \textit{if } p \textit{ odd} \\ \dfrac{p}{2q} & \textit{if } p \textit{ even,} \end{cases}$$

Example 6. *Let* $\frac{5}{7} \in \mathbb{Q}_{odd}$ *and* $\frac{5}{2} \in \mathbb{Q}_{even}$ *we have:*

$$\mathcal{O}\left(\frac{5}{7}\right) : \frac{5}{7} \to \frac{11}{7} \to \frac{20}{7} \to \frac{10}{7} \to \frac{5}{7}$$

and

$$\mathcal{O}\left(\frac{5}{2}\right) : \frac{5}{2} \to \frac{17}{2} \to \frac{53}{2} \to \frac{161}{2} \to \frac{485}{2} \to \ldots \to \frac{3^k 5 + (3^k - 1)}{2} \to \ldots \to \infty$$

We can observe that the extension of the conjecture on the set of rationals is false, since we have found a fraction with a divergent orbit, The first objective of the work is to show that there are no divergent orbits in $\mathbb{Q}_{odd}$*..*

We define the following generalization of the Collatz function.

Definition 10 (The Col_q map)**.** *Let* $q \in \mathbb{Z}$*, we define the* $q-$*Collatz function defined by* $Col_q : \mathbb{Z} \to \mathbb{Z}$ *given by*

$$Col_q(n) = \begin{cases} \dfrac{3n+q}{2} & \text{if } u \text{ odd} \\ \dfrac{n}{2} & \text{if } u \text{ even}, \end{cases}$$

Now, we will demonstrate the compatibility of this generalization

Proposition 9 ($Col - Col_q$ equivalence)**.** *Let* $\dfrac{p}{q} \in \mathbb{Q}_{odd}$. *Then for all integer numbers* $k \geq 0$ *we have*

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$$

Proof. We let's observe that

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{\frac{p}{q}+1}{2} & \text{if } p = 1 \pmod 2 \\ \dfrac{p}{2q} & \text{if } p = 0 \pmod 2 \end{cases} = \frac{1}{q}\begin{cases} \dfrac{3p+q}{2} & \text{if } p = 1 \pmod 2 \\ \dfrac{p}{2} & \text{if } p = 0 \pmod 2 \end{cases} = \frac{1}{q}Col_q(p)$$

Suppose first that $(q, 3) = 1$. This fraction is irreducible. Indeed, we have that $(3p+q, q) = (3p, q) = 1$. Then the parity of the fraction depends only on the numerator since there is no possibility of simplification that changes the parity of the numerator, and we can continue with the iteration for all k since the irreducibility of the iterations only depends on the initial fraction is irreducible. Then we have

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$$

Now to suppose that $(q, 3) \neq 1$, for this case, the resulting fraction is not irreducible. However, as we are going to prove below, this does not change the parity of the orbits, so the formula would continue to be valid for this case. Suppose that, $q = 3^l v$ with $(v, 2) = (v, p) = 1$ and let $Col_q^k(p) = \dfrac{3^b p + 3^l vN}{2^a}$. We will divide this proof into two parts.

Case one $b \leq l$**:** We are going to prove the statement by induction. To $k = 1$

$$\frac{1}{3^l v}Col_{3^l v}(p) = \frac{1}{3^l v}\left(\frac{3p+3^l v}{2}\right) = \frac{3\left(\frac{p}{3^l v}\right)+1}{2} = Col\left(\frac{p}{3^l v}\right)$$

Now suppose that the statement is true for k, observe before continuing that the expressions $\left(\dfrac{3^b p + 3^l vN}{2^a}\right)$ and $\left(\dfrac{p + 3^{l-b} vN}{2^a}\right)$ have the same parity. Indeed,

$$\frac{3^b p + 3^l vN}{2^a} = 3^b\left(\frac{p + 3^{l-b} vN}{2^a}\right)$$

if the expression on the left-hand side is even, if and only if $\left(\dfrac{p + 3^{l-b} vN}{2^a}\right)$ it is even. On the other hand, if the left side is odd, $\left(\dfrac{p + 3^{l-b} vN}{2^a}\right)$ must be odd and if $\left(\dfrac{p + 3^{l-b} vN}{2^a}\right)$ is odd, since the product of odd is odd, the left side is odd, so the expressions have the same parity.

1. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is odd. Expanding the left-hand side of the proposition,

$$
\begin{aligned}
\frac{1}{3^l v} Col^{k+1}(p) = \frac{1}{3^l v}\psi\left(\frac{3^b p + 3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^{b+1} p + 3^{l+1} vN + 2^a 3^l v}{2^{a+1}}\right) \\
&= \frac{1}{3^{l-(b+1)} v}\left(\frac{p + 3^{l+1-(b+1)} vN + 2^a 3^{l-(b+1)} v}{2^{a+1}}\right) \\
&= \frac{1}{3^{l-(b+1)} v}\left(\frac{p + 3^{l-b} vN + 2^a 3^{l-(b+1)} v}{2^{a+1}}\right)
\end{aligned}
$$

developing the right-hand side of the proposition,

$$
\begin{aligned}
Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right) \\
&= Col\left(\frac{1}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^a}\right)\right) \\
&= \frac{1}{2}\left(\frac{3}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^a}\right) + 1\right) \\
&= \frac{1}{2}\left(\frac{1}{3^{l-(b+1)} v}\left(\frac{p + 3^{l-b} vN + 2^a 3^{l-(b+1)} v}{2^a}\right)\right) \\
&= \frac{1}{3^{l-(b+1)} v}\left(\frac{p + 3^{l-b} vN + 2^a 3^{l-(b+1)} v}{2^{a+1}}\right)
\end{aligned}
$$

We conclude in this case that both parts are equal

2. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is even. Expanding the left-hand side of the proposition,

$$
\begin{aligned}
\frac{1}{3^l v} Col^{k+1}(p) = \frac{1}{3^l v}\theta\left(\frac{3^b p + 3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^{a+1}}\right) \\
&= \frac{1}{3^{l-p} v}\left(\frac{p + 3^{l-b} vN}{2^{a+1}}\right)
\end{aligned}
$$

developing the right-hand side of the proposition,

$$
\begin{aligned}
Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right) \\
&= Col\left(\frac{1}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^a}\right)\right) \\
&= \frac{1}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^{a+1}}\right)
\end{aligned}
$$

We conclude in this case that both parts are equal. Since in both cases it gave equality, we conclude that the proposition is true.

Case two $b \geq l$: We are going to prove the statement by induction. To $k = 1$

$$\frac{1}{3v}Col_{3v}(p) = \frac{1}{3v}\left(\frac{3p+3v}{2}\right) = \frac{3\left(\frac{p}{3}\right)+1}{2} = Col\left(\frac{p}{3v}\right)$$

Now suppose that the statement is true for k, observe before continuing that the expressions $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ and $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ have the same parity. Indeed,

$$\frac{3^b p + 3^l vN}{2^a} = 3^l\left(\frac{3^{b-l} p + vN}{2^a}\right)$$

if the expression on the left-hand side is even, if and only if $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ it is even. On the other hand, if the left side is odd, $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ must be odd and if $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ is odd since the product of odd is odd, the left side is odd, so the expressions have the same parity.

1. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is odd. Expanding the left-hand side of the proposition,

$$\begin{aligned}\frac{1}{3^l v}Col^{k+1}(p) = \frac{1}{3^l v}\psi\left(\frac{3^b p + 3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^{b+1} p + 3^{l+1} vN + 2^a 3^l v}{2^{a+1}}\right)\\ &= \frac{3^{b+1-l} p + 3vN + 2^a v}{2^{a+1} v}\end{aligned}$$

developing the right-hand side of the proposition,

$$\begin{aligned}Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right)\\ &= Col\left(\frac{3^{b-l} p + vN}{2^a v}\right)\\ &= \frac{1}{2}\left(3\left(\frac{3^{b-l} p + vN}{2^a v}\right) + 1\right)\\ &= \frac{3^{b+1-l} p + 3vN + 2^a v}{2^{a+1} v}\end{aligned}$$

We conclude in this case that both parts are equal.

2. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is even. Expanding the left-hand side of the proposition,

$$\begin{aligned}\frac{1}{3^l v}Col^{k+1}(p) = \frac{1}{3^l v}\theta\left(\frac{3^b p + 3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^{a+1}}\right)\\ &= \frac{3^{b-l} p + vN}{2^{a+1} v}\end{aligned}$$

developing the right-hand side of the proposition,

$$\begin{aligned} Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right) \\ &= Col\left(\frac{3^{b-l} p + vN}{2^a v}\right) \\ &= \frac{3^{b-l} p + vN}{2^{a+1} v} \end{aligned}$$

We conclude in this case that both parts are equal. Since in both cases it gave equality, we conclude that the proposition is true.

□

Other proof by induction:

Proof. We proceed by induction on k.
Base Case ($k = 0$): By definition:

$$\mathrm{Col}^0\left(\frac{p}{q}\right) = \frac{p}{q} = \frac{1}{q}\cdot p = \frac{1}{q}\mathrm{Col}_q^0(p).$$

Inductive Step ($k \to k+1$): Assume that for some $k \geq 0$:

$$\mathrm{Col}^k\left(\frac{p}{q}\right) = \frac{1}{q}\mathrm{Col}_q^k(p).$$

Let $x_k = \mathrm{Col}_q^k(p)$. We consider two cases based on the parity of x_k:
Case 1: x_k is odd, Using the inductive hypothesis:

$$\mathrm{Col}^{k+1}\left(\frac{p}{q}\right) = \mathrm{Col}\left(\mathrm{Col}^k\left(\frac{p}{q}\right)\right) = \mathrm{Col}\left(\frac{x_k}{q}\right).$$

Since x_k is odd and q is odd, $\dfrac{x_k}{q}$ is odd in $\mathbb{Q}_{\text{odd}}$. Thus:

$$\mathrm{Col}\left(\frac{x_k}{q}\right) = \frac{3\left(\frac{x_k}{q}\right)+1}{2} = \frac{3x_k+q}{2q} = \frac{1}{q}\left(\frac{3x_k+q}{2}\right) = \frac{1}{q}\mathrm{Col}_q^{k+1}(p).$$

Case 2: x_k is even, since x_k is even, $\dfrac{x_k}{q}$ is even in $\mathbb{Q}_{\text{odd}}$. Thus:

$$\mathrm{Col}\left(\frac{x_k}{q}\right) = \frac{\frac{x_k}{q}}{2} = \frac{x_k}{2q} = \frac{1}{q}\left(\frac{x_k}{2}\right) = \frac{1}{q}\mathrm{Col}_q^{k+1}(p).$$

In both cases, we have $\mathrm{Col}^{k+1}\left(\frac{p}{q}\right) = \frac{1}{q}\mathrm{Col}_q^{k+1}(p)$. By induction, the proposition holds for all $k \geq 0$. □

We will define a coding function for the Collatz q-functions and demonstrate that they produce the same coding as the fractions with denominator q.

We are going to consider the set of sequences 0 and 10 that we will denote by Σ_2^* and we formally define it as

$$\Sigma_2^* = \left\{\{\xi_j\}_{j=1}^{\infty}, \xi_j \in \{0,10\}\right\}$$

this set can be seen as a subset of the set of sequences 0 and 1 where after the entry 1 enters 0. Let's consider the following application: $Cod^k : \mathbb{Q}_{odd} \to \Sigma_2^*$ defined by

$$Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^k 0^\infty$$

with

$$\xi_j = \begin{cases} 10 & \text{if} \quad Col^{j-1}\left(\frac{p}{q}\right) \text{ is odd} \\ 0 & \text{if} \quad Col^{j-1}\left(\frac{p}{q}\right) \text{ is even} \end{cases}$$

and

Proposition 10 (Second Cod invariance). *Let* $\frac{p}{q} \in \mathbb{Q}$ *an irreducible fraction with* $(q,2) = 1$ *and* $Cod^k : \mathbb{Z} \to \Sigma_2^*$ *defined by*

$$Cod_q^k(p) = \{\xi_j\}_{j=1}^k 0^\infty, \textit{ with } \xi_i \in \{10, 0\}$$

with

$$\xi_j = \begin{cases} 10 & \textit{if} \quad Col_q^{j-1}(p) = 1 \pmod 2 \\ 0 & \textit{if} \quad Col_q^{j-1}(p) = 0 \pmod 2 \end{cases}$$

then we have

$$Cod^k\left(\frac{p}{q}\right) = Cod_q^k(p)$$

Proof. By proposition 9 we have

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q} Col_q^k(p)$$

Since q it is odd, then, we have coding of $Col^k\left(\frac{p}{q}\right)$ and $Col_q^k(p)$ must be the same. □

We will now establish the initial connection between sets of integers and coding. Specifically, we will demonstrate that all elements within the integer set S share the same coding.

Proposition 11 (First characterization of $\mathbb{E}(S)$). *Let* $p \in \mathbb{Z}$ *and* $S(x) \in \langle \theta, \psi^q \rangle$ *of length* k *with* $(q,2) = 1$, *then*

$$p \in \mathbb{E}(S) \textit{ if and only if } Cod_q^k(p) = Cod_q(S)$$

Proof. Let $p \in \mathbb{E}(S)$ and $S(x) = \bigodot_{i=1}^{k} s_i(x) \in \langle \theta, \psi^q \rangle$ then by definition $S(p) \in \mathbb{Z}$ by Proposition 1 we have $\bigodot_{i=1}^{l} s_i(p) \in \mathbb{Z}$ with $l \leq k$, then $Cod(S) = Cod_q^k(p)$.

Suppose that $Cod_q^k(p) = Cod_q(S)$ then $\{p, s_1(p), s_2 \circ s_1(p), \ldots, \bigodot_{i=1}^{k-1} s_i(p), \bigodot_{i=1}^{k} s_i(p) = S(p)\} \in \mathbb{Z}^{k+1}$, then $p \in \mathbb{E}(S)$. □

We show below the second connection between the integer sets and the encoding. Specifically, we demonstrate that all values p within the integer set S_q indeed have the same coding as the corresponding fraction $\frac{p}{q}$.

Proposition 12 (second characterization of $\mathbb{E}(S)$)**.** *Let* $p \in \mathbb{Z}$ *and* $S_q(x) \in \langle\theta, \psi^q\rangle$ *of length* k *with* $(q,2) = 1$*, then we have:*

$$p \in \mathbb{E}(S_q) \text{ if and only if } Cod^k\left(\frac{p}{q}\right) = Cod_q(S_q)$$

Proof. Let $S_q(x) \in \langle\theta, \psi^q\rangle$ of length k such that $Cod^k(p) = Cod_q(S_Q)$, for the proposition 9, we have

$$qCol^k\left(\frac{p}{q}\right) = Col_q^k(p) = S_q(p)$$

then $Cod^k\left(\frac{p}{q}\right) = Cod_q(S_q)$ finally by the proposition 11, we have $p \in \mathbb{E}(S_q)$ if and only if $Cod^k\left(\frac{p}{q}\right) = Cod_q(S_q)$. ☐

The following proposition demonstrates that for a given rational number, we can generate a family of rationals that share the same encoding. This suggests that there exist many rationals with the same k-th encoding

Example 7. *Let's consider the coding* $\xi = 10010100$*, we want to find rational numbers* $\frac{p}{q}$ *such that* $Cod^5\left(\frac{p}{q}\right) = \xi$*, we have that the function* $S(x) \in \langle\theta, \psi\rangle$ *with coding* ξ *is* $S(x) = \dfrac{27x+29}{32}$

1. $p = 1$*, we have to calculate some solution of the entire set of* $S(x) = \dfrac{27x+29}{32}$*. We have* $x_0 = 25$*, then* $\mathcal{O}_5(25) = \{25, 38, 19, 29, 44, 22\}$ *then* $Cod^5(25) = \xi$*.*
2. $p = 7$*, we have to calculate some solution of the entire set of* $S_7(x) = \dfrac{27x+29\cdot 7}{32}$*. We have* $x_0 = 15$*, then* $\mathcal{O}_5\left(\frac{15}{7}\right) = \left\{\frac{15}{7}, \frac{26}{7}, \frac{13}{7}, \frac{23}{7}, \frac{38}{7}, \frac{19}{7}\right\}$ *then* $Cod^5\left(\frac{15}{7}\right) = \xi$*.*
3. $p = 27$*, we have to calculate some solution of the entire set of* $S_{27}(x) = \dfrac{27x+29\cdot 27}{32}$*. We have that* $x_0 = 3$ *, then* $\mathcal{O}_5\left(\frac{3}{27}\right) = \left\{\frac{3}{27}, \frac{2}{3}, \frac{1}{3}, 1, 2, 1\right\}$ *then we have that* $Cod^5\left(\frac{3}{27}\right) = \xi$*.*

Proposition 13 (Invariance property of Coding of rational)**.** *Let* $\frac{p}{q} \in \mathbb{Q}$ *an irreducible fraction with* $(q,2) = 1$*,* A_k *=numbers from 0 to* $Cod^k\left(\frac{p}{q}\right)$ *and* $T \in \mathbb{Z}$ *then*

$$Cod^k\left(\frac{p}{q}\right) = Cod^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right)$$

.

Proof. Let $S(x) \in \langle\theta, \psi^q\rangle$ such that $Cod(S) = Cod_q^k(p)$ then $S(p) = Col_q^k(p) \in \mathbb{Z}$ this implies

$$\mathbb{E}(S) = p + 2^{A_k}T \Rightarrow Cod_q^k(p + 2^{A_k}T) = Cod_q^k(p)$$

then

$$Col^k\left(\frac{p+2^{A_k}T}{q}\right) = Col^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right) = Cod^k\left(\frac{p}{q}\right)$$

□

As we have seen so far, we can characterize the entire set S from its encoding. Exploiting this property, we generalize the entire set S to encompass all fractions sharing the same encoding. We will call the $k-$Coding set.

Definition 11 (The $k-$Coding set)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$, we define the k-th coding set of $\{\xi_j\}_{j=1}^{\infty}$*

$$Cod^k\{\xi_j\}_{j=1}^{\infty} = \left\{ \frac{p}{q} \in \mathbb{Q}_{odd}, Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k}0^{\infty} \right\}$$

The encoding set also exhibits the property of Monotonicity, similar to the integer set of S.

Proposition 14 (Monotonicity of the Coding set)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$ then*

$$Cod^{k+1}\{\xi_j\}_{j=1}^{\infty} \subset Cod^k\{\xi_j\}_{j=1}^{\infty}.$$

Proof. Let $\frac{p}{q} \in Cod^{k+1}\{\xi_j\}_{j=1}^{\infty}$ by definition $Cod^{k+1}\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k+1}0^{\infty}$ then trivially we have $Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k}0^{\infty}$, then $\frac{p}{q} \in Cod^k\{\xi_j\}_{j=1}^{\infty}$. □

Definition 12 (The Coding set)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$, we define the Coding Set of $\{\xi_j\}_{j=1}^{\infty}$*

$$Cod\{\xi_j\}_{j=1}^{\infty} := \bigcap_{k\in\mathbb{N}} Cod^k\{\xi_j\}_{j=1}^{\infty}.$$

Similarly, the behavior of the solutions of Diophantine equations, in which knowing a particular solution allows us to determine other solutions, is reflected in the coding set. This connection is illustrated in the following proposition.

Proposition 15 (Generating property)**.** *Let $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$, A_k =numbers from 0 to $\{\xi_j\}_{j=1}^{k}$ and $\frac{p}{q}, \frac{t}{r} \in Cod^k\{\xi_j\}_{j=1}^{\infty}$ then exist $T \in \mathbb{Z}$ such that*

$$\frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr}$$

Proof. Let $\frac{p}{q}, \frac{t}{r} \in Cod^k\{\xi_j\}_{j=1}^{\infty}$ and $S_k(x) = \frac{3^b x + N}{2^{A_k}} \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=1}^{k}$, now consider $S_k^q \in \langle\theta, \psi^q\rangle$ and $S_k^r \in \langle\theta, \psi^r\rangle$ such that $Cod_q^k(S_k^q) = Cod_r^k(S_k^r) = \{\xi_j\}_{j=1}^{k}$ by proposition 12 we have $S_k^q(p), S_k^r(t) \in \mathbb{Z}$ the latter is equivalent

$$\frac{3^b p + qN}{2^{A_k}}, \frac{3^b t + rN}{2^{A_k}} \in \mathbb{Z}$$

We are going to prove that pr and qt are elements of $\mathbb{E}(S_k^{qr})$ with $s^r(x) = \frac{3^b x + qrN}{2^{A_k}}$. Indeed,

$$\frac{3^b pr + qrN}{2^{A_k}} = r\left(\frac{3^b p + qN}{2^{A_k}}\right) \in \mathbb{Z}$$

and

$$\frac{3^b qt + qrN}{2^{A_k}} = q\left(\frac{3^b t + rN}{2^{A_k}}\right) \in \mathbb{Z}$$

then

$$pr = qt + 2^{A_k}T \text{ with } T \in \mathbb{Z} \Rightarrow \frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr} \text{ with } T \in \mathbb{Z}$$

□

Now, we will present the main theorem of this section, establishing that the encoding of a rational number is unique.

Theorem 5 (Uniqueness of the full coding on $\mathbb{Q}_{odd}$)**.** *Let* $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$*. If it exists* $\frac{p}{q} \in \mathbb{Q}_{odd}$ *such that* $Cod\left(\frac{p}{q}\right) := \lim_{k\to\infty} Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$ *then it is unique.*

Proof. Let A_k =numbers from 0 to $\{\xi_j\}_{j=1}^{k}$. Suppose there is another element, $\frac{t}{r} \in \mathbb{Q}_{odd}$ such than $Cod\left(\frac{t}{r}\right) = \lim_{k\to\infty} Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$ by proposition 15 exist $\{T_k\}_{k\in\mathbb{N}} \in \mathbb{Z}$ such that

$$\frac{t}{r} = \frac{p}{q} + \frac{2^{A_k}T}{rq} \text{ with } T \in \mathbb{Z} \text{ and } k \in \mathbb{N}$$

Since $\frac{p}{q} \neq \frac{t}{r}$ then $T_k \in \mathbb{Z}$ for all $k \in \mathbb{N}$. So

$$rq\left(\frac{t}{r} - \frac{p}{q}\right) = 2^{A_k}T_k \text{ then } T_k = \frac{rq\left(\frac{t}{r} - \frac{p}{q}\right)}{A_k} \to 0 \text{ as } k \to \infty$$

which is a contradiction. □

Part II: Analytic Theory of Infinite Systems of Diophantine Equations

6. The G_0, G_∞ and G_1 Sets

Let $\xi \in \Sigma_2^*$. Let $b_k$the quantity of 1 of $\{\xi_j\}_{j=1}^{k}$. Define the function $\frac{3^{b_k(\xi)}}{2^{a_{k+1}(\xi)}}$. This function corresponds to the slope of the function S_k such that $Cod^k(S_k) = \{\xi_j\}_{j=1}^{k}$. Let us consider three subsets that will be relevant to study the non-existence of divergent orbits. G_1, G_∞ and G_1 which correspond to the subset of the sequences such that $\frac{3^{b_k(\xi)}}{2^{a_{k+1}(\xi)}}$ converges to $0, \infty$ and some real respectively.

6.1. Summary of Propositions in the Section

1. **Definition 13:** Sets G_0, G_∞ and G_1.
2. **Lemma 6:** Characterization of G_0 and G_∞ through accumulation points of $\frac{3^k}{2^{a_{k+}}}$.
3. **Proposition 16:** Let $\xi \in G_1$ then exists $M, m > 0$ such that $m < \frac{3^k}{2^{a_{k+1}}} < M$.

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4. **Lemma 7:** Let $0 < \varepsilon < \dfrac{L}{7}$, then ε satisfies the following inequalities $L + \varepsilon < \dfrac{3}{2}(L - \varepsilon)$ and $\dfrac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$ if $\theta_{k+2} \geq 2$.
5. **Proposition 17:** Let $\xi \in G_1$, then $\dfrac{3^k}{2^{a_{k+1}}}$ is not convergent.

6.2. The G_0, G_∞, and G_1 Sets

In this section, we classify the sequences in Σ_2^* based on the asymptotic behavior of their cumulative exponents. This classification allows us to partition the space into three distinct sets: G_0, G_∞, and G_1. These sets will play a crucial role in determining the convergence properties of the associated dynamical systems.

Definition 13 (The G_0, G_∞, and G_1 sets)**.** *Let $\xi \in \Sigma_2^*$ be defined as $\xi = 0^{\theta_1} \prod_{j=2}^{\infty} 10^{\theta_j}$ with $\theta_1 \geq 0$ and $\theta_k > 0$ for $k > 1$. Alternatively, if ξ has a null tail, let $\xi = 0^{\theta_1} \prod_{j=2}^{K} 10^{\theta_j} \prod_{j=K+1}^{\infty} 0^{\theta_j}$ with $\theta_j = 1$ for $j > K$. Let $a_n = \sum_{j=1}^{n} \theta_j$. We define the following subsets of Σ_2^*:*

$$G_0 = \left\{ \xi \in \Sigma_2^* : \liminf_{k\to\infty} \frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)} \quad \text{or} \quad \xi \text{ has a Null Tail} \right\},$$

$$G_\infty = \left\{ \xi \in \Sigma_2^* : \limsup_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)} \quad \text{or} \quad \xi \text{ has a Null Tail} \right\},$$

and

$$G_1 = \left\{ \xi \in \Sigma_2^* : \text{There exists a subsequence } \{k_j\}_{j\in\mathbb{N}} \text{ such that } \lim_{j\to\infty} \frac{a_{k_j+1}}{k_j} = \frac{\ln(3)}{\ln(2)} \right\}.$$

To illustrate these definitions, we present specific examples representing each category based on the growth rate of their exponents.

Example 8. *1. Let $\xi_1 \in \Sigma_2^*$ such that $a_k = 3k + (-1)^k$. Since the limit of the ratio is 3, and $3 > \frac{\ln 3}{\ln 2} \approx 1.58$, we have $\xi_1 \in G_0$.*
2. Let $\xi_2 \in \Sigma_2^*$ such that $a_k = \lfloor k + \ln(k) \rfloor$. Since the linear coefficient is 1, and $1 < 1.58$, we have $\xi_2 \in G_\infty$.
3. Let $\xi \in \Sigma_2^*$ such that $a_k = \left\lfloor k \frac{\ln(3)}{\ln(2)} \right\rfloor$. The ratio converges exactly to the critical threshold, so $\xi \in G_1$.

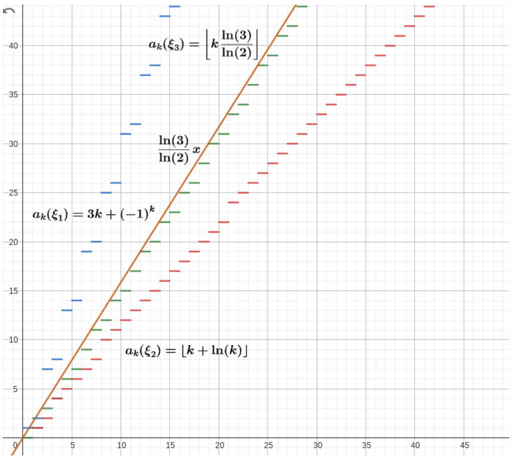

Figure 1. *Geometric interpretation: Excluding Null Tails, sequences in G_0 have their counting function a_k asymptotically above the line $y = \frac{\ln(3)}{\ln(2)}x$. Sequences in G_∞ lie below this line. Sequences in G_1 oscillate around or tend towards the line.*

The following result provides an equivalent characterization of the elements in G_0 and G_∞ in terms of the limit behavior of the sequence $\frac{3^k}{2^{a_{k+1}}}$. It is important to note that this characterization applies specifically to sequences without a null tail, as the null tail cases are included in the sets by definition due to their functional convergence properties.

Lemma 6. *Let $\xi \in \Sigma_2^*$ such that ξ does not have a null tail. Then:*

1. $\xi \in G_0$ *if and only if* $\lim\limits_{k\to\infty} \dfrac{3^k}{2^{a_{k+1}}} = 0$.
2. $\xi \in G_\infty$ *if and only if* $\lim\limits_{k\to\infty} \dfrac{3^k}{2^{a_{k+1}}} = \infty$.

Proof. 1. (⇒) Assume $\xi \in G_0$ and ξ is not a null tail. Then, by definition, $\liminf_{k\to\infty} \frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)}$. There exist $K, \varepsilon > 0$ such that $\frac{a_{k+1}}{k} > \frac{\ln(3)}{\ln(2)} + \varepsilon$ for all $k > K$. Thus:

$$\frac{3^k}{2^{a_{k+1}}} = \left(\frac{3}{2^{\frac{a_{k+1}}{k}}}\right)^k < \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}}\right)^k = \left(\frac{3}{3\cdot 2^{\varepsilon}}\right)^k = 2^{-k\varepsilon} \to 0 \text{ as } k \to \infty.$$

(⇐) Conversely, suppose $\frac{3^k}{2^{a_{k+1}}} \to 0$. If we assume $\xi \notin G_0$, then $\liminf_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)}$ (since ξ is not a null tail). Let $\frac{a_{k_l+1}}{k_l}$ be a subsequence converging to the liminf. There exist $K, \varepsilon > 0$ such that $\frac{a_{k_l+1}}{k_l} < \frac{\ln(3)}{\ln(2)} - \varepsilon$ for $l > K$. Then:

$$\frac{3^{k_l}}{2^{a_{k_l+1}}} > \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}-\varepsilon}}\right)^{k_l} = 2^{k_l\varepsilon} \to \infty \text{ as } l \to \infty,$$

which contradicts the hypothesis that the limit is 0.

2. The proof for G_∞ follows an analogous argument by reversing the inequalities.

□

Having characterized the asymptotic behavior of G_0 and G_∞ as tending to zero or infinity, we now turn our attention to the boundary set G_1. Unlike the previous sets, sequences in G_1 maintain a delicate balance. The following proposition establishes that elements in this set exhibit a bounded behavior, staying strictly away from both zero and infinity along the subsequences that define their critical density.

Proposition 16. *Let $\xi \in G_1$ and consider a subsequence $\{a_{k_l}\}$ such that $\lim\limits_{l\to\infty} \dfrac{a_{k_l+1}}{k_l} = \dfrac{\ln 3}{\ln 2}$. Then there exist $M, m > 0$ such that $m < \dfrac{3^{k_l}}{2^{a_{k_l+1}}} < M$ on $\mathbb{R}$.*

Proof. Let $\xi \in G_1$ such that $\lim\limits_{l\to\infty} \dfrac{a_{k_l+1}}{k_l} = \dfrac{\ln(3)}{\ln(2)}$. We aim to prove that there exist $M, m > 0$ satisfying $m < \dfrac{3^{k_l}}{2^{a_{k_l+1}}} < M$. Suppose, for the sake of contradiction, that such bounds do not exist. If there exists a subsequence such that $\dfrac{3^{k_l}}{2^{a_{k_l+1}}} \to 0$ as $l \to \infty$, then by the logic of Lemma 6, this would imply $\lim\limits_{l\to\infty} \dfrac{a_{k_l+1}}{k_l} > \dfrac{\ln(3)}{\ln(2)}$, which contradicts the hypothesis. Conversely, if the sequence diverges to ∞, it implies $\lim\limits_{l\to\infty} \dfrac{a_{k_l+1}}{k_l} < \dfrac{\ln(3)}{\ln(2)}$, which again contradicts the definition of the subsequence in G_1. Therefore, the sequence must be bounded. □

To rigorously demonstrate that this sequence oscillates rather than converging to a single value, we require a technical estimate regarding the algebraic steps of the iteration. The following lemma provides the necessary inequalities to show that the sequence cannot remain trapped within an arbitrarily small neighborhood of a limit point, due to the discrete nature of the multiplicative jumps.

Lemma 7. *Let $L > 0$ and $0 < \varepsilon < \frac{L}{7}$. Then ε satisfies the following inequalities:*

1. $L + \varepsilon < \frac{3}{2}(L - \varepsilon)$,
2. $\frac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$ *provided that* $\theta_{k+2} \geq 2$.

Proof. We verify that any ε satisfying $0 < \varepsilon < \frac{L}{7}$ holds for both inequalities.

1. For the first inequality: $L + \varepsilon < \frac{3}{2}(L - \varepsilon)$. Expanding and simplifying, we obtain $\varepsilon < \frac{L}{5}$. Since $\varepsilon < \frac{L}{7}$ and $\frac{L}{7} < \frac{L}{5}$, this inequality is strictly satisfied.
2. For the second inequality, we first establish the bound:

$$\frac{1}{7} \leq \frac{2^{\theta_{k+2}} - 3}{3 + 2^{\theta_{k+2}}}$$

for $\theta_{k+2} \geq 2$. Indeed, this is equivalent to $3 + 2^{\theta_{k+2}} \leq 7(2^{\theta_{k+2}} - 3)$, which simplifies to $24 \leq 6 \cdot 2^{\theta_{k+2}}$, or $4 \leq 2^{\theta_{k+2}}$, which holds for all $\theta_{k+2} \geq 2$.

Now, rearranging the target inequality $\frac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$, we require:

$$\varepsilon < L \cdot \frac{2^{\theta_{k+2}} - 3}{3 + 2^{\theta_{k+2}}}.$$

Since we chose $\varepsilon < \frac{L}{7}$ and we proved that $\frac{1}{7}$ is the minimum value of the right-hand side (attained at $\theta_{k+2} = 2$), the condition holds for all $\theta_{k+2} \geq 2$.

□

With the boundedness established and the jump inequalities in place, we can now state the main result regarding the dynamics of G_1. Unlike G_0 and G_∞, which exhibit definite asymptotic trends, the following proposition proves that the associated sequence for G_1 oscillates perpetually, failing to converge to any real number.

Proposition 17. *Let $\xi \in G_1$. Then the sequence* $\left\{ \frac{3^k}{2^{a_{k+1}}} \right\}_{k \in \mathbb{N}}$ *does not converge in* $\mathbb{R}$.

Proof. Suppose, for the sake of contradiction, that the limit exists. Let $\xi = 0^{\theta_1} \prod_{i=2}^{\infty} 10^{\theta_i}$ (implying no null tail, as $\xi \in G_1$) such that $\lim_{k\to\infty} \frac{a_{k+1}}{k} = \frac{\ln(3)}{\ln(2)}$. Assume there exists $L > 0$ such that $\lim_{k\to\infty} \frac{3^k}{2^{a_{k+1}}} = L$. Let $0 < \varepsilon < \frac{L}{7}$. By the definition of the limit, there exists $K > 0$ such that for all $k > K$, $L - \varepsilon < \frac{3^k}{2^{a_{k+1}}} < L + \varepsilon$.

Consider the recursive relation:

$$\frac{3^{k+1}}{2^{a_{k+2}}} = \frac{3}{2^{\theta_{k+2}}} \cdot \frac{3^k}{2^{a_{k+1}}}.$$

Applying the bounds, we get:

$$\frac{3}{2^{\theta_{k+2}}}(L - \varepsilon) < \frac{3^{k+1}}{2^{a_{k+2}}} < \frac{3}{2^{\theta_{k+1}}}(L + \varepsilon).$$

Using Lemma 7, we analyze the next step based on the value of θ_{k+2}:

1. If $\theta_{k+2} = 1$, Lemma 7 implies $L + \varepsilon < \frac{3}{2}(L - \varepsilon)$. Thus, the term jumps above the upper bound $L + \varepsilon$.
2. If $\theta_{k+2} \geq 2$, Lemma 7 implies $\frac{3}{2^{\theta_{k+2}}}(L + \varepsilon) < L - \varepsilon$. Thus, the term drops below the lower bound $L - \varepsilon$.

In either case, $\frac{3^{k+1}}{2^{a_{k+2}}} \notin (L - \varepsilon, L + \varepsilon)$, which contradicts the assumption of convergence. Therefore, the sequence is not convergent. □

7. Extension of the Collatz Function to $\mathbb{Z}_2$

In this section, we will study the extension of the Collatz function to the ring of 2-adic integers $\mathbb{Z}_2$, as proposed by Lagarias in [10]. Analogously to the real case, we will define the dyadic integer sets and the coding set. We will prove that given a coding sequence, there exists a unique dyadic integer with this coding. Furthermore, we will show that this extension is topologically conjugate to the shift function on Σ_2^* and we will use this result to prove that codings in G_1 correspond to unstable orbits.

7.1. Summary of Propositions in the Section

1. **Lemma 8:** Equivalence between the parity of rational numbers and their dyadic representation.
2. **Definition 14:** Extension of the Collatz function to the set of dyadic numbers, and definitions of the dyadic integer set and coding set.
3. **Proposition 18:** Characterization of the dyadic integer set.
4. **Proposition 19:** Establishes that the Coding set and the Dyadic Integer Set are equivalent.
5. **Proposition 20:** Establishes that for any given coding, there exists a unique dyadic number with that coding.
6. **Theorem 6:** The Collatz function on the set of dyadic numbers is topologically conjugate to the Shift map.
7. **Corollary 1:** The periodic points of the Collatz function in $\mathbb{Z}_2$ are dense.
8. **Proposition 21:** The periodic sequences of G_0 correspond to positive periodic points of the Collatz function, and the periodic sequences of G_∞ correspond to negative periodic points.

7.2. Extension of the Collatz Function to $\mathbb{Z}_2$

We begin by extending the Collatz function to the set $\mathbb{Z}_2$. In order for the extension to be compatible with the results obtained in the previous sections, we will first show that the parity of the elements of $\mathbb{Q}_{odd}$ is preserved in $\mathbb{Z}_2$.

Lemma 8. *Let $\beta = \sum_{i=0}^{\infty} \delta_i 2^i \in \mathbb{Z}_2$ be the dyadic representation of $\frac{p}{q} \in \mathbb{Q}_{odd}$. Then p is even if and only if $\delta_0 = 0$, and p is odd if and only if $\delta_0 = 1$.*

Proof. Let p be an even number. We have:

$$\frac{p}{q} = \frac{2k}{q} = \delta_0 + 2M \quad \text{for any } M \in \mathbb{Z}_2.$$

Thus, $2k = q\delta_0 + 2Mq$, which implies $q\delta_0 = 2(k - Mq)$. Since q is an odd number (invertible in $\mathbb{Z}_2$), δ_0 must be even, so $\delta_0 = 0$.

Conversely, let p be an odd number. We have:

$$\frac{p}{q} = \frac{2k+1}{q} = \delta_0 + 2M \quad \text{for any } M \in \mathbb{Z}_2.$$

Thus, $2k + 1 = q\delta_0 + 2Mq$. Since the left side is odd, $q\delta_0$ must be odd. Since q is odd, $\delta_0 = 1$. □

With the parity well-defined, we can now consider the following extension of the Collatz function on $\mathbb{Z}_2$.

Definition 14. *Let* $Col : \mathbb{Z}_2 \to \mathbb{Z}_2$ *be given by:*

$$Col(\beta) = \begin{cases} \dfrac{3\beta+1}{2} & \text{if } \beta \equiv 1 \pmod 2 \\ \\ \dfrac{\beta}{2} & \text{if } \beta \equiv 0 \pmod 2 \end{cases}$$

We define the k-coding of β *as* $Cod^k(\beta) = \{\eta_i\}_{i=1}^k$*, where* $\eta_i = 0$ *if* $Col^{i-1}(\beta) \equiv 0 \pmod 2$ *and* $\eta_i = 10$ *if* $Col^{i-1}(\beta) \equiv 1 \pmod 2$.

Let $\xi \in \Sigma_2^*$*. We define the k-Coding set of* ξ *as:*

$$Cod^k(\xi) = \left\{\beta \in \mathbb{Z}_2 : Cod^k(\beta) = \{\xi_i\}_{i=1}^k\right\}$$

Let $S \in \langle \theta, \psi \rangle$*. We define the dyadic integer set of* S *as:*

$$\mathbb{D}(S) = \{\beta \in \mathbb{Z}_2 : S(\beta) \in \mathbb{Z}_2\}$$

Example 9. *Let* $\beta = \overline{0110}11 \in \mathbb{Z}_2$*. The orbit is:*

$$\mathcal{O}(\overline{0110}11) = \{\overline{0110}11, \overline{0110}1001, \overline{0011}110, \overline{0011}11, \overline{0110}111, \dots\}$$

Next, we will show the version in $\mathbb{Z}_2$ of the results seen in previous Sections. The following Proposition characterizes the set of dyadic integers of $S_k \in \langle \theta, \psi \rangle$ analogously to the integer set case.

Proposition 18. *Let* $\xi = 0^{\theta_1} \prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ *and* $S_k \in \langle \theta, \psi \rangle$ *such that* $Cod(S_k) = 0^{\theta_1} \prod_{j=2}^{k+1} 10^{\theta_j}$*. If* $\beta_0 \in \mathbb{D}(S_k)$*, then* $\mathbb{D}(S_k) = \beta_0 + 2^{a_{k+1}}\mathbb{Z}_2$*.*

Proof. Let $\beta_0 \in \mathbb{D}(S_k)$. First, we show $\beta_0 + 2^{a_{k+1}}\mathbb{Z}_2 \subset \mathbb{D}(S_k)$. Indeed, let S_k be defined by:

$$S_k(x) = \frac{3^k x + N_k}{2^{a_{k+1}}}$$

Then for any $\alpha \in \mathbb{Z}_2$:

$$S_k(\beta_0 + 2^{a_{k+1}}\alpha) = \frac{3^k(\beta_0 + 2^{a_{k+1}}\alpha) + N_k}{2^{a_{k+1}}} = \frac{3^k\beta_0 + N_k}{2^{a_{k+1}}} + 3^k\alpha \in \mathbb{Z}_2$$

since $S_k(\beta_0) \in \mathbb{Z}_2$ and $3^k\alpha \in \mathbb{Z}_2$.

Now we show $\mathbb{D}(S_k) \subset \beta_0 + 2^{a_{k+1}}\mathbb{Z}_2$. Let $\beta \in \mathbb{D}(S_k)$, so $3^k\beta + N_k \equiv 0 \pmod{2^{a_{k+1}}}$. Since β_0 satisfies the same congruence:

$$\begin{aligned} 3^k\beta + N_k - (3^k\beta_0 + N_k) &\equiv 0 \pmod{2^{a_{k+1}}} \\ 3^k(\beta - \beta_0) &\equiv 0 \pmod{2^{a_{k+1}}} \end{aligned}$$

Since 3 is invertible in $\mathbb{Z}_2$, we have $\beta - \beta_0 \equiv 0 \pmod{2^{a_{k+1}}}$. Therefore:

$$\beta - \beta_0 = 2^{a_{k+1}}\alpha, \text{ with } \alpha \in \mathbb{Z}_2 \implies \beta = \beta_0 + 2^{a_{k+1}}\alpha.$$

□

The following proposition states that the dyadic integer set is exactly the coding set of S_k.

Proposition 19. *Let* $\xi = 0^{\theta_1}\prod_{j=2}^{\infty}10^{\theta_j} \in \Sigma_2^*$ *and* $S_k \in \langle\theta,\psi\rangle$ *such that* $Cod(S_k) = 0^{\theta_1}\prod_{j=2}^{k+1}10^{\theta_j}$. *Then* $\mathbb{D}(S_k) = Cod^{a_{k+1}}(\xi)$.

Proof. Let $\beta \in Cod^{a_{k+1}}(\xi)$. By definition, we have $Cod^{a_{k+1}}(\beta) = 0^{\theta_1}\prod_{j=2}^{k+1}10^{\theta_j}$. We can rewrite β as $\beta = \beta_k + 2^{a_{k+1}}\alpha_k$ for any $\alpha_k \in \mathbb{Z}_2$, where $\beta_k = \beta \pmod{2^{a_{k+1}}}$. We claim that $\beta_k \in Cod^{a_{k+1}}(\xi)$. Indeed:

$$Col^l(\beta) = Col^l(\beta_k) + 2^{a_{k+1}-l}\alpha_k' \quad \text{with } l \le a_{k+1}.$$

The parity on the right side depends only on $Col^l(\beta_k)$ because $2^{a_{k+1}-l}$ is even for $l < a_{k+1}$. Thus, β_k must have the same a_{k+1}-coding as β. By Proposition 11, we have $\beta_k \in \mathbb{E}(S_k)$ (viewed as a natural number). So $S_k(\beta) = S_k(\beta_k) + 3^k\alpha_k \in \mathbb{Z}_2$, which implies $\beta \in \mathbb{D}(S_k)$.

Conversely, let $\beta \in \mathbb{D}(S_k)$ and let $\beta_k = \beta \pmod{2^{a_{k+1}}}$. Then $\beta_k \in \mathbb{D}(S_k)$ by Proposition 18. Since β_k is a natural number and $S_k(\beta_k) \in \mathbb{Z}$ (the denominator divides the numerator), we have that $\beta_k \in \mathbb{E}(S_k)$. By Proposition 11, we have that $\beta_k \in Cod^{a_{k+1}}(\xi)$. Since $\beta = \beta_k + 2^{a_{k+1}}\alpha_k$ for some $\alpha_k \in \mathbb{Z}_2$, applying the Collatz map l times with $l \le a_{k+1}$, we get:

$$Col^l(\beta) = Col^l(\beta_k) + 2^{a_{k+1}-l}\alpha_k'.$$

Since the remainder term is always even for all $l < a_{k+1}$, the parity of each iteration only depends on $Col^l(\beta_k)$. Thus, $\beta \in Cod^{a_{k+1}}(\xi)$. ☐

In Theorem 5, we saw that given $\xi \in \Sigma_2^*$, if there exists a rational whose encoding is exactly ξ, then it is the only rational solution. However, we could not guarantee the existence of such a number. The following Proposition guarantees us that there exists a solution in the set of dyadic numbers.

Proposition 20. *Let* $\xi \in \Sigma_2^*$. *Then there exists a unique* $\beta \in \mathbb{Z}_2$ *such that* $Cod(\beta) = \xi$.

Proof. Let $\xi = 0^{\theta_1}\prod_{j=2}^{\infty}10^{\theta_j} \in \Sigma_2^*$. By Lemma 36, we have $-\pi^1(\xi) \in \mathbb{Z}_2$. Now we are going to prove that $-\pi^1(\xi)$ is the solution.

Claim: $-\pi^1(\xi) \in Cod^{a_{k+1}}(\xi)$ for all $k \in \mathbb{N}$.

Indeed, let $S_k \in \langle\theta,\psi\rangle$ such that $Cod(S_k) = 0^{\theta_1}\prod_{j=2}^{k+1}10^{\theta_j}$. Then $S_k(x) = \dfrac{3^k x + N_k}{2^{a_{k+1}}}$. We know that $S_k(-\pi_k^1(\xi)) = 0$ for all $k \in \mathbb{N}$. Then:

$$-\pi^1(\xi) = -\pi_k^1(\xi) - \sum_{j=k+1}^{\infty}\frac{2^{a_j}}{3^j}$$

Applying S_k:

$$\begin{aligned} S_k(-\pi^1(\xi)) &= S_k(-\pi_k^1(\xi)) - \frac{3^k}{2^{a_{k+1}}}\sum_{j=k+1}^{\infty}\frac{2^{a_j}}{3^j} \\ &= 0 - \sum_{j=k+1}^{\infty}\frac{2^{a_j-a_{k+1}}}{3^{j-k}}. \end{aligned}$$

Since $a_j \ge a_{k+1}$ for $j > k$, the powers of 2 are non-negative integers. Also, $1/3 \in \mathbb{Z}_2$. Thus, the infinite sum converges in $\mathbb{Z}_2$, so $S_k(-\pi^1(\xi)) \in \mathbb{Z}_2$.

By Proposition 19, we have $-\pi^1(\xi) \in Cod^{a_{k+1}}(\xi)$ for all $k \in \mathbb{N}$. Since $Cod^k(\xi) \subset Cod^{k-1}(\xi)$, we have:

$$-\pi^1(\xi) = -\lim_{k\to\infty} \pi^1_k(\xi) \in \bigcap_{k\in\mathbb{N}} Cod^{a_{k+1}}(\xi) = Cod(\xi).$$

To prove uniqueness, suppose there exists another dyadic integer α such that it is also in $Cod(\xi)$. Then for all $k \in \mathbb{N}$:

$$\alpha, -\pi^1(\xi) \in Cod(S_k) \implies \alpha \equiv -\pi^1(\xi) \pmod{2^{a_{k+1}}}.$$

Thus,

$$\|\pi^1(\xi) - \alpha\|_2 < 2^{-a_{k+1}} \to 0 \text{ as } k \to \infty.$$

Therefore, $\alpha = -\pi^1(\xi)$. □

The existence of solutions to the equation $Cod(\beta) = \xi$ in the dyadic numbers does not guarantee the existence of rational solutions. This will depend primarily on whether the dyadic solution can be represented as a rational number or, more generally, as a real number. Based on the nature of this solution, we can determine whether or not a rational solution exists.

7.3. Topological Conjugation

The Shift map ω on Σ_2^* is defined as the operation that removes the leading symbol of the sequence. The following theorem states that the Collatz function on $\mathbb{Z}_2$ is dynamically equivalent to the Shift map on Σ_2^* and that the function $-\pi^1$ acts as a homeomorphism between these two spaces. A similar result can be found in [10], where the Shift function is defined on $\mathbb{Z}_2$ directly, rather than via symbolic coding.

Theorem 6 (Col is topologically conjugate to ω)**.** *Let $\omega : \Sigma_2^* \to \Sigma_2^*$ be given by:*

$$\omega\left(\{\xi_j\}_{j=1}^{\infty}\right) = \begin{cases} \{\xi_{j+2}\}_{j=1}^{\infty} & \text{if } \xi_1 = 1 \\ \{\xi_{j+1}\}_{j=1}^{\infty} & \text{if } \xi_1 = 0 \end{cases}$$

The map Col is topologically conjugate to ω, meaning the following diagram is commutative:

$$\begin{tikzcd} \Sigma_2^* \arrow[r, "\omega"] \arrow[d, "-\pi^1"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma_2^* \arrow[d, "-\pi^1"] \\ \mathbb{Z}_2 \arrow[r, "Col"'] & \mathbb{Z}_2 \end{tikzcd}$$

and $-\pi^1$ is a homeomorphism.

Proof. We first prove that the diagram is commutative.

Let $\{\xi_j\}_{j=1}^{\infty} = 10^{\theta_1}10^{\theta_2}\ldots = \prod_{j=1}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ with $\theta_j \in \mathbb{N}$. Writing this way, we get an explicit form for the function $a_k = \sum_{j=1}^{k} \theta_j$. If $a_1 = 0$ (meaning the first block is trivial or structure starts differently), we have:

$$
\begin{aligned}
Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) &= Col\left(-\sum_{k=1}^{\infty}\frac{2^{a_k}}{3^k}\right)\\
&= \frac{1}{2}\left(-3\sum_{k=1}^{\infty}\frac{2^{a_k}}{3^k}+1\right)\\
&= \frac{1}{2}\left(-\sum_{k=1}^{\infty}\frac{2^{a_k}}{3^{k-1}}+1\right)\\
&= \frac{1}{2}\left(-2^{a_1}-\sum_{k=2}^{\infty}\frac{2^{a_k}}{3^{k-1}}+1\right)\\
&= \frac{1}{2}\left(-1-\sum_{k=2}^{\infty}\frac{2^{a_k}}{3^{k-1}}+1\right) \quad (\text{since } a_1 = 0 \implies 2^{a_1} = 1)\\
&= -\sum_{k=2}^{\infty}\frac{2^{a_k-1}}{3^{k-1}}\\
&= -\sum_{k=1}^{\infty}\frac{2^{a_{k+1}-1}}{3^k}.
\end{aligned}
$$

On the other hand, applying ω:

$$
-\pi^1 \circ \omega\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) = -\pi^1\left(0^{\theta_1-1}\prod_{j=2}^{\infty}10^{\theta_j}\right) = -\sum_{k=1}^{\infty}\frac{2^{a_{k+1}-1}}{3^k}
$$

where both parts are equal.

Now suppose that $\{\xi_j\}_{j=1}^{\infty} = 0^{\theta_1}10^{\theta_2}1\ldots = \prod_{j=1}^{\infty} 0^{\theta_j}1 \in \Sigma_2^*$ with $\theta_j \in \mathbb{N}$ and $\theta_1 > 0$. Then $\xi_1 = 0$.

$$
Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = Col\left(-\sum_{k=1}^{\infty}\frac{2^{a_k}}{3^k}\right) = -\sum_{k=1}^{\infty}\frac{2^{a_k-1}}{3^k}
$$

And for the shift:

$$
(-\pi^1)\circ\omega\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = (-\pi^1)\circ\left(0^{\theta_1-1}\prod_{j=1}^{\infty}0^{\theta_j}1\right) = -\sum_{k=1}^{\infty}\frac{2^{a_k-1}}{3^k}
$$

where again both parts are equal. Then we conclude that the diagram is commutative.

Bijectivity: Now we prove that $-\pi^1$ is a bijection. Let $Cod : \mathbb{Z}_2 \to \Sigma_2^*$ be given by $\xi_j = 0$ if $Col^{j-1}(\beta) \equiv 0 \pmod 2$ and $\xi_j = 10$ if $Col^{j-1}(\beta) \equiv 1 \pmod 2$.

1. $Cod \circ (-\pi^1) = Id_{\Sigma_2^*}$: By Corollary 20, we have $Cod(-\pi^1(\xi)) = \xi$.
2. $-\pi^1 \circ Cod = Id_{\mathbb{Z}_2}$: Let $\beta \in \mathbb{Z}_2$ and $\xi = Cod(\beta)$. Then $\beta \in Cod^k(\xi)$ for all $k \in \mathbb{N}$. Also, $-\pi^1(\xi) \in Cod^k(\xi)$ for all k. Let A_k be the number of zeros in $Cod^k(\xi)$. There exists $\gamma_k \in \mathbb{Z}_2$ such that $\beta = -\pi^1(\xi) + 2^{A_k}\gamma_k$. Thus $\beta \equiv -\pi^1(\xi) \pmod{2^{A_k}}$. Since $A_k \to \infty$ as $k \to \infty$, we have $\|\beta - (-\pi^1(\xi))\|_2 \leq 2^{-A_k} \to 0$. Thus $\beta = -\pi^1(Cod(\beta))$.

Uniform Continuity: Let $D : \Sigma_2^* \times \Sigma_2^* \to [0,\infty)$ be the symbolic metric defined by:

$$
D(\xi,\eta) = \sum_{j=1}^{\infty}\frac{1}{2^j}\Delta(\xi_j,\eta_j), \quad \text{with } \xi_j,\eta_j \in \{0,10\},
$$

where $\Delta(\xi_j,\eta_j) = 1$ if $\xi_j \neq \eta_j$ and 0 otherwise. The space (Σ_2^*, D) is a complete metric space where $D(\xi,\eta) < 2^{-r}$ if and only if $\xi_j = \eta_j$ for all $j < r$.

- $-\pi^1$ **is uniformly continuous:** Let $\varepsilon > 0$. Choose $A \in \mathbb{N}$ such that $2^{-A} < \varepsilon$. Let $\delta = 2^{-(A+2)}$. If $D(\xi, \eta) < \delta$, then the first $A+1$ blocks of ξ and η are identical. Let b_{A+1} be the total length of these blocks. Then $a_k(\xi) = a_k(\eta)$ for $k \leq b_{A+1}$. This implies that the partial sums match modulo 2^{A+1}.
$$-\pi^1(\xi) \equiv -\pi^1(\eta) \pmod{2^{A+1}} \implies \|\pi^1(\xi) - \pi^1(\eta)\|_2 \leq 2^{-(A+1)} < \varepsilon.$$
- Cod **is uniformly continuous:** Let $\varepsilon > 0$. Choose A such that $2^{-A} < \varepsilon$. Let $\delta = 2^{-A}$. If $\|\alpha - \beta\|_2 < \delta$, then $\alpha \equiv \beta \pmod{2^A}$. This implies their first A codings are identical, so $Cod^A(\alpha) = Cod^A(\beta)$. Therefore, $D(Cod(\alpha), Cod(\beta)) \leq 2^{-A} < \varepsilon$.

  Therefore, $-\pi^1$ is a homeomorphism and Col is topologically conjugate to ω. □

7.4. Periodic Points Analysis

As a first consequence of the topological conjugation established in the previous theorem, we can determine the structure of the periodic points of the Collatz function. Since the periodic points of the shift map are well-understood, we can transfer this property to $\mathbb{Z}_2$.

Corollary 1. *Let* $\mathrm{Per}(\omega)$ *be the set of periodic points of* ω*. Then we have* $\overline{-\pi^1(\mathrm{Per}(\omega))} = \mathbb{Z}_2$*. In other words, the periodic points of the Collatz function are dense in* $\mathbb{Z}_2$*.*

Proof. This is a direct consequence of the continuity of the homeomorphism $-\pi^1$ and the fact that the periodic sequences of the Shift function are dense in the symbolic space Σ_2^*. □

While the density tells us about the distribution of these points, it does not distinguish between their arithmetic properties. The topological conjugation implies that $\pi^1(\xi)$ is a periodic point in $\mathbb{Z}_2$ if and only if ξ is a periodic sequence. The following proposition establishes a crucial link between the asymptotic growth sets (G_0, G_∞) and the sign of the rational number represented by these periodic points.

Proposition 21. *Let* $\xi \in \Sigma_2^*$ *be a periodic sequence and let* $-\pi^1(\xi) \in \mathbb{Q} \subset \mathbb{Z}_2$ *be its corresponding rational number. Then:*

1. *If* $\xi \in G_0$*, then* $-\pi^1(\xi)$ *is a positive rational number.*
2. *If* $\xi \in G_\infty$*, then* $-\pi^1(\xi)$ *is a negative rational number.*

Proof. Let $\xi = 10^{\theta_2}10^{\theta_3}\ldots10^{\theta_K}10^{\theta_{K+1}}10^{\theta_2}10^{\theta_3}\ldots = \prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ with $\theta_{K+2} = \theta_2$, i.e., periodic of period K. Considering that $\theta_1 = 0$, we have $a_k = \sum_{j=1}^{k} \theta_j$. Let us first show that $a_{rK+l} = ra_{K+1} + a_l$ with $l \leq K$ for all $r \in \mathbb{N}$:

$$\begin{aligned} a_{rK+l} &= \sum_{j=1}^{rK+l} \theta_j \\ &= (\theta_2 + \theta_3 + \ldots + \theta_{K+1}) + (\theta_{K+2} + \theta_{K+3} + \ldots + \theta_{2K+1}) \\ &\quad + \ldots + (\theta_{rK+2} + \theta_{rK+3} + \ldots + \theta_{rK+l}) \\ &= (\theta_2 + \theta_3 + \ldots + \theta_{K+1}) + (\theta_2 + \theta_3 + \ldots + \theta_{K+1}) \\ &\quad + \ldots + (\theta_2 + \theta_3 + \ldots + \theta_l) \\ &= ra_{K+1} + a_l. \end{aligned}$$

We are going to show that $-\pi^1(\xi)$ is rational.

$$
\begin{aligned}
-\pi^1(\xi) &= -\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} \\
&= -\left\{\frac{1}{3} + \frac{2^{a_2}}{3^2} + \ldots + \frac{2^{a_K}}{3^K}\right\} - \left\{\frac{2^{a_{K+1}}}{3^{K+1}} + \frac{2^{a_{K+2}}}{3^{K+2}} + \ldots + \frac{2^{a_{2K}}}{3^{2K}}\right\} \\
&\quad - \ldots - \left\{\frac{2^{a_{rK+1}}}{3^{rK+1}} + \frac{2^{a_{rK+2}}}{3^{rK+2}} + \ldots + \frac{2^{a_{(r+1)K}}}{3^{(r+1)K}}\right\} - \ldots \\
&= -\left\{\frac{1}{3} + \frac{2^{a_2}}{3^2} + \ldots + \frac{2^{a_K}}{3^K}\right\} - \left\{\frac{2^{a_{K+1}}}{3^{K+1}} + \frac{2^{a_{K+1}+a_2}}{3^{K+2}} + \ldots + \frac{2^{a_{K+1}+a_K}}{3^{2K}}\right\} \\
&\quad - \ldots - \left\{\frac{2^{ra_{K+1}}}{3^{rK+1}} + \frac{2^{ra_{K+1}+a_2}}{3^{rK+2}} + \ldots + \frac{2^{ra_{K+1}+a_K}}{3^{(r+1)K}}\right\} - \ldots \\
&= -\left\{\frac{1}{3} + \frac{2^{a_2}}{3^2} + \ldots + \frac{2^{a_K}}{3^K}\right\} - \frac{2^{a_{K+1}}}{3^K}\left\{\frac{1}{3} + \frac{2^{a_2}}{3^2} + \ldots + \frac{2^{a_K}}{3^K}\right\} \\
&\quad - \ldots - \left(\frac{2^{a_{K+1}}}{3^K}\right)^r \left\{\frac{1}{3} + \frac{2^{a_2}}{3^2} + \ldots + \frac{2^{a_K}}{3^K}\right\} - \ldots \\
&= -\left\{\frac{1}{3} + \frac{2^{a_2}}{3^2} + \ldots + \frac{2^{a_K}}{3^K}\right\} \sum_{r=0}^{\infty} \left(\frac{2^{a_{K+1}}}{3^K}\right)^r \\
&= -\left\{\frac{1}{3} + \frac{2^{a_2}}{3^2} + \ldots + \frac{2^{a_K}}{3^K}\right\} \left(\frac{1}{1 - \frac{2^{a_{K+1}}}{3^K}}\right) \\
&= \left(\frac{1}{1 - \frac{2^{a_{K+1}}}{3^K}}\right)(-\pi^1_K(\xi)) \\
&= \frac{3^{K-1} + 2^{a_2} \cdot 3^{K-2} + \ldots + 2^{a_K}}{2^{a_{K+1}} - 3^K} \in \mathbb{Q},
\end{aligned}
$$

which corresponds to a rational number.

For the case $\xi = 0^{\theta_1} \prod_{j=2}^{\infty} 10^{\theta_j}$ with $\theta_1 > 0$, we take $\eta = \prod_{j=1}^{\infty} 10^{\theta_j}$ (where the periodic part starts immediately). We have that η is purely periodic, so:

$$
-\pi^1(\eta) = -\sum_{j=1}^{\infty} \frac{2^{a_j(\eta)}}{3^j} = \frac{3^{K-1} + 2^{a_2(\eta)} \cdot 3^{K-2} + \ldots + 2^{a_K(\eta)}}{2^{a_{K+1}(\eta)} - 3^K}.
$$

Then:

$$
\begin{aligned}
-\pi^1(\xi) = -\pi^1(0^{\theta_1}\eta) = -2^{\theta_1}\left(\frac{1}{3} + \sum_{j=2}^{\infty} \frac{2^{a_j(\eta)}}{3^j}\right) \\
&= -\, 2^{\theta_1}\left\{\frac{1}{3} + \frac{2^{a_2(\eta)}}{3^2} + \ldots + \frac{2^{a_K(\eta)}}{3^K}\right\}\left(\frac{1}{1 - \dfrac{2^{a_{K+1}(\eta)}}{3^K}}\right) \\
&= -\left\{\frac{2^{a_1(\xi)}}{3} + \frac{2^{a_2(\xi)}}{3^2} + \ldots + \frac{2^{a_K(\xi)}}{3^K}\right\}\left(\frac{1}{1 - \dfrac{2^{(a_{K+1}-a_1)(\xi)}}{3^K}}\right) \in \mathbb{Q}
\end{aligned}
$$

where its sign depends on the denominator $1 - \dfrac{2^{a_{K+1}-a_1}}{3^K}$.

1. If $\xi \in G_0$, then $\frac{3^K}{2^{a_{K+1}-a_1}} < 1$. So $1 < \frac{2^{a_{K+1}-a_1}}{3^K}$. The denominator is negative. Thus, $-\pi^1(\xi) = -(\text{Positive}) \times \frac{1}{\text{Negative}} = \text{Positive}$.
2. If $\xi \in G_\infty$, then $\frac{3^K}{2^{a_{K+1}-a_1}} > 1$. So $1 > \frac{2^{a_{K+1}-a_1}}{3^K}$. The denominator is positive. Thus, $-\pi^1(\xi) = -(\text{Positive}) \times \frac{1}{\text{Positive}} = \text{Negative}$.

☐

8. Real Function π^1 and π^2 Function

Let's define a new function π^1 defined on $\pi^1 : \Sigma_2^* \to [0,\infty]$ given by $\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$, unlike $\pi^1 : \Sigma_2^* \to \mathbb{Q}_2$ case ξ does not have a null tail and $\sum_{j=1}^{K} \frac{2^{a_j}}{3^j}$ when it has a null tail with index J. It is not always convergent. Does this mean that when it is divergent, then there is no solution to the encoding problem? The answer is no, for example if we take the encoding 100100100... which corresponds to the encoding of 1, however the function $\pi^1(\xi)$ is divergent. As we will show in the next section, when it is convergent, it is in fact the only solution to the encoding problem. In addition to the real function π^1, we will define the function π^2 which unlike π^1 which is a series, this is a function on the natural numbers to the rational numbers. We will show that the function $\pi^1 : \Sigma_2^* \to [0,\infty]$ is convergent if and only if $\xi \in G_\infty$ and that the function π^2 is bounded if and only if $\xi \in G_0$.

8.1. Summary of Propositions in the Section

1. **Definition 15:** We will give the definition of the functions π^1 and π^2.
2. **Proposition 22** Characterization of G_0 and G_∞ through functions π^1 and π^2.
    (a) $\xi \in G_\infty$ if and only if $\pi^1(\xi) < \infty$.
    (b) $\xi \in G_0$ if and only if $\pi^2(\xi)(k)$ is bounded.
3. **Lemma 11:** Let $\liminf_{k\to\infty} \frac{a_{k+1}}{k} = \lambda \in (0,\infty]$. Then exist $T > 0$ such that if $T < k - j$ we have $\frac{a_{k+1} - a_j}{k - j} \geq \lambda$.
4. **Lemma 12:** Let $k, j \in \mathbb{N}$, if $k > j$ then, we have $\frac{a_{k+1} - a_j}{k - j} \geq 1$.
5. **Corollary 2:** Let $\xi \in \Sigma_2^*$. If exist a sub-sequence such that $\lim_{j\to\infty} \frac{a_{k_l+1}(\xi)}{k_l} > \frac{\ln(3)}{\ln(2)}$. Then exist $\Omega > 0$ such that $\pi^2(\xi)(k_l) < \Omega$.

8.2. The π^1 and π^2 Functions

In this section, we define the auxiliary functions π^1 and π^2, which play a fundamental role in characterizing the asymptotic behavior of the sequences. These functions map symbolic sequences to real numbers and sequences of rational numbers, respectively.

8.3. Definitions

Definition 15 (The π^1 and π^2 functions)**.** *Let $\xi \in \Sigma_2^*$ be a sequence without a null tail, given by $\xi = 0^{\theta_1} \prod_{j=2}^{\infty} 10^{\theta_j}$, with $\theta_1 \geq 0$ and $\theta_k > 0$ for $k > 1$. Let $a_k = \sum_{j=1}^{k} \theta_j$.*

We define the function $N_k : \Sigma_2^* \to \mathbb{N}_0$ by:

$$N_k(\{\xi_j\}_{j=1}^{\infty}) = \begin{cases} 3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k} & \text{if } k > 1 \\ 0 & \text{if } k = 1 \end{cases}$$

We define $\pi^1_k : \Sigma^*_2 \to \mathbb{Q}$ *by:*

$$\pi^1_k(\xi) = \frac{N_k(\xi)}{3^k} = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$$

The π^1*-function,* $\pi^1 : \Sigma^*_2 \to \mathbb{R} \cup \{\infty\}$*, is defined as the limit:*

$$\pi^1(\xi) = \lim_{k\to\infty} \frac{N_k(\xi)}{3^k} = \sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$$

The π^2*-function,* $\pi^2 : \Sigma^*_2 \to \{f : \mathbb{N} \to \mathbb{Q}\}$*, maps a sequence* ξ *to a function of k, defined by:*

$$\pi^2(\xi)(k) = \frac{N_k(\xi)}{2^{a_{k+1}(\xi)}} = \frac{3^k}{2^{a_{k+1}}}\pi^1_k(\xi) = \frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$$

Null Tail Case: *Let* $\xi \in \Sigma^*_2$ *be a sequence with a null tail starting at index* $K \geq 0$*, given by* $\xi = 0^{\theta_1}\prod_{j=2}^{K} 10^{\theta_j} \prod_{j=K+1}^{\infty} 0$*, with* $\theta_1 \geq 0$ *and* $\theta_j > 0$ *for* $j > 1$*. We define* $\pi^1_k, \pi^1 : \Sigma^*_2 \to \mathbb{R} \cup \{\infty\}$ *as:*

$$\pi^1_k(\xi) = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} \quad \text{if } k < K$$
$$\pi^1_k(\xi) = \pi^1(\xi) = \sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j} \quad \text{if } k \geq K$$

And π^2 *is defined by:*

$$\pi^2(\xi)(k) = \frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} \quad \text{if } k < K$$
$$\pi^2(\xi)(K+k) = \frac{N_K(\xi)}{2^{a_K+k}} = \frac{3^K}{2^{a_K+k}}\pi^1_K(\xi) = \frac{3^{K-1}}{2^{a_K+k}} \sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j} \quad \text{for } k \geq 0$$

We illustrate these definitions with the following examples, which highlight the different convergence behaviors.

Example 10. *We provide examples of the functions* π^1 *and* π^2 *for various sequences:*

1. *Let* $\xi_0 = 00101001000000\ldots$*. This is a null tail sequence.*
   (a) $\pi^1(\xi_0) = \frac{2^2}{3} + \frac{2^3}{9} + \frac{2^5}{27}$.
   (b) *For* $k \geq 4$, $\pi^2(\xi_0)(k) = \frac{27}{2^{k+2}}\left\{\frac{2^2}{3} + \frac{2^3}{9} + \frac{2^5}{27}\right\}$*. Thus,* $\lim_{k\to\infty} \pi^2(\xi_0)(k) = 0$.
2. *Let* $\xi_1 = 100100100\ldots = \prod_{j=1}^{\infty} 100 \in \Sigma^*_2$*. Here* $\theta_j = 2$ *for all j.*
   (a) $\pi^1(\xi_1) = \sum_{j=1}^{\infty} \frac{2^{2(j-1)}}{3^j} = \frac{1}{3}\sum_{j=1}^{\infty} \frac{2^{2j}}{3^j} = \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{4}{3}\right)^j = \infty$.
   (b) $\pi^2(\xi_1)(k) = \frac{3^k}{2^{2k}} \sum_{j=1}^{k} \frac{2^{2(j-1)}}{3^j} = \frac{3^{k-1}}{2^{2k}} \sum_{j=1}^{k-1} \frac{2^{2j}}{3^j}$

$$= \frac{3^{k-1}}{2^{2k}}\left\{\frac{\frac{2^{2k}}{3^k} - 1}{\frac{4}{3} - 1}\right\} = \left\{1 - \left(\frac{3}{4}\right)^k\right\}.$$

3. *Let* $\xi_2 = 10101010\ldots = \prod_{j=1}^{\infty} 10 \in \Sigma_2^*$. *Here* $\theta_j = 1$.

   (a) $\pi^1(\xi_2) = \sum_{j=1}^{\infty} \frac{2^{j-1}}{3^j} = \frac{1}{3}\sum_{j=1}^{\infty}\left(\frac{2}{3}\right)^j = 1.$

   (b) $\pi^2(\xi_2)(k) = \frac{3^k}{2^k}\sum_{j=1}^{k}\frac{2^{j-1}}{3^j} = \frac{3^{k-1}}{2^k}\left\{\frac{\frac{2^k}{3^k}-1}{\frac{2}{3}-1}\right\} = \left\{\left(\frac{3}{2}\right)^k - 1\right\} \to \infty.$

4. *Let* ξ_3 *be a sequence such that* $a_k = \frac{k(k+1)}{2}$. *Then* $\frac{2^{a_j}}{3^j} \to \infty$, *so* $\pi^1(\xi_3) = \infty$. *However, for* π^2*:*

$$\pi^2(\xi_3)(k) = \frac{3^k}{2^{\frac{k(k+1)}{2}}}\sum_{j=1}^{k}\frac{2^{\frac{j(j+1)}{2}-1}}{3^j} < 1.$$

5. *Let* $\xi_4 = 101000101000\ldots$.

   (a) $\pi^1(\xi_4) = \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\}\sum_{j=0}^{\infty}\left(\frac{2^4}{3^2}\right)^j \to \infty.$

   (b) $\pi^2(\xi_4)(1) = \frac{1}{2}, \pi^2(\xi_4)(2) = \frac{5}{2^4},$

   $\pi^2(\xi_4)(2k+1) = \frac{3^{2k+1}}{2^{4k+1}}\left(\frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\}\sum_{j=1}^{k-1}\left(\frac{2^4}{3^2}\right)^j\right)$

   and $\pi^2(\xi_4)(2k) = \frac{3^{2k}}{2^{4(k+1)}}\left(\frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\}\sum_{j=1}^{k-2}\left(\frac{2^4}{3^2}\right)^j + \frac{2^{4k+1}}{3^{2k}}\right)$

8.4. Characterization of G_0 and G_∞

To characterize the sets G_0 and G_∞, we need some technical results relating the growth of the exponents a_k to the behavior of the ratio $\frac{a_{k+1}-a_j}{k-j}$.

Lemma 9. *Let* $\liminf_{k\to\infty}\frac{a_{k+1}}{k} = \lambda \in (0,\infty]$. *Then there exists* $T > 0$ *such that for any length* $d > T$, *the average slope of the exponents is bounded below by* λ*:*

$$\inf_{k-j=d}\frac{a_{k+1}-a_j}{k-j} \geq \lambda - \epsilon.$$

Specifically, there exists $T > 0$ *such that if* $k - j > T$, *we have* $\frac{a_{k+1}-a_j}{k-j} \geq \lambda$.

Proof. Let $d = k - j$. By the definition of the limit inferior of the average density:

$$\liminf_{(k-j)\to\infty}\frac{a_{k+1}-a_j}{k-j} = \liminf_{d\to\infty}\frac{a_{d+1}}{d} = \lambda.$$

for any $\epsilon > 0$, there exists $T \in \mathbb{N}$ such that for all $d > T$:

$$\frac{a_{d+1}}{d} > \lambda - \epsilon.$$

for sufficiently long segments $k - j > T$, the average density of exponents respects the global lower limit. Thus, we can assert $\frac{a_{k+1}-a_j}{k-j} \geq \lambda$ locally for large enough separation. □

Lemma 10. *Let $k, j \in \mathbb{N}$. If $k > j$, then we have:*

$$\frac{a_{k+1} - a_j}{k - j} \geq 1.$$

Proof. Let $\xi \in \Sigma_2^*$. Writing explicitly, we have $\xi = 0^{\theta_1} \prod_{i=2}^{\infty} 10^{\theta_i}$ with $\theta_1 \geq 0$ and $\theta_i \geq 1$ for $i > 1$. We can write $a_k = \sum_{i=1}^{k} \theta_i$. Suppose $k > j$. Since the minimum value that θ_i can take for $i \geq 2$ is 1, we have:

$$\frac{a_{k+1} - a_j}{k - j} = \frac{\sum_{i=j+1}^{k+1} \theta_i}{k - j} \geq \frac{1 \cdot (k + 1 - j)}{k - j} > \frac{k - j}{k - j} = 1.$$

□

The following proposition provides a complete characterization of the sets G_0 and G_∞ based on the behavior of π^1 and π^2.

Proposition 22 (Characterization of the G_0 and G_∞ sets)**.** *Let $\xi \in \Sigma_2^*$ $\pi^1 : \Sigma_2^* \to \mathbb{R}$ and $\pi^2 : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q}\}$. Then*

1. *$\xi \in G_\infty$ if and only if $\pi^1(\xi) < \infty$.*
2. *$\xi \in G_0$ if and only if $\pi^2(\xi)(k)$ is bounded.*

Proof. *Proof of the first statement.* Is obvious for the case of null tails with index J, since we have π^1 it is automatically finite, and as we see in the examples π^2 would be of the form $\frac{3^{k_0}}{2^k} A \to 0$ when $k \to \infty$ which implies that π^2 is finite. So we are going to assume that ξ has no tail null.

Suppose that $\xi \in G_\infty$, then by Lemma 6 we have $\limsup_{j\to\infty} \frac{a_j}{j} \leq \limsup_{j\to\infty} \frac{a_{j+1}}{j} < \frac{\ln(3)}{\ln(2)}$, so, there exists $J, \varepsilon > 0$ such that for all $j > J$ we have that $\frac{a_j}{j} < \frac{\ln(3)}{\ln(2)} - \varepsilon$

$$\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} = \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} \frac{2^{a_j}}{3^j} = \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} \left(\frac{2^{\frac{a_j}{j}}}{3} \right)^j$$

$$< \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} \left(\frac{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}{3} \right)^j = \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \sum_{j=J+1}^{\infty} 2^{-\varepsilon j}$$

$$= \sum_{j=1}^{J} \frac{2^{a_j}}{3^j} + \frac{2^{-\varepsilon(J+1)}}{1 - 2^{-\varepsilon}} < \infty$$

Let's suppose $\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} < \infty$, then

$$\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} = \frac{1}{3} \sum_{j=1}^{\infty} \frac{2^{a_j}}{3^{j-1}} = \frac{1}{3} \sum_{j=0}^{\infty} \frac{2^{a_{j+1}}}{3^j}$$

so $\lim_{j\to\infty} \frac{2^{a_{j+1}}}{3^j} = 0$ then $\xi \in G_\infty$.

☐ **of the first statement.**

Proof of the second statement. Suppose $\pi^2(\xi)(j)$ is bounded, we will prove that $\frac{3^k}{2^{a_{k+1}(\xi)}}$ converges to 0. Suppose $\frac{3^k}{2^{a_{k+1}(\xi)}} > \varepsilon > 0$ for $k > K$ for any $K \in \mathbb{N}$. Then we have

$$\pi^2(\xi)(k) = \frac{N_k}{2^{a_{k+1}}} = \frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$$

We have that the sum on the right is divergent.

$$\frac{3^k}{2^{a_{k+1}}} \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} > \varepsilon \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} = \infty$$

which generates a contradiction to the fact that $\pi^2\{\xi\}(k)$ is bounded.

To demonstrate the other implication, let us consider the following lemmas:

Lemma 11. *Let* $\liminf_{k\to\infty} \frac{a_{k+1}}{k} = \lambda \in (0, \infty]$. *Then exist* $T > 0$ *such that if* $T < k - j$ *we have*

$$\frac{a_{k+1} - a_j}{k - j} \geq \lambda.$$

Proof. Let $\liminf_{k\to\infty} \frac{a_{k+1}}{k} = \lambda \in (0, \infty]$ and $d = k - j$, then we have

$$\liminf_{(k-j)\to\infty} \frac{a_{k+1} - a_j}{k - j} = \liminf_{d\to\infty} \frac{a_{j+d+1} - a_j}{d} = \liminf_{d\to\infty} \frac{a_{d+1}}{d} = \lambda$$

On the other hand, by definition of lower limit, we have

$$\liminf_{(k-j)\to\infty} \frac{a_{k+1} - a_j}{k - j} = \lim_{t\to\infty} \left\{ \inf_{(k-j)>t} \frac{a_{k+1} - a_j}{k - j} \right\}$$

Then exist $T > 0$ such that if $T < k - j$ we have

$$\frac{a_{k+1} - a_j}{k - j} \geq \inf_{(k-j)>T} \frac{a_{k+1} - a_j}{k - j} = \lambda$$

☐

Lemma 12. *Let* $k, j \in \mathbb{N}$, *if* $k > j$ *then, we have* $\frac{a_{k+1} - a_j}{k - j} \geq 1$.

Proof. Let $\xi \in \Sigma_2^*$ writing explicitly, we have $\xi = 0^{\theta_1} \prod_{i=2}^{\infty} 10^{\theta_i}$ with $\theta_1 \geq 0$ and $\theta_i > 0$ for $i > 1$, then we can write:

$$a_k = \sum_{i=1}^{k} \theta_i$$

Suppose $k > j$. Since the minimum value that θ can take is 1, we have

$$\frac{a_{k+1}-a_j}{k-j} = \frac{\sum_{i=j}^{k+1}\theta_i}{k-j} \geq \frac{k+1-j}{k-j} > \frac{k-j}{k-j} = 1$$

□

By Claim 3 and 4 we have, exist $T, \varepsilon \in \mathbb{N}$ such that if $k - T > j$ we have

$$\frac{a_{k+1}-a_j}{k-j} > \frac{\ln(3)}{\ln(2)} + \varepsilon$$

Then by claim 5, Let $k > T$ so

$$\begin{aligned}
\frac{3^k}{2^{a_{k+1}}}\sum_{j=1}^{k}\frac{2^{a_j}}{3^j} &= \sum_{j=1}^{k}\frac{2^{a_j - a_{k+1}}}{3^{j-k}} \\
&= \sum_{j=1}^{k}\left(\frac{3}{2^{\frac{a_{k+1}-a_j}{k-j}}}\right)^{k-j} \\
&= \sum_{j=1}^{k-T}\left(\frac{3}{2^{\frac{a_{k+1}-a_j}{k-j}}}\right)^{k-j} + \sum_{j=k-T+1}^{k}\left(\frac{3}{2^{\frac{a_{k+1}-a_j}{k-j}}}\right)^{k-j} \\
&< \sum_{j=1}^{k-T}\left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}}\right)^{k-j} + \sum_{j=k-T+1}^{k}\left(\frac{3}{2}\right)^{k-j} \\
&< \sum_{j=1}^{k}\left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}}\right)^{k-j} + \sum_{j=k-T+1}^{k}\left(\frac{3}{2}\right)^{k-j} \\
&< \frac{1}{2^{\varepsilon k}}\sum_{j=1}^{k}2^{\varepsilon j} + \left(\frac{3}{2}\right)^{k}\sum_{j=k-T+1}^{k}\left(\frac{2}{3}\right)^{j} \\
&= \frac{1}{2^{\varepsilon k}}\left(\frac{2^{\varepsilon} - 2^{\varepsilon(k+1)}}{1-2^{\varepsilon}}\right) + \left(\frac{3}{2}\right)^{k}\left\{\frac{\left(\frac{2}{3}\right)^{k+1} - \left(\frac{2}{3}\right)^{k-T+1}}{\frac{2}{3}-1}\right\} \\
&< \frac{\frac{2^{\varepsilon}}{2^{\varepsilon k}} - 2^{\varepsilon}}{1-2^{\varepsilon}} + \frac{\frac{2}{3} - \left(\frac{3}{2}\right)^{T-1}}{\frac{2}{3}-1} \\
&\leq \frac{2^{\varepsilon}}{2^{\varepsilon}-1} + 3\left\{\left(\frac{3}{2}\right)^{T-1} - \frac{2}{3}\right\}
\end{aligned}$$

Let $M = \frac{2^{\varepsilon}}{2^{\varepsilon}-1} + 3\left\{\left(\frac{3}{2}\right)^{T-1} - \frac{2}{3}\right\} > 0$. Then we have $\frac{3^k}{2^{a_{k+1}}}\sum_{j=1}^{k}\frac{2^{a_j}}{3^j} < M$. Then we conclude that π^2 is bounded.

□ **of the second statement.**

□

Corollary 2. *Let $\xi \in \Sigma_2^*$. Suppose that there exists a sub-sequence such that $\lim\limits_{j\to\infty} \frac{a_{k_l+1}(\xi)}{k_l} > \frac{\ln(3)}{\ln(2)}$. Then there exists $M > 0$ such that $\pi^2(\xi)(k_l) < M$ for all $l \in \mathbb{N}$.*

Proof. Let $j < k_l$ then

$$\lim_{k_l-j\to\infty} \frac{a_{k_l+1}-a_j}{k_l-j} = \lim_{k_l-1\to\infty} \frac{a_{k_l+1}-a_0}{k_l-1} = \lim_{l\to\infty} \frac{a_{k_l+1}}{k_l} > \frac{\ln(3)}{\ln(2)}$$

then exist $T, \varepsilon > 0$ such that if $k_l - j > T$ we have

$$\frac{a_{k_l+1}-a_j}{k_l-j} > \frac{\ln(3)}{\ln(2)} + \varepsilon$$

Using the lemma 12 we have

$$\begin{aligned}
\pi^2(\xi)(k_l) &= \frac{3^{k_l}}{2^{a_{k_l+1}}} \sum_{j=1}^{k_l} \frac{2^{a_j}}{3^j} \\
&= \sum_{j=1}^{k_l} \frac{2^{a_j - a_{k_l+1}}}{3^{j-k_l}} \\
&= \sum_{j=1}^{k_l} \left(\frac{3}{2^{\frac{a_{k_l+1}-a_j}{k_l-j}}} \right)^{k_l-j} \\
&= \sum_{j=1}^{k_l-T} \left(\frac{3}{2^{\frac{a_{k_l+1}-a_j}{k_l-j}}} \right)^{k_l-j} + \sum_{j=k_l-T+1}^{k_l} \left(\frac{3}{2^{\frac{a_{k_l+1}-a_j}{k_l-j}}} \right)^{k_l-j} \\
&< \sum_{j=1}^{k_l-T} \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}} \right)^{k_l-j} + \sum_{j=k_l-J+1}^{k_l} \left(\frac{3}{2}\right)^{k_l-j} \\
&< \sum_{j=1}^{k_l} \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}} \right)^{k_l-j} + \sum_{j=k_l-J+1}^{k_l} \left(\frac{3}{2}\right)^{k_l-j} \\
&< \frac{1}{2^{\varepsilon k_l}} \sum_{j=1}^{k_l} 2^{\varepsilon j} + \left(\frac{3}{2}\right)^{k_l} \sum_{j=k_l-J+1}^{k_l} \left(\frac{2}{3}\right)^{j} \\
&= \frac{1}{2^{\varepsilon k_l}} \left(\frac{2^{\varepsilon} - 2^{\varepsilon(k_l+1)}}{1-2^{\varepsilon}} \right) + \left(\frac{3}{2}\right)^{k_l} \left\{ \frac{\left(\frac{2}{3}\right)^{k_l+1} - \left(\frac{2}{3}\right)^{k_l-J+1}}{\frac{2}{3}-1} \right\} \\
&\leq \frac{2^{\varepsilon}}{2^{\varepsilon}-1} + 3\left\{ \left(\frac{3}{2}\right)^{J-1} - \frac{2}{3} \right\} = M
\end{aligned}$$

Therefore $\pi^2(\xi)(k_l) < M$ □

9. The Sigma Function

In this section, we immerse ourselves in the study of Diophantine equations of the form $2^k y - ax = n$, where a, k, n are integers. Solving these equations in the domain of integers x and y is a problem in number theory. Usually, these types of Diophantine equations are solved using Euclid's algorithm or some similar technique, even by trial and error. However, these techniques begin to have a high degree of complexity for very large values. This mainly complicates when we want to study the behavior of the minimum positive values since in this case, we are interested in asymptotic solutions. We introduce the sigma function, symbolized as $\sigma_a(n)$ to address this challenge. This function, whose detailed analysis will constitute the core of our research, plays a fundamental role in the quest for specific solutions to the aforementioned Diophantine equations. Particularly noteworthy is the sigma function's remarkable property of delivering solutions that are closest to zero in the context of these equations.

9.1. Summary of Propositions in the Section

1. **Definition 16:** Definition of the sigma function.
2. **Theorem 7:** Establish that $\sigma_a^k(n)$ and $\frac{1}{a}(2^k\sigma_a^k(n) - n)$ are solutions of the Diophantine equation $2^k y - ax = n$. Additionally, $\frac{1}{a}(2^k\sigma_a^k(n) - n)$ is the minimum non-negative integer value.
3. **Corollary 3:** Establishes that the minimum value grows based on the number of times the sigma function takes odd values.
4. **Corollary 4:** $\sigma_a^k(n) - \sigma_{-a}^k(n) = a$
5. **Corollary5:** Let $n \in \mathbb{Z}$, then $\sigma^a(n) = \left\lceil \frac{n}{2^a} \right\rceil$ and $\sigma_{-1}^a(n) = \left\lfloor \frac{n}{2^a} \right\rfloor$
6. **Proposition 23:** Establishes inequalities that estimate the values of the sigma function
7. **Proposition 24:** It establishes the periods for the periodic points.
8. **Proposition 25:** Establish algebraic properties of additivity, dependent on the parity of the addends
9. Proposition 26 Let $a, n, p, q \in \mathbb{N}$ with a odd number. We have
$$\sigma_a^n(p+q) = \sigma_a^n(p) + \sigma_a^n(q) - a\delta \quad \text{with } \delta \in \{0,1\}$$
10. **Lemma 13** Let $a \in \mathbb{N}$ odd number, then $\sigma_a^k(aK) = a\sigma^k(K)$
11. **Proposition 27:** Let $a \in \mathbb{N}$ odd number and $n, m \in \mathbb{Z}$ such that $m = n \pmod{a}$, then $\sigma_a^k(m) = \sigma_a^k(n) \pmod{a}$
12. **Corollary 6:** Let $0 < n < 3^b$ with $(n, 3^b) = 1$ then n is a periodic point of periodic $\mathrm{Ord}_{3^b}(2) = 2 \cdot 3^{b-1}$ for σ_{3^b}. In particular, if $a = 3^b$ and $(u, 3) = 1$ then u is periodic point of periodic $\mathrm{Ord}_{3^b}(2) = 3^{b-1}2$. Let $\gamma = 3^{b-1}$ then
$$\sigma_{3\gamma}^{2\gamma}(u) = u$$
13. **Proposition 28:** Let $b, n \in \mathbb{N}$ such that $n < 3^b$ and $(n, 3^b) = 1$, $\gamma = 3^{b-1}$ then $\sigma_{3\gamma}^{a}(n) + \sigma_{3\gamma}^{a+\gamma}(n) = 3\gamma$
14. **Corollary 7:** Let $b, n \in \mathbb{N}$ such that $n < 3^b$ and $(n, 3^b) = 1$, $\gamma = 3^{b-1}$ then $\sigma_{3\gamma}^{\gamma}(n) = 3\gamma - n$.
15. **Proposition 29:** We consider the function sigma as a function of $\mathbb{Z}/a\mathbb{Z}$ in $\mathbb{Z}/a\mathbb{Z}$, then it is a group additive automorphism. i.e.
$$\sigma_a^k(m+n) = \sigma_a^k(m) + \sigma_a^k(n) \pmod{a}$$
16. **Proposition 30** Establish that the sigma function is homogeneous modulo a.
17. **Definition 19:** Extension of the sigma function on $\mathbb{Q}_{odd}$
18. **Lemma 14:** The extension of the sigma function to $\mathbb{Q}_{odd}$ is well-defined.

19. **Definition 20:** Characteristic Function
20. **Lemma 15:** Establishes an invariance in the coding of the orbits of the sigma function.
21. **Proposition 31:** Establishes homogeneity properties of the extension of the sigma function.
22. **Proposition 46** Let $r \in \mathbb{Q}_{odd}$, then $\sigma^n(r)$ is periodic if and only if $r \in [0,1]$.
23. **Proposition 33** Algebraic properties of the Extension of the Sigma function.
24. **Proposition 34:** Let $a, n, p, q, P, Q \in \mathbb{N}$ with a, P, Q odd number. We have

$$\sigma_a^n\left(\frac{p}{P} + \frac{q}{Q}\right) = \sigma_a^n\left(\frac{p}{P}\right) + \sigma_a^n\left(\frac{q}{Q}\right) - a\Delta \quad \text{with } \Delta \in \{0,1\}$$

25. **Definition ??:** Definition of dyadic numbers.
26. **Proposition ??:** Characterization of the dyadic representation of rational numbers.
27. **Definition 21:** Definition of Cod-Sigma function.
28. **Lemma 16:** Invariant coding lemma for Cod-Sigma function.
29. **Corollary 8:** Let $r_1, r_2 \in \mathbb{Z}$. Then $Cod\sigma_{3^b}^a(r_1 + r_2) = Cod\sigma_{3^b}^a(r_1) + Cod\sigma_{3^b}^a(r_2) \pmod{2}^a$
30. **Proposition 35** $Cod\sigma_{3^b} : \mathbb{Z} \to \mathbb{Z}_2$ is linear.
31. **Lemma 17:** Rational equivalence of the Cod-Sigma function.
32. **Definition 22:** we will say that $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$ has a null tail of index J if $J > 0$ the smallest index such that $j > J$ we have $\xi_j = 0$.
33. **Proposition 36:** $-\pi^1(\xi) \in \mathbb{Z}_2$ and $Cod\sigma(\pi^1(\xi)) = -\pi^1(\xi) \in \mathbb{Z}_2$.
34. **Lemma 18:** Let $S_k \in \langle \theta, \psi^q \rangle$ with $Cod(S_k) = 0^{\theta_1}10^{\theta_2} \dots 10^{\theta_k}10^{\theta_{k+1}}$. Then $\rho_0(S_k) = Cod\sigma^{a_{k+1}}\left(q\frac{N_k}{3^k}\right)$
35. **Proposition 37** Let $\{S_k\}_{k\in\mathbb{N}} \in \langle \theta, \psi^q \rangle$ such that $\xi = Cod\{S_k\}_{k\in\mathbb{N}}$. Then $\rho_0(S_k) = Cod\sigma^{A_k}\left(q\pi^1(\xi)\right)$, where A_k is the quantity of 0 of $Cod(S_k)$.
36. **Corollary 9** Let $\{S_k\}_{k\in\mathbb{N}} \in \langle \theta, \psi^q \rangle$ such that $\xi = Cod\{S_k\}_{k\in\mathbb{N}}$. Then $\rho_1(S_k) = -Cod\sigma_{-1}^{A_k}\left(q\pi^1(\xi)\right)$, where A_k is the quantity of 0 of $Cod(S_k)$.

9.2. The Sigma Function

We introduce the Sigma function, a discrete dynamical system analogous to the Collatz function. The primary distinction lies in the arithmetic operation: while Collatz involves multiplication by 3, the Sigma function relies on parity-dependent adjustments without this multiplier, specifically tailored to analyze divisibility by 2 relative to a parameter a.

Definition 16 (The Sigma function)**.** *Let $x, a \in \mathbb{Z}$ such that $(a,2) = 1$. We define the sigma function $\sigma_a : \mathbb{Z} \to \mathbb{Z}$*

$$\sigma_a(x) = \begin{cases} \dfrac{x+a}{2} & \text{if } x = 1 \pmod{2} \\ \\ \dfrac{x}{2} & \text{if } x = 0 \pmod{2} \end{cases}$$

In the following theorem, we explore solutions to the Diophantine equation $2^k y - ax = n$, where a, k, and n are integers. This equation arises frequently in number theory, particularly in the study of Diophantine equations. We'll demonstrate that the sigma function provides particular solutions for y, shedding light on the behavior of solutions in both positive and negative domains. Additionally, we'll establish formulas for the smallest non-negative solution ρ_0 and the largest non-positive solution ρ_1 for the variable x, offering valuable insights into the structure of solutions to this equation.

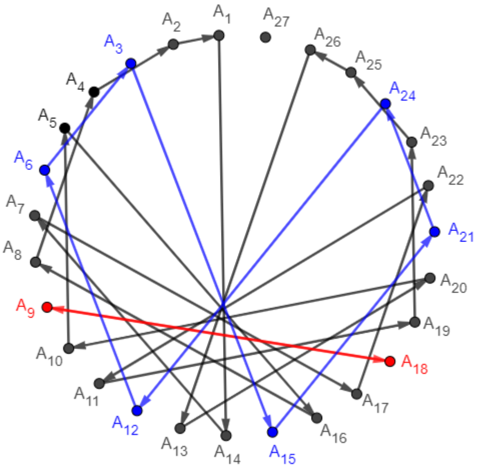

Figure 2. Orbits of σ_{27} for $0 < x \leq 27$.

Theorem 7 (Theorem on Diophantine Solutions)**.** *Let $a, k \in \mathbb{N}$ with $(a, 2) = 1$ and $n \in \mathbb{Z}$. Consider the Diophantine Equation $2^k y - ax = n$. Then a particular solution for y is given by*

$$\sigma_a^k(n) \text{ and } \sigma_{-a}^k(n).$$

where $\sigma_a^k(n) = \underbrace{\sigma_a \circ \ldots \circ \sigma_a(n)}_{k-times}$ and $\sigma_{-a}^k(n) = \underbrace{\sigma_{-a} \circ \ldots \circ \sigma_{-a}(n)}_{k-times}$. Furthermore. Let ρ_0 be the smallest non-negative solution for x, then

$$\rho_0 = \frac{1}{a}(2^k \sigma_a^k(n) - n)$$

let ρ_1 be the largest non-positive solution for x if then

$$\rho_1 = \frac{1}{a}(2^k \sigma_{-a}^k(n) - n)$$

Proof. We can write the sigma function as

$$\sigma_{\pm a}(n) = \frac{n \pm a\delta(n)}{2} \text{ where } \delta(n) = \begin{cases} 1 & \text{if } n = 1 \pmod 2 \\ 0 & \text{if } n = 0 \pmod 2 \end{cases}$$

Since the sigma function is defined on the set of integers in the integers, we have that its k-th composition is also an integer value: Let $\delta_j = \delta(\sigma_a^j(n))$ then

$$\sigma_a^k(n) = \cfrac{\cfrac{\cfrac{n+\delta_0}{2}}{\ddots} + a\delta_k}{2} = \frac{n + aL}{2^k} \in \mathbb{Z} \text{ where } L = \sum_{j=1}^{k-1} \delta_j 2^j$$

and let $\varepsilon_j = \delta(\sigma_{-a}^j(n))$ then

$$\sigma_{-a}^k(n) = \cfrac{\cfrac{\cfrac{n-a\varepsilon_0}{2}}{\ddots} - a\varepsilon_{k-1}}{2} = \frac{n - aU}{2^k} \in \mathbb{Z} \text{ where } U = \sum_{j=1}^{k-1} \varepsilon_j 2^j$$

replacing the k-th iteration sigma function σ_a in the equation $2^k y - ax = n$ and solving for x_0, we have

$$x_0 = \frac{1}{a}(2^k \sigma_a^k(n) - n) = L \in \mathbb{Z}$$

and replacing the k-th iteration sigma function σ_{-a} in the equation $2^k y - ax = n$ and solving for x_0, we have

$$x_0 = \frac{1}{a}(2^k \sigma_{-a}^k(n) - n) = -U \in \mathbb{Z}$$

For the positive case, we have that $0 \leq L = \sum_{j=1}^{k-1} \delta_j 2^j \leq 2^k - 1$, then due to the uniqueness of solutions in $[0, 2^k) \cap \mathbb{Z}$, L corresponds to the non-negative minimum value and for the negative case we have $0 \geq -U - \sum_{j=1}^{k-1} \varepsilon_j 2^j \geq -(2^k - 1)$, again due to uniqueness of solutions in $(2^k, 0] \cap \mathbb{Z}$, we have that $-U$ is the maximum non-positive solution. ☐

Example 11. *Let us consider the following Diophantine equation* $16y - 7x = 45$ *then*

$$\sigma_7^4(45) = 5 \text{ and } \frac{1}{7}(16 \cdot 5 - 45) = 5$$

are solutions of the equation.

We will demonstrate that this minimum value increases every time $\sigma_a^k(n)$ is an odd number. This result is crucial for understanding how the parity of the sigma function influences the structure of non-negative solutions of the associated Diophantine equation.

Corollary 3 (Monotonicity relation)**.** *Let* $a \in \mathbb{N}$ *such that* $(a, 2) = 1$ *and* $S_a(x) = \dfrac{ax + n}{2^k}$ *and* $\rho_0(S_a)$ *the minimum non-negative value of* $S_a(x)$*. Then* $\rho_0(S_a)$ *increases every time* $\sigma_a^k(n)$ *is an odd number. In particular* $\rho_0(S_a) = \sum_{j=1}^{k-1} \delta_j 2^j$ *with* $\delta_j = 0$ *if* $\sigma_a^{j-1}(n)$ *is even and* $\delta_j(n) = 1$ *if* σ_a^{j-1} *is odd.*

Proof. Let $S_a(x) = \dfrac{ax + n}{2^k}$ and $\delta_j = \delta(\sigma_a^j(n))$ then by Theorem 7 we have

$$\sigma_a^k(n) = \frac{1}{2^k}\left\{ n + a \sum_{j=1}^{k-1} \delta_j 2^j \right\}$$

then $S_a^{-1}(\sigma_a^k(n)) = \rho_0(S_a) = \sum_{j=1}^{k-1} \delta_j 2^j$. So, we have that every time $\delta_j = 1$, the minimum positive integer value increases, and this only happens when $\sigma_a^{j-1}(n)$ is odd. ☐

In the following corollary, we explore the relationship between the sigma functions $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$ in the context of the Diophantine equation $2^k y - ax = n$.

Corollary 4 (Relation between $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$)**.** *Let* $a, k \in \mathbb{N}$ *and* $n \in \mathbb{Z}$*. Consider the Diophantine Equation,* $2^k y - ax = n$*, then*

$$\sigma_a^k(n) - \sigma_{-a}^k(n) = a$$

Proof. By definition, we have that ρ_0 is the nearest non-negative solution to 0, and ρ_1 is the nearest non-positive solution to 0, which means that ρ_0 and ρ_1 are consecutive solutions. Therefore, $\rho_0 - 2^k = \rho_1$. then we have

$$2^k = \rho_0(S) - \rho_1(S) = \frac{1}{a}\left\{2^k\sigma_a^k(n) - n\right\} - \frac{1}{a}\left\{2^k\sigma_{-a}^k(n) - n\right\} = \frac{1}{a}2^k\sigma_a^k(n) - \frac{1}{a}2^k\sigma_{-a}^k(n)$$

Therefore $\sigma_a^k(n) - \sigma_{-a}^k(n) = a$ $\square$

Corollary 5. *Let $n \in \mathbb{Z}$, then $\sigma^a(n) = \left\lceil \frac{n}{2^a} \right\rceil$ and $\sigma_{-1}^a(n) = \left\lfloor \frac{n}{2^a} \right\rfloor$*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{x+n}{2^a}$. The integer set of f is determined by equation $x + n = 0 \pmod{2}^a$, then $\rho_0(f) = -n + 2^a\left\lceil \frac{n}{2^a} \right\rceil$ then

$$f(\rho_0) = \sigma^a(n) \text{ so } f(\rho_0) = \frac{-n + 2^a\left\lceil \frac{n}{2^a} \right\rceil + n}{2^a} = \left\lceil \frac{n}{2^a} \right\rceil = \sigma^a(n)$$

On the other hand by Corollary 4

$$\sigma_{-1}^a(n) = \sigma^a(n) - 1 = \left\lceil \frac{n}{2^a} \right\rceil - 1 = \left\lfloor \frac{n}{2^a} \right\rfloor$$

$\square$

In the following proposition, we examine the inequalities and estimations for the sigma function $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$, where n is an integer. We show that the sigma function lies in the interval $\left[\frac{n}{2^k}, \frac{n}{2^k} + a\right)$ for $\sigma_a^k(n)$, and in the interval $\left[\frac{n}{2^k} - a, n\right)$ for $\sigma_{-a}^k(n)$. These inequalities are fundamental to understand the range of values the sigma function can take in the context of the considered Diophantine equations.

Proposition 23 (Inequality and estimation of the sigma function)**.** *Let $a, k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then,*

$$\frac{n}{2^k} \leq \sigma_a^k(n) < \left(\frac{n}{2^k} + a\right) \text{ and } \frac{n}{2^k} - a < \sigma_{-a}^k(n) \leq \frac{n}{2^k}$$

Proof. For $\sigma_a^k(n)$ we have two possible extreme paths, either we always get even or we always get odd, for the first case we would always have division by 2

$$\frac{n}{2^k} \leq \sigma_a^k(n)$$

for the second we would have

$$\sigma_a^k(n) \leq \frac{n + a(1 + 2 + \ldots + 2^{k-1})}{2^k} = \frac{n}{2^k} + a\left(\frac{1 + 2 + \ldots + 2^{k-1}}{2^k}\right) = \frac{n}{2^k} + a\left(\frac{2^k - 1}{2^k}\right) < \frac{n}{2^k} + a$$

For $\sigma_a^k(n)$, regardless of the cases, we always get a less stringent value to the initial value. If it is always even, we will have that it is always divided by 2, now in the case that it is always odd we have

$$\sigma_{-a}^k(n) \geq \frac{n - a(1 + 2 + \ldots + 2^{k-1})}{2^k} = \frac{n}{2^k} - a\left(\frac{1 + 2 + \ldots + 2^{k-1}}{2^k}\right) > \frac{n}{2^k} - a$$

and clearly, we have

$$\frac{n}{2^k} > \frac{n}{2^k} - a$$

□

9.3. *Properties of the Sigma Function*

In this section, we investigate the algebraic and dynamical properties of the sigma function. As previously mentioned, the following proposition establishes that for values strictly between 0 and a, the sigma function exhibits periodic behavior. The period of these orbits is intimately related to the multiplicative order of 2 modulo the divisors of a. To formalize this relationship, we first recall the necessary number-theoretic definitions.

Definition 17. *Let a and n be integers with $\gcd(a,n)=1$. The* ***multiplicative order*** *of a modulo n, denoted $\mathrm{Ord}_n(a)$, is the smallest positive integer k such that*

$$a^k = 1 \pmod{n}.$$

The following are fundamental properties of the multiplicative order:

1. ***Divides any exponent satisfying the congruence:*** *If $a^m = 1 \pmod{n}$, then $\mathrm{Ord}_n(a) \mid m$.*
2. ***Divides $\varphi(n)$:*** *By Euler's theorem, $\mathrm{Ord}_n(a) \mid \varphi(n)$, where ϕ is Euler's totient function.*
3. ***Order modulo a composite number:*** *If $n = n_1 n_2 \cdots n_k$ with $\gcd(n_i, n_j) = 1$ for $i \neq j$, then*

$$\mathrm{Ord}_n(a) = \mathrm{lcm}\big(\mathrm{Ord}_{n_1}(a), \mathrm{Ord}_{n_2}(a), \ldots, \mathrm{Ord}_{n_k}(a)\big).$$

4. ***Primitive roots:*** *If $\mathrm{Ord}_n(a) = \varphi(n)$, then a is called a primitive root modulo n. These exist only for $n = 2, 4, p^\alpha$, or $2p^\alpha$, where p is an odd prime.*

Definition 18. *The* ***Carmichael function****, denoted by $\lambda(a)$, is defined as the smallest positive integer k such that*

$$n^k = 1 \pmod{a}$$

for all integers n satisfying $\gcd(a,n)=1$. That is, $\lambda(a)$ is the exponent of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. The Carmichael function can be calculated with the following formula

$$\lambda(a) = \begin{cases} \varphi(a) & \text{if } a \text{ is prime} \\ \\ [\varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})] & \text{if } a = \displaystyle\prod_{i=1}^{r} p_i^{\alpha_i} \text{ with } p_i \text{ prime and } \alpha_i \in \mathbb{N} \end{cases}$$

The following proposition describes the dynamics of the Sigma function; basically, all integers end up in some cycle between 1 and a.

Proposition 24 (Periodicity of periodic orbits). *Let $a \in \mathbb{N}$ The sigma function σ_a has the following properties,*

1. *The only fixed points are a and 0.*
2. *Let $u \in \mathbb{N}$ such that $u < a$ then, its orbit by is periodic with period $\mathrm{Ord}_{a/(a,u)}(2)$ is less than*

$$\lambda\left(\frac{a}{(a,u)}\right) = \begin{cases} \varphi\left(\dfrac{a}{(a,u)}\right) & \text{if } \dfrac{a}{(a,u)} \text{ is prime} \\ \\ [\varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})] & \text{if } \dfrac{a}{(a,u)} = \displaystyle\prod_{i=1}^{r} p_i^{\alpha_i} \text{ with } p_i \text{ prime and } \alpha_i \in \mathbb{N} \end{cases}$$

where φ is the Euler's totient function.

In particular, all points terminate in some periodic orbit (including periodic points) between 0 and a. When $a = 3^b$ and u are co-prime with 3, then the period of the orbit of u is $\mathrm{Ord}_{3^r}(2) = 3^{b-1}2$.

Proof. we have

1. Let $\sigma_a(u) = u$, if u is odd, then $\dfrac{u+a}{2} = u$ which implies $u = a$. If u is even, we have, $\dfrac{u}{2} = u$ which implies $u = 0$.
2. Let $u \in \mathbb{N}$ such that $u < a$ and $\sigma_a^k(u) = u$, so

$$\frac{u+aL}{2^k} = u \text{ for any } L = \sum_{j=1}^{k-1} \delta_j 2^j \text{ where } \delta_j \in \{0,1\}$$

Then

$$u + aL = 2^k u \Rightarrow aL = (2^k-1)u \Rightarrow (2^k-1)u = 0 \pmod{a}$$

suppose that $(u,a) = 1$, this implies that u is an invertible $\pmod{a}$ then, the equation is equivalent

$$2^k = 1 \pmod{a}$$

Let $k_0 = \lambda(a)$, then

$$(2^{k_0}-1)u = 0 \pmod{a} \Rightarrow (2^{k_0}-1)u = L_0 a \text{ for any } L_0 \in \mathbb{N}$$

$$(2^{k_0}-1) = L_0 \frac{a}{u}$$

as $\dfrac{a}{u} > 1$ then $(2^{k_0}-1) > L_0$, which is the necessary and sufficient condition for L_0 to admit decomposition in base 2 up to the power $k_0 - 1$ which implies that there exist $\delta_j \in \{0,1\}$ such that $L_0 = \sum_{j=1}^{k_0-1} \delta_j 2^j$.

Now suppose that $(u,a) = d > 1$, then we divide $aL = (2^k-1)u$ by, d

$$\frac{a}{d}L = (2^k-1)\frac{u}{d} \text{ where } \left(\frac{a}{d}, \frac{a}{d}\right) = 1$$

then the development is completely analogous to the first case.

In particular, when $a = 3^b$ and u are co-prime with 3, then the period of the orbit of u is $\mathrm{Ord}_{3^r}(2) = 3^{b-1}2$.

☐

Let us observe that for the equation $\sigma_a^k(u) = u$ to have a solution it is necessary and sufficient that $u < a$ since the function is monotonically decreasing for $u > a$.

Example 12. *For $a = 7$, let $n \in \mathbb{N}$ such that $(n,7) = 1$. Then the smallest solution to the equation $2^k = 1 \pmod{7}$ is $k = 3$, so any n coprime to 7 has period 3. For example, we have $\mathcal{O}(5) = \{5,6,3\}$ and $\mathcal{O}(4) = \{4,2,1\}$.*

For $a = 15$, the smallest k such that $2^k = 1 \pmod{15}$ is $k = 4$. Since $15 = 3 \cdot 5$, we compute the order of 2 modulo 3 and 5:

- $\mathrm{Ord}_3(2) = 2$ *because* $2^2 = 1 \pmod{3}$,
- $\mathrm{Ord}_5(2) = 4$ *because* $2^4 = 1 \pmod{5}$.

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Then, $\mathrm{Ord}_{15}(2) = \mathrm{lcm}(2,4) = 4$*. So any* n *coprime to* 7 *has period* 4*. For example, we have* $\mathcal{O}(1) = \{1,8,4,2\}$ *and* $\mathcal{O}(7) = \{7,11,13,14\}$*. On the other hand for* 3 *we have* $\mathrm{Ord}_{15/(15,3)}(2) = \mathrm{Ord}_5 = 4$*, then* 3 *have periodic* 4*, indeed* $\mathcal{O}(3) = \{3,9,12,6\}$ *and for* 5 *we have* $\mathrm{Ord}_{15/(15,5)}(2) = \mathrm{Ord}_3 = 2$*, then* 5 *have periodic* 2*, indeed* $\mathcal{O}(5) = \{5,10\}$*.*

Proposition 25. *Let* $a \in \mathbb{Z}$ *odd number and* $\sigma_a : \mathbb{Z} \to \mathbb{Z}$*, then we have*

1. *if* n, m *are even numbers, then* $\sigma_a(n+m) = \sigma_a(n) + \sigma_a(m)$*.*
2. *if* n *is an even number and* m *is an odd number, then* $\sigma_a(n+m) = \sigma_a(n) + \sigma_a(m)$*.*
3. *if* n, m *are odd numbers, then* $\sigma_a(n+m) = \sigma_a(n) + \sigma_a(m) - a$*.*

Proof. Let $m.n \in \mathbb{Z}$, then we have

1. If m, n are even, we have
$$\sigma_a(m+n) = \frac{m+n}{2} = \frac{m}{2} + \frac{n}{2} = \sigma_a(m) + \sigma_a(n)$$
2. If m is even and n is odd, we have
$$\sigma_a(m+n) = \frac{m+n+a}{2} = \frac{m}{2} + \frac{n+a}{2} = \sigma_a(m) + \sigma_a(n)$$
3. If m, n are odd, we have
$$\sigma_a(m+n) = \frac{m+n}{2} = \frac{m}{2} + \frac{n}{2} + a - a = \frac{m+a}{2} + \frac{n+a}{2} - a = \sigma_a(m) + \sigma_a(n) - a.$$

□

Proposition 26. *Let* $a, n, p, q \in \mathbb{N}$ *with* a *odd number. We have*

$$\sigma_a^n(p+q) = \sigma_a^n(p) + \sigma_a^n(q) - a\delta \quad \text{with } \delta \in \{0,1\}$$

Proof. Let us prove by induction that

$$\sigma_a^n(p+q) = \sigma_a^n(p) + \sigma_a^n(q) - a\delta \quad \text{with } \delta \in \{0,1\}$$

For the case $n = 1$, we have, by the proposition, that if at least one of the summands is even, then we have linearity. However, in the case where both summands are odd, we obtain linearity minus a. Therefore,

$$\sigma_a(p+q) = \sigma_a(p) + \sigma_a(q) - a\delta \text{ with } \delta \in \{0,1\}$$

Now suppose that the proposition holds for $n = k$, that is,

$$\sigma_a^k(p+q) = \sigma_a^k(p) + \sigma_a^k(q) - a\delta$$

Then for $n = k+1$, we have

$$\sigma_a^{k+1}(p+q) = \sigma\left(\sigma_a^k(p+q)\right) = \sigma\left(\sigma_a^k(p) + \sigma_a^k(q) - a\delta\right)$$

Suppose that $\sigma_a^k(p)$ and $\sigma_a^k(q)$ are even, then:

$$= \sigma_a^{k+1}(p) + \sigma_a^{k+1}(q) + \sigma_a(-a\delta) = \sigma_a^{k+1}(p) + \sigma_a^{k+1}(q)$$

Now suppose that $\sigma_a^k(p)$ is even and $\sigma_a^k(q)$ is odd, then

$$= \sigma_a^{k+1}(p) + \sigma(\sigma_a^k(q) - a\delta) = \sigma_a^{k+1}(p) + \sigma_a^{k+1}(q) + \sigma_a(-a\delta) - a = \sigma_a^{k+1}(p) + \sigma_a^{k+1}(q) - a$$

Finally, suppose that both $\sigma_a^k(p)$ and $\sigma_a^k(q)$ are odd, then

$$= \sigma_a^{k+1}(p) + \sigma_a(\sigma_a^k(q) - a\delta)$$

Since $\sigma_a^k(q) - a\delta$ is even, then

$$= \sigma_a^{k+1}(p) + \sigma_a^{k+1}(q) + \sigma(-a\delta) - a = \sigma_a^{k+1}(p) + \sigma_a^{k+1}(q) - a$$

Therefore, we have that for all $n \in \mathbb{N}$,

$$\sigma_a^n(p+q) = \sigma_a^n(p) + \sigma_a^n(q) - a\delta \quad \text{with } \delta \in \{0,1\}$$

□

9.4. The Sigma Function Modulo a

In this section, we address the linearity of the sigma function modulo a. Proposition 25 establishes the addition rules for the sigma function under different parity conditions of the involved numbers. We will see in Corollary 29 that the sigma function modulo a acts as an automorphism of $\mathbb{Z}/a\mathbb{Z}$. Furthermore, Proposition 30 establishes a relationship between $\sigma_a^k(m)$ and $\sigma_a^k(1)$.

This next corollary states that the sigma function, seen as a function on the set $\mathbb{Z}/a\mathbb{Z}$ taking values in $\mathbb{Z}/a\mathbb{Z}$, preserves the group structure under modular addition.

Lemma 13. *Let $a \in \mathbb{N}$ be an odd number. Then $\sigma_a^k(aK) = a\sigma_1^k(K)$, where σ_1 denotes the sigma function with parameter 1 (the ceiling function).*

Proof. By induction. For $k = 1$, we have:

$$\sigma_a(aK) = \frac{aK + a\delta(aK)}{2}$$

Since a is an odd number, the parity of aK depends only on K (i.e., $\delta(aK) = \delta(K)$). Therefore:

$$\frac{aK + a\delta(aK)}{2} = \frac{aK + a\delta(K)}{2} = a\left(\frac{K + \delta(K)}{2}\right) = a\sigma_1(K)$$

Suppose the proposition is true for k. Then:

$$\begin{aligned}
\sigma_a^{k+1}(K) &= \sigma_a(\sigma_a^k(aK)) \\
&= \sigma_a(a\sigma_1^k(K)) \quad \text{(by inductive hypothesis)} \\
&= \frac{a\sigma_1^k(K) + a\delta(a\sigma_1^k(K))}{2} \\
&= a\left(\frac{\sigma_1^k(K) + \delta(\sigma_1^k(K))}{2}\right) \\
&= a\sigma_1(\sigma_1^k(K)) = a\sigma_1^{k+1}(K)
\end{aligned}$$

□

Now we show that the function $\sigma_a^k : \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/a\mathbb{Z}$ is well-defined for all $k \in \mathbb{N}$.

Proposition 27. *Let $a \in \mathbb{N}$ be an odd number and $n, m \in \mathbb{Z}$ such that $m \equiv n \pmod{a}$. Then $\sigma_a^k(m) \equiv \sigma_a^k(n) \pmod{a}$.*

Proof. Let $n, m \in \mathbb{Z}$ such that $m \equiv n \pmod{a}$. Then there exists $K \in \mathbb{Z}$ such that $m = n + aK$. By the quasi-linearity property, we have:

$$\sigma_a^k(m) = \sigma_a^k(n + aK) = \sigma_a^k(n) + \sigma_a^k(aK) - a\Delta$$

where Δ is an integer correction term arising from the sum of parities. By Lemma 13, we have $\sigma_a^k(aK) = a\sigma_1^k(K)$. Thus:

$$\sigma_a^k(m) = \sigma_a^k(n) + a\sigma_1^k(K) - a\Delta \equiv \sigma_a^k(n) \pmod{a}$$

□

Corollary 6. *Let $0 < n < 3^b$ with $(n, 3^b) = 1$. Then n is a periodic point with period $\mathrm{Ord}_{3^b}(2) = 2 \cdot 3^{b-1}$ for σ_{3^b}. In particular, if $a = 3^b$ and $(u, 3) = 1$, then u is a periodic point with period $\mathrm{Ord}_{3^b}(2) = 2 \cdot 3^{b-1}$. Let $\gamma = 3^{b-1}$, then:*

$$\sigma_{3^b}^{2\gamma}(u) = u$$

Proof. We know that the number of elements in $(\mathbb{Z}/3^b\mathbb{Z})^\times$ is $\phi(3^b) = 2 \cdot 3^{b-1}$. Moreover, this is a cyclic multiplicative group since 2 is a primitive root modulo 3^b. This means that every element of the multiplicative group $(\mathbb{Z}/3^b\mathbb{Z})^\times$ is congruent to some power of 2. Therefore, there exists a unique $\eta \in \mathbb{N}$ with $\eta \leq 2 \cdot 3^{b-1}$ such that:

$$n \equiv 2^\eta \pmod{3^b}$$

Since $\sigma_{3^b}(x) \equiv x \cdot 2^{-1} \pmod{3^b}$, iterating k times corresponds to multiplying by 2^{-k}. Thus, by Proposition 27, for any k:

$$\sigma_{3^b}^k(n) \equiv n \cdot 2^{-k} \equiv 2^{\eta-k} \pmod{3^b}$$

The condition for the orbit to return to n is $2^{\eta-k} \equiv 2^\eta \pmod{3^b}$, which implies $2^{-k} \equiv 1 \pmod{3^b}$. The smallest positive k satisfying this is the order of 2 modulo 3^b, which is $2 \cdot 3^{b-1}$. □

An interesting particular situation arises when we consider pairs of numbers whose sum equals a. In this case, the behavior of the Sigma function exhibits a remarkable symmetry that contrasts with the general subadditive behavior previously discussed. Specifically, if two numbers add up to a, the sum of the values obtained by iterating the Sigma function on each number individually remains constant and equal to a through all iterations.

Indeed, let $m, n \in \mathbb{N}$ such that $m + n = a$. Since a is an odd number, it follows that either m or n is even, but not both. By Proposition 25, the Sigma function is linear whenever at least one of the arguments is even. Furthermore, Proposition 24 asserts that a itself is a fixed point of the Sigma function. Therefore, applying the function σ_a iteratively k times, we obtain the identity:

$$\sigma_a^k(m) + \sigma_a^k(n) = a.$$

This observation reveals the existence of a family of pairs whose images under the Sigma function remain connected through an additive identity. Moreover, in the specific case where $a = 3^b$, we can characterize these pairs more explicitly, as they share the same cyclic orbit under iteration.

Proposition 28. *Let $b, n \in \mathbb{N}$ such that $n < 3^b$ and $(n, 3^b) = 1$. Let $\gamma = 3^{b-1}$. Then:*

$$\sigma_{3^b}^{\gamma}(n) + \sigma_{3^b}^{\gamma+2\gamma}(n) = 3^b$$

(Note: Since the period is 2γ, this simplifies to $\sigma_{3^b}^{\gamma}(n) + n = 3^b$).

Proof. Let n be coprime to and less than 3^b. Consider the equation for k:

$$n + \sigma_{3^b}^k(n) = 3^b$$

Let $T_k(n) = \sum_{j=0}^{k-1} \delta_j 2^j$ and $\sigma_{3^b}^k(n) = \dfrac{n + 3^b T_k}{2^k}$. Then the equation $n + \sigma_{3^b}^k(n) = 3^b$ implies modulo 3^b:

$$n + \frac{n}{2^k} \equiv 0 \pmod{3^b} \implies n(1 + 2^{-k}) \equiv 0 \pmod{3^b}$$

Since $(n, 3^b) = 1$, we can divide by n:

$$2^k \equiv -1 \pmod{3^b}$$

We know that the order of 2 is $2\gamma = 2 \cdot 3^{b-1}$. The solution to $2^k \equiv -1$ occurs at half the period, so $k = \gamma = 3^{b-1}$.

Now we verify that this k yields a valid integer solution (i.e., that the numerator does not "overflow" the necessary bounds improperly). We require $T_\gamma \leq 2^\gamma - 1$. Indeed:

$$T_\gamma = \frac{2^\gamma(3^b - n) - n}{3^b} = 2^\gamma - n\left(\frac{2^\gamma + 1}{3^b}\right)$$

Claim: $\frac{2^{3^{b-1}}+1}{3^b} \geq 1$ for all $b \in \mathbb{N}$.

Let $f(x) = \frac{2^{3^{x-1}}+1}{3^x}$. We have:

1. $f(1) = \frac{2^1+1}{3} = 1$ and $f(2) = \frac{2^3+1}{9} = 1$.
2. The derivative $f'(x) > 0$ for $x > 1.6$. Thus, the function is monotonically increasing for integers $b \geq 2$.

Therefore, the term $\dfrac{2^\gamma + 1}{3^b}$ is a positive integer greater than or equal to 1. Consequently, $T_\gamma < 2^\gamma$, ensuring the existence of the binary coefficients. Thus, $n + \sigma_{3^b}^{\gamma}(n) = 3^b$. □

As a consequence of the above Proposition, we have:

Corollary 7. *Let $b, n \in \mathbb{N}$ such that $n < 3^b$ and $(n, 3^b) = 1$. Let $\gamma = 3^{b-1}$. Then:*

$$\sigma_{3^b}^{\gamma}(n) = 3^b - n$$

Proof. By Proposition 24 (periodicity) and Proposition 28:

$$\sigma_{3^b}^{\gamma}(n) + n = 3^b \implies \sigma_{3^b}^{\gamma}(n) = 3^b - n$$

□

We have seen that the function σ_a is, in general, a subadditive function where the error is a multiple of a. Therefore, the function σ_a is linear in $\mathbb{Z}/a\mathbb{Z}$.

Proposition 29 (Linearity modulo a)**.** *Consider the sigma function as a map from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/a\mathbb{Z}$. It is a group additive automorphism, i.e.,*

$$\sigma_a^k(m+n) \equiv \sigma_a^k(m) + \sigma_a^k(n) \pmod{a}$$

Proof. From the quasi-linearity proposition, we know that $\sigma_a(m+n) = \sigma_a(m) + \sigma_a(n) - a\delta$, where $\delta \in \{0,1\}$. Taking modulo a, the term $a\delta$ vanishes, yielding the result. $\square$

This proposition establishes the concept of homogeneity modulo a for the sigma function. It relates the value of $\sigma_a^k(m)$ to $m\sigma_a^k(1)$ under modular arithmetic.

Proposition 30 (Homogeneity mod a)**.** *Let $a, k, m \in \mathbb{N}$ such that $(a,2) = 1$. Then:*

$$\sigma_a^k(m) \equiv m\sigma_a^k(1) \pmod{a}$$

Proof. Using Proposition 29:

$$\sigma_a^k(m) = \sigma_a^k(\underbrace{1+\cdots+1}_{m \text{ times}}) \equiv \sum_{j=1}^{m} \sigma_a^k(1) \equiv m\sigma_a^k(1) \pmod{a}$$

$\square$

9.5. Extension of the Sigma Function to $\mathbb{Q}_{odd}$

We now extend the domain of the sigma function to the set of rational numbers with odd denominators.

Definition 19 ($\mathbb{Q}_{odd}$-extension of the sigma function)**.** *Let $\frac{u}{v}, \frac{x}{y} \in \mathbb{Q}_{odd}$ with u an odd integer. We define the sigma function $\sigma_{u/v} : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$ as:*

$$\sigma_{\frac{u}{v}}\left(\frac{x}{y}\right) = \begin{cases} \frac{1}{2}\left(\frac{x}{y} + \frac{u}{v}\right) & \text{if } x \equiv 1 \pmod{2}, \\ \frac{x}{2y} & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

We provide a numerical interpretation of this extension. Recall that in the integer case, the sigma function generates the smallest non-negative solution to the Diophantine equation $2^k y - ax = n$ via the relation $\frac{ax+n}{2^k} = \sigma_a^k(n)$. We can generalize this logic to fractions with odd denominators. Specifically, the extension satisfies the following relation for some $r \in \mathbb{Z}$:

$$\frac{yur + vx}{2^k} = vy\,\sigma_{\frac{u}{v}}^k\left(\frac{x}{y}\right)$$

Dividing by vy, this is equivalent to:

$$\frac{\frac{u}{v}r + \frac{x}{y}}{2^k} = \sigma_{\frac{u}{v}}^k\left(\frac{x}{y}\right)$$

In other words, the extension of the sigma function yields the fraction that solves the equation:

$$\frac{u}{v}r + \frac{x}{y} = 2^k t, \quad \text{with } r \in \mathbb{Z} \text{ and } t \in \mathbb{Q}_{odd}.$$

Lemma 14. *The extension of the sigma function to $\mathbb{Q}_{odd}$ is well-defined.*

Proof. Let $\mathbb{Q}_{\text{odd}} = \{p/q \in \mathbb{Q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, (q,2) = 1\}$. Let the parameter of the sigma function be $a = u/v$, where $u, v \in \mathbb{Z}$ are both odd integers. The extended sigma function $\sigma_a : \mathbb{Q}_{\text{odd}} \to \mathbb{Q}_{\text{odd}}$ is defined for an input $x/y \in \mathbb{Q}_{\text{odd}}$ (where y is odd) as:

$$\sigma_a\left(\frac{x}{y}\right) = \begin{cases} \frac{1}{2}\left(\frac{x}{y}+\frac{u}{v}\right) & \text{if } x \text{ is odd,} \\ \frac{x}{2y} & \text{if } x \text{ is even.} \end{cases}$$

We must verify two conditions:

1. The value is independent of the representative fraction for x/y and $a = u/v$.
2. The image $\sigma_a(x/y)$ remains in $\mathbb{Q}_{\text{odd}}$.

1. Independence of Representation: Let $x/y \in \mathbb{Q}_{\text{odd}}$ and $a = u/v$ (with u, v odd). Consider alternative representations $x/y = (xk_1)/(yk_1)$ and $a = u/v = (uk_2)/(vk_2)$, where k_1, k_2 are odd integers.

Since k_1 is odd, xk_1 shares the same parity as x. Similarly, since k_2 is odd and u is odd, uk_2 is odd. The denominators yk_1 and vk_2 remain odd. Let $a' = (uk_2)/(vk_2)$.

Case 1: x is odd. Then xk_1 is also odd.

$$\sigma_{a'}\left(\frac{xk_1}{yk_1}\right) = \frac{1}{2}\left(\frac{xk_1}{yk_1}+\frac{uk_2}{vk_2}\right) = \frac{1}{2}\left(\frac{x}{y}+\frac{u}{v}\right) = \sigma_a\left(\frac{x}{y}\right).$$

Case 2: x is even. Then xk_1 is also even.

$$\sigma_{a'}\left(\frac{xk_1}{yk_1}\right) = \frac{(xk_1)/(yk_1)}{2} = \frac{xk_1}{2yk_1} = \frac{x}{2y} = \sigma_a\left(\frac{x}{y}\right).$$

Thus, the definition is independent of the representation.

2. Closure in $\mathbb{Q}_{odd}$: Let $a = u/v$ (with u, v odd) and $x/y \in \mathbb{Q}_{\text{odd}}$ (with y odd).

Case 1: x is odd.

$$\sigma_a\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{x}{y}+\frac{u}{v}\right) = \frac{xv+uy}{2vy}.$$

Since x, y, u, v are all odd:

- xv and uy are products of odd integers, hence odd.
- Their sum $xv + uy$ is even. Let $xv + uy = 2K$ for some $K \in \mathbb{Z}$.
- The denominator vy is odd.

Substituting back:

$$\sigma_a\left(\frac{x}{y}\right) = \frac{2K}{2vy} = \frac{K}{vy}.$$

Since vy is odd, the result is in $\mathbb{Q}_{\text{odd}}$.

Case 2: x is even. Let $x = 2x'$ for some $x' \in \mathbb{Z}$.

$$\sigma_a\left(\frac{x}{y}\right) = \frac{x}{2y} = \frac{2x'}{2y} = \frac{x'}{y}.$$

Since y is odd, the result is in $\mathbb{Q}_{\text{odd}}$.

We conclude that σ_a is well-defined. □

9.6. Properties of the Extension of the Sigma Function

The introduction of the sigma function extended to odd rationals is crucial for understanding its behavior in a broader domain. This extension, defined on the set $\mathbb{Q}_{odd}$, allows us to explore the algebraic and arithmetic properties of the sigma function in a more general context. In this section, we delve into this extension and explore its implications, focusing on how the sigma function modifies its behavior when applied to fractions with odd denominators. Additionally, we present an important lemma that establishes an invariant relationship between the characteristic function δ and the sigma function, providing a deeper understanding of how the sigma function preserves certain properties under different transformations.

Definition 20 (Characteristic Function)**.** *We define the characteristic function* $\delta : \mathbb{Q}_{odd} \to \{0,1\}$ *given by*

$$\delta\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } p = 1 \pmod 2 \\ 0 & \text{if } p = 0 \pmod 2 \end{cases}$$

The Invariant Coding Lemma, stated in Lemma 15, establishes a fundamental relationship between the characteristic function δ and the sigma function under certain conditions. Specifically, it asserts that for co-prime integers u and v, with u being odd, the characteristic function δ remains invariant under iterations of the sigma function. This means that the parity of the output of $\sigma^j_u(v)$ is the same as the parity of $\sigma^j_1\left(\frac{v}{u}\right)$ for all non-negative integers j. Furthermore, if v is odd, the lemma demonstrates that the parity of $\sigma^j_u(v)$ is identical to the parity of $\sigma^j_{\frac{u}{v}}(1)$ for all non-negative integers j.

Lemma 15 (Invariant Characteristic Function Lemma)**.** *Let* $u, v \in \mathbb{Z}$ *with* u *not null, such that* $(u, 2) = 1$ *then*

1. $\delta\left(\sigma^j_u(v)\right) = \delta\left(\sigma^j_1\left(\frac{v}{u}\right)\right)$ *for* $j \geq 0$.
2. *if* v *is odd then* $\delta(\sigma^j_u(v)) = \delta\left(\sigma^j_{\frac{u}{v}}(1)\right)$ *for* $j \geq 0$.

Proof. We have, **first statement:** Let $\sigma^k_u(v) = \frac{v + uT^u_k(v)}{2^a}$ where $T^u_k(v) = \sum_{j=1}^{k-1} \delta(\sigma^j_u(v))2^j$ and $\sigma^j_1\left(\frac{v}{u}\right) = \frac{\frac{v}{u} + T^1_k\left(\frac{v}{u}\right)}{2^a}$ where $T^1_k\left(\frac{v}{u}\right) = \sum_{j=1}^{k-1} \delta\left(\sigma^j_1\left(\frac{v}{u}\right)\right)2^j$. We will prove by induction that $\delta\left(\sigma^j_u(v)\right) = \delta\left(\sigma^j_1\left(\frac{v}{u}\right)\right)$ For $j = 0$, Since if v is odd (or even) then $\frac{v}{u}$ is odd (or even) then

$$\delta\left(\sigma^0_u(v)\right) = \delta(v) = \delta\left(\frac{v}{u}\right) = \delta\left(\sigma^0_1\left(\frac{v}{u}\right)\right).$$

Suppose $\delta\left(\sigma^j_u(v)\right) = \delta\left(\sigma^j_1\left(\frac{v}{u}\right)\right)$ for $j \leq k$, then $T^u_k(v) = T^1_k\left(\frac{v}{u}\right)$, then we have

$$\sigma^k_u(v) = \frac{v + uT^u_k(v)}{2^a} = \frac{v + uT^1_k\left(\frac{v}{u}\right)}{2^a} = u\left(\frac{\frac{v}{u} + T^1_k\left(\frac{v}{u}\right)}{2^a}\right) = u\sigma^k_1\left(\frac{v}{u}\right)$$

Since u is odd, we have that $\sigma^k_u(v)$ and $\sigma^k_1\left(\frac{v}{u}\right)$ have the same parity, then $\delta\left(\sigma^{k+1}_u(v)\right) = \delta\left(\sigma^{k+1}_1\left(\frac{v}{u}\right)\right)$

Second statement Let $\sigma^k_u(v) = \frac{v + uT^u_k(v)}{2^a}$ where $T^u_k(v) = \sum_{j=1}^{k-1} \delta(\sigma^j_u(v))2^j$ and $\sigma^j_{\frac{u}{v}}(1) = \frac{1 + T^{\frac{u}{v}}_k(1)}{2^a}$ where $T^{\frac{u}{v}}_k(1) = \sum_{j=1}^{k-1} \delta\left(\sigma^j_{\frac{u}{v}}(1)\right)2^j$. We will prove by induction that $\delta\left(\sigma^j_u(v)\right) = \delta\left(\sigma^j_{\frac{u}{v}}(1)\right)$

For $j = 0$, Since v is odd

$$\delta\left(\sigma^0_u(v)\right) = \delta(v) = \delta(1) = \delta\left(\sigma^0_{\frac{u}{v}}(1)\right).$$

Suppose $\delta\left(\sigma^j_u(v)\right) = \delta\left(\sigma^j_{\frac{u}{v}}(1)\right)$ for $j \leq k$, then $T^u_k(v) = T^{\frac{u}{v}}_k(1)$, then we have

$$\sigma^k_u(v) = \frac{v + uT^u_k(v)}{2^a} = \frac{v + uT^{\frac{u}{v}}_k(1)}{2^a} = v\left(\frac{1 + \frac{u}{v}T^{\frac{u}{v}}_k(1)}{2^a}\right) = v\sigma^k_{\frac{u}{v}}(1)$$

Since v is odd, we have that $\sigma_u^k(v)$ and $\sigma_{\frac{u}{v}}^k(1)$ have the same parity, then

$$\delta\left(\sigma_u^{k+1}(v)\right) = \delta\left(\sigma_{\frac{u}{v}}^{k+1}(1)\right)$$

□

In the following proposition, we demonstrate homogeneity properties that leave the coding of the orbits of the sigma function invariant.

Proposition 31 (homogeneity)**.** *Let $u, v \in \mathbb{Z}$ with u not null, such that $(u,2) = (v,2) = 1$ then we have*

1. $\sigma_u^k(v) = u\sigma_1^k\left(\frac{v}{u}\right)$.
2. $\sigma_u^k(v) = v\sigma_{\frac{u}{v}}^k(1)$.

Proof. We have **First statement:** Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=1}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_1^j\left(\frac{v}{u}\right) = \dfrac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}$ where $T_k^1\left(\frac{v}{u}\right) = \sum_{j=1}^{k-1} \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)2^j$. Then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^1\left(\frac{v}{u}\right)}{2^a} = u\left(\frac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}\right) = u\sigma_1^k\left(\frac{v}{u}\right)$$

Second statement: Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=1}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_{\frac{u}{v}}^j(1) = \dfrac{1 + T_k^{\frac{u}{v}}(1)}{2^a}$ where $T_k^{\frac{u}{v}}(1) = \sum_{j=1}^{k-1} \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)2^j$. Then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^{\frac{u}{v}}(1)}{2^a} = v\left(\frac{1 + T_k^{\frac{u}{v}}(1)}{2^a}\right) = v\sigma_{\frac{u}{v}}^k(1)$$

□

Having extended the Sigma function to the set of odd rationals, $\mathbb{Q}_{odd}$, it is natural to investigate whether the properties previously established in the integer setting still hold in this broader context.

Proposition 32. *Let $r \in \mathbb{Q}_{odd}$, then $\sigma^n(r)$ is periodic if and only if $r \in [0,1]$.*

Proof. Let $r = \dfrac{p}{q} \in [0,1]$. Then

$$\sigma^n\left(\frac{p}{q}\right) = \frac{1}{q}\sigma_q^n(p)$$

Since $p \leq q$, we have that $\sigma_q^n(p)$ is periodic, and therefore $\sigma^n\left(\frac{p}{q}\right)$ is periodic. On the other hand, if $\sigma^n\left(\frac{p}{q}\right)$ is periodic, then there exists a $K > 0$ such that $\sigma^K\left(\frac{p}{q}\right) = \frac{p}{q}$. Suppose that $\frac{p}{q} \notin [0,1]$. Then

$$\left|\sigma\left(\frac{p}{q}\right)\right| < \frac{p}{q}$$

since if p is even we average $\dfrac{p}{q}$ with 0, and if p is odd we average $\dfrac{p}{q}$ with 1. In either case, we always obtain a value smaller than the initial one. Therefore, the only way for $\dfrac{p}{q}$ to be periodic is if it is within the interval $[0,1]$. ☐

Despite the presence of denominators, the recursive structure of $\sigma_{\frac{a}{A}}$ ensures that the behavior observed in the integer case persists, with the additive corrections involving $\frac{a}{A}$ appearing under similar conditions. The essential mechanism driving the quasi-linearity of the function, namely the cancellation of terms when one summand is even, extends directly to the rational setting, while the parity considerations remain encoded in the numerators.

Proposition 33 (Algebraic properties of the Extension of the Sigma function)**.** *Let $a \in \mathbb{Q}_{odd}$ with odd numerator and $\sigma_a : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$, satisfies the following identities*

1. *if $\dfrac{n}{q}, \dfrac{m}{p}$ are even fractions, then $\sigma_a\left(\dfrac{n}{q}+\dfrac{m}{p}\right) = \sigma_a\left(\dfrac{n}{q}\right)+\sigma_a\left(\dfrac{m}{p}\right)$.*
2. *if $\dfrac{n}{q}$ is an even fraction and $\dfrac{m}{p}$ is an odd fraction, then $\sigma_a\left(\dfrac{n}{q}+\dfrac{m}{p}\right) = \sigma_a\left(\dfrac{n}{q6.2}\right)+\sigma_a\left(\dfrac{m}{p}\right)$.*
3. *if $\dfrac{n}{q}, \dfrac{m}{p}$ are odd fractions, then $\sigma_a\left(\dfrac{n}{q}+\dfrac{m}{p}\right) = \sigma_a\left(\dfrac{n}{q}\right)+\sigma_a\left(\dfrac{m}{p}\right) - a$.*

Proof. Let's proved first for $a = 1$. Let $\dfrac{n}{q}, \dfrac{m}{p} \in \mathbb{Q}_{odd}$ and let $\beta : \mathbb{Z}\times\mathbb{Z} \to \{0,1\}$ given by $\beta(n,m) = 0$ if n or m is even fraction and $\beta(n,m) = 1$ if n and m are odd fraction.

$$\begin{aligned}
pq\sigma\left(\frac{n}{q}+\frac{m}{p}\right) &= pq\sigma\left(\frac{n}{q}+\frac{m}{p}\right)\\
&= pq\sigma\left(\frac{np+mq}{pq}\right)\\
&= \sigma_{pq}\left(pq\frac{np+mq}{pq}\right)\\
&= \sigma_{pq}(np+mq)\\
&= \sigma_{pq}(np)+\sigma_{pq}(mq) - pq\beta(n,m)\\
&= pq\sigma\left(\frac{np}{pq}\right)+pq\sigma\left(\frac{mq}{pq}\right) - pq\beta(n,m)\\
&= pq\sigma\left(\frac{n}{q}\right)+pq\sigma\left(\frac{m}{p}\right) - pq\beta(n,m)
\end{aligned}$$

Dividing everything by pq, we have

$$\sigma\left(\frac{n}{q}+\frac{m}{p}\right) = \sigma\left(\frac{n}{q}\right)+\sigma\left(\frac{m}{p}\right) - \beta(n,m)$$

Now let a an odd fraction with odd numerator, then we have $\frac{1}{a}$ is odd fraction, then multiplying by $\frac{1}{a}$ does not change the parity of $\frac{n}{q}$ or $\frac{m}{q}$. Then we have

$$\begin{aligned}
&\sigma\left(\frac{1}{a}\frac{n}{q}+\frac{1}{a}\frac{m}{p}\right)=\sigma\left(\frac{1}{a}\frac{n}{q}\right)+\sigma\left(\frac{1}{a}\frac{m}{p}\right)-\beta(n,m) \text{ multiplying by } a\\
&a\sigma\left(\frac{1}{a}\frac{n}{q}+\frac{1}{a}\frac{m}{p}\right)=a\sigma\left(\frac{1}{a}\frac{n}{q}\right)+a\sigma\left(\frac{1}{a}\frac{m}{p}\right)-a\beta(n,m)\\
&\sigma_a\left(\frac{n}{q}+\frac{m}{p}\right)=\sigma_a\left(\frac{n}{q}\right)+\sigma_a\left(\frac{m}{p}\right)-a\beta(n,m)
\end{aligned}$$

□

Proposition 34. *Let $a,n,p,q,P,Q\in\mathbb{N}$ with a,P,Q odd number. We have*

$$\sigma_a^n\left(\frac{p}{P}+\frac{q}{Q}\right)=\sigma_a^n\left(\frac{p}{P}\right)+\sigma_a^n\left(\frac{q}{Q}\right)-a\Delta \quad \textit{with } \Delta\in\{0,1\}$$

Proof. Let $a,n,p,q,P,Q\in\mathbb{N}$ with a,P,Q odd number. We have

$$\sigma_a^n\left(\frac{p}{P}+\frac{q}{Q}\right)=\frac{PQ}{PQ}\sigma_a^n\left(\frac{p}{P}+\frac{q}{Q}\right)=\frac{PQ}{PQ}\sigma_a^n\left(\frac{pQ+qP}{PQ}\right)=\frac{1}{PQ}\sigma_{aPQ}^n(pQ+qP)$$

By Proposition 26:

$$=\frac{1}{PQ}\left(\sigma_{aPQ}^n(pQ)+\sigma_{aPQ}^n(qP)-aPQ\Delta\right) \text{ with } \Delta\in\{0,1\}$$

$$=\frac{1}{PQ}\left(Q\sigma_{aP}^n(p)+P\sigma_{aQ}^n(q)-aPQ\Delta\right)=\sigma_a^n\left(\frac{p}{P}\right)+\sigma_a^n\left(\frac{q}{Q}\right)-a\Delta$$

□

9.7. Coding of Sigma Function

Having established the framework of p-adic numbers, specifically the 2-adic integers ($\mathbb{Z}_2$), we can now apply this structure to the dynamics of the sigma function. We associate a symbolic sequence to the orbit of any rational number in $\mathbb{Q}_{odd}$.

Definition 21. *Let $\frac{p}{q}\in\mathbb{Q}_{odd}$ and let $g\in\mathbb{Z}$ be an odd number. We define the Coding of $\frac{p}{q}$ under σ as $Cod\sigma_g\left(\frac{p}{q}\right)=\prod_{j\in\mathbb{N}_0}\delta_j\in\mathbb{Z}_2$ defined by*

$$\delta_j=\begin{cases}1 & \textit{if} \quad \sigma_g^j\left(\frac{p}{q}\right) \textit{ is odd}\\[2ex] 0 & \textit{if} \quad \sigma_g^j\left(\frac{p}{q}\right) \textit{ is even}\end{cases}$$

and the finite k-coding as $Cod\sigma_g^k\left(\frac{p}{q}\right)=\ldots0000\delta_k\ldots\delta_0$.

This coding scheme possesses invariance properties that allow us to simplify the analysis of orbits by scaling the arguments.

Lemma 16 (Invariant coding lemma)**.** *Let $u,v\in\mathbb{Z}$ with u non-zero, such that $(u,2)=1$. Then:*

1. *$Cod\sigma^j_u(v) = Cod\sigma^j\left(\frac{v}{u}\right)$ for $j \geq 0$.*
2. *If v is odd, then $Cod\sigma^j_u(v) = Cod\sigma^j_{\frac{u}{v}}(1)$ for $j \geq 0$.*

Proof. This is a reformulation of Lemma 15. □

This structural relationship leads to a surprising property: the coding function behaves linearly with respect to addition in $\mathbb{Z}_2$.

Corollary 8. *Let $r_1, r_2 \in \mathbb{Z}$. Then*

$$Cod\sigma^a_{3^b}(r_1 + r_2) = Cod\sigma^a_{3^b}(r_1) + Cod\sigma^a_{3^b}(r_2) \pmod{2^a}.$$

Proof. Let $r = r_1 + r_2$ and let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ be given by $\mathcal{H}(x) = \frac{3^b x + r}{2^a}$. By Proposition 25, we have

$$\frac{3^b x + r}{2^a} = \sigma^a_{3^b}(r) = \sigma^a_{3^b}(r_1) + \sigma^a_{3^b}(r_2) - 3^b u \text{ with } u \in \mathbb{Z}$$

Then

$$\rho_0 = \frac{2^a \sigma^a_{3^b}(r_1) - r_1}{3^b} + \frac{2^a \sigma^a_{3^b}(r_2) - r_2}{3^b} - 2^a u$$

On the other hand, we have that $\frac{2^a \sigma^a_{3^b}(r_j) - r_j}{3^b} = \rho_0\left(\frac{3^b x + r_j}{2^a}\right)$, so $\frac{2^a \sigma^a_{3^b}(r_1) - r_1}{3^b}(\text{ base } 2) = Cod\sigma(r_j)$. Thus,

$$Cod\sigma^a_{3^b}(r_1 + r_2) = Cod\sigma^a_{3^b}(r_1) + Cod\sigma^a_{3^b}(r_2) \pmod{2^a}.$$

□

Extending the previous corollary to the infinite limit, we confirm the linearity of the coding function on the entire domain $\mathbb{Z}$.

Proposition 35. *Let $r_1, r_2 \in \mathbb{Z}$. Then $Cod\sigma_{3^b} : \mathbb{Z} \to \mathbb{Z}_2$ is linear, i.e.,*

$$Cod\sigma_{3^b}(r_1 + r_2) = Cod\sigma_{3^b}(r_1) + Cod\sigma_{3^b}(r_2) \in \mathbb{Z}_2.$$

Proof. By Corollary 8, we have $Cod\sigma^n_{3^b}(r_1 + r_2) = Cod\sigma^n_{3^b}(r_1) + Cod\sigma^n_{3^b}(r_2) \pmod{2^n}$ for all $n \in \mathbb{N}$, which is equivalent to

$$\|Cod\sigma^n_{3^b}(r_1 + r_2) - Cod\sigma^n_{3^b}(r_1) + Cod\sigma^n_{3^b}(r_2)\|_2 < 2^{-n} \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$Cod\sigma_{3^b}(r_1 + r_2) = Cod\sigma_{3^b}(r_1) + Cod\sigma_{3^b}(r_2) \in \mathbb{Z}_2.$$

□

To demonstrate the utility of this linearity, we calculate the codings for several distinct affine maps.

Example 13. *1. Let $S(x) = \frac{9x + 5}{2^4}$. We have*

$$\sigma^4_9(5) = \sigma^4_9(3) + \sigma^4_9(2) \pmod{9}$$

(a) $\mathcal{O}\sigma_9^4(3) = \{3,6,3,6,3\}$ *then* $Cod\sigma_9^4(3) = 0101.$

(b) $\mathcal{O}(\sigma_9^4(2)) = \{2,1,5,7,8\}$ *then* $Cod\sigma_9^4(2) = 1110.$

$$\begin{aligned} Cod\sigma_9^4(5) &= Cod\sigma_9^4(3) + Cod\sigma_9^4(2) \pmod{2^4} \\ &= 1110 + 0101 \pmod{2^4} \\ &= 0011 \end{aligned}$$

$\mathcal{O}(\sigma_9^4(5)) = \{5,7,8,4,2\}$ *then* $Cod\sigma_9^4(5) = 0011$. *Then we have* $\rho_0(S) = 1 + 2 = 3.$

$$S(3) = \frac{9 \cdot 3 + 5}{2^4} = 2$$

2. *Let* $S(x) = \dfrac{9x + 3 + 2^{\theta_1}}{2^{\theta_1+\theta_2}}$ *with* θ_1 *even. We have*

$$\begin{aligned} \sigma_9^{\theta_1+\theta_2}(3 + 2^{\theta_1}) &= \sigma_9^{\theta_1+\theta_2}(3) + \sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \pmod 9 \\ &= 3\sigma_3^{\theta_1+\theta_2} + \sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \pmod 9 \end{aligned}$$

Then

$$\begin{aligned} Cod\sigma_9^{\theta_1+\theta_2}(3 + 2^{\theta_1}) &= Cod\sigma_9^{\theta_1+\theta_2}(3) + Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \pmod{2^{\theta_1+\theta_2}} \\ &= Cod\sigma_3^{\theta_1+\theta_2} + Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \pmod{2^{\theta_1+\theta_2}} \end{aligned}$$

(a) $\mathcal{O}(\sigma_3^{\theta_1+\theta_2}) = \{1,2,1,2,1,2,\ldots\}$ *then* $Cod\sigma_3^{\theta_1+\theta_2} = \underbrace{\ldots 010101}_{\theta_1+\theta_2}.$

(b) $\mathcal{O}(\sigma_9^{\theta_1+\theta_2}(2^{\theta_1})) = \{2^{\theta_1}, 2^{\theta_1-1}, \ldots, 1, 5, 7, 8, 4, 2, 1, 5, 7, 8, 4, 2, 1, \ldots\}$ *then*

$$Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) = \underbrace{\ldots 000111000111}_{\theta_2}\underbrace{\ldots 00000}_{\theta_1}$$

Then we have

$$\begin{aligned} Cod\sigma_9^{\theta_1+\theta_2}(3 + 2^{\theta_1}) &= \underbrace{\ldots 000111000111}_{\theta_2}\underbrace{\ldots 00000}_{\theta_1} + \underbrace{\ldots 010101}_{\theta_1+\theta_2} \pmod{2^{\theta_1+\theta_2}} \\ &= \underbrace{\ldots 000111000111000111100}_{\theta_2}\underbrace{\ldots 1010101010}_{\theta_1} \pmod{2^{\theta_1+\theta_2}} \\ &= Cod\sigma_9^{\theta_2-2}00Cod\sigma_3^{\theta_1} \pmod{2^{\theta_1+\theta_2}} \end{aligned}$$

Then we have $\rho_0(S)(base\ 2) = Cod\sigma_9^{\theta_2-2}00Cod\sigma_3^{\theta_1} \pmod{2^{\theta_1+\theta_2}}$.

3. *Let* $S_n(x) = \dfrac{3^n x + 4^n - 3^n}{4^n}$

$$\sigma_{3^n}^{2n}(4^n - 3^n) = \sigma_{3^n}^{2n}(4^n) + \sigma_{3^n}^{2n}(-3^n) = 1 - 3^n \underbrace{\sigma_{-1}^{2n}}_{=0} = 1 \pmod{3^n}$$

On the other hand

(a) $\mathcal{O}(\sigma_{3^n}^{2n}(2^{2n})) = \{2^{2n}, 2^{2n-1}, 2^{2n-2}, \ldots, 1\}$ *then* $Cod\sigma_{3^n}^{2n}(2^{2n}) = \underbrace{\ldots 00000}_{2n}.$

(b) $\mathcal{O}(\sigma_{3^n}^{2n}(3^n)) = \{3^n, 3^n, 3^n, 3^n \ldots\}$ *then* $Cod\sigma_{3^n}^{2n}(3^n) = \underbrace{\ldots 11111111}_{2n}.$

Then we have

$$\begin{aligned}
&\underbrace{0\ldots0}_{2n}-\underbrace{1\ldots1}_{2n} \pmod{2^{2n}}\\
=&1\underbrace{0\ldots0}_{2n}-0\underbrace{1\ldots1}_{2n} \pmod{2^{2n}}\\
=&1 \pmod{2^{2n}}\\
=&1
\end{aligned}$$

Then

$$Cod\sigma_{3^n}^{2n}(4^n-3^n)=00\ldots001$$

That is, $Cod\sigma_{3^n}^{2n}(4^n-3^n)$ has a constant coding equal to 1. This is natural, since S_n is stable.

We now explore the specific form of the coding for powers of 3 and integer multiples, revealing a simple rational equivalence in $\mathbb{Z}_2$.

Lemma 17 (Rational equivalence of the Cod-Sigma function)**.** *Let $b\in\mathbb{N}$. Then we have*

$$Cod\sigma_{3^b}=-\frac{1}{3^b}\in\mathbb{Z}_2.$$

Furthermore, let $u\in\mathbb{N}$; then

$$Cod\sigma_{3^b}(u)=-\frac{u}{3^b}\in\mathbb{Z}_2.$$

Proof. Let $u=\overline{\gamma_b}\in\mathbb{Z}_2$ with $\gamma_b=Cod\sigma_{3^b}^{2\cdot 3^{b-1}}$. Let $u=x$; then multiplying by $2^{2\cdot 3^{b-1}}$ we have $u\underbrace{0\ldots0}_{2\cdot 3^{b-1}}=2^{2\cdot 3^{b-1}}x$, and subtracting, we have

$$\begin{aligned}
u-u\underbrace{0\ldots0}_{2\cdot 3^{b-1}}&=-(2^{2\cdot 3^{b-1}}-1)x\\
\gamma_b&=-(2^{2\cdot 3^{b-1}}-1)x
\end{aligned}$$

On the other hand we have that $\gamma_b=\dfrac{2^{2\cdot 3^{b-1}}-1}{3^b}$, then

$$\frac{2^{2\cdot 3^{b-1}}-1}{3^b}=-(2^{2\cdot 3^{b-1}}-1)\text{ then we have }x=-\frac{1}{3^b}$$

Another way to prove it is:

$$\begin{aligned}
u = \sum_{n=0}^{\infty} u_n 2^n &= \left\{ \sum_{n=0}^{2\cdot 3^{b-1}-1} u_n 2^n \right\} + \left\{ \sum_{n=2\cdot 3^{b-1}}^{4\cdot 3^{b-1}-1} u_n 2^n \right\} + \left\{ \sum_{n=4\cdot 3^{b-1}}^{6\cdot 3^{b-1}-1} u_n 2^n \right\} \dots \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + 2^{2\cdot 3^{b-1}} Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + \left(2^{2\cdot 3^{b-1}}\right)^2 Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + \dots \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} \left\{ \sum_{n=0}^{\infty} \left(2^{2\cdot 3^{b-1}}\right)^n \right\} \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} \left\{ \frac{1}{1 - 2^{2\cdot 3^{b-1}}} \right\} \\
&= -\left\{ \frac{1 - 2^{2\cdot 3^{b-1}}}{3^b} \right\} \left\{ \frac{1}{1 - 2^{2\cdot 3^{b-1}}} \right\} \\
&= -\frac{1}{3^b}
\end{aligned}$$

Now we demonstrate the second part. Let $v \in \mathbb{N}$.

$$Cod\sigma_{3^b}^{k}(v) = -v \left\{ \frac{1 - 2^{2\cdot 3^{b-1} k}}{3^b} \right\} = -\frac{v}{3^b} + \frac{v 2^{2\cdot 3^{b-1} k}}{3^b}$$

On the other hand,

$$\left\| Cod\sigma_{3^b}^{k}(v) - \left(-\frac{v}{3^b} \right) \right\|_2 \leq 2^{-2\cdot 3^{b-1} k} \to 0 \text{ as } k \to \infty.$$

Therefore, $Cod\sigma_{3^b}(v) = -\dfrac{v}{3^b}$. □

Before presenting the relationship between the coding and the π^1 function, we define the concept of a null tail, which corresponds to sequences that eventually become zero.

Definition 22 (Null Tail). *We say that $\{\xi_j\}_{j=1}^{\infty} \in \Sigma_2^*$ has a null tail of index J if $J > 0$ is the smallest index such that for all $j > J$, we have $\xi_j = 0$.*

Using this definition, we can link the partial sums π_k^1 and the function π^1 directly to the coding σ in the context of $\mathbb{Z}_2$.

Proposition 36. *Let $\xi = 0^{\theta_1} \prod_{j=2}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ with $\theta_1 \geq 0$ and $\theta_k > 0$ for $k > 1$ and $a_n = \sum_{j=1}^{n} \theta_j$. We define $\pi_k^1(\xi) = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$. Let $\xi \in \Sigma_2^*$ with null tail of index K given by $\xi = 0^{\theta_1} \prod_{j=2}^{K} 10^{\theta_j} \prod_{j=K+1}^{\infty} 0^{\theta_j}$ with $\theta_1 \geq 0$, $\theta_k > 0$ for $K \geq k > 1$ and $\theta_j = 1$ for $j > K$. We define $\pi_k^1(\xi) = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j}$ if $k < K$ and $\pi_k^1(\xi) = \sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j}$ if $k \geq K$. We define the π^1 function given by $\pi^1 : \Sigma_2^* \to \mathbb{Z}_2$ and defined by $\pi^1(\xi) = \sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ if ξ does not have a null tail and $\pi^1(\xi) = \sum_{j=1}^{K-1} \frac{2^{a_j}}{3^j}$ if ξ has a null tail of index K. Then we have $\pi^1(\xi), \pi_k^1(\xi) \in \mathbb{Z}_2$ and*

$$Cod\sigma(\pi^1(\xi)) = -\pi^1(\xi) \in \mathbb{Z}_2.$$

Proof. Without loss of generality, let's assume that ξ does not have a null tail.

Claim 1: Let $\xi \in \Sigma_2^*$ and $k \in \mathbb{N}$; then we have $Cod\sigma\left(\pi_k^1(\xi)\right) = -\pi_k^1(\xi) \in \mathbb{Z}_2$.
Indeed, let $k \in \mathbb{N}$ and $\xi \in \Sigma_2^*$. By Lemma 16, Proposition 35, and Lemma 17, we have

$$Cod\sigma\left(\pi_k^1(\xi)\right) = Cod\sigma\left(\sum_{j=1}^{k}\frac{2^{a_j}}{3^j}\right) = Cod\sigma_{3^k}\left(\sum_{j=1}^{k}2^{a_j}3^{k-j}\right)$$

$$= \sum_{j=1}^{k}Cod\sigma_{3^k}\left(2^{a_j}3^{k-j}\right) = \sum_{j=1}^{k}Cod\sigma_{3^j}(2^{a_j}) = -\sum_{j=1}^{k}\frac{2^{a_j}}{3^j} = -\pi_k^1(\xi)$$

Since $-\frac{2^{a_j}}{3^j} \in \mathbb{Z}_2$, it follows that $-\pi_k^1(\xi) \in \mathbb{Z}_2$.

Claim 2: $\lim\limits_{k\to\infty}\pi_k^1(\xi) = \pi^1(\xi) \in \mathbb{Z}_2$.

Indeed, we have the following equivalence on $\mathbb{Q}_2$ (Proposition 3.3, page 76 of [8]):

$$\pi^1(\xi) = \sum_{n=1}^{\infty}\frac{2^{a_n}}{3^n} \in \mathbb{Q}_2 \text{ if and only if } \lim_{k\to\infty}\frac{2^{a_k}}{3^k} = 0 \text{ on } \mathbb{Q}_2.$$

On the other hand,

$$\left\|\frac{2^{a_k}}{3^k}\right\|_2 = \|Cod\sigma_{3^b}(2^{a_k})\|_2 = \|Cod\sigma_{3^b}0^{a_k}\|_2 = 2^{-a_k} \to 0 \text{ since } a_k \text{ is increasing.}$$

Therefore $\pi^1(\xi) = \sum\limits_{n=1}^{\infty}\frac{2^{a_n}}{3^n} \in \mathbb{Q}_2$. Furthermore, we have that $\pi_k^1(\xi)$ is a Cauchy sequence on $\mathbb{Z}_2$. Indeed,

$$\left\|\pi_{k+1}^1(\xi) - \pi_k^1(\xi)\right\|_2 = \left\|\frac{2^{a_{k+1}}}{3^{k+1}}\right\|_2 = 2^{-a_{k+1}} \to 0 \text{ as } k \to \infty$$

and by Proposition 2.10, page 59 of [8], we also have that $\mathbb{Z}_2$ is a complete metric space; therefore, $\lim\limits_{k\to\infty}\pi_k^1(\xi) = \pi^1(\xi) \in \mathbb{Z}_2$. □

Finally, we apply these results to find the minimum positive integer value ρ_0 for the affine maps, expressing it in terms of the coding.

Lemma 18. *Let $S_k \in \langle\theta, \psi^q\rangle$ with $Cod(S_k) = 0^{\theta_1}10^{\theta_2}\ldots10^{\theta_k}10^{\theta_{k+1}}$. Then $\rho_0(S_k) = Cod\sigma^{a_{k+1}}\left(q\frac{N_k}{3^k}\right)$.*

Proof. Let $S_k \in \langle\theta, \psi^q\rangle$. Then

$$\rho_0(S_k) = \frac{2^{a_{k+1}}}{3^k}\sigma_{3^k}^{a_{k+1}}(N_k) - \frac{qN_k}{3^k} = \frac{2^{a_{k+1}}}{3^k}\left\{\frac{N_k}{2^{a_{k+1}}} + \frac{3^k Cod\sigma^{a_{k+1}}\left(\frac{qN_k}{3^k}\right)}{2^{a_{k+1}}}\right\} - \frac{qN_k}{3^k}$$

$$= Cod\sigma_{3^k}^{a_{k+1}}(qN_k) = Cod\sigma^{a_{k+1}}\left(\frac{qN_k}{3^k}\right).$$

□

As a consequence of the next proposition, we have that if $-\pi^1$ is a negative or non-integer number, the minimum value diverges, since we have that the dyadic representation of these numbers always has an infinite amount of numbers.

Proposition 37. *Let $\{S_k\}_{k\in\mathbb{N}} \in \langle\theta, \psi^q\rangle$ such that $\xi = Cod\{S_k\}_{k\in\mathbb{N}}$. Then $\rho_0(S_k) = Cod\sigma^{A_k}\left(q\pi^1(\xi)\right)$, where A_k is the quantity of zeros of $Cod(S_k)$.*

Proof. Let us assume without loss of generality that ξ does not have a null tail. Let $\xi = Cod(\{S_k\}_{k\in\mathbb{N}} = 0^{\theta_1}\prod_{j=2}^{\infty} 10^{\theta_j}$. Thus,

$$\left\|Cod\sigma\left(q\pi^1(\xi)\right) - Cod\sigma^{a_{k+1}}\left(q\frac{N_k}{3^k}\right)\right\|_2 = \left\|q\pi^1(\xi) - q\pi^1_k(\xi)\right\|_2 \le 2^{-a_{k+1}}, \text{ since } \pi^1_k(\xi) = \pi^1(\xi) \pmod{2^{a_{k+1}}}.$$

Then $Cod\sigma^{a_{k+1}}\left(\frac{qN_k}{3^k}\right) = Cod\sigma\left(q\pi^1(\xi)\right) \pmod{2^{a_{k+1}}}$. ☐

We can also express ρ_1, derived from ρ_0, using the coding with a shift.

Corollary 9. *Let $\{S_k\}_{k\in\mathbb{N}} \in \langle\theta, \psi^q\rangle$ such that $\xi = Cod\{S_k\}_{k\in\mathbb{N}}$. Then $\rho_1(S_k) = -Cod\sigma_{-1}^{A_k}\left(q\pi^1(\xi)\right)$, where A_k is the quantity of zeros of $Cod(S_k)$.*

Proof. Since $\rho_0(S_k) - \rho_1(S_k) = 2^{A_k}$, we have $\rho_1(S_k) = \rho_0(S_k) - 2^{A_k} = Cod\sigma^{A_k}(q\pi^1(\xi)) - 2^{A_k} = -Cod\sigma_{-1}^{A_k}(q\pi^1(\xi))$. ☐

The section concludes with examples illustrating the calculation of ρ_0 for various map families.

Example 14. *1. Let $S_a(x) = \frac{3x+1}{2^{2a}}$. We have $Cod\sigma\left(\frac{1}{3}\right) = \ldots 101010101$ so $\rho_0(S_a) = \sum_{n=0}^{a} 2^{2n} = \frac{4^{a+1}-1}{3}$.*

2. Let $H_a(x) = \frac{9x+1}{2^{6a}}$. We have $Cod\sigma\left(\frac{1}{9}\right) = \ldots 111000111000111$, so $\rho_0(H_{6a}) = 7\sum_{n=0}^{a-1} 2^{6n} = \frac{64^a - 1}{9}$.

3. Let $J_k(x) = \frac{3^k x + (4^k - 3^k)}{4^k}$. Then

$$Cod\sigma\left(\frac{4^k - 3^k}{3^k}\right) = Cod\sigma\left(\frac{4^k}{3^k}\right) + Cod\sigma\left(\frac{-3^k}{3^k}\right) = Cod\sigma\left(\frac{1}{3^k}\right)0^{4k} + 1,$$

so $\rho_0(J_k) = 1$.

10. Coding of Set G_∞

Now we are going to prove that there is a complete metric on G_∞. We will use this result to prove that if $\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right) \in \mathbb{Q}_{odd}$ then $Cod\left(\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right)\right) = \{\xi_j\}_{j=1}^{\infty}$ and in the case that $\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right) \in \mathbb{R}\setminus\mathbb{Q}$, then there is no rational r such that $Cod(r) = \{\xi_j\}_{j=1}^{\infty}$. We also show that the parity of the Collatz function on $\pi^1(G_\infty)$ depends solely on the first term. Building upon this insight, we extend the Collatz function to $\pi^1(G_\infty)$ and conclude the section by showing that the Collatz function is topologically conjugate to the Shift function in Σ_2^*. We will use this result to establish that the set of periodic orbits is dense.

10.1. Summary of Propositions in the Section

1. **Lemma 19:** Established that when the function $d(\pi^1(\xi), \pi^1(\eta)) = \sum_{j\in\mathbb{N}} \frac{1}{3^j}|2^{a_j(\xi)} - 2^{a_j(\eta)}| \le \frac{1}{3^r}$ then ξ and η share at least the first $k-1$ terms.
2. **Proposition 38:** Establishes that $(\pi^1(G_\infty), d)$ is a complete metric space.
3. **Corollary 10:** Established that the $k-$coding set is an open set.

4. **Theorem 8:** Established that the full coding set is a singleton set or an empty set depending on whether $\pi^1(\xi)$ is rational or not.
5. **Proposition 39:** $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ is continuous.
6. **Corollary 11:** The $\pi^1 : G_\infty \to \pi^1(G_\infty) \subset \mathbb{R}$ is a continuous function with the usual metric of $\mathbb{R}$.
7. **Theorem 8:** Let $\{\xi_j\}_{j=1}^\infty \in G_\infty$, then if $\pi^1(\{\xi_j\}_{j=1}^\infty)$ it is rational, then the $-\pi^1(\{\xi_j\}_{j=1}^\infty) = -\frac{p}{q}$ only rational that satisfies $Cod\left(-\frac{p}{q}\right) = \{\xi_j\}_{j=1}^\infty$, in particular $\pi^1(\{\xi_j\}_{j=1}^\infty) \in \mathbb{Q}_{odd}$. If $\pi^1(\{\xi_j\}_{j=1}^\infty)$ it is irrational, then there is no rational $\frac{p}{q}$ such that $Cod\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^\infty$.
8. **Proposition 40:** It establishes that the parity of $\pi^1(\xi)$ depends only on the first term of the series.
9. **Definition 23:** Defines an extension of the Collatz function on all $\pi^1(G_\infty)$.
10. **Proposition 41:** The Collatz functions are continuous.
11. **Proposition 42:** The Collatz function on $\pi^1(G_\infty)$ is topological conjugacy to Shift map on Σ_2^*
12. **Corollary 12:** It is stable that the periodic points of the Collatz function in $(\pi^1(G_\infty), d)$ are dense.
13. **Proposition 43 :** Let $\xi \in G_\infty$ and $\pi^1 : \Sigma_2^* \to \mathbb{Z}_2$ and $\pi^1 : \Sigma_2^* \to \pi^1(G_\infty) \subset \mathbb{R}$, then if $-\pi^1(\xi) = \alpha \in \mathbb{Z}_2$ and $-\pi^1(\xi) = \beta \in \mathbb{Q}$, then $\alpha = \beta$.

10.2. $\pi^1(G_\infty)$ as Complete Metric Space

To ensure the coherent definition of a metric in $\pi^1(G_\infty)$, we need to "complete" the missing terms of the series to enable the calculation of the difference $|2^{a_k(\xi)} - 2^{a_k(\eta)}|$ for all $k \in \mathbb{N}$, irrespective of whether ξ or η has a null tail. To accomplish this, we define that when the sequence of 1s in ξ ends, the function a_k will take on the value $-\infty$. Hence, we have $|2^{a_k(\xi)} - 2^{a_k(\eta)}| = |2^{-\infty} - 2^{a_k(\eta)}| = 2^{a_k(\eta)}$ from the index of ξ. Let $\xi \in \Sigma_2^*$ with null tail with index J. We will write a short description.

$$\pi^1\left(\{\xi\}_{j=1}^\infty\right) := \sum_{j=1}^{J-1} \frac{2^{a_k}}{3^j} + \sum_{j=J+1}^{\infty} \frac{2^{-\infty}}{3^j}$$

In the following lemma, we are going to introduce a new function, which, as we will see later, corresponds to a metric in the space $\pi^1(G_\infty)$. Additionally, we will present another result that we will examine more closely in this section and essentially indicates to us that, since the parity of $\pi^1(\xi)$ depends only on the first term, we can interpret this in the following way: If two sequences are arbitrarily close, then they share the first terms of their encoding. This is of great importance for understanding the behavior of the orbits of the Collatz function, since, if we consider the Euclidean metric in $\mathbb{Q}$ or that of the absolute value, we observe the phenomenon that even though two numbers are arbitrarily close, their dynamics are completely different. One may converge to a cycle in a few iterations, while the other may take a very long time.

Lemma 19 (Convergence and Coincidence Lemma)**.** *Let $\xi, \eta \in G_\infty$, then*

1. $\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right|$ *is well defined.*
2. $\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_j(\xi)} - 2^{a_j(\eta)}\right| \leq \frac{1}{3^r} \Rightarrow a_j(\xi) = a_j(\eta)$ *for all $j < r$. In addition, we have to $\xi_{j-1} = \eta_j$ for all $j < r$.*

Proof. Let $\xi, \eta \in G_\infty$, then we have

Claim Let, $a, b \in \mathbb{N}$ then $|2^a - 2^b| < 2^{\max\{a,b\}}$. If $a = b$, then $0 < 2^a$. Suppose that $a > b$ then

$$|2^a - 2^b| = 2^a\left|1 - \frac{1}{2^{a-b}}\right| < 2^a.$$

□ **of the Claim.**

Now we prove that it is well-defined, by *Claim* we have:

$$d\left(\sum_{j=1}^{\infty}\frac{2^{a_j(\xi)}}{3^j},\sum_{j=1}^{\infty}\frac{2^{a_j(\eta)}}{3^j}\right)=\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|<\sum_{j=1}^{\infty}\frac{2^{\max\{a_j(\xi),a_j(\eta)\}}}{3^j}$$

as $\frac{2^{a_j(\xi)}}{3^j}$ and $\frac{2^{a_j(\eta)}}{3^j}$ converge to 0, then for $n>\mathbb{N}$ exist $J\in\mathbb{N}$ such that if $j>J$ we have

$$\left|\frac{2^{a_j(\xi)}}{3^j}\right|,\left|\frac{2^{a_j(\eta)}}{3^j}\right|<\frac{1}{n}$$

Then for $j>J$ we also have

$$\frac{2^{\max\{a_j(\xi),a_j(\eta)\}}}{3^j}<\frac{1}{n}$$

Then we have that $\frac{2^{\max\{a_j(\xi),a_j(\eta)\}}}{3^j}$ also converges to 0. Then by Proposition 22 we have $\sum_{j=1}^{\infty}\frac{2^{\max\{a_j(\xi),a_j(\eta)\}}}{3^j}<\infty$. To prove the statement, we will consider whether the sequences ξ and η in Σ_2^* have a null tail or not.

Let us first assume that the sequence does not have a null tail, then if we have

$$\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|\leq\frac{1}{3^r}$$

All terms less than r must be null. Suppose there exists some non-zero term between 1 and $r-1$, then we have that

$$\frac{1}{3^{r-1}}\leq\sum_{j=1}^{r-1}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|\leq\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|\leq\frac{1}{3^r}\text{ Then }r-1\geq r$$

which is absurd. Then we have that $\sum_{j=1}^{r-1}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|=0$. Which implies that

$$a_j(\xi)=a_j(\eta)\text{ for all }j<r.$$

Now we will prove that the sequences coincide up to r.

$$\xi=0^{\theta_1^1}\prod_{i=2}^{\infty}10^{\theta_i^1}\text{ and }\eta=0^{\theta_1^2}\prod_{i=2}^{\infty}10^{\theta_i^2}$$

writing this way, we have to

$$a_j(\xi)=\sum_{i=1}^{j}\theta_i^1\text{ and }a_j(\eta)=\sum_{i=1}^{j}\theta_i^2$$

then

$$\theta_j^1=a_j(\xi)-a_{j-1}(\xi)=a_j(\eta)-a_{j-1}(\eta)=\theta_j^2\text{ for all }1\leq j<r$$

which means that ξ and η share the first $(r-1)-10^{\theta_j}$ blocks Now suppose that ξ has a null tail of index $I+1$ and η has no tail null. Then we have

$$\sum_{j=1}^{I}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|+\sum_{j=I+1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\eta)}\right|\leq\frac{1}{3^r}$$

if $r \leq I$ then we have the previous case, then $a_j(\xi) = a_j(\eta)$ for all $j < r$. Now if $r > I$ we have

$$\sum_{j=1}^{I}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|+\sum_{j=I+1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\eta)}\right|\leq\frac{1}{3^r}$$
$$\Rightarrow\sum_{j=1}^{I}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|+\sum_{j=I+1}^{r-1}\frac{1}{3^j}\left|2^{a_j(\eta)}\right|=0$$

The latter makes sense if η also has a null tail of index $I+1$, then

$$\xi_j = \eta_j \text{ for all } j \in \mathbb{N}$$

In particular $\xi_j = \eta_j$ for all $j < r$. Finally, suppose that ξ and η have a null tail of index $I+1$ and $L+1$ respectively, without loss of generality we can assume that $I \leq L$. Then

$$\sum_{j=1}^{I}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|+\sum_{j=I+1}^{L}\frac{1}{3^j}\left|2^{a_j(\eta)}\right|+\sum_{j=L+1}^{\infty}\frac{1}{3^j}\left|2^{-\infty}\right|\leq\frac{1}{3^r}$$

1. If $1 \leq r \leq I$ then all terms with an index less than r are null and in particular we have $a_j(\xi) = a_j(\eta)$ for $j < r$. and as we already saw in the proofs above, this implies that $\xi_j = \eta_j$ for all $j < r$.
2. If $r \geq I$. Then $\sum_{j=I+1}^{L}\frac{1}{3^j}\left|2^{a_j(\eta)}\right| = 0$ we have that $I = J$ therefore $\xi = \eta$. particular we have $a_j(\xi) = a_j(\eta)$ for $j < r$.

□

Now we are going to show that the function we defined above is a complete metric on $\pi^1(G_\infty)$.

Proposition 38 (Metric space complete)**.** *Let* $d : \pi^1(G_\infty) \times \pi^1(G_\infty) \to [0, \infty)$ *given by*

$$d\left(\sum_{j=1}^{\infty}\frac{2^{a_j(\xi)}}{3^j},\sum_{j=1}^{\infty}\frac{2^{a_j(\eta)}}{3^j}\right)=\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|$$

Then $(\pi^1(G_\infty), d)$ *is a metric space complete.*

Proof. Let's prove that d is a metric through the axioms of metric:

1. $d(\pi^1(\xi), \pi^1(\eta)) = 0$ if and only if $\pi^1(\xi) = \pi^1(\eta)$ for all $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$: Trivially we have that if $\pi^1(\xi) = \pi^1(\eta)$, then

$$d\left(\sum_{j=1}^{\infty}\frac{2^{a_j(\xi)}}{3^j},\sum_{j=1}^{\infty}\frac{2^{a_j(\xi)}}{3^j}\right)=\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\xi)}\right|=0.$$

Let $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$ such that

$$d\left(\sum_{j=1}^{\infty}\frac{2^{a_j(\xi)}}{3^j},\sum_{j=1}^{\infty}\frac{2^{a_j(\eta)}}{3^j}\right)=\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_j(\xi)}-2^{a_j(\eta)}\right|=0$$

by lemma 19 we have

$$\sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| \leq \frac{1}{3^r} \Rightarrow a_j(\xi) = a_j(\eta) \text{ for } j < r$$

In particular, for $r \to \infty$ we have $\xi = \eta$.

2. $d(\pi^1(\xi), \pi^1(\eta)) = d(\pi^1(\eta), \pi^1(\xi))$ for all $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$:

$$\sum_{j=1}^{k} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| = \sum_{j=1}^{k} \frac{1}{3^j} \left| 2^{a_j(\eta)} - 2^{a_j(\xi)} \right|$$

then

$$\sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| = \sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\eta)} - 2^{a_j(\xi)} \right|$$

3. $d(\pi^1(\xi), \pi^1(\eta)) \leq d(\pi^1(\xi), \pi^1(\kappa)) + d(\pi^1(\kappa), \pi^1(\eta))$ for all, $\pi^1(\xi), \pi^1(\eta), \pi^1(\kappa) \in \pi^1(G_\infty)$

$$\sum_{j=1}^{k} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| \leq \sum_{j=1}^{k} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\kappa)} \right| + \sum_{j=1}^{k} \frac{1}{3^j} \left| 2^{a_j(\kappa)} - 2^{a_j(\eta)} \right|$$

then

$$\sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\eta)} \right| \leq \sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi)} - 2^{a_j(\kappa)} \right| + \sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\kappa)} - 2^{a_j(\eta)} \right|$$

then $(\pi^1(G_\infty), d)$ is a metric space. Now we are going to prove that it is a complete metric space. Let $\{\{\pi^1(\xi_j^k)\}_{j=1}^{\infty}\}_{k=1}^{\infty}$ be a Cauchy sequence on $\pi^1(G_\infty)$ then for any $\varepsilon > 0$ exist $K > 0$ such that

$$d(\{\pi^1(\xi_j^n)\}_{j=1}^{\infty}, \{\pi^1(\xi_j^m)\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi^n)} - 2^{a_j(\xi^m)} \right| < \varepsilon \text{ for all } n, m > K$$

Let $r > 0$ such that $\sum_{j=1}^{\infty} \frac{1}{3^j} \left| 2^{a_j(\xi^n)} - 2^{a_j(\xi^m)} \right| < \frac{1}{3^r} < \varepsilon$ by lemma 19 we have

$$\xi_j^n = \xi_j^m \text{ for all } j < r$$

On the other hand let $D : \Sigma_2^* \times \Sigma_2^* \to [0, \infty)$ the symbolic metric of two symbols given by

$$D(\xi, \eta) = \sum_{j=1}^{\infty} \frac{1}{2^j} \Delta(\xi_j, \eta_j)$$

with

$$\Delta(\xi_j, \eta_j) = \begin{cases} 1 & \text{if} \quad \xi_j \neq \eta_j \\ 0 & \text{if} \quad \xi_j = \eta_j \end{cases}$$

The space (Σ_2^*, D) is a complete metric space with the property that if two sequences are arbitrarily close if and only if their first terms are equal.

$$D(\xi, \eta) < \frac{1}{2^r} \text{ if and only if } \xi_j = \eta_j \text{ for all } j < r.$$

then given a Cauchy sequence in $\pi_1(G_\infty)$ by the observation above we obtain a Cauchy sequence in Σ_2^* and the latter being complete there is a $\xi \in \Sigma_2^*$ such that

$$\xi^n \to \xi \text{ as } n \to \infty$$

We will now prove that ξ is in G_∞ and that the sequence $\pi_1(\xi^n)$ converges to $\pi_1(\xi)$.

1. Let's prove that G_∞ is complete. Let $\{\xi_j\}_{j\in\mathbb{N}}$ a Cauchy sequence on G_∞ and let $\lim\limits_{j\to\infty} \xi_j = \xi \in \Sigma_2^*$. Let's show that $\xi \in G_\infty$. Let $R = a_j + b_j$, then by definition we have exist $N > 0$ such that $D(\xi_n, \xi_m) < \frac{1}{R}$ for all $m, n > N$, so we have ξ_m and ξ_n have the first R terms equal, Suppose ξ has no null tail then $b_j = j$ and

   $$\lim_{j\to\infty} \frac{3^j}{2^{a_{j+1}}} = \infty \text{ so } \xi \in G_\infty$$

   If ξ has a null tail, then by definition it is in G_∞.
2. Let's prove that $\pi^1(G_\infty)$ is complete. Let $\pi^1(\xi_k)$ a Cauchy sequence in $\pi^1(G_\infty)$. Let $r \in \mathbb{N}$, then exist N such that

   $$d(\pi^1(\xi_n), \pi^1(\xi_m)) < \frac{1}{3^r} \text{ for all } m, n > N$$

   by Lemma 19 we have $D(\xi_n, \xi_m) < \frac{1}{2^r}$ for all $m, n > N$. In other words we have that $\{\xi_j\}$ is a Cauchy sequence in G_∞. As we proved in point 1, we have that G_∞ is a complete metric subspace of Σ_2^*, so exist $\xi \in G_\infty$ such that $\lim\limits_{j\to\infty} \xi_j = \xi$. Let us now demonstrate that $\lim\limits_{j\to\infty} \pi^1(\xi_j) = \pi^1(\xi)$

   $$\begin{aligned} d(\pi^1(\xi_j), \pi^1(\xi)) &= \sum_{k\in\mathbb{N}} \frac{1}{3^k}\left|2^{a_k(\xi_j)} - 2^{a_k(\xi)}\right| \\ &= \underbrace{\sum_{k=1}^{r} \frac{1}{3^k}\left|2^{a_k(\xi_j)} - 2^{a_k(\xi)}\right|}_{=0} + \sum_{k=j+1}^{\infty} \frac{1}{3^k}\left|2^{a_k(\xi_j)} - 2^{a_k(\xi)}\right| \\ &\le \sum_{k=r+1}^{\infty} \frac{1}{3^k} 2^{\max\{a_k(\xi_j), a_k(\xi)\}} \to 0 \text{ as } r \to \infty \end{aligned}$$

   Therefore $\pi^1(G_\infty)$ it is a complete space.

then we can conclude that the metric space $(\pi^1(G_\infty), d)$ is complete. ☐

With this metric, we have that the coding sets are open sets.

Corollary 10 (Cod^k is an open set)**.** *Let $\xi \in G_\infty$ and $k \in \mathbb{N}$. Then $Cod^k(\xi)$ is an open set on $(\pi^1(G_\infty), d)$*

Proof. Let $\xi \in G_\infty$ and $u \in \pi^1(G_\infty)$ such that $u \in Cod^k(\xi)$. Let us consider v in $\pi^1(G_\infty)$ such that $d(u, v) < \frac{1}{3^{k+1}}$, by definition, exist $\mu, \tau \in G_\infty$ such that $\pi^1(\mu) = u$ and $\pi^1(\tau) = v$. By Lemma 19 have $\mu_j = \tau_j$ for $j \le k$, then we have that $v \in Cod^k(\xi)$. Therefore, then $B\left(u, \frac{1}{3^{k+1}}\right)$ i.e. the ball of radius $\frac{1}{3^{k+1}}$ and center u is a subset of $Cod^k(\xi)$, therefore $Cod^k(\xi)$ is an open set. ☐

Proposition 39. *$-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ is continuous.*

Proof. Let $\xi \in G_\infty$ and $\{\eta_j\}_{l\in\mathbb{N}}$ a sequence on G_∞ such that $D(\eta_j,\xi) < \frac{1}{2^r}$, so $a_k(\eta_j) = a_k(\xi)$ for all $j < r$, then we have

$$d(\pi^1(\eta_j),\pi^1(\xi)) = \sum_{k=r}^{\infty} \frac{|2^{a_k(\eta_j))} - 2^{a_k(\eta))}|}{3^k} \to 0 \text{ as } r \to \infty$$

Therefore $d(\pi^1(\eta_j),\pi^1(\xi)) \to 0$ as $D(\eta_j,\xi) \to 0$, that is to say that $-\pi^1$ es $\square$

Corollary 11. *The $\pi^1 : G_\infty \to \pi^1(G_\infty) \subset \mathbb{R}$ is a continuous function with the usual metric of $\mathbb{R}$.*

Proof. Let $\xi \in G_\infty$ a sequence $\{\xi_j\}$ on G_∞ such that $\xi_j \to \xi$. By continuity of $\pi^1 : G_\infty \to \pi^1(G_\infty)$ we have

$$\begin{aligned}
\left|\pi^1(\xi_j) - \pi^1(\xi)\right| &= \left|\sum_{j\in\mathbb{N}} \frac{2^{a_j(\xi_j)}}{3^j} - \sum_{j\in\mathbb{N}} \frac{2^{a_j(\xi)}}{3^j}\right| \\
&\leq \sum_{j\in\mathbb{N}} \frac{1}{3^j}\left|2^{a_j(\xi_j)} - 2^{a_j(\xi)}\right| \\
&= d(\pi^1(\xi_j),\pi^1(\xi)) \to 0 \text{ as } D(\xi_j,\xi) \to 0
\end{aligned}$$

$\square$

Theorem 8 (Asymptotic Solutions Theorem). *Let $\{\xi_j\}_{j=1}^{\infty} \in G_\infty$, then if $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ it is rational, then the $-\pi^1(\{\xi_j\}_{j=1}^{\infty}) = -\frac{p}{q}$ only rational that satisfies $Cod\left(-\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$, in particular $\pi^1(\{\xi_j\}_{j=1}^{\infty}) \in \mathbb{Q}_{odd}$. If $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ it is irrational, then there is no rational $\frac{p}{q}$ such that $Cod\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$.*

Proof. Let $\xi \in G_\infty$, We have by definition that $\pi^1(\xi) = \lim_{k\to\infty} \pi^1_k(\xi) \in \mathbb{Q}$. First, we will prove that this limit also makes sense in $(\pi^1(G_\infty), d)$. We have:

Claim 1: $d(\pi^1_k(\xi),\pi^1(\xi)) \to 0$ as $k \to \infty$.

Indeed, we have

$$d(\pi^1_k(\xi),\pi^1(\xi)) = \underbrace{\sum_{j=1}^{n} \frac{1}{3^j}|2^{a_j(\xi)} - 2^{a_j(\xi)}|}_{0} + \sum_{j=k+1}^{\infty} \frac{2^{a_j(\xi)}}{3^j} = \sum_{j=k+1}^{\infty} \frac{2^{a_j(\xi)}}{3^j}$$

By Proposition 22, we have $\sum_{j=k+1}^{\infty} \frac{2^{a_j(\xi)}}{3^j} \to 0$ as $k \to \infty$, then $d(\pi^1_k(\xi),\pi^1(\xi)) \to 0$ as $k \to \infty$ $\square$

We will now prove, using the completeness of $(\pi^1(G_\infty), d)$, that $\pi^1(\xi) \in \pi^1(G_\infty)$:

Claim 2: $\pi^1(\xi) \in \pi^1(G_\infty)$.

We can be rewritten,

$$\pi^1_k(\xi) = \pi^1\left(\{\xi_j\}_{j=1}^{k}000\ldots\right)$$

let's prove that $\pi^1_k(\xi)$ is a Cauchy sequence in $\pi^1(G_\infty)$. Let $\varepsilon > 0$ and $n, m \in \mathbb{N}$ with $n < m$, then

$$d(\pi^1_n(\xi),\pi^1_m(\xi)) = \underbrace{\sum_{j=1}^{n} \frac{1}{3^j}|2^{a_j(\xi)} - 2^{a_j(\xi)}|}_{0} + \sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j} = \sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j}$$

Since $\xi \in G_\infty$ we have by Proposition 22 $\pi^1(\xi)$ is convergent, then $\sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j} \to 0$ as $n, m \to \infty$. Therefore, exists $N \in \mathbb{N}$ such that $n, m > N$ we have $\sum_{j=n+1}^{m} \frac{2^{a_j(\xi)}}{3^j} < \varepsilon$. Then the sequence is Cauchy and since $\pi^1(G_\infty)$ is a complete metric space, we have that $\pi^1(\xi) = \lim_{k\to\infty} \pi_k^1(\xi) \in \pi^1(G_\infty)$.

☐ **of the Claim.**

Claim 3: Let $\xi = 0^{\theta_1}10^{\theta_2}\ldots10^{\theta_{k+1}}\ldots \in \Sigma_2^*$, then $-\pi_k^1(\xi) \in Cod^{a_{k+1}}(\xi)$.

Indeed, let $S_k \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = 0^{\theta_1}10^{\theta_2}\ldots10^{\theta_{k+1}}$ by Proposition 12 we have

$$S_k(-\pi_k^1\left(\{\xi_j\}_{j=1}^{\infty}\right) = \frac{3^k(-\pi_k^1\left(\{\xi_j\}_{j=1}^{\infty}\right) + N_k}{2^{a_{k+1}}} = \frac{3^k\left(-\frac{N_k}{3^k}\right) + N_k}{2^{a_{k+1}}} = 0.$$

In other hand, we have

$$\frac{3^k\left(-\frac{N_k}{3^k}\right) + N_k}{2^{a_{k+1}}} = \frac{3^k(-N_k) + 3^k(N_k)}{3^k 2^{a_{k+1}}} = \frac{1}{3^k}\left(\frac{3^k(-N_k) + 3^k(N_k)}{2^{a_{k+1}}}\right) = \frac{1}{3^k} S^{3^k}(-N_k)$$

then $-N_k \in \mathbb{E}(S^{3^k})$, by Proposition 6 we have $Cod_{3^k}(S_k^{3^k}) = Cod(S_k)$. Therefore $Cod^{a_{k+1}}\left(\frac{-N_k}{3^k}\right) = Cod^{a_{k+1}}(-\pi_k^1(\xi)) = Cod_{3^k}(S^{3^k}) = Cod(S_k)$.

☐ **of the Claim.**

Claim 4: $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$.

Let $L + 1 \in \mathbb{N}$, we have by *Claim 1*, exist $K \in \mathbb{N}$ such that if $k > K > L + 1$ so $d(\pi_k^1(\xi), \pi^1(\xi)) < \frac{1}{3^{L+1}}$, By Lemma 19 we have to share the first L terms of the coding. On the other hand, by *Claim 3*, we have that $-\pi_k^1(\xi) \in Cod^{a_{k+1}}(\xi)$ and since we have $a_{k+1} > K > L + 1$, due to the Monotonicity of Cod, we have $-\pi_k^1(\xi) \in Cod^L(\xi)$. Therefore $-\pi^1(\xi) \in Cod^L(\xi)$.

☐ **of the Claim.**

Claim 5: $\pi^1(\xi) \in Cod(\xi)$.

Since $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$, we have to

$$-\pi^1(\xi) \in \bigcap_{k\in\mathbb{N}} Cod^k(\xi) = Cod(\xi).$$

☐ **of the Claim.**

Claim 6: $Cod\left(-\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right)\right) = \{\xi_j\}_{j=1}^{\infty}$.

We first show that $\pi^1(\xi) \in \mathbb{Q}_{odd}$. Let's assume that $\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right)$ converges to a fraction with an even denominator; then its coding is $1111\ldots$. However, $1111\ldots$ is not an element of Σ_2^*, which leads to a contradiction with *claim 4*. Then we have $\pi^1(\xi) \in \mathbb{Q}_{odd}$ and, by Theorem 5, we have $Cod\left(-\pi^1\left(\{\xi_j\}_{j=1}^{\infty}\right)\right) = \{\xi_j\}_{j=1}^{\infty}$.

☐ **of the Claim.**

For the next part of the proposition, we will leverage the results presented in Section 9. In this section, we introduce the Sigma function along with its main properties and applications in solving linear Diophantine equations. It serves as an alternative to classical methods for solving this type of equation.

Claim 7: If $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ it is irrational, then there is no rational then there is no rational $\dfrac{p}{q}$ such that $cod\left(\dfrac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$.

We will prove that there is no rational solution, By the theorem 4 we have that if there is another rational solution it must be a minimum positive integer value or a maximum negative integer value for $S_k(x) \in \langle \theta, \psi^q \rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=1}^{k+1} = 0^{\theta_1}10^{\theta_2}\dots0^{\theta_k}10^{\theta_{k+1}}$ for unique q not null, by proposition 5 we have

$$S_k(x) = \frac{3^k x + qN_k(\{\xi_j\}_{j=1}^{\infty})}{2^{a_{k+1}}}$$

by Propositions 7 and 31 the minimum positive integer value is

$$\begin{aligned}\rho_0(S_k) = \frac{1}{3^k}\left(2^{a_{k+1}}\sigma_{3^k}^{a_{k+1}}(qN_k) - qN_k\right) &= 2^{a_{k+1}}\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty}) - q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})\\ &= Cod\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})\end{aligned}$$

and the maximum negative integer value is

$$\begin{aligned}\rho_1(S_k) = \frac{1}{3^k}\left(2^{a_{k+1}}\sigma_{-3^k}^{a_{k+1}}(qN_k) - qN_k\right) &= \frac{1}{3^k}\left(2^{a_{k+1}}\sigma_{3^k}^{a_{k+1}}(qN_k) - qN_k\right) - 2^{a_{k+1}}\\ &= Cod\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty}) - 2^{a_{k+1}}\end{aligned}$$

On the other hand, by Proposition 37 we have that the coding of the

$$\sigma^{a_{k+1}}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty}) = \sum_{j=1}^{a_{k+1}-1} \delta_j 2^j \text{ with } \delta_j \in \{0,1\}.$$

Since $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ is irrational, then the dyadic expansion of $q\pi^1(\{\xi_j\}_{j=1}^{\infty})$ will never have a tail of 0 or 1 for all $q \in \mathbb{Z}$, so $Cod\sigma^{a_k}(q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$ has an infinite number of non-zero digits for all $q \in \mathbb{Z}$. In particular we have to $\rho_0(S_k) \to \infty$. Now for $\rho_1(S_k)$, we have

$$\rho_1(S_k) = \sum_{j=1}^{a_{k+1}-1} \delta_j 2^j - 2^{a_{k+1}} = \sum_{j=1}^{a_{k+1}-1} \delta_j 2^j - \sum_{j=1}^{a_{k+1}-1} 2^j = \sum_{j=1}^{a_{k+1}-1} (\delta_j - 1)2^j$$

for this sum to be finite it is necessary exist $J > 0$ such that δ_j are all 1 for $j > J$, however as $q\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$ is irrational for all $q \in \mathbb{Z}$, then there are infinitely many terms of δ_j that are null, then this sum is divergent.

Then if $\pi_k^1(\{\xi_j\}_{j=1}^{\infty})$ is irrational then there is no rational $\dfrac{p}{q}$ such that $cod\left(\dfrac{p}{q}\right) = \{\xi_j\}_{j=1}^{\infty}$.

☐ **of the Claim.**

Example 15. *1. Let* $\xi_1 = 101010\ldots \in \Sigma_2^*$, *then* $\pi^1(\xi) = 1$ *(see Example 10). Therefore* $Cod(-1) = 101010\ldots$.

2. *Let* $\xi_2 = 1010010100\ldots \in \Sigma_2^*$, *then*

$$\begin{aligned}
\pi^1(\xi_2) &= \frac{1}{3} + \frac{2}{3^2} + \frac{2^3}{3^3} + \frac{2^4}{3^4} + \frac{2^6}{3^5} + \frac{2^7}{3^6} + \frac{2^9}{3^7} + \ldots \\
&= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \frac{2^3}{3^2}\left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \left(\frac{2^3}{3^2}\right)^2\left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \ldots \\
&= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\}\sum_{j=1}^{\infty}\left(\frac{2^3}{3^2}\right)^j \\
&= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\}\frac{1}{1 - \frac{2^3}{3^2}} = 5
\end{aligned}$$

Therefore $Cod(-5) = 1010010100\ldots$

Proposition 40 (Parity Preservation Proposition)**.** *Let* $\{\xi_j\}_{j=1}^{\infty} \in G_\infty$ *such that* $\pi^1(\{\xi_j\}_{j=1}^{\infty}) \in \mathbb{Q}_{odd}$, *then*

$$Col\left(-\pi^1(\{\xi_j\}_{j=1}^{\infty})\right) = \begin{cases} \dfrac{-3\pi^1(\{\xi_j\}_{j=1}^{\infty}) + 1}{2} & \text{if } a_1 = 0 \\ -\dfrac{1}{2}\pi^1(\{\xi_j\}_{j=1}^{\infty}) & \text{if } a_1 > 0 \end{cases}$$

or equivalent if $\pi^1(\{\xi_j\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty}\frac{2^{a_j}}{3^j}$ *then*

$$Col\left(-\sum_{j=1}^{\infty}\frac{2^{a_j}}{3^j}\right) = \begin{cases} -\sum\limits_{j=1}^{\infty}\dfrac{2^{a_{j+1}}}{3^j} & \text{if } a_1 = 0 \\ -\sum\limits_{j=1}^{\infty}\dfrac{2^{a_j-1}}{3^j} & \text{if } a_1 > 0 \text{ or } -\infty \end{cases}$$

Proof. Let's prove that the parity of $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ only depends on the first term

$$\pi^1(\{\xi_j\}_{j=1}^{\infty}) = \frac{2^{a_1}}{3} + \sum_{j=2}^{\infty}\frac{2^{a_j}}{3^j} = \frac{2^{a_1}}{3} + 2\left(\sum_{j=2}^{\infty}\frac{2^{a_j-1}}{3^j}\right)$$

Claim: The series $\sum_{j=2}^{\infty}\frac{2^{a_j-1}}{3^j}$ cannot converge to a fraction with an even denominator.

Let us assume by contradiction that have $\sum_{j=2}^{\infty}\frac{2^{a_j-1}}{3^j} = \frac{p}{2q}$ with $(q,p) = (p,2) = 1$. Let $\{\xi_j\}_{j=1}^{l}0\ldots \in G_\infty$ such that $\pi^1\{\xi_j\}_{j=1}^{l}0\ldots = \frac{2^{a_1}}{3}$. Let $\eta \in G_\infty$ given by $\eta = \{\xi_j\}_{j=l+1}^{\infty}$ so $\pi^1(\eta) = \pi^1\{\xi_j\}_{j=l+1}^{\infty} = \sum_{j=2}^{\infty}\frac{2^{a_j-1}}{3^j} = \frac{p}{2q}$ and since η are in G_∞, this generates a contradiction to the Theorem 8.

Let $\sum_{j=2}^{\infty}\frac{2^{a_j-1}}{3^j} = \frac{p}{q}$ with $(p,q) = (q,2) = 1$. We have:

$$\pi^1(\{\xi_j\}_{j=1}^{\infty}) = \frac{2^{a_1}}{3} + \sum_{j=2}^{\infty}\frac{2^{a_j}}{3^j} = \frac{2^{a_1}}{3} + 2\left(\sum_{j=2}^{\infty}\frac{2^{a_j-1}}{3^j}\right) = \frac{2^{a_1}}{3} + \frac{2p}{q} = \frac{2^{a_1}q + 6p}{3q}$$

if $a_1 = 0$ then $2^0 q + 6p = q + 6p$ is odd, since q is odd. Then $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ is odd, and if $a_1 > 0$ then $2^{a_1} q + 6p$ is even. Then $\pi^1(\{\xi_j\}_{j=1}^{\infty})$ is even. In the case that $a_1 = -\infty$ we have that $\xi = 0$ or equivalent $\xi_j = 0$ for all $j \in \mathbb{N}_0$, so $\pi^1(0) = 0$, then $Col(0) = \dfrac{0}{2} = 0$. □

Definition 23 (Extension Collatz functions on $-\pi^1(G_\infty)$)**.** *We defined $Col : -\pi^1(G_\infty) \to -\pi^1(G_\infty)$ by*

$$Col\Big(-\pi^1(\{\xi_j\}_{j=1}^{\infty})\Big) = \begin{cases} \dfrac{-3\pi^1(\{\xi_j\}_{j=1}^{\infty}) + 1}{2} & \text{if } a_1 = 0 \\[2ex] -\dfrac{1}{2}\pi^1(\{\xi_j\}_{j=1}^{\infty}) & \text{if } a_1 > 0 \end{cases}$$

or equivalent if $\pi^1(\{\xi_j\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \dfrac{2^{a_j}}{3^j}$ then

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}\right) = \begin{cases} -\displaystyle\sum_{j=1}^{\infty} \frac{2^{a_{j+1}-1}}{3^j} & \text{if } a_1 = 0 \\[2ex] -\displaystyle\sum_{j=1}^{\infty} \frac{2^{a_j-1}}{3^j} & \text{if } a_1 > 0 \text{ or } -\infty \end{cases}$$

The extension of the Collatz function on -$\pi^1(G_\infty)$ is continuous.

Proposition 41 (Collatz function is continuous)**.** *$Col : -\pi^1(G_\infty) \to -\pi^1(G_\infty)$ is continuous.*

Proof. Let us consider the metric induced in $-\pi^1(G_\infty)$ by $-id : \pi^1(G_\infty) \to -\pi^1(G_\infty)$, that is, $d_{-\pi^1(G_\infty)}(-u, -v) = d(u, v)$ on $\pi^1(G_\infty)$, we will use the same notation for both metrics.

Let $u \in -\pi^1(G_\infty)$ and $\{u_j\}_{j\in\mathbb{N}}$ sequence of $-\pi^1(G_\infty)$ such that $u_j \to u$. Let $\xi, \xi_j \in G_\infty$ such that $Cod(u) = \xi$ and $Cod(\xi_j) = u_j$.

$$d(u_j, u) = \sum_{k\in\mathbb{N}} \frac{|2^{a_k(\xi_j)} - 2^{a_k(\xi)}|}{3^k} \to 0 \text{ as } j \to \infty.$$

then if $a_1 > 0$ so

$$d(Col(u_j), Col(u)) = \sum_{k\in\mathbb{N}} \frac{|2^{a_k(\xi_j)-1} - 2^{a_k(\xi)-1}|}{3^k} \to 0 \text{ as } j \to \infty.$$

and if $a_0 = 1$, so

$$\begin{aligned} d(Col(u_j), Col(u)) &= \sum_{k\in\mathbb{N}} \frac{|2^{a_{k+1}(\xi_j)-1} - 2^{a_{k+1}(\xi)-1}|}{3^k} \\ &= \underbrace{\frac{|2^{a_1(\xi_j)} - 2^{a_1(\xi)}|}{3}}_{=0} + \frac{3}{2}\sum_{k\in\mathbb{N}} \frac{|2^{a_{k+1}(\xi_j)} - 2^{a_{k+1}(\xi)}|}{3^{k+1}} \\ &= \frac{1}{2}\sum_{k\in\mathbb{N}} \frac{|2^{a_k(\xi_j)} - 2^{a_k(\xi)}|}{3^k} \to 0 \text{ as } j \to \infty. \end{aligned}$$

Therefore Col is continuous. □

10.3. Topological Conjugation

In a dynamic system, there is a well-studied dynamics in the space of sequences of two symbols, known as the shift map. This map acts on the sequences by eliminating the first term. It is known that with the metric D, this map is continuous, and its periodic orbits form a dense set. In the following proposition, we will show that the extension of the Collatz function on $-\pi^1(G_\infty)$ is, in fact, topologically conjugate to the dynamics of the Shift map.

Proposition 42. *Let us consider the following function* $\omega : \Sigma_2^* \to \Sigma_2^*$ *given by,*

$$\omega\left(\{\xi_j\}_{j=1}^{\infty}\right) = \begin{cases} \{\xi_{j+2}\}_{j=1}^{\infty} & \text{if } \xi_1 = 10 \\ \{\xi_{j+1}\}_{j=1}^{\infty} & \text{if } \xi_1 = 0 \end{cases}$$

Then Col and ω *are Topologically Conjugacy.*

Proof. We are going to prove that this diagram is commutative

$$\begin{tikzcd}
G_\infty \arrow[r, "\omega"] \arrow[d, "-\pi^1"'] & G_\infty \arrow[d, "-\pi^1"] \\
-\pi^1(G_\infty) \arrow[r, "Col"'] & -\pi^1(G_\infty)
\end{tikzcd}$$

and that $-\pi^1$ is a homeomorphism.

Claim 1: The diagram is commutative. Suppose that $\{\xi_j\}_{j=1}^{\infty} = 0^{\theta_1}10^{\theta_2}10^{\theta_3}\ldots = 0^{\theta_1}\prod_{j=2}^{\infty}10^{\theta_j} \in G_\infty$ with $\theta_j \in \mathbb{N}$. In this way, we get an explicit form for the function $a_k = \sum_{j=1}^{k}\theta_j$. If $a_1 = 0$ we have

$$\begin{aligned}
Col \circ (-\pi^1)\left(\prod_{j=2}^{\infty}10^{\theta_j}\right) &= Col\left(-\sum_{k=2}^{\infty}\frac{2^{a_k(\{\xi_j\}_{j=2}^{\infty})}}{3^k}\right) \\
&= \frac{1}{2}\left(-3\sum_{k=2}^{\infty}\frac{2^{a_k(\{\xi_j\}_{j=2}^{\infty})}}{3^k}+1\right) \\
&= \frac{1}{2}\left(-\sum_{k=3}^{\infty}\frac{2^{a_k(\{\xi_j\}_{j=2}^{\infty})}}{3^{k-1}}+1\right) \\
&= \frac{1}{2}\left(-1-\sum_{k=2}^{\infty}\frac{2^{a_k(\{\xi_j\}_{j=2}^{\infty})}}{3^{k-1}}+1\right) \\
&= -\sum_{k=2}^{\infty}\frac{2^{a_k(\{\xi_j\}_{j=2}^{\infty})-1}}{3^{k-1}} \\
&= -\sum_{k=2}^{\infty}\frac{2^{a_{k+1}(\{\xi_j\}_{j=1}^{\infty})-1}}{3^{k}}
\end{aligned}$$

and

$$-\pi^1 \circ \omega\left(\prod_{j=2}^{\infty}10^{\theta_j}\right) = -\pi^1\left(0^{\theta_2-1}\prod_{j=3}^{\infty}10^{\theta_j}\right) = -\sum_{k=2}^{\infty}\frac{2^{a_{k+1}(\{\xi_i\}_{i=2}^{\infty})-1}}{3^k}$$

where both parts are equal. On the other hand. suppose that $\{\xi_j\}_{j=1}^{\infty} = 0^{\theta_1}10^{\theta_2}1\ldots = \prod_{j=1}^{\infty} 0^{\theta_j}1 \in G_\infty$ with $\theta_j \in \mathbb{N}$ and $\theta_1 > 0$. We have

$$Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = Col\left(-\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})}}{3^k}\right) = -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})-1}}{3^k}$$

and

$$(-\pi^1) \circ \omega\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = (-\pi^1) \circ \left(\theta^{\theta_1 - 1}\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=1}^{\infty})-1}}{3^k}$$

where again both parts are equal. Then we conclude that the diagram is commutative.
Claim 2: $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ **with It is a bijective function**. Let $Cod : -\pi^1(G_\infty) \to G_\infty$ let us prove that $Cod \circ -\pi^1 = Id_{G_\infty}$ and $-\pi^1 \circ Cod = Id_{-\pi^1(G_\infty)}$.

1. $Cod \circ -\pi^1 = Id_{G_\infty}$: Let $\xi = \xi_1\xi_2\xi_3\ldots \in G_\infty$ with $\xi_j \in \{0, 10\}$. Since the parity of $-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ depends only on the first term, if ξ starts with 0 then $a_1 > 0$, then $-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ is even then the first term of its coding is 0, and if ξ starts with 1 then $a_1 = 0$ then $-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ is odd then the first term of coding is 10. By applying the Collatz function, we obtain the same result as applying a translation of the terms of ξ. Indeed
   (a) if $a_1 = 0$

   $$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{2^{a_{j+1}-1}}{3^j} = -\pi^1(\xi_2\xi_3\ldots).$$

   (b) if $a_1 > 0$

   $$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{2^{a_j-1}}{3^j} = -\pi^1(\xi_2\xi_3\ldots).$$

   Then applying the function $-\pi^1$. Then we can repeat the same procedure and we recover ξ. Therefore

   $$Cod(-\pi^1(\xi)) = \xi$$

2. $-\pi^1 \circ Cod = Id_{-\pi^1(G_\infty)}$: Let $u \in -\pi^1(G_\infty)$ and $\xi \in G_\infty$ such that $-\pi^1(\xi) = u$. On the other hand we have $Cod(-\pi^1(\xi)) = Cod(u)$ and $Cod(-\pi^1(\xi)) = \xi$ then $Cod(u) = \xi$, applying $-\pi^1$ on both sides we have $-\pi^1(Cod(u)) = -\pi^1(\xi) = u$.

☐ **of the Claim.**

Claim 3: $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ **is continuous.** Consequence of the Proposition 39.

☐ **of the Claim.**

Claim 4: $Cod : -\pi^1(G_\infty) \to G_\infty$ **is continuous.** Let $u \in -\pi^1(G_\infty)$ and let $\varepsilon > 0$. Take $r \in \mathbb{N}$ such that $\frac{1}{2^r} < \varepsilon$. then by Lemma 19 we have

$$d(u,v) < \frac{1}{3^r} \Rightarrow Cod(u)_j = Cod(v)_j \text{ for all } j < r \Rightarrow D(Cod(u), Cod(v)) < \frac{1}{2^r} < \varepsilon$$

☐ **of the Claim.**

Therefore, $-\pi^1$ is a homeomorphism and therefore a topological conjugation. ☐

Corollary 12 (The Periodic orbit of Collatz function)**.** *The set of periodic orbits of Collatz function is dense in* $\pi^1(G_\infty)$

Proof. Direct consequence of the proposition 42 ☐

As a consequence of the uniqueness of solutions to the coding problem in $\mathbb{Z}_2$ and in $-\pi^1(G_\infty)$, we have that the numeric value of the function $-\pi^1 : G_\infty \to \mathbb{Z}_2$ and $-\pi^1 : G_\infty \to -\pi^1(G_\infty) \subset \mathbb{R}$ coincide.

Proposition 43. *Let* $\xi \in G_\infty$ *and* $\pi^1 : \Sigma_2^* \to \mathbb{Z}_2$ *and* $\pi^1 : \Sigma_2^* \to \pi^1(G_\infty) \subset \mathbb{R}$*, then if* $-\pi^1(\xi) = \alpha \in \mathbb{Z}_2$ *and* $-\pi^1(\xi) = \beta \in \mathbb{Q}$*, then* $\alpha = \beta$*.*

Proof. By the Proposition we know that there exists a unique $\alpha \in \mathbb{Z}_2$ such that $Cod(\alpha) = \xi$. On the other hand, we have that $-\sum_{j\in\mathbb{N}} \frac{2^{a_j}}{3^j} = \beta \in \mathbb{R}$ also satisfies $Cod(\beta) = \xi$. If $\beta \in \mathbb{Q}$ (with odd denominator), then β can be embedded in $\mathbb{Z}_2$, hence by uniqueness we have $\beta = \alpha$. ☐

11. The G_1 Is Positive Unstable

Having successfully characterized the sets G_0 and G_∞ via the asymptotic behavior of the auxiliary functions π^1 and π^2, we now direct our analysis to the critical set G_1. The primary objective of this section is to prove that G_1 is positive instability and, fundamentally, to demonstrate that no rational number admits a coding sequence ξ belonging to this set.

To establish these results, we construct the proof through three analytical stages. First, we introduce the fix function, which maps a coding sequence to the fixed point of its associated affine transformation within $\langle \theta, \psi \rangle$. We demonstrate that the asymptotic behavior of fix(ξ) acts as a discriminator between the sets: unlike G_0 and G_∞, the function is unbounded for $\xi \in G_1$. Second, we establish a necessary condition for positive stability through the concept of *minimality*, proving that any positively stable system must be reducible to a minimal form. Finally, we connect these dynamical properties with the arithmetic structure of $\mathbb{Q}_{odd}$ by analyzing the stopping time $T(r)$ and the 2-adic expansion of rational numbers. This synthesis reveals that the unbounded nature of the fix function in G_1 is incompatible with the arithmetic constraints of rational orbits, thereby confirming the instability of the set.

11.1. Summary of Propositions in the Section

1. **Definition 24:** Definition of the fix function.
2. **Proposition 44:** Establishes the asymptotic behavior of fix(ξ) for the sets G_∞, G_0, and G_1.
3. **Definition 25:** Definition of minimal and reducible to minimal affine maps S.
4. **Definition:** Definition of minimal and reducible to minimal sequences of maps $\{S_j\}_{j\in\mathbb{N}}$.
5. **Proposition 45:** If $\{S_j\}_{j\in\mathbb{N}}$ is positively stable, then it is reducible to minimal.
6. **Lemma 20:** Relates the stopping time $T(r)$ to the exponents a_j for minimal sequences.
7. **Proposition 46:** Establishes the relationship between the stopping time $T(r)$ and the non-periodic part of the 2-adic expansion of $-r$.
8. **Proposition 47:** Ensures that if integer approximations eventually fall into the minimal region $(0,1)$, the rational sequence will inevitably follow suit.

9. **Proposition 49:** Let $\xi \in G_1$. Then $\{S_k\} \in \langle \theta, \psi^q \rangle$ associated with ξ is negatively unstable.
10. **Proposition 48:** If $\{S_j\}_{j\in\mathbb{N}}$ is positively stable and the perturbation term decays sufficiently fast, then it is strictly minimal.
11. **Proposition 50:** Proves that $\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}}\pi_k^1(\xi) = 0$ for $\xi \in G_0 \cup G_1 \cup G_\infty$.
12. **Lemma 21:** Provides a strict lower bound for the exponential incremental quotient ($|e^x - 1| > R_0|x|$), crucial for applying transcendental bounds.
13. **Proposition 51:** Proves that the normalized limit of the fix function is zero: $\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \mathrm{fix}_k(\xi) = 0$.
14. **Proposition 52:** Demonstrates that the topological proximity of periodic sequences forces their 2-adic non-periodic parts to be nested.
15. **Theorem 9:** Main theorem establishing that for $\xi \in G_1$, no rational number p/q exists such that $Cod(p/q) = \xi$, proving that G_1 is positive unstable.

11.2. The Fix *Function*

Definition 24. *Let $\xi \in \Sigma_2^*$ be a sequence without a null tail. We define the* fix *function,* $\mathrm{fix} : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{R}\}$*, by the mapping* $\mathrm{fix}(\xi) : \mathbb{N} \to \mathbb{R}$*:*

$$\mathrm{fix}(\xi)(k) = -\left(\frac{1}{1 - \frac{2^{a_{k+1}-a_1}}{3^k}}\right)\sum_{j=1}^{k}\frac{2^{a_j}}{3^j} = -\left(\frac{1}{1 - \frac{2^{a_{k+1}-a_1}}{3^k}}\right)\pi_k^1(\xi).$$

This function associates with each sequence ξ the fixed point of the map $S \in \langle \theta, \psi \rangle$ whose first k coding bits correspond to $\{\xi_j\}_{j=1}^k$. The asymptotic behavior of this function allows us to distinguish between the sets G_0, G_∞, and G_1, as shown in the following proposition.

Proposition 44. *Let $\xi \in \Sigma_2^*$ and $\pi^1 : \Sigma_2^* \to \mathbb{R}$ and $\pi^2, \mathrm{fix} : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{R}\}$. Then we have:*

1. *if $\xi \in G_\infty$ then exist $\lim_{k\to\infty} \mathrm{fix}(\xi)(k)$ and $\lim_{k\to\infty} \mathrm{fix}(\xi)(k) = -\pi^1(\xi)$.*
2. *if $\xi \in G_0$ then $\mathrm{fix}(\xi)$ is bounded and if $\pi^2(\xi)(k_j)$ is a subsequence convergent then $\lim_{j\to\infty} \mathrm{fix}(\xi)(k_j) = \lim_{j\to\infty} \pi^2(\xi)(k_j)$.*
3. *if $\xi \in G_1$ then exist a subsequence of a_k such that $\lim_{l\to\infty} |\mathrm{fix}(k_l)| = \infty$.*

Proof. Let $\xi \in \Sigma_2^*$, then:

1. If $\xi \in G_\infty$ and let $\pi^1 : \Sigma_2^* \to \mathbb{R}$ by Proposition 22 we have

$$\begin{aligned}
&\left|\mathrm{fix}(\xi)(k) - (-\pi_k^1(\xi))\right| \\
&= \left|-\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right)\pi_k^1(\xi) + \pi_k^1(\xi)\right| = |\pi_k^1(\xi)|\left|\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) - 1\right| \\
&< \Omega\left|\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}}\right) - 1\right| \to 0 \text{ as } k \to \infty, \text{ since } \lim_{k\to\infty}\frac{2^{a_{k+1}}}{3^k} = 0
\end{aligned}$$

then $\lim_{k\to\infty} \mathrm{fix}(\xi) = -\lim_{k\to\infty} \pi_k^1(\xi) = -\pi^1(\xi) \in \mathbb{R}$

2. If $\xi \in G_0$ then $|\pi^2(\xi)| < \Omega$. On the other hand, we have the following.

$$
\begin{aligned}
|\operatorname{fix}(\xi)(k) - \pi^2(\xi)(k)| &= \left| -\left(\frac{1}{1 - \frac{2^{a_{k+1}}}{3^k}} \right) \pi^1_k(\xi) - \frac{3^k}{2^{a_{k+1}}} \pi^1_k(\xi) \right| \\
&= \left| \frac{3^k}{2^{a_{k+1}}} \pi^1_k(\xi) \right| \left| \frac{\frac{2^{a_{k+1}}}{3^k}}{1 - \frac{2^{a_{k+1}}}{3^k}} + 1 \right| \\
&< \Omega \left| \frac{\frac{2^{a_{k+1}}}{3^k}}{1 - \frac{2^{a_{k+1}}}{3^k}} + 1 \right| \to 0 \text{ as } k \to \infty \text{ since } \frac{\frac{2^{a_{k+1}}}{3^k}}{1 - \frac{2^{a_{k+1}}}{3^k}} \to -1
\end{aligned}
$$

Then if $\pi^2(\xi)(k_j)$ is a convergent subsequence, then $\lim_{j\to\infty} \pi^2(\xi)(k_j) = \lim_{j\to\infty} \operatorname{fix}(\xi)(k_j)$. Now we will demonstrate that $\operatorname{fix}(\xi)$ is bounded. Let $K > 0$ such that $|\operatorname{fix}(\xi)(k) - \pi^2(\xi)(k)| < 1$ then

$$
|\operatorname{fix}(\xi)(k)| < \max_{k \le K}\{\operatorname{fix}(\xi)(k)\} + \Omega + 1
$$

3. If $\xi \in G_1$, then exist a subsequence of a_k such that $\lim_{l\to\infty} \frac{a_{k_l+1}}{k_l} = \frac{\ln 3}{\ln 2}$ so exist $\varepsilon > 0$ such that

$$
\left| \frac{1}{1 - \frac{2^{a_{k_l+1}}}{3^{k_l}}} \right| > \varepsilon
$$

Suppose that exist a subsequence $\{a_{k_{l_j}}\}_{l\in\mathbb{N}}$ such that $\left| \frac{1}{1 - \frac{2^{a_{k_{l_j}+1}}}{3^{k_{l_j}}}} \right| \to 0$ as $l \to \infty$, this is equivalent to $\left| 1 - \frac{2^{a_{k_{l_j}+1}}}{3^{k_{l_j}}} \right| \to \infty$ as $l \to \infty$ so

$$
\lim_{l\to\infty} \frac{2^{a_{k_{l_j}+1}}}{3^{k_{l_j}}} = \infty
$$

This contradicts Proposition 16 where it states that there exist $M, m > 0$ such that $m < \frac{3^{k_l}}{2^{a_{k_l+1}}} < M$ for all $l \in \mathbb{N}$.
Therefore, we have

$$
|\operatorname{fix}(k_l)| = \left| \left(\frac{1}{1 - \frac{2^{a_{k_l+1}}}{3^{k_l}}} \right) \pi^1_{k_l}(\xi) \right| > \varepsilon \pi^1_{k_l}(\xi)
$$

Let us denote by $\gamma_{k_l} = \varepsilon \pi^1_{k_l}(\xi)$, as $\pi^1_{k_l}(\xi) = \sum_{j=1}^{k_l} \frac{2^{a_j}}{3^j}$, then we have that γ_{k_l} is monotonically increasing, on the other hand as $\xi \notin G_\infty$, then by Proposition 22 we have that $\pi^1_{k_l}(\xi) \to \infty$ as $l \to \infty$

☐

11.3. Minimality

With the analytical properties of $\text{fix}(\xi)$ established, we now proceed to link these properties to the dynamical stability of the system. To do this, we need to introduce the concept of minimality for the affine maps involved.

Definition 25. *Let $S : \mathbb{R} \to \mathbb{R}$ be given by $S(x) = \dfrac{qx+N}{2^a}$. We say that S is **minimal** if:*

$$0 < \sigma_q^a(N) < q$$

*and we say that S is **reducible to minimal** if exist $u < 2^a$ such that $u = N \pmod{2^a}$:*

$$0 < \sigma_q^a(u) < q$$

Example 16. *Let $S : \mathbb{R} \to \mathbb{R}$ given by $S(x) = \dfrac{3x+11}{4}$. This is not minimal and reducible to minimal, since $\sigma_3^2(11) = 5 \geq 3$ and $11 = 3 \pmod 4$ with $\sigma_3^2(3) = 3 \geq 3$*

Definition 26. *Let $\{S_j\}_{j\in\mathbb{N}} \in \langle\theta, \psi\rangle$. Then $\{S_j\}_{j\in\mathbb{N}}$ is minimal (or reducible to minimal) if there exists $J > 0$ such that S_j is minimal (or reducible to minimal) for all $j > J$.*

Example 17. *Let $S_k : \mathbb{R} \to \mathbb{R}$ given by $S_k(x) = \dfrac{3^k x + 3^k + 2^k}{2^k}$. This is not minimal and reducible to minimal, since $\sigma_{3^k}^k(3^k + 2^k) = 3^k + 1 \geq 3$ and $3^k + 2^k = 3^k \pmod{2^k}$ with $\sigma_{3^k}^k(3^k) = 3^k \geq 3^k$*

The importance of minimality lies in its relationship with positive stability. As the following proposition shows, if a sequence of functions constitutes a positively stable system, it must eventually exhibit this reducibility property. This provides a necessary condition for stability.

Proposition 45. *Let $\{S_j\}_{j\in\mathbb{N}} \in \langle\theta, \psi^q\rangle$. If $\{S_j\}_{j\in\mathbb{N}}$ is positively stable, then $\{S_j\}_{j\in\mathbb{N}}$ is reducible to minimal.*

Proof. Suppose that $\{S_j\}_{j\in\mathbb{N}}$ is **not reducible to minimal**. This means there exists a **subsequence** of $\{S_j\}_{j\in\mathbb{N}}$ such that $\sigma_{3^j}^{a_{j+1}}(qN_j \pmod{2^{a_{j+1}}}) \geq 3^j$. Let $u_j = qN_j \pmod{2^{a_{j+1}}}$ such that $u_j < 2^{a_{j+1}}$ then $qN_j = u_j + 2^{a_{j+1}} v_j$ with $v_j \in \mathbb{Z}$. Calculating the minimal value ρ_0:

$$\begin{aligned}
\rho_0(S_j) &= 2^{a_{j+1}} \sigma^{a_{j+1}}\left(\frac{qN_j}{3^j}\right) - \frac{qN_j}{3^j} \\
&= 2^{a_{j+1}} \sigma^{a_{j+1}}\left(\frac{u_j + 2^{a_{j+1}} v_j}{3^j}\right) - \frac{u_j + 2^{a_{j+1}} v_j}{3^j} \\
&= 2^{a_{j+1}}\left(\sigma^{a_j}\left(\frac{u_j}{3^j}\right) + \sigma^{a_{j+1}}\left(\frac{2^{a_{j+1}} v_j}{3^j}\right)\right) - \frac{u_j + 2^{a_{j+1}} v_j}{3^j} \\
&= 2^{a_{j+1}}\left(\sigma^{a_{j+1}}\left(\frac{u_j}{3^j}\right) + \frac{v_j}{3^j}\right) - \frac{u_j + 2^{a_{j+1}} v_j}{3^j} \\
&= 2^{a_{j+1}} \sigma^{a_{j+1}}\left(\frac{u_j}{3^j}\right) - \frac{u_j}{3^j} \\
&\geq 2^{a_{j+1}} - \frac{u_j}{3^j} > 2^{a_{j+1}} - \frac{2^{a_{j+1}}}{3^j} \\
&= 2^{a_{j+1}}\left(1 - \frac{1}{3^j}\right) \to \infty \text{ as } j \to \infty.
\end{aligned}$$

Thus $\{S_j\}_{j\in\mathbb{N}}$ is **positively unstable**. □

To quantify the "distance" of an element from stability, we introduce the stopping time function $T(r)$. This function measures the number of iterations required for an orbit to enter the unit interval $[0,1]$.

Lemma 20. *Let* $r \in \mathbb{Q}_{odd}$. *Define* $T(r) = \min\{n \in \mathbb{N}_0 : \sigma^n(r) \in [0,1]\}$. *If* $\{S_j\}_{j\in\mathbb{N}} \in \langle \theta, \psi^q \rangle$ *with* $Cod(S_{j+1}) = Cod(S_j) \pmod{2^{a_j}}$ *is Minimal, then there exists* $J > 0$ *such that* $T\left(\frac{qN_j}{3^j}\right) \le a_j$ *for all* $j > J$.

Proof. Since $\{S_j\}_{j\in\mathbb{N}}$ is Minimal, exists $J > 0$ such that $\{S_j\}_{j\in\mathbb{N}}$ is minimal. This implies:

$$\sigma^{a_j}\left(\frac{N_j}{3^j}\right) < 1 \quad \text{for all } j > J.$$

Thus, the orbit enters $[0,1]$ within a_j steps. ☐

This stopping time $T(r)$ is not arbitrary; it is intimately related to the structure of the 2-adic expansion of r.

Proposition 46. *Let* $r \in \mathbb{Q}_{odd}$ *and let* $-r = \sum_{j=0}^{J} \delta_j 2^j + \sum_{j=J+1}^{\infty} \delta_j 2^j \in \mathbb{Z}_2$ *be its 2-adic expansion, where the first sum corresponds to the* ***non-periodic part*** *of* $-r$ *and the second sum corresponds to the* ***periodic part*** *of* $-r$. *Then* $T(r) - 1 = J$.

Proof. The proof relies on the fact that $Cod\sigma(r)$ is periodic if and only if $r \in [0,1] \cap \mathbb{Q}$. Suppose $Cod\sigma(r)$ is periodic:

$$\begin{aligned} Cod\sigma(r) &= \overline{n_{k-1}n_{k-2}\dots n_0} = (n_0 + \cdots + n_{k-1}2^{k-1}) \sum_{i=0}^{\infty} (2^k)^i \\ &= \frac{1}{1-2^k}(n_0 + \cdots + n_{k-1}2^{k-1}) \in [-1,0] \cap \mathbb{Q}. \end{aligned}$$

Since $Cod\sigma(r) = -r$, then implies $r \in [0,1] \cap \mathbb{Q}$. On the other hands, if $r \in [0,1] \cap \mathbb{Q}$, then $\sigma^j(r)$ is periodic, so $Cod\sigma(r)$ is periodic. Thus, the non-periodic part of the 2-adic expansion of $-r$ corresponds exactly to the iterations required to enter the periodic region. Hence $T(r) - 1 = J$. ☐

Having established the connection between the stopping time $T(r)$ and the 2-adic expansion, we must now understand how sequences of rational approximations behave dynamically. The next proposition ensures that if the integer approximations eventually fall into the minimal region $(0,1)$, the rational sequence will inevitably follow suit.

Proposition 47. *Let* $\{q_k\}$ *be a sequence of rational numbers on* $\mathbb{Q}_{odd}$ *such that* $q_{k+1} = q_k \pmod{2^{a_{k+1}}}$ *and* $\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} q_k = 0$. *Let* $Q_k \in \mathbb{Q}_{odd}$ *such that* $|Q_k| < 2^{a_{k+1}}$ *and* $Q_k = q_k \pmod{2^{a_{k+1}}}$. *If exist* $K_0 > 0$ *such that* $0 < \sigma^{a_{k+1}}(Q_k) < 1$ *for all* $k > K_0$. *Then there exists* $K > 0$ *such that* $0 < \sigma^{a_{k+1}}(q_n) < 1$ *for all* $k > K$.

Proof. We have $q_n = Q_n + 2^{a_{k+1}} R_k$ with $R_k \in \mathbb{Q}_{odd}$. Since $\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} q_k = 0$, then

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} Q_k + \lim_{k\to\infty} R_k = 0 \text{ then } \lim_{k\to\infty} R_k = 0$$

Since $0 < \sigma^{a_{k+1}}(Q_k) < 1$, exist $\delta, \varepsilon > 0$ such that $\delta < \sigma^{a_{k+1}}(Q_k) < 1 - \varepsilon$. Let $\eta < \min\{\delta, \varepsilon\}$, exist $K > 0$ such that $-\eta < R_k < \eta$ for all $k > K$.

$$\sigma^{a_{k+1}}(q_k) = \sigma^{a_{k+1}}(Q_k + 2^{a_{k+1}} R_k) = \sigma^{a_{k+1}}(Q_k) + R_k \in (\delta - \eta, 1 - \varepsilon + \eta) \subset (0,1)$$

□

Building on this convergence behavior, we can now tighten our necessary condition for stability. If the perturbation term decays sufficiently fast relative to the 2-adic denominator, a positively stable sequence is not just reducible to minimal, but is strictly minimal.

Proposition 48. *Let* $\{S_j\}_{j\in\mathbb{N}} \in \langle\theta,\psi^q\rangle$. *If* $\{S_j\}_{j\in\mathbb{N}}$ *is positively stable and* $\lim_{j\to\infty}\frac{1}{2^{a_{j+1}}}N_j = 0$, *then* $\{S_j\}_{j\in\mathbb{N}}$ *is minimal.*

Proof. Suppose that $\{S_j\}_{j\in\mathbb{N}}$ is **not minimal**. This means there exists a **subsequence** of $\{S_j\}_{j\in\mathbb{N}}$ such that $\sigma_{3^j}^{a_j}(qN_j) \geq 3^j$. Calculating the minimal value ρ_0:

$$\rho_0(S_j) = 2^{a_{j+1}}\sigma^{a_{j+1}}\left(\frac{qN_j}{3^j}\right) - \frac{qN_j}{3^j} \geq 2^{a_{j+1}} - \frac{qN_j}{3^j} \to \infty \text{ as } j \to \infty.$$

Thus $\{S_j\}_{j\in\mathbb{N}}$ is **positively unstable**. □

11.4. G_1 Is Unstable

Before delving into the main challenge of this section proving the positive instability of G_1 we first address its negative stability. Establishing that sequences in G_1 are negatively unstable is relatively straightforward. It relies directly on the real-valued divergence of the partial sums $\pi_k^1(\xi)$ and the elementary invariant properties of the Collatz map on negative integers.

In stark contrast, demonstrating that G_1 is positively unstable is significantly more complex. The positive case forces us to confront the delicate interaction between the real-valued explosion of the orbits and their 2-adic topological boundaries. To achieve this, we will later need to introduce several new dynamical concepts, such as the minimality of affine maps, stopping times, and transcendental bounds. We begin with the simpler negative case.

Proposition 49. *Let* $\xi \in G_1$. *Then* $\{S_k\} \in \langle\theta,\psi^q\rangle$ *associated with* ξ *is negatively unstable.*

Proof. Assume for contradiction that the sequence $\{S_k\}$ is negatively stable. By definition, this implies that the maximal non-positive integer solution $\rho_1(S_k)$ becomes constant for all sufficiently large k. That is, $\rho_1(S_k) = p$ for some $k > K$, where $p \in \mathbb{Z}^-$.

Consequently, its k-th Collatz iteration is exactly given by the evaluation of the $S_k(p)$:

$$Col^k(p) = S_k(p) = \frac{3^k p + N_k}{2^{a_{k+1}}} = \frac{3^k}{2^{a_{k+1}}}\left(p + \pi_k^1(\xi)\right)$$

We now analyze this orbit in the real numbers $\mathbb{R}$. Since $\xi \in G_1$, we established in Proposition 44 that the real sequence of partial sums diverges, meaning $\pi_k^1(\xi) \to \infty$ as $k \to \infty$. Because p is a fixed negative integer constant, there exists an iteration index K^* such that for all $k > K$:

$$p + \pi_k^1(\xi) > 0$$

Since the scaling factor $\frac{3^k}{2^{a_{k+1}}}$ is strictly positive, it guarantees that:

$$Col^k(p) = S_k(p) > 0 \quad \text{for all } k > K$$

This implies that the Collatz orbit of the negative integer p eventually becomes strictly positive. However, it is a foundational property of the Collatz function on integers that the orbit of any strictly negative integer remains strictly negative forever. Specifically:

- If $q \leq -1$ is even, $Col(q) = \frac{q}{2} \leq -\frac{1}{2} \implies Col(q) \leq -1$.

- If $q \leq -1$ is odd, $Col(q) = \frac{3q+1}{2} \leq \frac{-3+1}{2} = -1$.

Therefore, $Col^k(p) \leq -1$ for all $k \in \mathbb{N}$, making it impossible for the orbit to cross into the positive reals. □

Note that this contradiction does not arise for sequences in G_∞, because in that case $\pi_k^1(\xi)$ converges to a finite real limit L. If $-p > L$, the term $p + \pi_k^1(\xi)$ remains negative indefinitely, allowing the orbit to stay in the negative domain.

Proving that G_1 is negatively unstable turned out to be, in retrospect, a rather friendly endeavor. It sufficed to observe an elementary property: the Collatz function itself acts as an insurmountable wall that traps negative integers, preventing them from crossing over into the positive reals.

The positive case, however, is an entirely different challenge. In the domain of natural numbers, we do not have such a convenient "barrier"; orbits can bounce, grow, and shrink freely. To prove that none of these orbits can stabilize at our critical threshold, we will need a much finer analytical scalpel.

This is where the machinery we have been preparing comes into play. We already have on the table the fix function (which reveals how the real magnitude explodes) and the concept of "minimality" for affine maps. What we will do next is make these pieces interact.

In the following results, we will connect this minimality with "stopping times," which will allow us to measure exactly how long it takes for an orbit to enter a region of strict contraction. From there, we will see how these times rigidly dictate the size of the non-periodic part of the 2-adic expansion.

But to seal this proof and deal with the delicate logarithmic resonances that occur exactly at the critical threshold, we will invoke a profound result from transcendental number theory: the **Baker-Wüstholz Theorem (1993)** on linear forms in logarithms. This powerful tool will guarantee that the exponential perturbations do not decay fast enough to evade our analysis.

In the end, all these pieces will come together to prove something fascinating: the constant oscillation in G_1, backed by the Baker-Wüstholz bounds, generates a relentless avalanche of binary carries. Because these carries cannot be absorbed by the periodic part of the expansion, they accumulate infinitely. This pressure inexorably forces any possible initial integer solution to explode towards infinity, thereby confirming, once and for all, the absolute positive instability of the system.

Proposition 50. *Let $\xi \in G_0 \cup G_1 \cup G_\infty$ and $\pi^1 : \Sigma_2^* \to \mathbb{R}$. Then*

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \pi_k^1(\xi) = 0.$$

Proof. First, suppose that $\xi \in G_\infty$. Then, by the Proposition 22, we have $\pi^1(\xi) < \infty$, hence

$$\frac{1}{2^{a_{k+1}}} \pi_k^1(\xi) \to 0 \cdot \pi^1(\xi) = 0 \text{ as } k \to \infty.$$

Now suppose that $\xi \in G_0$. Again, by the Proposition 22, there exists $M > 0$ such that

$$\pi_k^2(\xi) = \frac{3^k}{2^{a_{k+1}}} \pi_k^1(\xi) < M, \quad \text{so } \frac{1}{2^{a_{k+1}}} \pi_k^1(\xi) < \frac{M}{3^k} \to 0 \text{ as } k \to \infty.$$

Suppose now that $\xi \in G_1$. By definition there exists a subsequence $\lim\limits_{k\to\infty}\frac{a_k}{k}=\frac{\ln 3}{\ln 2}$, then exist $J, \varepsilon > 0$ such that $\frac{a_k}{k} \leq \frac{\ln 3}{\ln 2} + \varepsilon$ for all $k > J$, then

$$\frac{1}{2^{a_k+1}}\sum_{j=1}^{k}\frac{2^{a_j}}{3^j}=\frac{1}{2^{a_k+1}}\sum_{j=1}^{k}\left(\frac{2^{\frac{a_j}{j}}}{3}\right)^j=\frac{1}{2^{a_k+1}}\sum_{j=1}^{J}\left(\frac{2^{\frac{a_j}{j}}}{3}\right)^j+\frac{1}{2^{a_k+1}}\sum_{j=J+1}^{k}\left(\frac{2^{\frac{a_j}{j}}}{3}\right)^j$$

$$\leq \frac{1}{2^{a_k+1}}\sum_{j=1}^{J}\left(\frac{2^{\frac{a_j}{j}}}{3}\right)^j+\frac{1}{2^{a_k+1}}\sum_{j=1}^{k}\left(\frac{2^{\frac{\ln 3}{\ln 2}+\varepsilon}}{3}\right)^j$$

$$=\frac{1}{2^{a_k+1}}\sum_{j=1}^{J}\left(\frac{2^{\frac{a_j}{j}}}{3}\right)^j+\frac{1}{2^{a_k+1}}\sum_{j=1}^{k}2^{\varepsilon j}=\frac{1}{2^{a_k+1}}\sum_{j=1}^{J}\left(\frac{2^{\frac{a_j}{j}}}{3}\right)^j+\frac{2^{\varepsilon}}{2^{a_k+1}}\frac{1-2^{\varepsilon k}}{1-2^{\varepsilon}}\to 0 \text{ as } k\to\infty$$

□

To handle the critical case of G_1 in the subsequent analysis, we require a fine control over the exponential growth of the perturbations. The following technical lemma provides a strict lower bound for the exponential incremental quotient, which will be instrumental when applying transcendental number theory to our dynamical bounds.

Lemma 21. *For any* $L > 0$, *there exists a constant* $R_0 > 0$ *such that for all* x *satisfying* $0 < |x| < L$, *the following inequality holds:*

$$|e^x - 1| > R_0|x|.$$

Proof. Let $L > 0$ be fixed. We define the function $f : [-L, L] \to \mathbb{R}$ as the continuous extension of the incremental quotient of the exponential function at the origin:

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

By the Taylor series expansion $e^x = 1 + x + \frac{x^2}{2} + \mathcal{O}(x^3)$, we have $\lim\limits_{x\to 0} f(x) = 1 = f(0)$, ensuring f is continuous on the compact interval $[-L, L]$. Calculating the derivative for $x \neq 0$:

$$f'(x) = \frac{xe^x - (e^x - 1)}{x^2} = \frac{e^x(x-1)+1}{x^2}.$$

The function $g(x) = e^x(x-1)+1$ satisfies $g(0) = 0$ and $g'(x) = xe^x$. Since $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$, $g(x)$ has an absolute minimum at $x = 0$. Thus, $g(x) > 0$ for all $x \neq 0$, implying $f'(x) > 0$. Therefore, f is **strictly increasing** on $[-L, L]$.

The minimum value of f is attained at the lower bound $x = -L$. Let: Since $e^x - 1$ and x always share the same sign, $f(x)$ is always positive. Thus, $f(x) > R_0$ implies $\left|\frac{e^x - 1}{x}\right| > R_0$, concluding the proof. □

With this exponential bound in hand, we can now tackle the normalized limit of the fix function. This is arguably one of the most delicate steps, as it requires the Baker-Wüstholz theorem to resolve the tight logarithmic resonances that occur precisely when sequences hover at the critical threshold of G_1.

Proposition 51. *Let* $\xi \in G_0 \cup G_1 \cup G_\infty$ *and* $\pi^1 : \Sigma_2^* \to \mathbb{R}$*. Then*

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \operatorname{fix}_k(\xi) = 0.$$

Proof. Let us suppose that $\xi \in G_\infty$. Then, by Proposition 50, we have

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \pi_k^1(\xi) = 0$$

and hence

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \operatorname{fix}_k(\xi) = \lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \left(-\frac{1}{1-\dfrac{2^{a_{k+1}}}{3^k}} \right) \pi_k^1(\xi)$$

$$= \underbrace{\lim_{k\to\infty} \left(-\frac{1}{1-\dfrac{2^{a_{k+1}}}{3^k}} \right)}_{=-1} \lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \pi_k^1(\xi) = 0$$

Now suppose that $\xi \in G_0$. Then, by Proposition 50, we have that

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \pi_k^1(\xi) = 0,$$

and hence

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \operatorname{fix}_k(\xi) = \lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \left(-\frac{1}{1-\dfrac{2^{a_{k+1}}}{3^k}} \right) \pi_k^1(\xi)$$

$$= \underbrace{\lim_{k\to\infty} \left(-\frac{1}{1-\dfrac{2^{a_{k+1}}}{3^k}} \right)}_{=0} \lim_{k\to\infty} \frac{1}{2^{a_{k+1}}} \pi_k^1(\xi) = 0$$

Finally, suppose that $\xi \in G_1$. As we know, in this case there are many accumulation points. Let us assume that $3 \cdot 2^R$ is an accumulation point of the sequence $\dfrac{2^{a_{k+1}}}{3^k}$. Then we take

$$a_k = \frac{\ln 3}{\ln 2} k + R + Q(k) \in \mathbb{N}$$

with $R \in \mathbb{R}$ and $Q(k) \neq 0$ for all $k \in \mathbb{N}$, and such that

$$\lim_{k\to\infty} Q(k) = 0.$$

We express 2^{a_k} as follows: as:

$$2^{a_k} = 2^{\frac{\ln 3}{\ln 2} k} \cdot 2^R \cdot 2^{Q(k)} = 3^k \cdot 2^R \cdot \epsilon(k)$$

where we define the multiplicative perturbation $\epsilon(k) = 2^{Q(k)}$. Given that $Q(k) \to 0$, it follows that $\lim_{k\to\infty} \epsilon(k) = 1$.

then

$$\pi_k^1(\xi) = \sum_{j=1}^{k} \frac{2^{a_j}}{3^j} = \sum_{j=1}^{k} \frac{3^j \cdot 2^R \cdot \epsilon(j)}{3^j} = 2^R \sum_{j=1}^{k} \epsilon(j)$$

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The scale factor in the denominator of $\mathrm{fix}_k(\xi)$ is:

$$\frac{2^{a_{k+1}}}{3^k} = \frac{3^{k+1} \cdot 2^R \cdot \epsilon(k+1)}{3^k} = 3 \cdot 2^R \cdot \epsilon(k+1)$$

Substituting these terms into the original definition of the normalized limit:

$$\frac{1}{3^{k+1} \cdot 2^R \cdot \epsilon(k+1)} \left[-\frac{1}{1 - 3 \cdot 2^R \cdot \epsilon(k+1)} \left(2^R \sum_{j=1}^{k} \epsilon(j) \right) \right]$$

Simplifying the constant 2^R and reorganizing the denominator:

$$\frac{\sum_{j=1}^{k} \epsilon(j)}{3^{k+1} \cdot \epsilon(k+1)(3 \cdot 2^R \cdot \epsilon(k+1) - 1)}$$

Since $\lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \epsilon(j) = 1$, given $\delta > 0$ there exists $K > 0$ such that $\sum_{j=1}^{k} \epsilon(j) < k\delta$ for all $k > K$. Therefore, for $k > K$ we have:

$$\left| \frac{\sum_{j=1}^{k} \epsilon(j)}{3^{k+1} \cdot \epsilon(k+1)(3 \cdot 2^R \cdot \epsilon(k+1) - 1)} \right| < \left| \frac{k\delta}{3^{k+1} \cdot \epsilon(k+1)(3 \cdot 2^R \cdot \epsilon(k+1) - 1)} \right|.$$

First suppose that $3 \cdot 2^R \neq 1$. Then $\lim_{k\to\infty} \epsilon(k+1)\Big(3 \cdot 2^R \cdot \epsilon(k+1) - 1\Big) = 3 \cdot 2^R - 1 \neq 0$, and since $\lim_{k\to\infty} \frac{k}{3^k} = 0$, it follows that

$$\left| \frac{k\delta}{3^{k+1} \cdot \epsilon(k+1)(3 \cdot 2^R \cdot \epsilon(k+1) - 1)} \right| \to 0 \quad \text{as } k \to \infty,$$

Now consider the case $3 \cdot 2^R = 1$. In this situation,

$$\left| \frac{1}{2^{a_{k+1}}} \mathrm{fix}_k(\xi) \right| < \left| \frac{k\delta}{3^{k+1} \cdot \epsilon(k+1)(\epsilon(k+1) - 1)} \right|.$$

Since $\epsilon(k) = 2^{Q(k)}$ and $\lim_{k\to\infty} Q(k) = 0$, for $L > 0$ exist $k > k$ such that $|Q(k)| < L$ then by Lemma 21, there exists $W_0 > 0$ such that

$$|\epsilon(k+1) - 1| = |2^{Q(k+1)} - 1| = |e^{Q(k+1)\ln 2} - 1| > W_0 |Q(k+1)|$$

for k sufficiently large.

We observe that

$$Q(k) \ln 2 = a_k \ln 2 - (k-1) \ln 3,$$

which defines a linear form in the logarithms of the algebraic numbers 2 and 3:

$$\Lambda_k = a_k \ln 2 - (k-1) \ln 3.$$

Here $\Lambda_k = \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2$ with $\alpha_1 = 2$, $\alpha_2 = 3$, $\beta_1 = a_k \in \mathbb{Z}^+$ and $\beta_2 = -(k-1) \in \mathbb{Z}^-$.

We apply the **Baker-Wüstholz Theorem** [3], Since $a_k, k-1 > e$, existe $C > 0$ such that :

$$\log|\Lambda_k| > -C\ln\{\max\{a_k, k-1\}\},$$

We have $a_k \geq k$ for all $k \in \mathbb{N}$, then $\log|\Lambda_k| > -C\ln(a_k)$. Since $a_k = \dfrac{\ln 3}{\ln 2}(k-1) + Q(k)$, exist $c_0, K > 0$ such that $a_k > c_0(k-1)$ for $k > K$, then

$$\log|\Lambda_k| > -C\ln(a_k) > -C\ln(c_0(k-1))\},$$

This is equivalent to $|\Lambda_k| > W_1(k-1)^{-C}$, with $W_1 = e^{-C\ln(c_0)}$ and therefore

$$|Q(k)| = \frac{|\Lambda_k|}{\ln 2} > \frac{W_1(k-1)^{-C}}{\ln 2}.$$

Finally,

$$\left|\frac{k\delta}{3^{k+1}\cdot\epsilon(k+1)\cdot W_0\cdot Q(k+1)}\right| < \frac{k^{C+1}\delta\ln 2}{3^{k+1}\cdot\epsilon(k+1)\cdot W_0 W_1} \to 0 \quad \text{as } k\to\infty,$$

since the numerator has polynomial growth while the denominator has exponential growth. We conclude that

$$\lim_{k\to\infty}\frac{1}{2^{a_{k+1}}}\operatorname{fix}_k(\xi) = 0.$$

□

Having tamed the real-valued asymptotic behavior of the fixed points, we must now translate these bounds back into the 2-adic realm. The next proposition bridges the gap, demonstrating that the topological proximity of periodic sequences forces their 2-adic non-periodic parts to be nested.

Proposition 52. *Let $\xi \in \Sigma_2^*$ and $\{S_k\} \in \langle\theta, \psi^q\rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=1}^k$ and positively stable. Let $\{\eta_k\}_{k\in\mathbb{N}} \subset \Sigma_2^*$ periodic such that $D(\xi, \eta_k) \leq \dfrac{1}{2^{a_{k+1}}(\xi)}$. Then exist $K > 0$ such that that all the non-periodic part of the 2-adic expansion of $-\pi^1(\eta_k)$ is contained in the non-periodic part of the 2-adic expansion of $-\pi_k^1(\xi)$ for all $k > K$.*

Proof. We have that $\{S_k\}$ is positively stable, by Proposition 50 we have that

$$\lim_{k\to\infty}\frac{1}{2^{a_{k+1}}}\pi_k^1(\xi) = 0$$

and by Proposition 48, we obtain that $\{S_k\}$ is minimal. Hence, there exists $K_1 > 0$ such that $0 < \sigma^{a_{k+1}}(\pi_k^1(\xi)) < 1$ for $k > K_1$.

Since $\lim_{k\to\infty}\dfrac{1}{2^{a_{k+1}}}\pi_k^1(\xi) = 0$ we have exist $K_2 > 0$ such that $|\pi_k^1(\xi)| < 2^{a_{k+1}}$ for all $k > K_2$. Let $K_3 > 0$ such that $K_3 > \max\{K_1, K_2\}$, then $\pi_k^1(\xi) = \pi^1(\eta_k) \pmod{2^{a_{k+1}}}$ with $0 < \sigma^{a_{k+1}}(\pi_k^1(\xi)) < 1$ with $|\pi_k^1(\xi)| < 2^{a_{k+1}}$ for $k > K_3$ and by Proposition 51 we have that

$$\lim_{k\to\infty}\frac{1}{2^{a_{k+1}}}\pi^1(\eta_k) = 0$$

Hence by Proposition 47, then exist $K > 0$ such that $K > K_3$ and $0 < \sigma^{a_{k+1}}(\pi^1(\eta_k)) < 1$ for all $k > K$ and by Lemma 20 we have $T_k = T_k(\pi^1(\eta_k)) \leq a_{k+1}$ and by Proposition 46, we have that all the non-periodic part of the 2-adic expansion of $-\pi^1(\eta_k)$ is contained in the non-periodic part of the 2-adic expansion of $-\pi_k^1(\xi)$. □

We are finally in a position to assemble the pieces. By juxtaposing the explosive real-valued growth of the fix function in G_1 with the rigid 2-adic nesting of its non-periodic expansions, we can force a profound arithmetic contradiction. This leads us to the culmination of this section: the proof that G_1 sequences cannot correspond to any rational orbit.

Theorem 9. *Let $\xi \in G_1$, then not exist $\frac{p}{q} \in \mathbb{Q}^+$ such that $Cod\left(\frac{p}{q}\right) = \xi$. In particular, ξ is positive unstable.*

Proof. Let $q > 0$ and $\{S_k\} \in \langle \theta, \psi^q \rangle$ such that $Cod_q(S_k) = \xi \in G_1$. Assume for contradiction that $\{S_k\}$ is positively stable, so $\rho_0(S_k) = p$ for any $k > K$. Then by uniqueness $-\pi^1(\xi) = p \in \mathbb{Z}_2 \cap \mathbb{N}$.

We are going to consider two cases, the first case being if there exists $\eta_k \in G_0$ periodic such that $\eta_k \to \xi$ in the topology of Σ_2^* and the second case being if there exists $\eta_k \in G_\infty$ periodic such that $\eta_k \to \xi$ in the topology of Σ_2^*. By the continuity of $-\pi^1 : \Sigma_2^* \to \mathbb{Z}_2$, we have that $-\pi^1(\eta_k) = \mathrm{fix}_k(\eta_k) = \mathrm{fix}_k(\xi) \to -\pi^1(\xi)$ in the 2-adic topology. This is equivalent to

$$-\pi^1(\eta_k) = -\pi^1(\xi) = -\pi^1_k(\xi) \pmod{2^{a_{k+1}}} \text{ for all } k \in \mathbb{N}.$$

Let $-\pi^1(\eta_k) = \sum_{j=0}^{T_k-1} \delta_{jk}2^j + \sum_{j=T_k}^{\infty} \delta_{jk}2^j$, with $\sum_{j=0}^{T_k-1} \delta_{jk}2^j$ representing the **non-periodic part** and $\sum_{j=T_k}^{\infty} \delta_{jk}2^j$ representing the **periodic part** of the 2-adic expansion.

For the periodic part, assume the pattern repeats every $L = L(k)$ bits. Let r be the integer value of this repeating pattern. We can factor out 2^{T_k} and write the sum as a geometric series:

$$\begin{aligned} \sum_{j=T_k}^{\infty} \delta_{jk}2^j &= 2^{T_k}\left(r + r2^L + r2^{2L} + \dots\right) \\ &= 2^{T_k} \cdot r \sum_{n=0}^{\infty} (2^L)^n \end{aligned}$$

In the 2-adic metric, since $||2^L||_2 < 1$, the geometric series converges to $\frac{1}{1-2^L}$:

$$= \frac{r \cdot 2^{T_k}}{1 - 2^L}$$

Substituting this back into the original expression, we obtain the rational form:

$$-\pi^1(\eta_k) = \sum_{j=0}^{T_k-1} \delta_{jk}2^j - \frac{r \cdot 2^{T_k}}{2^L - 1}$$

Let us suppose that there exists some subsequence $\{\eta_k\}$ on G_0 such that $\eta_k \to \xi \in G_1$. Since $\xi \in G_1$, we established in Proposition 44 that the real magnitude $|\pi^1(\eta_k)| = |\,\mathrm{fix}(\xi)(k)|$ diverges to infinity. Since $-\pi^1(\eta_k)$ are positive rational numbers, we have $-\pi^1(\eta_k) \to \infty$ on $\mathbb{R}$, this occurs if and only if

$$\sum_{j=0}^{T_k-1} \delta_{jk}2^j > \frac{r \cdot 2^{T_k}}{2^L - 1} \text{ and } \sum_{j=0}^{T_k-1} \delta_{jk}2^j \to \infty$$

Since $\pi^1(\eta_k) = \pi^1(\eta_{k+1}) \pmod{2^{a_{k+1}}}$, then the number of 1-bits of the non-periodic part of the 2-adic expansion of $-\pi^1(\eta_k)$ must tend to infinity as $k \to \infty$ and by Proposition 52, we have

$$\rho_0(S_k) = q \sum_{j=0}^{a_{k+1}} \delta_{jk}2^j \geq q \sum_{j=0}^{T_k-1} \delta_{jk}2^j \to \infty.$$

which contradicts the positive stability of $\{S_k\}$.

Let us suppose that there exists a subsequence $\eta_k \in G_\infty$ such that $\eta_k \to \xi$ as $k \to \infty$, where $\xi \in G_1$. Since $\eta_k \in G_\infty$, it follows from Proposition 22 that $\pi^1(\eta_k) = \sum_{j=1}^{\infty} \frac{2^{a_k}}{3^j} \in \mathbb{R}$.

Suppose that there exists a sequence $\eta_k \in G_\infty$ such that $\pi^1(\eta_k) \in \mathbb{Z}$, that is, in the case where $f_k = 0$. Under these conditions, we have that

$$N_k = \operatorname{fix}_k(\xi)(2^{a_{k+1}} - 3^k) \in \mathbb{N}.$$

Therefore, we obtain:

$$\begin{aligned}
\rho_0(S_k) &= 2^{a_{k+1}}\sigma^{a_{k+1}}\left(\frac{N_k}{3^k}\right) - \frac{N_k}{3^k} \\
&= 2^{a_{k+1}}\sigma^{a_{k+1}}\left(\frac{\operatorname{fix}_k(\xi)(2^{a_{k+1}} - 3^k)}{3^k}\right) - \frac{\operatorname{fix}_k(\xi)(2^{a_{k+1}} - 3^k)}{3^k} \\
&= 2^{a_{k+1}}\sigma^{a_{k+1}}\left(\frac{\operatorname{fix}_k(\xi)2^{a_{k+1}}}{3^k} - \operatorname{fix}_k(\xi)\right) - \frac{\operatorname{fix}_k(\xi)2^{a_{k+1}}}{3^k} + \operatorname{fix}_k(\xi) \\
&= 2^{a_{k+1}}\sigma^{a_{k+1}}\left(\frac{\operatorname{fix}_k(\xi)2^{a_{k+1}}}{3^k}\right) + 2^{a_{k+1}}\sigma^{a_{k+1}}(-\operatorname{fix}_k(\xi)) - \frac{\operatorname{fix}_k(\xi)2^{a_{k+1}}}{3^k} + \operatorname{fix}_k(\xi) \\
&= \frac{\operatorname{fix}_k(\xi)2^{a_{k+1}}}{3^k} + 2^{a_{k+1}}\sigma^{a_{k+1}}(-\operatorname{fix}_k(\xi)) - \frac{\operatorname{fix}_k(\xi)2^{a_{k+1}}}{3^k} + \operatorname{fix}_k(\xi) \\
&= 2^{a_{k+1}}\sigma^{a_{k+1}}(-\operatorname{fix}_k(\xi)) + \operatorname{fix}_k(\xi)
\end{aligned}$$

Since $\operatorname{fix}_k(\xi) \in \mathbb{Z}$ we have $\sigma^{a_{k+1}}(-\operatorname{fix}_k(\xi)) = \left\lceil \frac{-\operatorname{fix}_k(\xi)}{2^{a_{k+1}}} \right\rceil$. By Proposition 51 exist $K > 0$ such that $\left\lceil \frac{-\operatorname{fix}_k(\xi)}{2^{a_{k+1}}} \right\rceil = 1$ for all $k > K$ we have

$$= 2^{a_{k+1}}\left\lceil \frac{-\operatorname{fix}_k(\xi)}{2^{a_{k+1}}} \right\rceil - \operatorname{fix}_k(\xi) = 2^{a_{k+1}} - \operatorname{fix}_k(\xi) \to \infty \text{ as } k \to \infty \text{ since } \operatorname{fix}_k(\xi) \ll 2^{a_{k+1}}$$

Suppose now that there exists a subsequence, such that $\pi^1(\eta_{k_j}) \notin \mathbb{Z}$ for all $j \in \mathbb{N}$.

Let $E_k = \lfloor -\pi^1(\eta_k) \rfloor = \sum_{j=0}^{\infty} \kappa_{jk} 2^j$ represent the integer part, and let $f_k = -\operatorname{frac}(\pi^1(\eta_k)) = \sum_{j=0}^{\infty} \varepsilon_{jk} 2^j \in (-1, 0) \cap \mathbb{Q}$, then the 2-adic expansion is purely periodic. Thus, we have:

$$-\pi^1(\eta_k) = E_k + f_k$$

Since $\|\pi^1(\eta_{k+1}) - \pi^1(\eta_k)\|_2 < 2^{-a_k}$, their 2-adic expansions share the first a_k terms. This implies that in each iteration we add new bits that remain fixed. Furthermore, because the sequence of linear functions associated with $-\pi^1(\eta_k)$ is minimal, the non-periodic part of $-\pi^1(\eta_k)$ is contained within the first a_k terms of its 2-adic expansion.

Let $W(-\pi^1(\eta_k)) = \sum_{j=0}^{T_k - 1} \delta_{jk}$ be the sum of the digits of the non-periodic part. We will show that this function diverges, meaning the number of bits equal to 1 is infinite.

Since $\pi^1(\eta_k)$ diverges in $\mathbb{R}$ for $\xi \in G_1$, there exists a subsequence k_i such that $\pi^1(\eta_{k_i})$ is monotonically increasing toward infinity. For simplicity, we continue denoting this by k.

The first a_{k+1} bits of each 2-adic expansion remain fixed in the next step; this is a consequence of convergence with respect to the 2-adic metric, thus, it suffices to show that at step $k+1$, the count of bits equal to 1 increases by at least one.

Let us first verify that at every iteration a new perturbation appears. Since $\|\pi^1(\eta_{k+1}) - \pi^1(\eta_k)\|_2 < 2^{-a_k}$, then

$$E_k + f_k = E_{k+1} + f_{k+1} \pmod{2^{a_k}}.$$

Let $E^0_{k+1}, f^0_{k+1} \in \mathbb{N}$ such that $E^0_{k+1}, f^0_{k+1} < 2^{a_k}$ and $E^0_{k+1} = E_{k+1} \pmod{2^{a_k}}$ and $f^0_{k+1} = f_{k+1} \pmod{2^{a_k}}$, then we have

$$E_k + f_k = E^0_{k+1} + f^0_{k+1} \pmod{2^{a_k}}$$

So exist $\gamma_{k+1} \in \mathbb{Q}_{odd}$ such that

$$E_{k+1} + f_{k+1} = E^0_{k+1} + f^0_{k+1} + 2^{a_k}\gamma_{k+1}.$$

Hence,

$$\gamma_{k+1} = \frac{1}{2^{a_k}}(E_{k+1} - E^0_{k+1}) + \frac{1}{2^{a_k}}(f_{k+1} - f^0_{k+1})$$

We evaluate the fractional component:

$$\frac{1}{2^{a_k}}\left(f_{k+1} - f^0_{k+1}\right) = \frac{1}{2^{a_k}}\left(\sum_{j=0}^{\infty} \varepsilon_{j(k+1)}2^j - \sum_{j=0}^{a_k-1} \varepsilon_{j(k+1)}2^j\right)$$

$$= \frac{1}{2^{a_k}}\left(\sum_{j=a_k}^{\infty} \varepsilon_{j(k+1)}2^j\right) = \sum_{i=0}^{\infty} \varepsilon_{(i+a_k)(k+1)}2^i$$

Since f_{k+1} is purely periodic, then its shifted sequence $\sum_{i=0}^{\infty} \varepsilon_{(i+a_k)(k+1)}2^i$ is also purely periodic. Hence we have that:

$$\sum_{i=0}^{\infty} \varepsilon_{(i+a_k)(k+1)}2^i \in (-1, 0) \cap \mathbb{Q}$$

Now, we evaluate the integer component. Since $\pi^1(\eta_k) \to \infty$ in $\mathbb{R}$, we have $-\pi^1(\eta_k) \to -\infty$, meaning that $E_{k+1} = \lfloor -\pi^1(\eta_{k+1}) \rfloor$ is a strictly negative integer. In $\mathbb{Z}_2$, the expansion of any negative integer is eventually periodic, ending in an infinite tail of 1s. Since $E^0_{k+1} = E_{k+1} \pmod{2^{a_k}}$ consists exactly of the first a_k bits of E_{k+1}, we have:

$$\frac{1}{2^{a_k}}(E_{k+1} - E^0_{k+1}) = \frac{1}{2^{a_k}}\left(\sum_{j=0}^{\infty} \kappa_{j(k+1)}2^j - \sum_{j=0}^{a_k-1} \kappa_{j(k+1)}2^j\right) = \sum_{i=0}^{\infty} \kappa_{(i+a_k)(k+1)}2^i$$

Because the original sequence of bits $\kappa_{j(k+1)}$ ends in an infinite tail of 1s, this shifted sequence also ends in an infinite tail of 1s. Thus, this shifted sum represents a strictly negative integer, meaning:

$$\frac{1}{2^{a_k}}(E_{k+1} - E^0_{k+1}) \in \mathbb{Z}^-$$

Since γ_{k+1} is the sum of a strictly negative integer and a fraction strictly between -1 and 0, it clearly follows that:

$$\gamma_{k+1} \notin (-1, 0)$$

Therefore, γ_{k+1} cannot be purely periodic, meaning it must be eventually periodic. This strictly implies the existence of a new perturbation (the growth of the non-periodic part).

In the worst-case scenario, a bit 1 carry is generated by this perturbation, creating a sequence of zeros. However, this carry cannot be infinite because the non-periodic part is finite and the carry does not enter the periodic part. By Proposition 34:

$$\sigma^j(E_k + f_k) = \sigma^j(E_k) + \sigma^j(f_k) - \Delta \text{ with } \Delta \in \{0, 1\}$$

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Since $\sigma^j(E_k) \in \mathbb{N}$, $\sigma^{T_k}(f_k) \in (0,1)$ and $\sigma^j(E_k + f_k) < 1$ then

$$\sigma^{T_k}(E_k + f_k) = \underbrace{\sigma^{T_k}(E_k)}_{=1} + \sigma^{T_k}(f_k) - 1 = \sigma^{T_k}(f_k)$$

This means that the T_k-th term of $E_k + f_k$ is equal to the T_k-th term of f_k. Therefore, if the carried 1 bit enters the periodic part, then it necessarily changes the first bit of the periodic part; we would then have that the periodic part begins after the T_k-th term, which would be a contradiction, since T_k marks the boundary between the periodic and the non-periodic part.

We conclude that:

$$W(-\pi^1(\eta_{k+1})) \geq W(-\pi^1(\eta_k)) + 1$$

This divergence implies that no natural number n can satisfy the coding for $\xi \in G_1$, since

$$\rho_0(S_k) = q \sum_{j=0}^{a_k} \delta_{jk} 2^j \geq q \sum_{j=0}^{T_{k-1}} \delta_{jk} 2^j \geq q W(-\pi^1(\eta_k)) \to \infty \text{ as } k \to \infty$$

Therefore $\dfrac{p}{q} = \infty$, which contradicts the assumption that ξ is positively stable. Then, since it is always possible to form some subsequence in G_0 or G_∞ that approximates any element in G_1, we have that G_1 is positively unstable. ☐

Part III: The non-existence of divergent orbits through an analysis of the codification of linear Diophantine dynamical systems.

12. The Problem of Divergence

In this section, we address the fundamental aspects of divergence of the Collatz function. The primary focus is on the behavior of sequences and orbits, especially those with divergent slopes and their stability properties. The main results are summarized in the following key theorems:

1. **Theorem 10:** This theorem states that all sequences S_k with a divergent slope are positively unstable, defining the sufficient condition under which a sequence becomes unstable.
2. **Theorem 11:** This theorem shows that all orbits with codings in G_0 are bounded.
3. **Theorem 12:** This theorem concludes that all natural numbers have bounded orbits, implying the non-existence of divergent orbits for natural numbers.

First, we examine the conditions under which the slope of a function S_k diverges, leading to instability. Next, we explore the boundedness of orbits coded within G_0 and G_∞, providing proofs and corollaries to support these findings. Finally, we demonstrate the non-existence of divergent orbits for natural numbers.

12.1. Summary of Propositions in the Section

1. **Theorem 10:** It is stated that all sequences S_k with a divergent slope are positively unstable.
2. **Theorem 11:** It is stated that all orbits with codings in G_0 are bounded.
3. **Corollary 2:** If exist a sub-sequence such that $\lim\limits_{j\to\infty} \dfrac{a_{k_l+1}(\xi)}{k_l} > \dfrac{\ln(3)}{\ln(2)}$. Then exist $\Omega > 0$ such that $\pi^2(\xi)(k_l) < \Omega$.
4. **Lemma 22** Let $\xi \in \Sigma_2^*$ such that $\limsup\limits_{n\to\infty} \dfrac{a_n}{n} \leq \dfrac{\ln(3)}{\ln(2)}$. Then not exist $n \in \mathbb{N}$ such that $Cod(n) = \xi$.
5. **Theorem 12:** There are no divergent orbits for the Collatz function on natural numbers.

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6. **Lemma 23:** Let $\frac{p}{q} \in \mathbb{Q}$ then $Cod\left(\frac{p}{q}\right) \in G_0 \cup G_\infty$. In particular, if there is $J > 0$ such that $Col_q^J(p) > 0$ then $Cod_q(p) \in G_0$ and if $Col_q^j(p) < 0$ for all $j \in \mathbb{N}$, then $Cod_q(p) \in G_\infty$.
7. **Theorem 13:** Consider the extension of Collatz's function on $\mathbb{Q}$, then all orbit fall into some cycle.

12.2. The Problem of Divergence

The following theorem shows that if the slope of the function S_k diverges, then so does the minimum value, this is because the only value that satisfies the encoding of S_k is negative.

Theorem 10 (Positively unstable Theorem). *Let* $\{S_j\}_{j=1}^{\infty} \subset \langle \theta, \psi^q \rangle$ *such that* $\frac{3^{b_j}}{2^{a_{j+1}}} \to \infty$, *then* $S_j(x)$ *is positively unstable for all* $q \in \mathbb{Z}$.

Proof. Let $\xi = Cod(\{S_j\}_{j=1}^{\infty})$. Since $\frac{3^{b_j}}{2^{a_{j+1}}} \to \infty$ then $\xi \in G_\infty$ by Proposition 22, 20 and 37 and Theorem 8. We have $-\pi^1(\xi) = -\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j} \in \mathbb{R}$ is the only value whose coding is ξ. On the other hand, regardless of rationality, this number is always negative. Therefore, the minimum value must necessarily be divergent. □

We are going to show a series of results referring to the bounds of the orbits of numbers whose coding is in G_0 and G_∞.

Theorem 11 (Bounded Orbit Theorem). *Let* $n \in \mathbb{Z}$ *such that* $Cod_q(n) \in G_0$ *then the orbit of* n *is bounded for all* $q \in \mathbb{N}$ *odd.*

Proof. Let $\{S_j\}_{j\in\mathbb{N}}$ on $\langle \theta, \psi^q \rangle$ such that $Cod_q\{S_j\}_{j\in\mathbb{N}} = Cod(n)$. Without loss of generality we can assume that n is positive, because in the case that n is negative we have that

$$\begin{aligned}\lim_{j\to\infty} Col_q^{a_{j+1}}(n) = \lim_{j\to\infty} S_{a_{j+1}}(n) = \lim_{j\to\infty} \frac{3^j n + qN_j}{2^{a_{j+1}}} \\ = \lim_{j\to\infty} \frac{3^j}{2^{a_{j+1}}} n + \lim_{j\to\infty} q\pi^2(Cod(n))(j) = \lim_{j\to\infty} q\pi^2(Cod(n))(j) \geq 0\end{aligned}$$

then eventually its orbit will fall into a non-negative number.
If the coding of $\{S_j\}_{j\in\mathbb{N}}$ has a null tail, the result is trivial. Suppose $\{S_j\}_{j\in\mathbb{N}}$ has no null tail, then:

$$Col_q^{a_{j+1}}(n) = S_{a_{j+1}}(n) = \frac{3^j n + qN_j}{2^{a_{j+1}}} = \frac{3^j}{2^{a_{j+1}}} n + q\pi^2(Cod(n))(j)$$

Since $\frac{3^j}{2^{a_{j+1}}} \to 0$ we have by Proposition 22 we have that $q\pi^2(Cod(n))(j)$ is bounded and let $M > 0$ such that $q\pi^2(Cod(n))(j) < M$ for all $j \in \mathbb{N}$. On the other hand, since $\frac{3^j}{2^{a_{j+1}}} \to 0$ we have $\frac{3^j}{2^{a_{j+1}}}$ is bounded and let $H > 0$ such that $\frac{3^j}{2^{a_{j+1}}} < H$ for all $j \in \mathbb{N}$, then

$$\frac{3^j}{2^{a_{j+1}}} n + q\pi^2(Cod(n))(j) < Hn + M$$

Therefore $Col_q^j(n) < \infty$. □

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Monks and Yazinski [14] also extend the results of Eliahou [6] (1993) and Lagarias [10] (1985) concerning the density of "odd" points in an orbit. Let $b_n(x)$ denote the number of ones in the first n digits of $x \in \Sigma_2^*$. If $x \in Q_{odd}$ eventually enters an n-periodic orbit, then

$$\frac{\ln(2)}{\ln(3+1/m)} \leq \lim_{n\to\infty} \frac{b_n(x)}{n} \leq \frac{\ln(2)}{\ln(3+1/M)}$$

where m, M are the least and greatest cyclic elements in the eventual cycle. If $x \in Q_{odd}$ diverges, then

$$\frac{\ln(2)}{\ln(3)} \leq \liminf_{n\to\infty} \frac{b_n(x)}{n}.$$

We will now show the main theorem of this work, where we finally show the non-existence of divergent orbits for every positive integer. We will show that the necessary and sufficient condition for an orbit to be divergent is

$$\limsup_{j\to\infty} \frac{a_{j+1}}{j} \leq \frac{\ln(3)}{\ln(2)}$$

which implies that the only solution if it exists must be

$$-\infty < -\pi^1(\xi) \leq -\frac{1}{3}.$$

Lemma 22. *Let* $\xi \in \Sigma_2^*$ *such that* $\limsup_{j\to\infty} \frac{a_{j+1}}{j} \leq \frac{\ln(3)}{\ln(2)}$. *Then not exist* $n \in \mathbb{N}$ *such that* $Cod(n) = \xi$.

Proof. Suppose that exist $n \in \mathbb{N}$ such that $Cod(n) = \xi$ such that $\limsup_{j\to\infty} \frac{a_{j+1}}{j} \leq \frac{\ln(3)}{\ln(2)}$ and let $\{S_j\} \in \langle\theta, \psi^q\rangle$ such that $Cod\{S_j\}_{j=1}^{\infty} = \xi$. if $Cod\{S_j\}_{j=1}^{\infty} \in G_1$ by Theorem 9 we have $\rho_0\{S_j\}_{j=1}^{\infty} \to \infty$ so $n = \infty$ which contradicts the hypothesis, so $Cod\{S_j\}_{j=1}^{\infty} \in G_\infty$. By Theorem 10 we have $\{S_j\}_{j=1}^{\infty}$ is positively unstable. Therefore then not exist $n \in \mathbb{N}$ such that $Cod(n) = \xi$. ☐

Theorem 12 (Divergent Orbits Theorem)**.** *There are no divergent orbits for the Collatz function on natural numbers.*

Proof. Let $n \in \mathbb{N}$ such that $Col^j(n) \to \infty$ as $j \to \infty$ and $\{S_j\}_{j=1}^{\infty} \subset \langle\theta, \psi\rangle$ such that $Cod\{S_j\}_{j=1}^{\infty} = Cod(n)$. We are going to prove that the necessary and sufficient condition for an orbit to be divergent is that the coding does not have a null tail and $\frac{3^j}{2^{a_{j+1}}} \to \infty$ as $j \to \infty$.

If $Cod\{S_j\}_{j=1}^{\infty}$ has a null tail, it means that from a certain iteration, the orbit of n must always be even, which implies that this orbit must be decreasing. This contradicts the fact that we have assumed that $Col^j(n) \to \infty$.

We can assume that $Cod\{S_j\}_{j=1}^{\infty}$ does not have a null tail

$$Col^{a_j}(n) = \frac{3^j n + N_j}{2^{a_{j+1}}} \to \infty \text{ then } \frac{3^j}{2^{a_{j+1}}} \to \infty \text{ or } \pi_k^2(Cod\{S_j\}_{j=1}^{\infty}) = \frac{N_j}{2^{a_{j+1}}} \to \infty$$

If $\frac{3^j}{2^{a_{j+1}}} \to \infty$, then by Proposition 22, we have that $Cod(n) \in G_\infty$. Thus, we have that $-\pi^1(Cod(n))$ is the only real number that satisfies the coding, and $-\pi^1(Cod(n)) \neq n$ since $-\pi^1(Cod(n)) < 0$, which contradicts the hypothesis that S_j is positively stable, since $\rho_0(S_j) \to \infty$ as $j \to \infty$.

If $\pi_k^2(Cod\{S_j\}_{j=1}^{\infty}) \to \infty$, by Proposition 22 we have that $Cod\{S_j\}_{j=1}^{\infty} \notin G_0$ then $\liminf_{j\to\infty} \frac{a_{j+1}}{j} \leq \frac{\ln(3)}{\ln(2)}$. Let's show now in fact $\limsup_{k\to\infty} \frac{a_{k+1}}{k} \leq \frac{\ln(3)}{\ln(2)}$. Suppose that exist $\{j_k\}_{k\in\mathbb{N}}$ such that $\lim_{j\to\infty} \frac{a_{j_k+1}}{j_k} > \frac{\ln(3)}{\ln(2)} + \varepsilon$ with $\varepsilon > 0$, so using the estimated bound in the demonstration of the Lemma 2, we have

$$\pi^2(\xi)(j_k) < \frac{2^{\varepsilon}}{2^{\varepsilon}-1} + 3\left\{\left(\frac{3}{2}\right)^{T-1} - \frac{2}{3}\right\} = M$$

Since $\frac{3^{j_k}}{2^{a_{j_k+1}}} \to 0$ as $j \to \infty$. So $\max_k\left\{\frac{3^{j_k}}{2^{a_{j_k+1}}}\right\} < \infty$ then we have

$$Col^{j_k}(n) = \frac{3^{j_k}}{2^{a_{j_k+1}}}n + \frac{N_{j_k}}{2^{j_k+1}} < \max_k\left\{\frac{3^{j_k}}{2^{a_{j_k+1}}}\right\}n + M$$

Since $Col^j(n) \in \mathbb{N}$ for all $j \in \mathbb{N}$, then exist $K \in \mathbb{N}$ such that $Col^{j_k}(n) = Col^{j_{k+K}}(n)$. So we have that the orbit of n must fall into a cycle, which implies that $\lim_{j\to\infty} \frac{3^j}{2^{a_{j+1}}} = 0$ which implies that $\pi^2(\xi)(j)$ is bounded, which contradicts the hypothesis that $Col^j(n) \to \infty$. Therefore $\limsup_{j\to\infty} \frac{a_{j+1}}{j} \leq \frac{\ln(3)}{\ln(2)}$. Therefore by Lemma 22 we have that cannot exist $n \in \mathbb{N}$ such that its orbit is divergent. □

Next we will present a more general result. Indeed, all rational ones have orbits that fall into some cycle.

Lemma 23. *Let* $\frac{p}{q} \in \mathbb{Q}$ *then* $Cod\left(\frac{p}{q}\right) \in G_0 \cup G_\infty$. *In particular, if there is* $J > 0$ *such that* $Col_q^J(p) > 0$ *then* $Cod_q(p) \in G_0$ *and if* $Col_q^j(p) < 0$ *for all* $j \in \mathbb{N}$, *then* $Cod_q(p) \in G_\infty$.

Proof. Let $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$. If $p > 0$ then we can repeat the same argument from the proof of the Theorem 12 to show that $Cod_q(p) \in G_0$. Now if $p < 0$. Suppose that $Cod_q(p) \notin G_0 \cup G_\infty$. Additionally, let us assume that the orbit of p is always negative, since if there exists $J > 0$ such that $Col^J(p) > 0$, then $Col^j(p) > 0$ for all $j > J$ and we can repeat the argument of the Theorem 12 again using $n = Col^J(p)$. Then assuming that the orbit of p is always negative we have

$$Col^{a_{j+1}}(p) = \frac{3^j}{2^{a_{j+1}}}p + \frac{N_j}{2^{a_{j+1}}} < 0 \text{ for all } j \in \mathbb{N}$$

so

$$\frac{N_j}{2^{a_{j+1}}} < -\frac{3^j}{2^{a_{j+1}}}p \text{ so we have } \frac{N_j}{3^j} = \pi_j^1(Cod_q(p)) < -p \text{ since } p < 0$$

Since π_j^1 is monotonous, then $\lim_{j\to\infty} \pi_j^1(Cod_q(p)) \in \mathbb{R}$ and by Proposition 22 we have that $Cod_q(p) \in G_\infty$ □

Theorem 13. *Consider the extension of Collatz's function on* $\mathbb{Q}$, *then if* $\frac{p}{q} \in \mathbb{Q}_{odd}$ *then all orbit fall into some cycle and if* $\frac{p}{q} \in \mathbb{Q}_{even}$ *then the orbit is divergent.*

Proof. By Lemma 23 we only have to analyses the cases where the coding is in G_0 or in G_∞. Let $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$. Suppose that $Cod\left(\frac{p}{q}\right) = \xi \in G_\infty$ and that ξ does not have a null tail, otherwise its orbit falls at point 0. On the other hand, $\frac{p}{q} = -\sum_{j\in\mathbb{N}} \frac{2^{a_j}}{3^j}$. since $\xi \in G_\infty$ exist $\varepsilon > 0$ such that $\limsup_{k\to\infty} \frac{a_{k+1}}{k} < \frac{\ln(3)}{\ln(2)} - \varepsilon$. Exist $J > 0$ such that $\frac{a_{j+1} - a_k}{j-k} < \frac{\ln(3)}{\ln(2)} - \varepsilon$ for all $k > J$. so

$$Col^{a_{k+1}}\left(-\pi^1(\xi)\right) = Col^{a_{k+1}}\left(-\sum_{j\in\mathbb{N}}\frac{2^{a_j}}{3^j}\right) = -\frac{3^k}{2^{a_{k+1}}}\sum_{j\in\mathbb{N}}\frac{2^{a_j}}{3^j} + \frac{N_k}{2^{a_{k+1}}}$$

$$= -\frac{3^k}{2^{a_{k+1}}}\sum_{j=1}^{k}\frac{2^{a_j}}{3^j} - \frac{3^k}{2^{a_{k+1}}}\sum_{j=k+1}^{\infty}\frac{2^{a_j}}{3^j} + \frac{N_k}{2^{a_{k+1}}} = -\frac{N_k}{2^{a_{k+1}}} - \frac{3^k}{2^{a_{k+1}}}\sum_{j=k+1}^{\infty}\frac{2^{a_j}}{3^j} + \frac{N_k}{2^{a_{k+1}}}$$

$$= -\frac{3^k}{2^{a_{k+1}}}\sum_{j=k+1}^{\infty}\frac{2^{a_j}}{3^j} = -\sum_{j=k+1}^{\infty}\frac{2^{a_j - a_{k+1}}}{3^{j-k}} = -\sum_{j=k+1}^{\infty}\left(\frac{2^{\frac{a_j - a_{k+1}}{j-k}}}{3}\right)^{j-k}$$

$$> -\sum_{j=k+1}^{\infty}\left(\frac{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}{3}\right)^{j-k} = -\sum_{j=k+1}^{\infty} 2^{-\varepsilon(j-k)} = -2^{\varepsilon k}\sum_{j=k+1}^{\infty} 2^{-\varepsilon j}$$

$$= -2^{\varepsilon k}\left\{\frac{2^{-\varepsilon(k+1)}}{1 - 2^{-\varepsilon}}\right\} = -\frac{2^{-\varepsilon}}{1 - 2^{-\varepsilon}}$$

Let $\Omega = \max\left\{\frac{2^{-\varepsilon}}{1-2^{-\varepsilon}}, Col^{a_{k+1}}\left(-\pi^1(\xi)\right) \text{ with } k < J\right\}$, so $Col^{a_{k+1}}\left(\frac{p}{q}\right) > -\Omega$ for all $k \in \mathbb{N}$. By Proposition 9 we have

$$Col^{a_{k+1}}\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^{a_{k+1}}(p) \text{ then } Col_q^{a_{k+1}}(p) > -q\Omega \text{ since } q > 0$$

Since the orbits of $-\pi^1(\xi)$ are always negative, we have to $Col_q^{a_{k+1}}(p) < 0$. Since the sub-orbit of p is bounded, and Col_q is defined on the integers, we have to necessarily have the orbit fall into some cycle.

Now suppose that $\xi \in G_0$, then by Theorem 11 we have the orbit is bounded, Without loss of generality we can assume that its orbit is positive, so its orbit must necessarily fall in some cycle. □

Corollary 13. *Let $\xi \in \Sigma_2^*$. The element ξ is semi-periodic if and only if $\pi^1(\xi) \in \mathbb{Z}_2$ is rational.*

Proof. If ξ is semi-periodic, then by the preceding Proposition 21, $\pi^1(\xi) \in \mathbb{Z}_2$ is rational.

Conversely, if $\pi^1(\xi) \in \mathbb{Z}_2$ is rational, then the orbit does not fall into any cycle, so the codification is semi-periodic. □

13. Conclusion

While the complete proof of the Collatz conjecture—specifically ruling out the existence of non-trivial periodic cycles—remains an open problem, this work successfully resolves two significant weak versions of the conjecture. First, we have rigorously proven the non-existence of divergent orbits for the natural numbers. Second, we have demonstrated that the extension of the Collatz function over the domain of rational numbers with odd denominators ($\mathbb{Q}_{odd}$) always inevitably falls into a periodic cycle.

The overarching proof of the non-existence of divergent orbits relies on a conceptually simple argument by contradiction, grounded in the asymptotic parity density of the orbits. However, formally validating this intuition required the development of a comprehensive theoretical framework. A substantial portion of this paper is dedicated to building this machinery, which shifts the perspective from analyzing mere sequences of numbers to studying infinite sequences of linear Diophantine equations.

The most formidable challenge within this theory was proving that these sequences of Diophantine equations admit no natural solutions within the critical threshold set G_1. The absolute crux of this demonstration was establishing the limit:

$$\lim_{k\to\infty} \frac{1}{2^{a_{k+1}}}\pi^1(\eta_k) = 0$$

Without this specific convergence result, it would have been analytically impossible to leverage the real-valued divergence of $\pi^1(\eta_k)$ to show that the number of '1' bits in the non-periodic part of the 2-adic expansion strictly and perpetually increases at each step. This infinite accumulation of binary carries is what ultimately forces the arithmetic contradiction.

To secure this limit and firmly bound the exponential perturbations at the G_1 threshold, we had to bridge discrete dynamics with transcendental number theory by invoking the **Baker-Wüstholz Theorem**. Without the precise bounds on linear forms in logarithms provided by this profound theorem, our framework would have fallen short; it would have merely established a logical equivalence to the divergent orbits conjecture, rather than a definitive proof of their non-existence.

Ultimately, this article highlights the deep and perhaps unexpected connections between symbolic dynamics, 2-adic topology, linear Diophantine systems, and transcendental number theory. We hope that the algebraic and topological tools developed here will not only provide a definitive answer to the problem of divergence in the Collatz function but also open new pathways for researching other discrete arithmetic dynamical systems.

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