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Article

Discovering Doily in PG(2,5)

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Abstract: W. L. Edge [2] proved that the internal points of a conic in PG(2,5), together with the collinear triples on the non-secant lines, form the Desargues configuration. M. Saniga [7] showed an intimate connection between Desargues configurations and the generalized quadrangles of order two, GQ(2,2), whose representation has been dubbed "the doily" by Stan Payne in 1973. In this paper we prove that the external points of a conic in PG(2,5), together with the collinear and non-collinear triples on the non-tangent lines, form the generalized quadrangle of order two.

Keywords: desargues configuration; generalized quadrangle of order two; projective plane of order five

2020 Mathematics Subject Classification. 51E20

1. Introduction and Motivation

W. L. Edge [2] proved that the internal points of a conic in PG(2,5) together with the non-secant lines form a Desargues configuration. M. Saniga [7] showed an intimate connection between Desargues configurations and the generalized quadrangle of order two, GD(2,2). The two results motivate the writing of this note. By using the Singer representation of PG(2,5), we provide a short proof of W. L. Edge result and, believing it is novel, we prove that the external points of a conic of PG(2,5) define the generalized quadrangle of order two, GD(2,2). The reason for deciding to conduct a detailed investigation of this special case is the charm of small projective planes, cf. [1,3,5,6,9,10].

2. The Singer representation of PG(2,5)

Let ω be a primitive element of $\mathbf{F}_{5^3}$ over $\mathbf{F}_5$ and let $f(x) = -a_0 - a_1x - a_2x^2 + x^3$ be its minimal polynomial over $\mathbf{F}_5$. The companion matrix $C(f)$ of f is given by

$$C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{pmatrix}$$

and it induces a Singer cycle γ of PG(2,5), cf. [8]. Let's us consider the minimal polynomial $f(x)=4+4x+x^3$ over $\mathbf{F}_5$. The companion matrix $C(f)$ of f

$$C(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

gives the 31 points of this plane as follows, cf. [4]. If the first point is $\omega^0 = (x_0, x_1, x_2) = (1 \quad 0 \quad 0)$, we get

$$\omega^1 = \omega^0 T = (1 \quad 0 \quad 0) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = (0 \quad 1 \quad 0);$$

$$\omega^2 = \omega^0 T^2 = \omega^1 T = (0 \quad 1 \quad 0) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = (0 \quad 0 \quad 1);$$

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$$\omega^3=\omega^0T^3=\omega^2T=\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 1 & 0\end{pmatrix};$$

$\omega^0=(1,0,0)$					
$\omega^1=(0,1,0)$	$\omega^2=(0,0,1)$	$\omega^3=(1,1,0)$	$\omega^4=(0,1,1)$	$\omega^5=(1,1,4)$	$\omega^6=(1,2,1)$
$\omega^7=(1,0,3)$	$\omega^8=(1,4,0)$	$\omega^9=(0,1,4)$	$\omega^{10}=(1,1,1)$	$\omega^{11}=(1,0,4)$	$\omega^{12}=(1,2,0)$
$\omega^{13}=(0,1,2)$	$\omega^{14}=(1,1,2)$	$\omega^{15}=(1,3,2)$	$\omega^{16}=(1,3,1)$	$\omega^{17}=(1,0,2)$	$\omega^{18}=(1,3,0)$
$\omega^{19}=(0,1,3)$	$\omega^{20}=(1,1,3)$	$\omega^{21}=(1,4,3)$	$\omega^{22}=(1,4,2)$	$\omega^{23}=(1,3,3)$	$\omega^{24}=(1,4,4)$
$\omega^{25}=(1,2,4)$	$\omega^{26}=(1,2,2)$	$\omega^{27}=(1,3,4)$	$\omega^{28}=(1,2,3)$	$\omega^{29}=(1,4,1)$	$\omega^{30}=(1,0,1)$

Let us denote the points represented by ω^i simply by i. So, the Singer group is isomorphic to the additive group Z_{31}, the integers modulo 31. Now select any line: for example, we choose the line $x_1=x_2$, which contains the points: $\ell_0=\{0,4,10,23,24,26\}$. The remaining lines of the plane are found by adding 1 to each point of the preceding line beginning with ℓ_0 and using addition modulo 31. For convenience, we represent the projective plane of order 5 as a set of orthogonal arrays of the affine plane of order 5 with the intersection point of the member of each parallel class indicated to the right of the row array and at the bottom of the column array. We do this by using the Singer difference set defining PG(2,5) as the line at infinity, designated by ℓ_∞. Thus, let $\ell_\infty=\{0,4,10,23,24,26\}$. The remaining lines of the plane are found by adding 1 to each point of the preceding line beginning with ℓ_∞ as ℓ_0 and using addition modulo 31. The pencil of lines on the point 4 is then intersected by the pencil of lines on the point 0 to form the first array. Thus, each row (column) plus its point at infinity represents a line of the plane. Now, let us take into account the Singer representation

1	2	9	1	1	
			3	9	
3	1	1	2	6	
	1	5	1		
8	3	2	1	2	4
	0	8	4	7	

1	6	1	2	3	
		6	9	0	
7	8	1	1	2	
		5	9	5	
2	5	2	1	3	1
0			4		0

1	5	1	2	2	
		1	5	7	
1	1	9	6	7	
4	8				
1	3	2	1	1	2
5	0	0	2	3	4

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1	7	5	1	2		2	1	2	1	9		1	1	1	2	3
2			6	2		1	7	7	2			7	9	6	8	
1	1	2	2	2		2	1	1	1	2		2	2	8	2	2
8	7	9	5	0		8	3	8	1	2		2	1			9
		0						2						2		
								3						6		

Moreover, by the Singer representation, since all conics in PG(2,5) are projectively equivalent, see [4], let us consider the conic $C=-\ell_\infty=\{-0,-4,-10,-23,-24,-26\}=\{0,5,7,8,21,27\}$,

1	2	9	1	1			1	6	1	2	3			1	**5**	1	2	**2**	
			3	9					6	9	0					1	5	**7**	
3	1	1	**2**	6			**7**	**8**	1	1	2			1	1	9	6	**7**	
	1	5	**1**						5	9	5			4	8				
8	3	2	1	**2**	4		2	**5**	2	1	3	1		1	3	2	1	1	2
	0	8	4	**7**			0			4		0		5	0	0	2	3	4
1	**7**	**5**	1	2			**2**	1	**2**	1	9			1	1	1	2	3	
2			6	2			**1**	7	**7**	2				7	9	6	8		
1	1	2	2	2			2	1	1	1	2			2	**2**	**8**	2	2	
8	7	9	5	0			8	3	8	1	2			2	**1**			9	
		0							2							2			
									3							6			

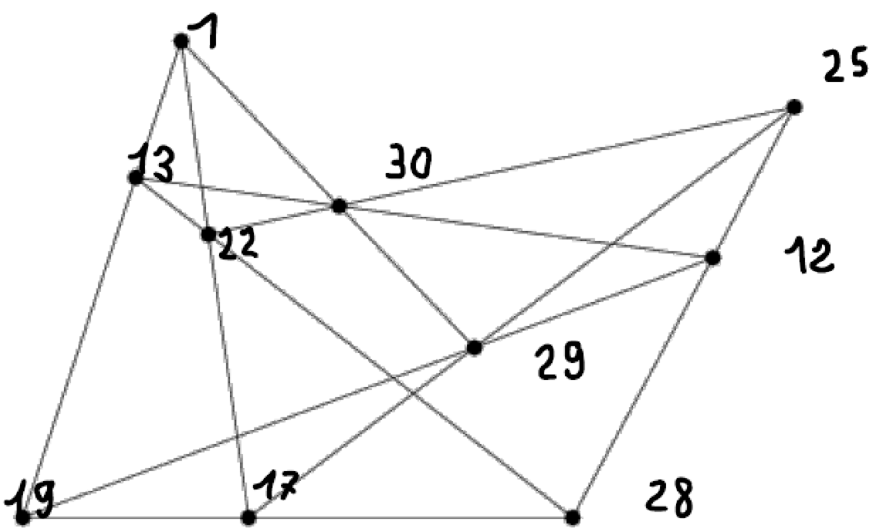
by taking into account the points not on C and not on the tangent lines, we get the 10–set I of the internal points of the conic $I=\{1,12,13,17,19,22,25,28,29,30\}$. Now, taking into account the triples on the external lines

1	2	9	**1**	**1**		**1**	6	1	**2**	**3**		**1**	5	1	**2**	2
			3	**9**				6	**9**	**0**				1	**5**	7

3	1	1	2	6		7	8	1	1	2		1	1	9	6	7	
	1	5	1					5	9	5		4	8				
8	3	2	1	2	4	2	5	2	1	3	1	1	3	2	1	1	2
	0	8	4	7		0			4		0	5	0	0	2	3	4
1	7	5	1	2		2	1	2	1	9		1	1	1	2	3	
2			6	2		1	7	7	2			7	9	6	8		
1	1	2	2	2		2	1	1	1	2		2	2	8	2	2	
8	7	9	5	0		8	3	8	1	2		2	1			9	
		0						2						2			
								3						6			

We get {{1,13,19},{1,17,22},{1,29,30},{12,13,30},{12,19,29},{12,25,28},{13,22,28},{17,19,28},{17,25,29},{22,25,30}}. Let us now consider the point-line incidence geometry (*I*,*T*) where the point-set *I* is the 10–set of the internal points of the conic *I*={1,12,13,17,19,22,25,28,29,30}, and the line-set *T* is the union of the triples of collinear points on the external lines:

T={{1,13,19},{1,17,22},{1,29,30},{12,13,30},{12,19,29},{12,25,28},{13,22,28},{17,19,28},{17,25,29},{22,25,30}}



A brief inspection of the above Figure confirms that the geometry (*I*,*T*) is the Desargues configuration, as W. L. Edge proved in [2].

Now, by taking into account the points non on *C*, but on the tangent lines,

1	2	9	1	1	1	6	1	2	3	1	5	1	2	2
			3	9			6	9	0			1	5	7

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3	**1**	**1**	2	**6**		7	8	**1**	1	2		**1**	**1**	**9**	**6**	7	
	1	**5**	1					**5**	9	5		**4**	**8**				
8	3	2	1	2	**4**	**2**	5	**2**	**1**	**3**	**1**	1	3	**2**	1	1	**2**
	0	8	4	7		**0**			**4**		**0**	5	0	**0**	2	3	**4**
1	7	5	1	2		2	1	2	1	9		1	1	**1**	2	3	
2			6	2		1	7	7	2			7	9	**6**	8		
1	1	2	2	2		2	1	**1**	1	2		2	2	8	2	2	
8	7	9	5	0		8	3	**8**	1	2		2	1			9	
		0						**2**						**2**			
								3						**6**			

We get the 15–set of external points of the conic E={2,3,4,6,9,10,11,14,15,16,18,20,23,24,26}, cf. [4]. Now, taking into account the triples on the external lines and the triples of non-collinear points of the triangles of the 2-lines:

1	**2**	**9**	1	1		1	**6**	**1**	2	3		1	5	1	2	2	
			3	9				**6**	9	0				1	5	7	
3	1	1	2	6		7	8	1	1	2		**1**	1	9	**6**	7	
	1	5	1					5	9	5		**4**	8				
8	3	2	1	2	**4**	2	5	2	**1**	**3**	**1**	**1**	3	**2**	1	1	**2**
	0	8	4	7		0			**4**		**0**	**5**	0	**0**	2	3	**4**
1	7	5	1	2		2	1	2	1	**9**		1	1	**1**	2	**3**	
2			6	2		1	7	7	2			7	9	**6**	8		
1	1	2	2	**2**		2	1	**1**	**1**	2		2	2	8	**2**	2	
8	7	9	5	**0**		8	3	**8**	**1**	2		2	1			9	
		0						**2**						**2**			
								3						**6**			

We get the sets T_1={{2,4,9},{2,6,26},{3,9,23},{3,16,24},{4,18,20},{6,10,16},{10,11,18},{11,14,23},{14,15,26},{15,20,24}} and T_2={{2,11,24},{3,18,26},{4,14,16},{6,20,23},{9,10,15}}

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1	**2**	**9**	1	1		1	**6**	1	2	3		1	5	**1**	2	2	
			3	9				6	9	0				**1**	5	7	
3	**1**	**1**	2	**6**		7	8	**1**	1	2		1	**1**	9	6	7	
	1	**5**	1					**5**	9	5		4	**8**				
8	3	2	**1**	2	**4**	**2**	5	2	1	3	**1**	1	3	2	1	1	**2**
	0	8	**4**	7		**0**			4		**0**	5	0	0	2	3	**4**
1	7	5	**1**	2		2	1	2	1	**9**		1	1	1	2	**3**	
2			**6**	2		1	7	7	2			7	9	6	8		
1	1	2	2	**2**		2	1	1	1	2		2	2	8	**2**	2	
8	7	9	5	**0**		8	3	8	1	2		2	1			9	
		0						**2**						**2**			
								3						**6**			

Let us now construct the point-line incidence geometry (E,L) where the point-set E is the 15–set of the external points of the conic, and the line-set $L=T_1\cup T_2$

A brief inspection of the under Figure confirms that this geometry is isomorphic to GD(2,2).

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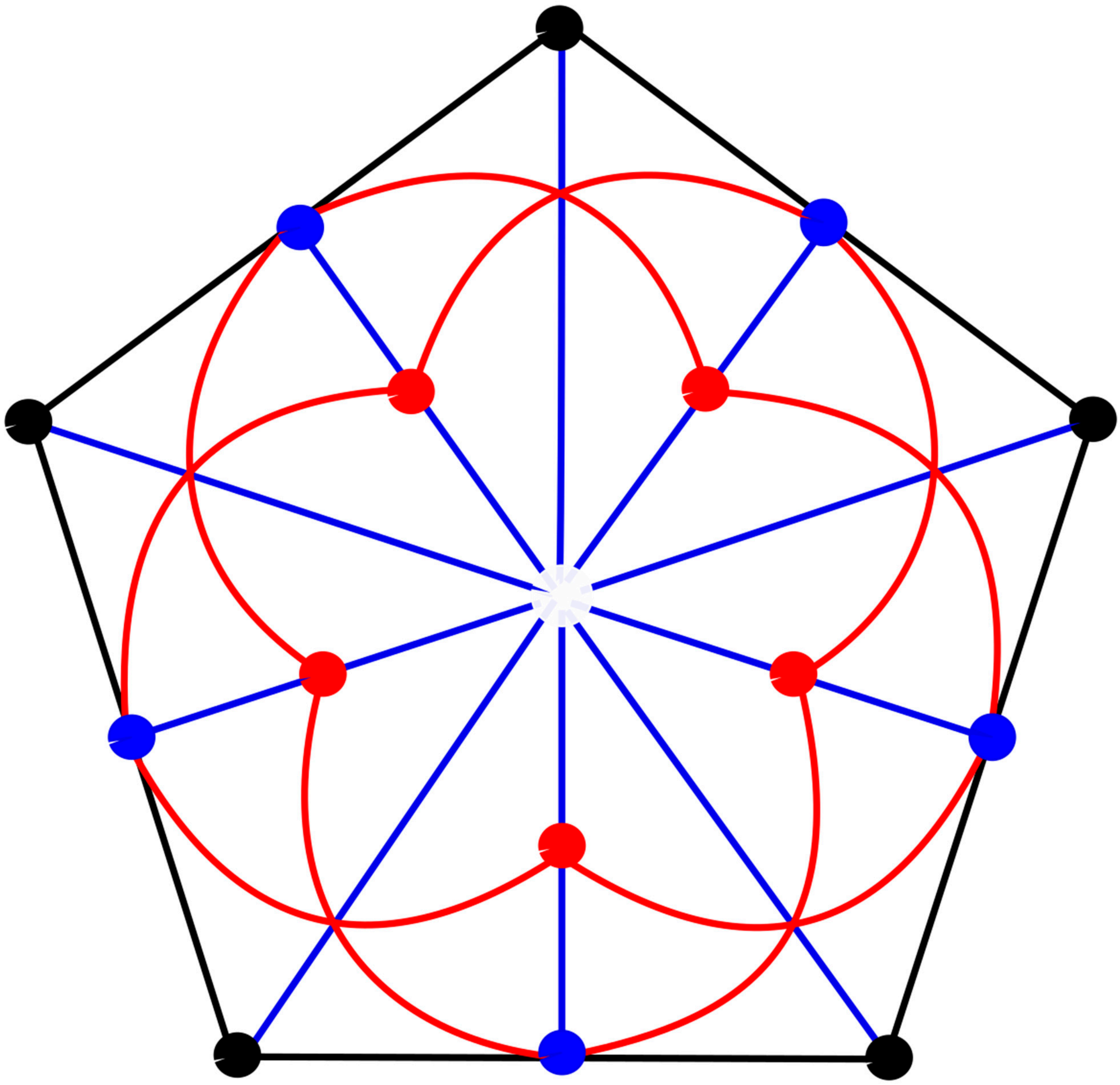

3. Conclusion

This note confirms the intimate connection between Desargues configurations and the generalized quadrangles of order two.

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