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Article

General Quantum Gravity: 'Metric Mechanics' and Generalized Quantum Gravitational Field Theory

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Abstract

Quantum Mechanics is sufficiently capable of proving quantum gravity by itself without considering actual Einsteinian General Relativistic formalism. Due to the non-applicability of Einsteinian relativity in quantum gravity, in this article, we have described gravity as a correspondence between General (Quantum) Relativity and Quantum Field Theory (QFT) by introducing a (quantum) quadratic form and a (quantum) metric tensor along with dynamic time t . Here, we have developed a Kline-Gordon-like equation and a Dirac-like equation in QFT, which are themselves actually nothing but the quantum gravitational field equations (analogous to Einstein's field equation in General Relativity) for bosons and fermions, respectively. Furthermore, we have developed a Generalized Quantum Gravitational Field Theory, where QFT is conjugated with gravity and Dark Energy (for inconstant cosmological constant), so that it can unify Standard Model with gravity and Dark Energy in 'General Unified Theory' as $SU(5) = SU(3)_C \times (SU(2)_L \oplus iSU(2)_G)$ through a Gravito-weak symmetry group. In addition, we have shown that unbounded operators, such as, i) the (quantum) relativistic mass and time, ii) the quantum scalar curvature and the proper time, iii) the (quantum) relativistic mass and its inversely stretched/shrank $(3+1)D$ curvilinear quantum spacetime, all in pairs are satisfied their individual Uncertainty Principles, i.e., they cannot have definite and constant values at the same time. We have also proved that the present theory of Quantum Gravity is 'multiplicatively renormalizable'.

Keywords: General Quantum Gravity; Quantum Differential Geometry; Gravito-weak Symmetry; General Unified Theory

1. Introduction

There is undoubtedly a strong conceptual link between Einstein's theory of gravity and quantum gauge theories [1]. But merging Einstein's theory of gravity and Quantum Mechanics is one of the main outstanding problems of modern physics [2]. Experimentalists often use the Newton-Schrödinger equations (NSEs) as the conceptual framework and technical platform for understanding the interaction of quantum matter with classical gravity at today's low energy (compared to Planck energy) and to compare their laboratory results, but NSE cannot be derived from General Relativity plus Quantum Field Theory (QFT) [3]. It is nowadays thought that gravity cannot be described by a local QFT of point particles and that a theory of Quantum Gravity must be fundamentally non-local [4]. But in the present article, we have shown that gravity not only can be derived by a local QFT of elementary particles quite naturally but it also preserves quantum phenomena along with a set of two separate (quantum) gravitational field equations, one for bosons and/or conjugated fermions and another for free fermions.

On a mathematical level the quantization of gravity, i.e. the description of spacetime by quantum states, leads to physically inconsistent results. Quantum mechanics is a linear non-deterministic theory, whereas General Relativity is a non-linear deterministic one. The problems may be resolved within a new framework of quantum gravity, where one should point out that the obtainable phenomena need not necessarily have to do with either quantum or gravity theory [5]. This is the main purpose of the

present article, which is based on identifying a correspondence between General (Quantum) Relativity and QFT. The aspect of the work highlights herein how, for a valid theory of Quantum Gravity, the quantization of the field can originate from a General Quantum Gravity framework rather than directly from Einsteinian General Relativity.

It is technically impossible to be accurate to combine constant velocity with accelerating velocity in a common framework. So, we mostly omit both Quantum Mechanical and General Relativistic computations in this article and simply follow a different approach to Quantum Mechanics by focusing on various forms of metric line elements (we should call such an approach as ‘Metric Mechanics’) and introducing a form of ‘Quantum Differential Geometry’.

A common practice to demonstrate Quantum Mechanics in curved spacetime is forcing QFT, which is developed from Special Relativity, to behave according to General Relativity (as we know, for example [6], or some references used in this article, or any other works on QFT in curved spacetime). In this article, we shall develop a completely opposite perspective to force General (Quantum) Relativity to behave according to QFT in $(3 + 1)D$ curvilinear quantum spacetime. In this way, we shall be able to develop quantum gravity without considering the Laplace-Beltrami operator defined on the background spacetime with metric $g_{\mu\nu}$, i.e., without considering a generalization of the d’Alambertian wave operator in Minkowski space to curved spacetime, for example.

Similarly, to construct QFT in this article, we shall not go to consider the orthodox Quantum Relativistic Klein-Gordon equation, which is developed from Special Relativity, as a contribution from the spin connection (as we know, for example [7]). On the other hand, in order to write Dirac equation in general relativity, we shall not use here the spinorial covariant derivative with the spinor affine connection coefficients for spin- $\frac{1}{2}$ particles (as we know, for example [7,8]). Instead, in this article, a Kline-Gordon-like equation and a Dirac-like equation in QFT, which are themselves actually nothing but the quantum gravitational field equation (analogous to Einstein’s field equation in General Relativity) for bosons and fermions, respectively, are originated quite naturally from ‘Metric Mechanics’ in $(3 + 1)D$ curvilinear quantum spacetime.

The structure of this article is as follows:

- 1. Part I: Simple Quantum Gravity & Nature of Dynamic Time,
- 2. Part II: General Quantum Gravity,
- 3. Part III: Discussion and Conclusion.

In overall scenario,

- 1. By introducing a form of ‘Quantum Differential Geometry’, we shall construct a universal quantum gravitational field equation, which is quite analogous to Einstein’s gravitational field equation.
- 2. Unlike orthodox Kline-Gordon equation or Dirac equation, which are developed from Special Relativity, all of the (unorthodox) equations in this article are technically in accelerating frames. We shall develop two sets of separate gravitational field equations, one set for bosons and/or conjugated fermions (Kline-Gordon-like equation) and another set for free fermions (Dirac-like equation) by using ‘Metric Mechanics’. Both of these equations satisfy the local non-vacuum quantum gravitational field equations in QFT, we should call it as Generalized Quantum Gravitational Field Theory.
- 3. We shall develop Gravito-weak symmetry group $SU(2)_L \oplus iSU(2)_G$, so as it can be applied to unify Standard Model with gravity and Dark Energy (for inconstant cosmological constant) in $(3 + 1)D$ curvilinear quantum spacetime as follows,

$$SU(5) = SU(3)_C \times (SU(2)_L \oplus iSU(2)_G),$$

which we call as the ‘General Unified Theory’.

- 4. Additionally, we shall show that the pairs of unbounded operators, such as,
 - A) the (quantum) relativistic mass m of a particle and time t ,

- B) i) the quantum scalar curvature $\sqrt{\mathcal{P}}$ for a fermion and the proper time τ ,
 ii) the quantum scalar curvature $\sqrt{\mathcal{P}}$ for a boson and the proper time τ ,
 C) i) the (quantum) relativistic mass m of a fermion and its inversely stretched/shrank $(3+1)D$ spacetime $(\Phi_\mu^F)^{-1} \mathcal{V}_{\mu\nu} dx^\nu$ due to stretching/shrinking parameters Φ_μ^F ,
 ii) the (quantum) relativistic mass m of a boson and its inversely stretched/shrank $(3+1)D$ spacetime $(\Phi_\mu^B)^{-1} \mathcal{V}_{\mu\nu} dx^\nu$ due to stretching/shrinking parameters Φ_μ^B ,

all are satisfying their Uncertainty Principles independently as follows:

- A') $(\Delta_\psi m)(\Delta_\psi t) \geq \frac{\hbar}{2}$,
 B') i) $\frac{1}{\sqrt{3}} (\Delta_\psi \sqrt{\mathcal{P}})(\Delta_\psi \tau) \geq -\mathcal{V}^{\mu\nu}$ for the (quantum) metric tensor $\mathcal{V}^{\mu\nu}$,
 ii) $\frac{1}{\sqrt{3}} (\Delta_\psi \sqrt{\mathcal{P}})(\Delta_\psi \tau) \geq \mathcal{V}^{\mu\nu}$ for the (quantum) metric tensor $\mathcal{V}^{\mu\nu}$, respectively,
 C') i) $(\Phi_\mu^F)^{-1} (\Delta_\psi m)(\Delta_\psi \mathcal{V}_{\mu\nu} dx^\nu) \geq \frac{\hbar}{2}$,
 ii) $(\Phi_\mu^B)^{-1} (\Delta_\psi m)(\Delta_\psi \mathcal{V}_{\mu\nu} dx^\nu) \geq \frac{\hbar}{2}$, respectively,

i.e., in pairs, they cannot have definite and constant values at the same instance. Additionally, the rootages of the Uncertainty Principles in C' immediately establish the direct and exclusively straightforward relations between the Einsteinian mass-energy relationship and Quantum Mechanics as,

- $E\psi \doteq mc^2\psi = i\hbar\Phi_\mu^F \mathcal{V}^{\mu\nu} \vec{\nabla}_\nu \psi$ for its $(3+1)D$ stretched/shrank spacetime due to stretching/shrinking parameters Φ_μ^F for a fermion,
- $E\psi \doteq mc^2\psi = i\hbar\Phi_\mu^B \mathcal{V}^{\mu\nu} \vec{\nabla}_\nu \psi$ for its $(3+1)D$ stretched/shrank spacetime due to stretching/shrinking parameters Φ_μ^B for a boson.

5. And finally, we shall prove that the General Quantum Gravity is actually 'multiplicatively renormalizable'.

Definitely, in this article, we are going to develop a different kind of Quantum Mechanics in $(3+1)D$ curvilinear quantum spacetime influenced by quantized gravity with different scope and nature of reality, where the dynamic temporal axis $(i\hbar)^{-1}x^0 = (i\hbar)^{-1}(ct)$ plays a significant role to achieve a scenario of quantum gravity. We shall see that General Quantum Gravity seems to defy partially both the concepts of Quantum Mechanics and General Relativity simultaneously by describing gravity as *the curvature of quantized spacetime caused by quantized mass and energy under the influence of quantized gravity*.

PART I: Simple Quantum Gravity & Nature of Dynamic Time

2. Simple Quantum Gravity

Let a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$ is associated with any quantum system. Let de Broglie's matter wave be described by a complex-valued wave function as,

$$\Psi(\vec{r}, t) = \frac{1}{(\sqrt{2\pi\hbar})^3} \int a(\hat{p}) \exp\left[\frac{i}{\hbar}(\hat{p} \cdot \vec{r} - Et)\right] d\hat{p}, \quad (1)$$

where time t is non-dynamic, homogeneous, isotropic and mathematically rigid, unique external background parameter to define how the probabilities of possible values for quantum observables change over time in a $3D$ quantum space.

Theorem 1 (Existence of $3D$ Schrödinger Equation with Total Energy $\hat{E}$ while Time t is Non-dynamic). *There always exists a $3D$ Schrödinger equation which represents the total energy $\hat{E}$.*

Proof. We know that momentum $\hat{\mathbf{p}}$ and angular frequency ω are related through the law of dispersion, that is [9],

$$\omega = \frac{\hat{E}}{\hbar} = \frac{\hat{\mathbf{p}}^2}{2\hbar m}, \quad (2)$$

which yields after integrating as,

$$2\hat{E}t - \hat{\mathbf{p}} \cdot \vec{r} = 0, \quad (3)$$

then for the mass-energy relationship from [10],

$$i\hbar\partial_t \rightarrow \hat{E} = \hbar\omega \equiv \frac{1}{2}mc^2 \doteq \frac{1}{2}E, \quad (4)$$

a suitable plane wave would be for total energy $\hat{E} = 2\hat{E}$ as,

$$\exp\left[\frac{i}{\hbar}(\hat{\mathbf{p}} \cdot \vec{r} - 2\hat{E}t)\right], \quad \text{i.e.,} \quad \exp\left[\frac{i}{\hbar}(\hat{\mathbf{p}} \cdot \vec{r} - \hat{E}t)\right]. \quad (5)$$

Technically, a wave packet, that satisfies $\exp\left[\frac{i}{\hbar}(\hat{\mathbf{p}} \cdot \vec{r} - \hat{E}t)\right]$, have to satisfy $\exp\left[\frac{i}{\hbar}(\hat{\mathbf{p}} \cdot \vec{r} - 2\hat{E}t)\right]$, too, for (2). By superposition of a set of such waves, a wave packet can be constructed as,

$$\Psi(\vec{r}, t) = \frac{1}{(\sqrt{2\pi\hbar})^3} \int a(\hat{\mathbf{p}}) \exp\left[\frac{i}{\hbar}(\hat{\mathbf{p}} \cdot \vec{r} - \hat{E}t)\right] d\hat{\mathbf{p}},$$

resulting the quantum operators as,

$$\hat{E} \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \text{and} \quad \hat{\mathbf{p}} \rightarrow -i\hbar \frac{\partial}{\partial \vec{r}}, \quad (6)$$

whence the expression,

$$\frac{1}{(\sqrt{2\pi\hbar})^3} \int \left(\frac{\hat{\mathbf{p}}^2}{m} - \hat{E}\right) a(\hat{\mathbf{p}}) \exp\left[\frac{i}{\hbar}(\hat{\mathbf{p}} \cdot \vec{r} - \hat{E}t)\right] d\hat{\mathbf{p}},$$

vanishes for $\hat{E} = (\hat{\mathbf{p}}^2/m)$, thus for (6), it satisfies,

$$-\frac{\hbar^2}{m} \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = 0 \quad \forall i, j \in \{1, 2, 3\}, \quad (7)$$

which is a three-dimensional Schrödinger equation representing total energy $\hat{E}$, while time t is non-dynamic by the definition. This completes the proof. $\square$ $\square$

Proposition 1 (Position-momentum Relation for Non-dynamic Time t [11]). Suppose $|\psi(t)\rangle$ is a solution to the Schrödinger equation (7) for a sufficiently nice potential $V(x, y, z)$ and for a sufficiently nice initial condition $|\psi(0)\rangle = |\psi_0\rangle$ when time t is non-dynamic. Then the expected position and expected three-momentum in the state $|\psi(t)\rangle$ satisfy,

$$\frac{d}{dt} \langle x \rangle_{|\psi(t)\rangle} = \frac{1}{m} \langle \hat{\mathbf{p}} \rangle_{|\psi(t)\rangle}. \quad (8)$$

Theorem 2. Three-dimensional Schrödinger equation (7) representing total energy $\hat{E}$ is achievable from a relation,

$$\hat{E}m - \hat{\mathbf{p}}^i \cdot \hat{\mathbf{p}}^j = 0,$$

while time t is non-dynamic.

Proof. For the kinetic energy,

$$\hat{E} = \frac{\hat{\mathbf{p}}^i \cdot \hat{\mathbf{p}}^j}{2m} \quad \Rightarrow \quad \hat{\mathbf{p}}^i \cdot \hat{\mathbf{p}}^j = 2 \hat{E} m \doteq E m \quad [\text{for (4)}], \quad (9)$$

thus, for (3) along with (8), suppose a suitable plane wave for (9) is as follows,

$$\begin{aligned} \exp \left[\frac{i}{\hbar} m \frac{d}{dt} (\hat{\mathbf{p}}^i \cdot x^j - \hat{E} t) \right] &= \exp \left[\frac{i}{\hbar} \hat{\mathbf{p}}^i \cdot \hat{\mathbf{p}}^j \right] \cdot \exp \left[-\frac{i}{\hbar} \hat{E} m \right] \\ &= \exp \left[\frac{i}{\hbar} \hat{\mathbf{p}}^i \cdot \hat{\mathbf{p}}^j \right] \cdot \exp \left[-\frac{i}{\hbar} \hat{\mathbf{p}}^i \cdot \hat{\mathbf{p}}^j \right] \\ &= 1. \end{aligned}$$

By superposition of a set of such waves, a wave packet can be constructed as,

$$\Psi(\vec{r}, t) = \frac{1}{(\sqrt{2\pi\hbar})^3} \int a(\hat{\mathbf{p}}) \exp \left[\frac{i}{\hbar} m \frac{d}{dt} (\hat{\mathbf{p}}^i \cdot x^j - \hat{E} t) \right] d\hat{\mathbf{p}}. \quad (10)$$

The partial derivatives occurring in (10) are,

$$\frac{\partial \Psi(\vec{r}, t)}{\partial t} = \frac{1}{(\sqrt{2\pi\hbar})^3} \int \frac{-i m d \hat{E}}{\hbar dt} a(\hat{\mathbf{p}}) \exp \left[\frac{i}{\hbar} m \frac{d}{dt} (\hat{\mathbf{p}}^i \cdot x^j - \hat{E} t) \right] d\hat{\mathbf{p}},$$

and,

$$\frac{\partial \Psi(\vec{r}, t)}{\partial x^j} = \frac{1}{(\sqrt{2\pi\hbar})^3} \int \frac{i m d \hat{\mathbf{p}}^i}{\hbar dt} a(\hat{\mathbf{p}}) \exp \left[\frac{i}{\hbar} m \frac{d}{dt} (\hat{\mathbf{p}}^i \cdot x^j - \hat{E} t) \right] d\hat{\mathbf{p}},$$

resulting the quantum operators are,

$$m \frac{d \hat{E}}{dt} \rightarrow i \hbar \frac{\partial}{\partial t}, \quad \text{and} \quad m \frac{d \hat{\mathbf{p}}^i}{dt} \rightarrow -i \hbar \frac{\partial}{\partial x^j}, \quad (11)$$

whence the expression,

$$\frac{1}{(\sqrt{2\pi\hbar})^3} \int \left(\frac{\hat{\mathbf{p}}^i \cdot \hat{\mathbf{p}}^j}{m} - E \right) a(\hat{\mathbf{p}}) \exp \left[\frac{i}{\hbar} m \frac{d}{dt} (\hat{\mathbf{p}}^i \cdot x^j - \hat{E} t) \right] d\hat{\mathbf{p}},$$

vanishes, and for using (11), it satisfies,

$$-\frac{\hbar^2}{m} \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = 0 \quad \forall i, j \in \{1, 2, 3\},$$

which is nothing but (7). This completes the proof. $\square$ $\square$

Remark 1 (Target of This Article). To combine gravity and Quantum Mechanics, we require a new understanding of time [12]. On the other hand, the total energy E in (4) is actually related to a $(3+1)D$ spacetime. So, we have to try to develop a $(3+1)D$ Schrödinger equation representing total energy $\hat{E}$ while time t is dynamic in a $(3+1)D$ curvilinear quantum spacetime by commencing the following.

Definition 1 (Dynamic Time t). Let us consider that time t is hereafter a variable, observer-dependent and a dynamic dimension to merge with 3D non-curvilinear quantum space to generate a general $(3+1)D$ curvilinear quantum spacetime coordinate so as the whole spacetime can curve.

Definition 2 ($\mathcal{W}^2$ Function). For the quantum energy operator $c^{-1} \hat{E} \rightarrow i \hbar (\partial / \partial (c t)) \equiv i \hbar \partial_0$ and three-momentum operator $\hat{\mathbf{p}}_i \rightarrow -i \hbar (\partial / \partial x^i) \equiv -i \hbar \partial_i \forall i \in \{1, 2, 3\}$ in Quantum Mechanics, let us consider a function as,

$$\begin{aligned} \mathcal{W}^2 \psi &= \frac{\hat{E}^2}{c^2} \psi - \sum_{i,j=1}^3 \hat{\mathbf{p}}_i \cdot \hat{\mathbf{p}}_j \psi \\ &= (i \hbar)^2 \frac{\partial^2}{\partial (c t)^2} \psi - \sum_{i,j=1}^3 (i \hbar)^2 \frac{\partial^2}{\partial x^i \partial x^j} \psi, \end{aligned} \quad (12)$$

where $\mathcal{W}$ may be the rest mass or the invariant mass of the system of particles, or equal to the mass of the decay particle, etc.

Remark 2 (Spacetime & Objective of This Article). Since (12) has an analogy to a $(3+1)D$ curvilinear (Minkowski-like) spacetime structure, then instead of Minkowskian spacetime $(c t, x^i) \forall i \in \{1, 2, 3\}$, by using (12) with little bravery, we may consider the quantum energy operator $c^{-1} \hat{E} \rightarrow i \hbar (\partial / \partial (c t)) \equiv i \hbar \partial_0$ and three-momentum operator $\hat{\mathbf{p}}_i \rightarrow -i \hbar (\partial / \partial x^i) \equiv -i \hbar \partial_i \forall i \in \{1, 2, 3\}$ as the contravariant ‘temporal’ and ‘spatial’ coordinates, respectively, of a hypothetical $(3+1)D$ curvilinear quantum spacetime. In addition, since $\mathcal{W}$ in (12) would be the rest mass or the invariant mass of the system of particles, or equal to the mass of the decay particle, etc., this so-called hypothetical $(3+1)D$ curvilinear quantum spacetime is **not** a perfect geometrically relevant $(3+1)D$ curvilinear spacetime similar to what we usually use to construct Einsteinian Relativity. So, from the very beginning of this article, we shall follow the term $\mathcal{W}$ as the Quantum Mechanical ‘footprint’ of a so-called ‘spacetime’ structure to achieve a General (Quantum) Relativistic approach to (quantum) gravitational field equation without considering actual Einsteinian General Relativistic formalism.

Definition 3 ((Quantum) Quadratic Form). Suppose a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^4)$ is associated with any quantum system while time t is dynamic in a hypothetical $(3+1)D$ curvilinear quantum spacetime for $\mathcal{W}$ in (12). Let us postulate that the existence of a single spacetime M , on which all events occur, with no particular coordinates attached to it. Let us consider a finite-dimensional complex vector space $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$. Consider the ‘four-momentum’ operator $\hat{\mathcal{P}}_\mu$ on $L^2(\mathbb{R}^4)$ for some state $\psi \in \mathcal{H} = L^2(\mathbb{R}^4)$, when,

$$\hat{\mathcal{P}}_\mu \rightarrow i \hbar \vec{\nabla}_\mu = \left(i \hbar \frac{\partial}{\partial (c t)}, -i \hbar \frac{\partial}{\partial x^i} \Big|_{i=1}^3 \right) = \left(c^{-1} \hat{E}, \hat{\mathbf{p}}_i \Big|_{i=1}^3 \right), \quad (13)$$

for $\mu \in \{0, 1, 2, 3\}$ and $i \in \{1, 2, 3\}$, with the energy operator $\hat{E} \rightarrow i \hbar \partial_t$ and the three-momentum operator $\hat{\mathbf{p}}_i \rightarrow -i \hbar \partial_i$. For a diffeomorphism $\underline{\mathbb{X}} : M \rightarrow L^2(\mathbb{R}^4)$, suppose an event at a point $m \in M$, is occurred at $\underline{\mathbb{X}}(m)$. Similarly, for another diffeomorphism $\Psi : M \rightarrow L^2(\mathbb{R}^4)$, the same event is considered to be occurred at $\Psi(m)$. From inertial reference frames, the changing coordinates is then given by the map $F : \Psi \circ \underline{\mathbb{X}}^{-1} : L^2(\mathbb{R}^4) \rightarrow L^2(\mathbb{R}^4)$. Let $Q : L^2(\mathbb{R}^4) \rightarrow L^2(\mathbb{R})$ be the (quantum) quadratic form defined by,

$$\begin{aligned} Q(i \hbar \partial_0, -i \hbar \partial_i) \psi &= (c^{-1} \hat{E} + \hat{\mathbf{p}}_i) \cdot (c^{-1} \hat{E} - \hat{\mathbf{p}}_i) \psi \\ &\equiv \left(i \hbar \frac{\partial}{\partial (c t)} - \sum_{i=1}^3 i \hbar \frac{\partial}{\partial x^i} \right) \cdot \left(i \hbar \frac{\partial}{\partial (c t)} + \sum_{j=1}^3 i \hbar \frac{\partial}{\partial x^j} \right) \psi \\ &= \left((i \hbar)^2 \frac{\partial^2}{\partial (c t)^2} - \sum_{i,j=1}^3 (i \hbar)^2 \frac{\partial^2}{\partial x^i \partial x^j} \right) \psi \doteq \mathcal{W}^2 \psi \end{aligned} \quad \text{[for (12)]}, \quad (14)$$

so that it can satisfy the following orthogonality conditions,

$$\begin{aligned} 1. & \quad (i\hbar)^2 \partial_0 \cdot \sum_{i=1}^3 \partial_i = 0, \\ 2. & \quad (i\hbar)^2 \sum_{i,j=1}^3 \partial_i \cdot \partial_j = 0 \quad \text{if } i \neq j, \\ 3. & \quad (i\hbar)^2 \sum_{i,j=1}^3 \partial_i \cdot \partial_j \neq 0 \quad \text{if } i = j, \end{aligned} \quad (15)$$

while,

$$\left\{ \left(i\hbar \frac{\partial}{\partial(ct)} - \sum_{i=1}^3 i\hbar \frac{\partial}{\partial x^i} \right), \left(i\hbar \frac{\partial}{\partial(ct)} + \sum_{j=1}^3 i\hbar \frac{\partial}{\partial x^j} \right) \right\} \in \mathbb{C}. \quad (16)$$

The last term of (14) becomes zero if their states are not entangled.

Definition 4 ((Quantum) Metric Tensor). Let a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^4)$ is associated with any quantum system. Let us consider a finite dimensional complex vector space $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^4)$. Suppose $(M^4, \mathcal{A})$ is a smooth manifold, that is, M^4 is an four-dimensional differentiable manifold and $\mathcal{A}$ is a (quantum) metric tensor, then,

$$\begin{aligned} \mathcal{A}_{\mu\nu} &= \mathcal{A}_{\alpha\beta} \left(\frac{i\hbar \vec{\nabla}_\mu}{i\hbar \vec{\nabla}_\alpha} \frac{i\hbar \vec{\nabla}_\nu}{i\hbar \vec{\nabla}_\beta} \right) = \mathcal{A}_{\alpha\beta} \left(\frac{\hat{\mathcal{P}}_\mu}{\hat{\mathcal{P}}_\alpha} \frac{\hat{\mathcal{P}}_\nu}{\hat{\mathcal{P}}_\beta} \right) \quad [\text{for (13)}] \\ &= \text{diag}[\mathcal{A}_{00}, \mathcal{A}_{ij}] = \text{diag}[1, -1, -1, -1], \end{aligned} \quad (17)$$

so as,

$$\mathcal{A}^{\mu\nu} \mathcal{A}_{\mu\nu} = \delta_\nu^\nu = 4 \quad [\text{for } (3+1)D \text{ spacetime}], \quad (18)$$

but $\mathcal{A}_{\mu\nu}$ is not a four-dimensional Lorentz metric $g_{\mu\nu} = \text{diag}[1, -1, -1, -1]$ since $\mathcal{A}_{\mu\nu}$ is in Hilbert space $\mathcal{H} = L^2(\mathbb{R}^4)$. Here, the metric tensor $\mathcal{A}_{\mu\nu}$ is symmetric, i.e., $\mathcal{A}_{\mu\nu} = \mathcal{A}_{\nu\mu}$, and $\det(\mathcal{A}_{\mu\nu}) \neq 0$. Components of its inverse matrix $\mathcal{A}^{-1}$ are themselves the components of matrix $\mathcal{A}$, namely, $\mathcal{A}_{\mu\nu} \mathcal{A}^{\mu\gamma} = \mathcal{A}^{\gamma\mu} \mathcal{A}_{\mu\nu} = \delta_\nu^\gamma$, where δ_ν^γ is the Kronecker delta.

Definition 5 ((Quantum) Line Elements). For the bilinear form $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ in (14), let us define a freely-falling inertial frame (an orthonormal line operator) in the neighborhood of a moving body in such a way that (using the summation convention),

$$\begin{aligned} Q(i\hbar \partial_0, -i\hbar \partial_i) \psi &= \mathcal{W}^2 \psi \quad [\text{for (14)}] \\ &\doteq \frac{\partial^2}{\partial \hat{s}^2} \psi \equiv (i\hbar)^2 \left(\frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^i \partial x^j} \right) \psi \end{aligned} \quad (19)$$

$$\doteq (i\hbar)^2 \mathcal{A}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi \equiv \mathcal{A}^{\mu\nu} \hat{\mathcal{P}}_\mu \hat{\mathcal{P}}_\nu \psi, \quad (20)$$

for $\mu = \nu$, where $\mathcal{A}^{\mu\nu}$ is the (quantum) metric tensor. Since $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^4)$, then (20) may yield a dual base $(3+1)D$ curvilinear quantum coordinate system $((i\hbar)^{-1}x^0, \dots, -(i\hbar)^{-1}x^3)$ for a point $p \in \mathcal{V}$, when,

$$(i\hbar)^{-1}x^0 = (i\hbar)^{-1}(ct), \quad (21)$$

so as it can get a line element $d\hat{s}^2$ in $\mathcal{V}$ as,

$$\begin{aligned} d\hat{s}^2 \Psi &= \mathcal{A}_{\mu\nu} d[(i\hbar)^{-1}x^\mu] d[(i\hbar)^{-1}x^\nu] \Psi \\ &= (i\hbar)^{-2} \mathcal{A}_{\mu\nu} dx^\mu dx^\nu \Psi. \end{aligned} \quad (22)$$

Proposition 2 (Position-momentum Relation for Dynamic Time t). Suppose $|\psi(t)\rangle$ is a solution to (20) in $(3+1)D$ curvilinear quantum spacetime for a sufficiently nice potential $V(\vec{r}, t)$ and for a sufficiently nice initial condition $|\psi(0)\rangle = |\psi_0\rangle$ when time t is dynamic. Then the expected position and expected four-momentum in the state $|\psi(t)\rangle$ satisfy,

$$\frac{d}{dt} \langle x \rangle_{|\psi(t)\rangle} = \frac{1}{m} \langle \hat{p} \rangle_{|\psi(t)\rangle}. \quad (23)$$

Since the mass-energy relationship (4) and since position and momentum satisfy (23) in the state $\Psi(\vec{r}, t)$, then we can yield (22) as,

$$\begin{aligned} (i\hbar)^2 m^2 \left(\frac{d\hat{s}}{dt} \right)^2 \Psi &= m^2 \mathcal{A}_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \Psi \\ &= m^2 \left\{ \frac{d(ct)}{dt} \right\}^2 \Psi - m^2 \sum_{i,j=1}^3 \frac{dx^i}{dt} \frac{dx^j}{dt} \Psi \\ &\quad \forall i, j \in \{1, 2, 3\} \end{aligned} \quad (24)$$

$$\begin{aligned} &= m^2 c^2 \Psi - \sum_{i,j=1}^3 (m v^i) (m v^j) \Psi \\ &\div 4 \frac{\hat{E}^2}{c^2} \Psi - \hat{p}^i \cdot \hat{p}^j \Psi \quad [\text{for (4) and (23)}] \end{aligned} \quad (25)$$

$$\text{or alternatively,} \quad \div m \frac{d}{dt} (\hat{E} t - \hat{p}^i \cdot x^j) \Psi \quad [\text{for (4)}], \quad (26)$$

where,

$$m = \eta \bar{m}, \quad (27)$$

is the mass by setting a phase factor η , such as $|\eta| = 1$, and $v^i = (dx^i/dt)$ is the ordinary three-dimensional velocity vector.

Note 1 Technically, any mechanics that satisfies (26) in $(3+1)D$ curvilinear quantum spacetime would not be the orthodox Quantum Mechanics, not even Theorem 2, due to dynamic time t in a hypothetical $(3+1)D$ curvilinear quantum spacetime. Hence, the mechanics we are going to study in this article would lead an unorthodox Quantum Mechanics, which unmistakably stand apart from its $3D$ orthodox version. In other words, $(3+1)D$ curvilinear quantum spacetime is locally orthodox Quantum Mechanical in $3D$ space for its global $(3+1)D$ curvilinear spacetime covering, i.e., it holds $3D \subset (3+1)D$ in quantum spacetime.

Let us demonstrate the non-applicability of Einsteinian energy-momentum relationship in the present article as follows.

Definition 6 (Energy-momentum Relation for Massive and Massless Particles). Comparing (24) and (25), we can yield the (quantum) relativistic relation for $i\hbar \partial_t \rightarrow \hat{E} = \hbar \omega \equiv \frac{1}{2} m c^2 \div \frac{1}{2} E$ from [10] as,

$$\begin{aligned} E^2 \Psi &= m^2 c^2 \mathcal{A}_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \Psi + c^2 \hat{p}^i \cdot \hat{p}^j \Psi \\ &= c^2 \mathcal{A}_{\mu\nu} \hat{p}^\mu \cdot \hat{p}^\nu \Psi + c^2 \hat{p}^i \cdot \hat{p}^j \Psi \quad [\text{for (23)}], \end{aligned} \quad (28)$$

which is neither equivalent to and nor deductive from the Einsteinian energy-momentum equation, $E^2 = m_0^2 c^4 + c^2 p^2$. (This is the fundamental reason why we shall not follow Einsteinian energy-momentum relationship in this article.) Again,

1. For the first term in the RHS of (28), the given object follows a curved path in $(3+1)D$ spacetime, its direction is constantly changing, which means it is accelerating, even if its speed remains constant.

2. We shall also consider in the Appendix A below that if a given object satisfies (22), so as satisfies (28) naturally, then its spacetime must satisfy a proper time, for example $\tau = (i\hbar)^{-1} \gamma_B^{-1} t$ in (A13) for a boson, or $\tau = (i\hbar)^{-1} \gamma_F^{-1} t$ in (A25) for a fermion, thus the curvature of spacetime also affects the flow of time.

Hence, the given object in (28) experiences time dilation, where its temporal velocity slows down relative to a distant observer, and its spatial velocity must increase to maintain a constant 'four-velocity' magnitude. Thus, the (quantum) relativistic equation (28) is definitely in an accelerating vector spacetime of a 'gravitating' body moving at accelerating velocities by the definition.

Theorem 3 (Existence of $(3+1)D$ Schrödinger Equation with Total Energy $\hat{E}$ while Time t is Dynamic). Prove that the total energy E of (28) yields a Schrödinger equation in a $(3+1)D$ curvilinear quantum spacetime as,

$$-\frac{\hbar^2}{2m} \square \psi + \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = 0,$$

where the d'Alembertian operator $\square = \mathcal{A}\psi^{\mu\nu} \partial_\mu \partial_\nu \forall \mu, \nu \in \{0, 1, 2, 3\}$, while time t is dynamic.

Proof. For a free relativistic particle we have from (28) as,

$$\begin{aligned} \frac{dE}{d\hat{p}^\nu} &= \frac{c^2 \mathcal{A}\psi_{\mu\nu} \hat{p}^\mu}{E} = \frac{\mathcal{A}\psi_{\mu\nu} \hat{p}^\mu}{m} \quad [\text{for } E = mc^2] \\ &= \mathcal{A}\psi_{\mu\nu} v^\mu \quad [\text{for (23)}], \end{aligned} \quad (29)$$

where $v^\mu = (dx^\mu/dt)$ is the four-dimensional velocity vector, lead to the group velocity v_g^μ of the associated wave packet as,

$$v_g^\mu = \frac{dE}{d\hat{p}^\nu} = \mathcal{A}\psi_{\mu\nu} v^\mu.$$

Integrating the first line of (29), if $\mu = \nu$ by the orthogonality condition of (15), we get,

$$E = \frac{\mathcal{A}\psi_{\mu\nu} \hat{p}^\mu \cdot \hat{p}^\nu}{2m}. \quad (30)$$

A relativistic free particle (let us follow [9], for example) of energy E is associated with a wave of angular frequency $\omega = E/\hbar$ and wavelength $\lambda = 2\pi\hbar/\hat{p}^\mu$. Hence, the propagation vector for the wave is $\mathbf{k} = \hat{p}^\mu/\hbar$ and of magnitude $k = 2\pi/\lambda$. Momentum and angular frequency are related through the law of dispersion, that is,

$$\omega = \frac{E}{\hbar} = \frac{\mathcal{A}\psi_{\mu\nu} \hat{p}^\mu \cdot \hat{p}^\nu}{2\hbar m}.$$

The phase velocity is,

$$v_{ph}^\mu = \frac{\omega}{k} = \frac{E}{2\hat{p}^\nu} = \frac{\mathcal{A}\psi_{\mu\nu} \hat{p}^\mu}{2m} = \frac{v_g^\mu}{2}. \quad (31)$$

A suitable plane wave is,

$$\exp\left[\frac{i}{\hbar} (\mathcal{A}\psi_{\mu\nu} \hat{p}^\mu \cdot x^\nu - \hat{E} t)\right]. \quad (32)$$

By superposition of a set of such waves, a four-dimensional wave packet can be constructed as,

$$\Psi(\vec{r}, t) = \frac{1}{(\sqrt{2\pi\hbar})^4} \int a(\hat{\mathbf{p}}) \exp\left[\frac{i}{\hbar}(\mathcal{V}_{\mu\nu} \hat{\mathbf{p}}^\mu \cdot x^\nu - \hat{E}t)\right] d\hat{\mathbf{p}}. \quad (33)$$

The partial derivatives occurring in (33) are,

$$\frac{\partial \Psi(\vec{r}, t)}{\partial t} = \frac{1}{(\sqrt{2\pi\hbar})^4} \int \frac{-i\hat{E}}{\hbar} a(\hat{\mathbf{p}}) \exp\left[\frac{i}{\hbar}(\mathcal{V}_{\mu\nu} \hat{\mathbf{p}}^\mu \cdot x^\nu - \hat{E}t)\right] d\hat{\mathbf{p}},$$

and,

$$\frac{\partial \Psi(\vec{r}, t)}{\partial x^\nu} = \frac{1}{(\sqrt{2\pi\hbar})^4} \int \frac{i\mathcal{V}_{\mu\nu} \hat{\mathbf{p}}^\mu}{\hbar} a(\hat{\mathbf{p}}) \exp\left[\frac{i}{\hbar}(\mathcal{V}_{\mu\nu} \hat{\mathbf{p}}^\mu \cdot x^\nu - \hat{E}t)\right] d\hat{\mathbf{p}},$$

resulting the quantum operators as,

$$\hat{E} \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \text{and} \quad \mathcal{V}_{\mu\nu} \hat{\mathbf{p}}^\mu \rightarrow -i\hbar \frac{\partial}{\partial x^\nu}, \quad (34)$$

$$\text{thus,} \quad \hat{\mathbf{p}}^\mu \rightarrow -i\hbar \frac{\partial}{\partial \mathcal{V}_{\mu\nu} \partial x^\nu}, \quad (35)$$

$$\text{or,} \quad \mathcal{V}_{\mu\nu} \rightarrow -i\hbar \frac{\partial}{\partial \hat{\mathbf{p}}^\mu} \cdot \frac{\partial}{\partial x^\nu}, \quad (36)$$

whence the expression,

$$\frac{1}{(\sqrt{2\pi\hbar})^4} \int \left(\frac{\mathcal{V}_{\mu\nu} \hat{\mathbf{p}}^\mu \cdot \hat{\mathbf{p}}^\nu}{2m} - \hat{E} \right) a(\hat{\mathbf{p}}) \exp\left[\frac{i}{\hbar}(\mathcal{V}_{\mu\nu} \hat{\mathbf{p}}^\mu \cdot x^\nu - \hat{E}t)\right] d\hat{\mathbf{p}},$$

satisfies a Schrödinger equation representing total energy in a $(3+1)D$ curvilinear quantum spacetime for (30) as,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial \mathcal{V}_{\mu\nu} \partial x^\mu \partial x^\nu} + \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = 0 \quad [\text{for (35)}],$$

$$\text{that is,} \quad -\frac{\hbar^2}{2m} \square \psi + \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = 0, \quad (37)$$

where the d'Alembertian operator $\square = \mathcal{V}^{\mu\nu} \partial_\mu \partial_\nu \forall \mu, \nu \in \{0, 1, 2, 3\}$, while time t is dynamic. This completes the proof. $\square$ $\square$

Remark 3 (Orthodox Kline-Gordon Equation). We can check that by inputting $i\hbar \partial_t \rightarrow \hat{E} = \hbar\omega \equiv \frac{1}{2} m c^2 \div \frac{1}{2} E$ from [10], the $(3+1)D$ Schrödinger equation (37) naturally and straightforwardly yields the orthodox Kline-Gordon equation $(\square + \hbar^{-2} m^2 c^2) \psi = 0$ in a $(3+1)D$ curvilinear quantum spacetime. Thus, (37) is actually a relativistic Schrödinger equation. This strongly establishes that **the present article is nothing but a different perspective of the fundamental Quantum Field Theory**.

Testable Prediction 1. Non-relativistic 3D Schrödinger equation masks its covering relativistic Schrödinger equation in a $(3+1)D$ curvilinear quantum spacetime deluding with the similarities in their expressions.

Claim 1. Non-dynamic temporal axis t in non-relativistic 3D must behave as dynamic in $(3+1)D$ curvilinear quantum spacetime.

Theorem 4. If the mass m behaves as a quantum operator then it can yield the relativistic Schrödinger equation (37) with total energy $\hat{E}$ in a $(3+1)D$ curvilinear quantum spacetime, while time t is dynamic.

Proof. Multiplying both sides of (24) by $d(t^2)/dx^\nu$, we can get that,

$$\begin{aligned} (i\hbar)^2 m \frac{d(d\hat{s}^2)}{dx^\nu} \Psi &= m \frac{d(t^2)}{dx^\nu} \mathcal{U}_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \Psi \\ &\equiv m \mathcal{U}_{\mu\nu} x^\mu \Psi, \\ \text{or alternatively,} \quad &\doteq \mathcal{U}_{\mu\nu} \hat{p}^\mu t \Psi \quad [\text{for (23)}], \end{aligned} \quad (38)$$

hence, a suitable plane wave for the last two lines of (38) yields,

$$\exp \left[\frac{i}{\hbar} (\mathcal{U}_{\mu\nu} \hat{p}^\mu t - m \mathcal{U}_{\mu\nu} x^\mu) \right].$$

By superposition of a set of such waves, while mirroring (32), a four-dimensional wave packet can be constructed as,

$$\Psi(\vec{r}, t) = \frac{1}{(\sqrt{2\pi\hbar})^4} \int a(\hat{p}) v^\nu \exp \left[\frac{i}{\hbar} (\mathcal{U}_{\mu\nu} \hat{p}^\mu t - m \mathcal{U}_{\mu\nu} x^\mu) \right] d\hat{p}, \quad (39)$$

where $v^\nu = (dx^\nu/dt)$ is the four-dimensional velocity vector. The partial derivatives occurring in (39) are,

$$\begin{aligned} \frac{\partial \Psi(\vec{r}, t)}{\partial t} &= \frac{1}{(\sqrt{2\pi\hbar})^4} \int \frac{i \mathcal{U}_{\mu\nu} \hat{p}^\mu \cdot v^\nu}{\hbar} a(\hat{p}) \times \\ &\quad \times \exp \left[\frac{i}{\hbar} (\mathcal{U}_{\mu\nu} \hat{p}^\mu t - m \mathcal{U}_{\mu\nu} x^\mu) \right] d\hat{p}, \end{aligned}$$

and,

$$\begin{aligned} \frac{\partial \Psi(\vec{r}, t)}{\partial x^\mu} &= \frac{1}{(\sqrt{2\pi\hbar})^4} \int \frac{-i m \mathcal{U}_{\mu\nu} v^\nu}{\hbar} a(\hat{p}) \times \\ &\quad \times \exp \left[\frac{i}{\hbar} (\mathcal{U}_{\mu\nu} \hat{p}^\mu t - m \mathcal{U}_{\mu\nu} x^\mu) \right] d\hat{p}, \end{aligned}$$

resulting the quantum operators as,

$$\mathcal{U}_{\mu\nu} \hat{p}^\mu \cdot v^\nu \rightarrow -i\hbar \frac{\partial}{\partial t}, \quad \text{and} \quad m \mathcal{U}_{\mu\nu} v^\nu \rightarrow i\hbar \frac{\partial}{\partial x^\mu}, \quad (40)$$

$$\text{thus, } \hat{p}^\mu \rightarrow -i\hbar \frac{\partial}{\mathcal{U}_{\mu\nu} \partial x^\nu}, \quad m \rightarrow i\hbar \frac{\partial^2 t}{\mathcal{U}_{\mu\nu} \partial x^\mu \partial x^\nu}, \quad (41)$$

$$\text{or, } \mathcal{U}_{\mu\nu} \rightarrow -i\hbar \frac{\partial}{\partial \hat{p}^\mu} \cdot \frac{\partial}{\partial x^\nu}, \quad \mathcal{U}_{\mu\nu} \rightarrow \frac{i\hbar}{m} \frac{\partial^2 t}{\partial x^\mu \partial x^\nu}, \quad (42)$$

where the LHS relations of (41) and (42) are nothing but the relations of (35) and (36). In addition, since,

$$\mathcal{U}_{\mu\nu} \hat{p}^\mu \cdot v^\nu \doteq \frac{\mathcal{U}_{\mu\nu} \hat{p}^\mu \cdot \hat{p}^\nu}{m} \quad [\text{for (23)}],$$

hence, the expression for (30),

$$\frac{1}{(\sqrt{2\pi\hbar})^4} \int \left(\frac{\mathcal{U}_{\mu\nu} \hat{p}^\mu v^\nu}{2} - \hat{E} \right) a(\hat{p}) v^\nu \exp \left[\frac{i}{\hbar} (\mathcal{U}_{\mu\nu} \hat{p}^\mu t - m \mathcal{U}_{\mu\nu} x^\mu) \right] d\hat{p},$$

satisfies the same relativistic Schrödinger equation (37) with total energy $\hat{E}$ in a $(3+1)D$ curvilinear quantum spacetime due to (41) and (42), while time t is dynamic. This completes the proof. $\square$ $\square$

Definition 7 (Generalized (Quantum) Metric Operators in $(3 + 1)D$ Curvilinear Quantum Mechanics). From (42), we can write down the generalized (quantum) metric operator equation in quantum spacetime as,

$$\mathcal{V}_{\mu\nu} \rightarrow -\frac{\partial}{\partial \hat{\mathbf{p}}^\mu} \cdot \left(i\hbar \frac{\partial}{\partial x^\nu} \right) \doteq -\partial_{\hat{\mathbf{p}}_\mu} \cdot \hat{\mathbf{p}}_\nu \quad [\text{for (13)}], \quad (43)$$

$$\mathcal{V}_{\mu\nu} \rightarrow \left(i\hbar \frac{\partial}{\partial x^\mu} \right) \cdot \frac{1}{m} \frac{\partial t}{\partial x^\nu} \doteq \hat{\mathbf{p}}_\mu \cdot \partial_{\hat{\mathbf{p}}_\nu} \quad [\text{for (23)}]. \quad (44)$$

The form $(i\hbar \frac{\partial}{\partial x^\mu})$ in (44) is selected due to its primal presence in (40). Without any doubt, (43) and (44) may yield their contravariant metric tensors as,

$$\mathcal{V}^{\mu\nu} \rightarrow -\hat{\mathbf{p}}_\mu \cdot \partial_{\hat{\mathbf{p}}_\nu}, \quad (45)$$

$$\mathcal{V}^{\mu\nu} \rightarrow \partial_{\hat{\mathbf{p}}_\mu} \cdot \hat{\mathbf{p}}_\nu, \quad (46)$$

respectively.

From Theorem 4, or Theorem 3, it is clear that the generalized (quantum) metric operators (43) and (45), or (44) and (46), are providing us a definition of Quantum Differential Geometry in $(3 + 1)D$ curvilinear quantum spacetime. In addition, $\mathcal{V}_{\mu\nu}$ in (43) and (44), or $\mathcal{V}^{\mu\nu}$ in (45) and (46), establishes that graviton is its own antiparticle.

Note 2 Here, $\mathcal{V}_{\mu\nu}$ in (43) and (44), as well as $\mathcal{V}^{\mu\nu}$ in (45) and (46), provide us their quantum operator forms, whereas, (17) is the tensor form of $\mathcal{V}_{\mu\nu}$ in a $(3 + 1)D$ curvilinear quantum spacetime though, (43), (44), (45), (46), including (17) all are numerically equivalent for $\mu = \nu$.

Definition 8 (Feebleness of Gravity). Expression of (43), for example, would possible to be rewritten by using (23) as,

$$\mathcal{V}_{\mu\nu} \rightarrow -i\hbar \frac{\partial}{\partial \hat{\mathbf{p}}^\mu} \cdot \frac{\partial}{\partial x^\nu} \doteq -i\hbar \frac{1}{m} \frac{\partial t}{\partial x^\mu} \cdot \frac{\partial}{\partial x^\nu}. \quad (47)$$

Insertion of $i\hbar \partial_t \rightarrow \hat{E} = \hbar\omega \equiv \frac{1}{2}mc^2 \doteq \frac{1}{2}E$ from [10] into $m^{-1}(\partial t / \partial x^\mu)$ factor of (47) yields,

$$\frac{1}{m} \frac{\partial t}{\partial x^\mu} = \frac{c}{2\hat{E}} \frac{\partial (ct)}{\partial x^\mu} = \frac{c}{2\hat{E}} \frac{\partial x^0}{\partial x^\mu} \quad [\text{for (21)}]. \quad (48)$$

Suppose the energy of a photon is $\hat{E}_\gamma$ and its wavelength is λ_γ^μ , if $\hbar c = \hat{E}_\gamma \lambda_\gamma^\mu$, then the insertion of (48) into (47) yields,

$$\mathcal{V}_{\mu\nu} \rightarrow -\frac{i\hbar c}{2\hat{E}} \frac{\partial x^0}{\partial x^\mu} \cdot \frac{\partial}{\partial x^\nu} \doteq -\frac{i\hat{E}_\gamma}{2\hat{E}} \frac{\partial \lambda_\gamma^\mu}{\partial x^\mu} \cdot \frac{\partial x^0}{\partial x^\nu}, \quad (49)$$

where $\partial x^0 / \partial x^\nu$ is the vector normal to the hypersurface x^0 . Extreme smallness of $\hat{E}_\gamma / \hat{E}$ in (49) is the basic reason for the feebleness of gravity emerging from $\mathcal{W}$ of (19) in comparison to other particle interactions, namely QED, Electroweak, or QCD. The situation does not altered in the case of $\mathcal{V}^{\mu\nu}$.

3. Nature of Dynamic Time

Note 3 [Essential Prerequisites] Before we proceed our study further, we **must** need some essential basic ideas of Quantum Field Theory in $(3 + 1)D$ curvilinear quantum spacetime where the temporal direction t is dynamic. We have developed them in the Appendix A below. We **must** need to make an excursion into it before proceeding further.

By using bosonic (A11) from Appendix A.1, we can develop a first order bosonic quantum equation in $(3 + 1)D$ curvilinear quantum spacetime where the temporal direction t is dynamic as follows.

Theorem 5 (Time-Space Duality in Bosonic Particle). For (A11), the dynamics of the quantum mechanical (bosonic) particle is describable by a first order quantum equation in $(3 + 1)D$ curvilinear quantum spacetime when time t is dynamic as,

$$i\hbar \frac{\partial}{\partial t} \psi = -i\hbar \Phi_i^B \partial_i \psi + V(\vec{r}, t) \psi \equiv \mathcal{H} \psi,$$

for a stretching/shrinking parameters $\Phi_i^B = [(\gamma_B^2 - 1) v^i]^{-1} c^2$, while $V(\vec{r}, t)$ is the external potential energy and $\mathcal{H}$ is the Hamiltonian (or total energy) operator.

Proof. Equalizing and then rearranging the third and fourth terms of (A11), we can get,

$$(i\hbar)^2 \partial_0^2 \psi = (i\hbar)^2 \gamma_B^2 \partial_0^2 \psi + (i\hbar)^2 \partial_i \partial_j \psi \quad \forall i, j \in \{1, 2, 3\}. \quad (50)$$

While the temporal axis is $(i\hbar)^{-1} x^0 = (i\hbar)^{-1} (ct)$, the dynamics of the quantum state is possible to be defined as follows by shifting one $i\hbar \partial_0$ from the LHS of (50) to the RHS of (50) as,

$$\begin{aligned} i\hbar \frac{\partial}{\partial (ct)} \psi &= (i\hbar)^{-1} \partial_0^{-1} \left[(i\hbar)^2 \gamma_B^2 \partial_0^2 + (i\hbar)^2 \partial_i \partial_j \right] \psi \\ &= (i\hbar) \left[\gamma_B^2 \partial_0 + \partial_0^{-1} \partial_i \partial_j \right] \psi = (i\hbar) \left[\gamma_B^2 \partial_0 + \frac{c}{v^i} \partial_j \right] \psi, \end{aligned} \quad (51)$$

where $\partial_0^{-1} \partial_i = c(v^i)^{-1}$, while $v^i = (\partial x^i / \partial t)$ is the three-dimensional velocity vector. Therefore, by introducing external potential energy $V(\vec{r}, t)$ and the Hamiltonian (or total energy) operator $\mathcal{H}$, we have an ordinary differential equation in $(3 + 1)D$ curvilinear quantum spacetime as,

$$i\hbar \frac{\partial}{\partial t} \psi = -i\hbar \Phi_i^B \partial_i \psi + V(\vec{r}, t) \psi \equiv \mathcal{H} \psi, \quad (52)$$

where $\Phi_i^B = [(\gamma_B^2 - 1) v^i]^{-1} c^2$ is a stretching/shrinking parameters (stretching at $v^i \ll c$, whereas shrinking at $v^i \gtrsim c$), while time t is dynamic in (A1), or more precisely in (22). In (52), t is a (positive) formation of space for $\hat{p} \rightarrow -i\hbar \partial_j$. This completes the proof. $\square$ $\square$

Let us develop a first order fermionic quantum equation in $(3 + 1)D$ curvilinear quantum spacetime (analogous to Theorem 5) as follows.

Theorem 6 (Time-Space Duality in Fermionic Particle). For (A29) from Appendix A.2, the dynamics of the quantum mechanical (fermionic) particle is describable by a first order quantum equation in $(3 + 1)D$ curvilinear quantum spacetime when time t is dynamic as,

$$i\hbar \frac{\partial}{\partial t} \psi = \frac{i\hbar}{2} \left\{ \leq^i \Phi_i^F \partial_j, \leq^j \Phi_j^F \partial_i \right\} \psi + V(\vec{r}, t) \psi \equiv \mathcal{H} \psi,$$

for the stretching/shrinking parameters,

$$\text{diag}(\Phi_i^F, \Phi_j^F) = \text{diag} \left(\left[(\gamma_F^2 - \mathbb{K}_4) v^i \right]^{-1} c^2, \left[(\gamma_F^2 - \mathbb{K}_4) v^j \right]^{-1} c^2 \right),$$

while $V(\vec{r}, t)$ is the external potential energy and $\mathcal{H}$ is the Hamiltonian (or total energy) operator.

Proof. The rearrangement of equation (A29) yields after using $\{\leq^\mu, \leq^\nu\} = 2\mathcal{V}^{\mu\nu} \mathbb{K}_4$ as,

$$\begin{aligned} (i\hbar)^2 \partial_0^2 \mathbb{K}_4 \psi &= (i\hbar)^2 \gamma_F^2 \partial_0^2 \psi - \frac{1}{2} (i\hbar)^2 \left\{ \leq^i, \leq^j \right\} \partial_i \partial_j \psi \\ &\quad \forall i, j \in \{1, 2, 3\}. \end{aligned} \quad (53)$$

The dynamics of the quantum state is possible to be defined by shifting one $i\hbar\partial_0$ in the LHS of (53) to the RHS of (53) as follows while the temporal axis is $(i\hbar)^{-1}x^0 = (i\hbar)^{-1}(ct)$ as,

$$\begin{aligned} i\hbar \frac{\partial}{\partial(ct)} \mathbb{K}_4 \psi &= (i\hbar)^{-1} \partial_0^{-1} \left[(i\hbar)^2 \gamma_F^2 \partial_0^2 - \frac{1}{2} (i\hbar)^2 \left\{ \leq^i, \leq^j \right\} \partial_i \partial_j \right] \psi \\ &= (i\hbar) \left[\gamma_F^2 \partial_0 - \frac{1}{2} \left\{ \leq^i, \leq^j \right\} \partial_0^{-1} \partial_i \partial_j \right] \psi \end{aligned} \quad (54)$$

$$\begin{aligned} &= (i\hbar) \left[\gamma_F^2 \partial_0 - \frac{1}{2} \left[\leq^i \leq^j \partial_0^{-1} \partial_i \partial_j + \leq^j \leq^i \partial_0^{-1} \partial_j \partial_i \right] \right] \psi \\ &= (i\hbar) \left[\gamma_F^2 \partial_0 - \frac{1}{2} \left[\leq^i \leq^j \frac{c}{v^i} \partial_j + \leq^j \leq^i \frac{c}{v^j} \partial_i \right] \right] \psi, \end{aligned} \quad (55)$$

while $\partial_0^{-1} \partial_i = (v^i)^{-1} c$, etc. Therefore, by introducing external potential energy $V(\vec{r}, t)$ and the Hamiltonian (or total energy) operator $\mathcal{H}$, we have a first order quantum equation for fermions in $(3+1)D$ curvilinear quantum spacetime as,

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi &= \frac{i\hbar}{2} \left[\leq^i \leq^j \Phi_i^F \partial_j + \leq^j \leq^i \Phi_j^F \partial_i \right] \psi + V(\vec{r}, t) \psi \equiv \mathcal{H} \psi \\ &= \frac{i\hbar}{2} \left\{ \leq^i \Phi_i^F \partial_j, \leq^j \Phi_j^F \partial_i \right\} \psi + V(\vec{r}, t) \psi \equiv \mathcal{H} \psi, \end{aligned} \quad (56)$$

for the stretching/shrinking parameters (stretching at $v^i \ll c$, whereas shrinking at $v^i \lesssim c$),

$$\text{diag}(\Phi_i^F, \Phi_j^F) = \text{diag} \left(\left[(\gamma_F^2 - \mathbb{K}_4) v^i \right]^{-1} c^2, \left[(\gamma_F^2 - \mathbb{K}_4) v^j \right]^{-1} c^2 \right),$$

while time t is dynamic for (A17), or more precisely due to (A15). Additionally, t is a (positive) formation of space for $\hat{p} \rightarrow -i\hbar\partial_j$, or $\hat{p} \rightarrow -i\hbar\partial_i$. This completes the proof. $\square$ $\square$

Definition 9 (Direction of Dynamic Time). *Another arrangement of (51) (or (55), but we discard it here for unnecessary repetition) is also possible (suppose, but not necessarily, for an antiparticle, for example) as follows,*

$$i\hbar \frac{\partial}{\partial t} \psi = i\hbar (\Phi_i^B)^\dagger \partial_j \psi + V(\vec{r}, t) \psi \equiv \mathcal{H} \psi, \quad (57)$$

for $(\Phi_i^B)^\dagger = [(1 - \gamma_B^2) v^i]^{-1} c^2$. Since $\gamma_B^2 \gg 1$ in $(\Phi_i^B)^\dagger$, then time t is a (positive) formation of space for $\hat{p}^\dagger \rightarrow +i\hbar\partial_j$ (suppose for antiparticle, for example) not depending on the direction of its associatory spatial coordinates, which is exactly opposite of the spatial coordinates of a particle in (52). So, in associate with fermionic particles (which is not discussed further here), this 'positive' temporal direction strongly supports that,

Claim 2. *Dynamic time flows in its forward direction not depending on its associatory spatial coordinates for (bosonic or fermionic) matter or antimatter.*

Definition 10 (Nature of Dynamic Time). *Since (52) or (57) are actually the relations of time-space duality and time is nothing but a (positive) formation due to a stretching/shrinking parameters $\Phi_i^{B,F}$ or $(\Phi_i^{B,F})^\dagger$ of its associatory space, thus, the duality in (52) or (57) wants to tell us that,*

Claim 3. *Dynamic time is not fundamental but emergent, i.e., dynamic time is not an independent physical entity. If space not exists then time not exists, too, but the reverse condition is not true since the stretching/shrinking parameters $\Phi_i^{B,F}$ or $(\Phi_i^{B,F})^\dagger$ satisfy v^i ranging from $v^i = 0$ to $v^i < c$. On the contrary, space is an independent physical entity in both (52) and (57).*

Remark 4. At $v^i = c$, the direction of time ‘relativistically’ reverses depending on its associatory spatial coordinates due to its stretching/shrinking parameters $\Phi_i^{B,F}$ or $(\Phi_i^{B,F})^\dagger$, but at $v^i > c$, the direction transforms into a complex number, which is technically unacceptable.

Testable Prediction 2. The transformation of stretching/shrinking parameters from $\Phi_i^{B,F}$ to $(\Phi_i^{B,F})^\dagger$ may yield a time-reflection at $v^i = c$, i.e., a uniform inversion of the temporal evolution of a signal, for example, photonic time-reflection [13], which arises when an abrupt change in the properties of the host material occurs uniformly in space.

Testable Prediction 3. The transformations of stretching/shrinking parameters, i.e.,

1. from Φ_i^B to $(\Phi_i^B)^\dagger$ for bosonic matter to antimatter and
2. from Φ_i^F to $(\Phi_i^F)^\dagger$ for fermionic matter to antimatter,

or their vice versa, i.e., antimatter to matter transformations, may break time-translation symmetry at $v^i = c$ in a medium depending on the three-velocity v^i for $i \in \{1, 2, 3\}$.

To remove the dynamic time t from (52) or (56), we can introduce a set of time-independent 3D wave equations for bosons or fermions, respectively, as follows.

Definition 11 (Time-independent 3D Mass-Momentum Wave Equation for Bosons). By introducing (8) into (52), we can yield,

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi &= \Phi_i^B m \frac{\partial x^j}{\partial t} \psi \quad [\text{for } i, j \in \{1, 2, 3\}], \\ \therefore \quad \left(-i\hbar \vec{\nabla}_j + \Phi_i^B m \right) \psi &= 0. \end{aligned}$$

It is a time-independent 3D mass-momentum wave equation for bosons.

Definition 12 (Time-independent 3D Mass-Momentum Wave Equation for Fermions). By introducing (8) into (56), we can yield,

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi &= -\frac{m}{2} \left\{ \not{\epsilon}^i \Phi_i^F \frac{\partial x^j}{\partial t}, \not{\epsilon}^j \Phi_j^F \frac{\partial x^i}{\partial t} \right\} \psi \quad [\text{for } i, j \in \{1, 2, 3\}], \\ \therefore \quad i\hbar \frac{\partial}{\partial x^j} \psi &= -\frac{m}{2} \left\{ \not{\epsilon}^i, \not{\epsilon}^j \right\} \Phi_i^F \psi = \Phi_i^F m \psi, \end{aligned}$$

since the Clifford algebra satisfies $\{\not{\epsilon}^\mu, \not{\epsilon}^\nu\} = \not{\epsilon}^\mu \not{\epsilon}^\nu + \not{\epsilon}^\nu \not{\epsilon}^\mu = 2\gamma^{\mu\nu} \mathbb{K}_4$ with $\gamma^{ij} = [-1, -1, -1]$, hence,

$$\left(-i\hbar \vec{\nabla}_j + \Phi_i^F m \right) \psi = 0.$$

It is a time-independent 3D mass-momentum wave equation for fermions.

PART II: General Quantum Gravity

4. Universal Quantum Gravitational Field Equations

By inputting the modified form of (34) as follows,

$$\begin{aligned} \gamma_{\mu\nu} \hat{p}^\mu &\rightarrow -i\hbar \frac{\partial}{\partial x^\nu} \quad \text{i.e.,} \quad \gamma_{\mu\nu} (m v^\mu) \rightarrow -i\hbar \vec{\nabla}_\nu \quad [\text{for (23)}] \\ &\doteq -\hat{\mathcal{P}}_\nu \quad [\text{for (13)}], \end{aligned} \tag{58}$$

in place of $(i\hbar)^2 \vec{\nabla}_\mu \vec{\nabla}_\nu$ of (A12) from the Appendix A.1 below and after that, by dropping m^2 from both of the terms of the resultant (A12), and finally multiplying it by $(\mathcal{V}^{\mu\nu})^2$, we can acquire a new form of (A12) as follows,

$$\begin{aligned} & \left[\mathcal{V}^{\mu\nu} (\mathcal{V}_{\mu\nu})^2 v^\nu v^\mu + (i\gamma_B c)^2 \right] \psi = 0, \\ \therefore & \left[\mathcal{V}^{\mu\nu} v^\nu v^\mu + \left(\frac{1}{4} i \mathcal{V}^{\mu\nu} \gamma_B c \right)^2 \right] \psi = 0 \quad [\text{for (18)}]. \end{aligned} \quad (59)$$

Multiplying both terms of (59) by $(i\hbar)^2 [\partial^2 t / (\partial x^\nu \partial x^\mu)]^2$, we can get,

$$\left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + \left\{ \frac{1}{4} i \mathcal{V}^{\mu\nu} \gamma_B c \left(\frac{\partial t}{\partial x^\nu} i\hbar \frac{\partial}{\partial x^\mu} \right) \right\}^2 \right] \psi = 0. \quad (60)$$

For $\hat{\mathcal{P}}_\mu \rightarrow i\hbar \frac{\partial}{\partial x^\mu}$ due to (13), the second term in (60) gives,

$$\begin{aligned} \left(\frac{\partial t}{\partial x^\nu} i\hbar \frac{\partial}{\partial x^\mu} \right) &= \left(\frac{\partial t}{\partial x^\nu} \hat{\mathcal{P}}_\mu \right) \doteq \left(\frac{\partial t}{\partial x^\nu} m \frac{\partial x^\mu}{\partial t} \right) \quad [\text{for (23)}] \\ &\equiv m \delta_\nu^\mu \doteq 4m \quad [\text{since } \mu = \nu], \end{aligned} \quad (61)$$

where δ_ν^μ is the Kronecker delta, thus, (60) yields,

$$\left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i \mathcal{V}^{\mu\nu} \gamma_B m c)^2 \right] \psi = 0. \quad (62)$$

Hence, (62) is still an analogy to Kline-Gordon-like equation (A12), but now containing a gravitational field within it without replacing partial derivatives with covariant derivatives (i.e., geometrically covariant generalization, as we know from, for example, [14] or [15]). Comparing (37) and (62) as both of them are originated from (22), we can get,

$$i\hbar \frac{\partial \psi}{\partial t} + \left(\frac{1}{\sqrt{2}} i \mathcal{V}^{\mu\nu} \gamma_B \sqrt{m} c \right)^2 \psi = 0.$$

It is a natural substitution of the relativistic Schrödinger equation (37), even the orthodox Kline-Gordon equation (according to Remark 3), as well as (62), too.

Remark 5. Conjugated spinorial (A31) in $(3+1)D$ curvilinear quantum spacetime from the Appendix A.2 below can also develop an equation containing a gravitational field within it as,

$$\left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi = 0, \quad (63)$$

which is quite analogous to (62).

Since $[\partial^2 / (\partial \hat{s}^2)] \psi = (i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi$ for (20), then (62) yields a positive mass relation with (quantum) metric tensor for bosons as follows,

$$\frac{\partial^2}{\partial \hat{s}^2} \psi = - (i \mathcal{V}^{\mu\nu} \gamma_B m c)^2 \psi, \quad (64)$$

and similarly, (63) yields a positive mass relation with (quantum) metric tensor for conjugated fermions as follows,

$$\frac{\partial^2}{\partial \hat{s}^2} \psi = - (i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi, \quad (65)$$

neither of which are technically possible to acquire from the orthodox Kline-Gordon equation.

Note 4 [Fermionic Scenario] In a similar way, let us consider fermions. Then, by inputting (58) in place of $i\hbar \vec{\nabla}_\mu$ in (A28) from the Appendix A.2 below, we can get the following relation after dropping m from both of the terms of the resultant (A28), and finally multiplying it by $\mathcal{V}^{\mu\nu}$, we can acquire a new form of (A28) as follows,

$$\begin{aligned} & [-\mathcal{V}^{\mu\nu} \mathcal{V}_{\mu\nu} v^\nu - \gamma_F c] \psi = 0, \\ \therefore & \left[-\mathcal{V}^{\mu\nu} v^\nu - \frac{1}{4} \mathcal{V}^{\mu\nu} \gamma_F c \right] \psi = 0 \quad [\text{for (18)}]. \end{aligned} \quad (66)$$

Avoiding any further repetition, multiplying both terms of (66) simply by $(i\hbar) [\partial^2 t / (\partial x^\nu \partial x^\mu)]$ from the left we can get,

$$\left[-i\hbar \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu - \frac{1}{4} \mathcal{V}^{\mu\nu} \gamma_F c \left(\frac{\partial t}{\partial x^\nu} i\hbar \frac{\partial}{\partial x^\mu} \right) \right] \psi = 0. \quad (67)$$

For $\hat{\mathcal{P}}_\mu \rightarrow i\hbar \frac{\partial}{\partial x^\mu}$ due to (13), the second term in (67) gives,

$$\begin{aligned} \left(\frac{\partial t}{\partial x^\nu} i\hbar \frac{\partial}{\partial x^\mu} \right) &= - \left(i\hbar \frac{\partial}{\partial x^\mu} \frac{\partial t}{\partial x^\nu} \right) \quad [\text{for anticommutativity}] \\ &= - \left(\hat{\mathcal{P}}_\mu \frac{\partial t}{\partial x^\nu} \right) \div - \left(m \frac{\partial x^\mu}{\partial t} \frac{\partial t}{\partial x^\nu} \right) \quad [\text{for (23)}] \\ &\equiv -m \delta_\nu^\mu \div -4m \quad [\text{since } \mu = \nu], \end{aligned}$$

thus, (67) yields,

$$\left[i\hbar \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu - \mathcal{V}^{\mu\nu} \gamma_F m c \right] \psi = 0, \quad (68)$$

i.e., using (A14) and (17) while admitting (27), the expanded form of (68) would be rewritten as,

$$\begin{aligned} & i\hbar \begin{pmatrix} \gamma^\mu & 0 & 0 & 0 \\ 0 & \gamma^\mu & 0 & 0 \\ 0 & 0 & -\tilde{\gamma}^\mu & 0 \\ 0 & 0 & 0 & -\tilde{\gamma}^\mu \end{pmatrix} \begin{pmatrix} \vec{\nabla}_\mu & 0 & 0 & 0 \\ 0 & \vec{\nabla}_\mu & 0 & 0 \\ 0 & 0 & \vec{\nabla}_\mu & 0 \\ 0 & 0 & 0 & \vec{\nabla}_\mu \end{pmatrix} \begin{pmatrix} \varphi & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi_c \end{pmatrix} - \\ & - \begin{pmatrix} \mathcal{V}^{00} & 0 & 0 & 0 \\ 0 & \mathcal{V}^{11} & 0 & 0 \\ 0 & 0 & \mathcal{V}^{22} & 0 \\ 0 & 0 & 0 & \mathcal{V}^{33} \end{pmatrix} \gamma_F \begin{pmatrix} m_D & 0 & 0 & 0 \\ 0 & m_D^* & 0 & 0 \\ 0 & 0 & m_M & 0 \\ 0 & 0 & 0 & m_M \end{pmatrix} c \begin{pmatrix} \varphi & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 \\ 0 & 0 & \xi_c & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix} = 0, \end{aligned}$$

so as we have the formalisms,

$$\begin{aligned} & \left[i\hbar \gamma^\mu \vec{\nabla}_\mu - \gamma_F m_D c \right] \varphi = 0, \\ & \left[i\hbar \gamma^\mu \vec{\nabla}_\mu + \gamma_F m_D^* c \right] \chi = 0, \\ & i\hbar (-\tilde{\gamma}^\mu) \vec{\nabla}_\mu \xi + \gamma_F m_M c \xi_c = 0, \\ & i\hbar (-\tilde{\gamma}^\mu) \vec{\nabla}_\mu \xi_c + \gamma_F m_M c \xi = 0, \end{aligned} \quad (69)$$

which make the appearance of $\mathcal{V}^{\mu\nu}$ in (68) quite harmless. In (69), γ^μ are the Dirac gamma-matrices and $\tilde{\gamma}^\mu$ are the Majorana gamma-matrices. The first two equations in (69) give us particle and antiparticle Dirac equations, respectively, whereas the remaining two equations therein give us neutrinos. Actually, (68) is an extended and/or modified version of the generalized relativistic field equation in Ref. [16]. Speaking straightforwardly, (69) is nothing but a ‘Special’-within-Special Relativistic QFT equation because of γ_F . Quite like Remark 3, here, (69) also establishes that *the present article*

is nothing but a different perspective of the fundamental Quantum Field Theory, if we accept γ_F additionally with mass m to achieve the targeted General (Quantum) Relativistic solutions, otherwise, by removing γ_F , the overall scenario naturally transforms itself into the orthodox Special Relativistic forms.

In addition, (68) also yields a positive mass relation with (quantum) metric tensor for (A16) as follows,

$$\frac{\partial}{\partial \hat{s}} \psi = \mathcal{V}^{\mu\nu} \gamma_F m c \psi. \quad (70)$$

Though (68) is still an analogy to Dirac-like equation (A28), but it is now accompanied by a gravitational field.

By the way, in (68), the coupling of a Dirac spinor to a background gravitational field is quite straightforward rather than considering the Lorentz connection (or gauge potential) usually followed by the authors of the contemporary literature, for example [17].

Although, both bosonic (62), or (63), and fermionic (68) fields are now recombining Quantum Mechanics with the gravitational field $\mathcal{V}^{\mu\nu}$ within one framework, but none of them at their present structural formations are able to give us the quantum gravitational field equations we are looking for. So, we need something else to investigate further, which will be elaborated in the next three sections, but initially, we start with the theorem as follows.

Theorem 7. For the quantum differential geometric curvature forms $\mathcal{P}_{\mu\nu}$ and $\mathcal{P}$ (i.e., tensor and scalar curvatures, respectively), there exists a non-vacuum quantum gravitational field equation in $(3+1)D$ curvilinear quantum spacetime in such a way that,

$$\left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi + \mathcal{V}_{\mu\nu} \Lambda' \psi = \kappa \mathcal{T}_{\mu\nu} \psi,$$

where (quantum) metric tensor $\mathcal{V}_{\mu\nu}$ is a property of $(3+1)D$ curvilinear Quantum Differential Geometry satisfying both (17) and (42) simultaneously, whereas $\Lambda' = (i\hbar)^2 \Lambda$ is the 'naïve' cosmological constant for the cosmological constant Λ and $\mathcal{T}_{\mu\nu} = (i\hbar)^2 T_{\mu\nu}$ is the quantum energy-momentum tensor for the General Relativistic energy-momentum tensor $T_{\mu\nu}$.

Using [18]. Let us consider a finite-dimensional complex vector space $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^4)$. Let $(M^4, \mathcal{V})$ be a smooth manifold and X, Y, Z are smooth vector fields on M^4 , where M^4 is an four-dimensional differentiable manifold and $\mathcal{V}$ is the (quantum) metric tensor satisfying (17) and also the both forms of (42) simultaneously. This $\mathcal{V}$ is either a positive-definite section of the bundle of symmetric (covariant) 2-tensors $T^*M \otimes_S T^*M$ or a positive-definite bilinear maps, $\mathcal{V}((i\hbar)^{-1}x) : T_{((i\hbar)^{-1}x)}M \times T_{((i\hbar)^{-1}x)}M \rightarrow \mathcal{V}$ for all $(i\hbar)^{-1}x \in M$. Here, $T^*M \otimes_S T^*M$ is the subspace of $T^*M \otimes T^*M$ generated by elements of the form $X \otimes Y + Y \otimes X$. Let $\{(i\hbar)^{-1}x^\mu\}_{\mu=0}^3$ be local coordinates in a neighborhood U of some point of M . For (20), in U the vector fields $\{i\hbar \vec{\nabla}_\mu\}_{\mu=0}^3 \doteq \{\hat{\mathcal{P}}_\mu\}_{\mu=0}^3$ due to (13) form a local basis for TM and the 1-forms $\{(i\hbar)^{-1}dx^\mu\}_{\mu=0}^3$ form a dual basis for T^*M , that is,

$$(i\hbar)^{-1}dx^\mu \cdot i\hbar \vec{\nabla}_\nu = \delta_\nu^\mu.$$

The metric may then be written in local coordinates as $\mathcal{V} = (i\hbar)^{-2} \mathcal{V}_{\mu\nu} dx^\mu \otimes dx^\nu$, where $\mathcal{V}_{\mu\nu} \doteq \mathcal{V}(i\hbar \vec{\nabla}_\mu, i\hbar \vec{\nabla}_\nu) \doteq \mathcal{V}(\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu)$ for (17).

Let $\nabla^{\mathcal{V}}$ denote the Levi-Civita connection of the metric $\mathcal{V}$. The Christoffel symbol is the components of the Levi-Civita connection and is defined in U by $\nabla_{(i\hbar \vec{\nabla}_\mu)}(i\hbar \vec{\nabla}_\nu) \doteq \nabla_{\hat{\mathcal{P}}_\mu}^k \hat{\mathcal{P}}_\nu \doteq \nabla_{\hat{\mathcal{P}}_\mu}^k \hat{\mathcal{P}}_k$ for (13), and for a permutations of the vector fields X, Y and Z as,

$$2\mathcal{A}\psi(\nabla_X Y, Z) = X(\mathcal{A}\psi(Y, Z)) + Y(\mathcal{A}\psi(X, Z)) - Z(\mathcal{A}\psi(X, Y)) + \\ + \mathcal{A}\psi([X, Y], Z) - \mathcal{A}\psi([X, Z], Y) - \mathcal{A}\psi([Y, Z], X),$$

along with $\left[(i\hbar\vec{\nabla}_\mu), (i\hbar\vec{\nabla}_\nu)\right] \doteq [\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu] = 0$, thus, using (44) and (46), we can show that they are given by the (quantum) Christoffel symbols as,

$$\begin{aligned} \Gamma_{\mu\nu}^k \psi &= \frac{1}{2} \mathcal{A}\psi^{k\ell} \left[(i\hbar\vec{\nabla}_\mu) \mathcal{A}\psi_{\nu\ell} + (i\hbar\vec{\nabla}_\nu) \mathcal{A}\psi_{\mu\ell} - (i\hbar\vec{\nabla}_\ell) \mathcal{A}\psi_{\mu\nu} \right] \psi \\ &= \frac{1}{2} \partial_{\hat{\mathcal{P}}_k} \cdot \hat{\mathcal{P}}_\ell \left[(i\hbar\vec{\nabla}_\mu) \hat{\mathcal{P}}_\nu \cdot \partial_{\hat{\mathcal{P}}_\ell} + (i\hbar\vec{\nabla}_\nu) \hat{\mathcal{P}}_\mu \cdot \partial_{\hat{\mathcal{P}}_\ell} - (i\hbar\vec{\nabla}_\ell) \hat{\mathcal{P}}_\mu \cdot \partial_{\hat{\mathcal{P}}_\nu} \right] \psi \\ &= \frac{1}{2} \partial_{\hat{\mathcal{P}}_k} \cdot \hat{\mathcal{P}}_\ell \left[\hat{\mathcal{P}}_\mu \cdot \hat{\mathcal{P}}_\nu \cdot \partial_{\hat{\mathcal{P}}_\ell} + \hat{\mathcal{P}}_\nu \cdot \hat{\mathcal{P}}_\mu \cdot \partial_{\hat{\mathcal{P}}_\ell} - \hat{\mathcal{P}}_\ell \cdot \hat{\mathcal{P}}_\mu \cdot \partial_{\hat{\mathcal{P}}_\nu} \right] \psi \quad [\text{for (13)}]. \end{aligned} \quad (71)$$

Suppose, the curvature (3,1)-tensor $\mathfrak{P}_{(3,1)}$ is defined by,

$$\mathfrak{P}_{(3,1)}(X, Y)Z\psi \doteq [\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z]\psi.$$

Let the tensor $\mathfrak{P}_t$ is the trace of $\mathfrak{P}_{(3,1)}$ curvature tensor as,

$$\mathfrak{P}_t(Y, Z)\psi \doteq \text{Tr}(X \mapsto \mathfrak{P}_{(3,1)}(X, Y)Z)\psi.$$

So as its components are defined by,

$$\mathcal{P}_{\mu\nu}\psi \doteq \mathfrak{P}_t((i\hbar\vec{\nabla}_\mu), (i\hbar\vec{\nabla}_\nu))\psi = \mathfrak{P}_t(\hat{\mathcal{P}}_\mu, \hat{\mathcal{P}}_\nu)\psi,$$

thus,

$$\begin{aligned} \mathcal{P}_{\mu\nu}\psi &= \left[(i\hbar\vec{\nabla}_\ell) \Gamma_{\mu\nu}^\ell - (i\hbar\vec{\nabla}_\nu) \Gamma_{\mu\ell}^\ell + \Gamma_{k\ell}^\ell \Gamma_{\mu\nu}^k - \Gamma_{k\nu}^\ell \Gamma_{\mu\ell}^k \right] \psi \\ &\doteq \left[\hat{\mathcal{P}}_\ell \Gamma_{\mu\nu}^\ell - \hat{\mathcal{P}}_\nu \Gamma_{\mu\ell}^\ell + \Gamma_{k\ell}^\ell \Gamma_{\mu\nu}^k - \Gamma_{k\nu}^\ell \Gamma_{\mu\ell}^k \right] \psi. \end{aligned} \quad (72)$$

Let the scalar curvature $\mathfrak{P}$ is the trace of $\mathfrak{P}_t$ tensor,

$$\mathfrak{P}\psi \doteq \sum_{a=0}^3 \mathfrak{P}_t(e_a, e_a)\psi,$$

i.e., in local coordinates, $\mathcal{P} = \mathcal{A}\psi^{\mu\nu} \mathcal{P}_{\mu\nu}$, where $e_a \in T_{((i\hbar)^{-1}x)}M^4$ is a unit vector spanning of a line $L \subset T_{((i\hbar)^{-1}x)}M^4$.

Now, we can easily show that the (twice) contracted second Bianchi identity is,

$$\mathcal{A}\psi^{\mu\nu} \nabla_\mu \mathcal{P}_{\nu\xi} \psi = \nabla_\xi \mathcal{P} \psi,$$

and being divergence-free which is,

$$\text{div} \left\{ \mathfrak{P}_t - \frac{1}{2} \mathcal{A}\psi_{\mu\nu} \mathfrak{P} \right\} = 0, \quad (73)$$

then, the non-vacuum quantum gravitational field equation yields (in its components) due to (73) along with the quantum energy-momentum tensor $\mathcal{T}_{\mu\nu} = (i\hbar)^2 T_{\mu\nu}$, where $T_{\mu\nu}$ is the General Relativistic energy-momentum tensor for the Einstein gravitational constant $\kappa = (8\pi G)/c^4$ as,

$$\left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{A}\psi_{\mu\nu} \mathcal{P} \right] \psi = \kappa \mathcal{T}_{\mu\nu} \psi. \quad (74)$$

Let us introduce the ‘naïve’ cosmological constant as $\Lambda' = (i\hbar)^2 \Lambda$ for the cosmological constant Λ , describing the pressure-like effect of Dark Energy, then the quantum gravitational field equation (74) would be rewritten as,

$$\left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi + \mathcal{V}_{\mu\nu} \Lambda' \psi = \kappa \mathcal{T}_{\mu\nu} \psi \quad [\text{for (43)}]. \quad (75)$$

This completes the proof. $\square$ $\square$

Remark 6 (Universality of (75)). *The non-vacuum quantum gravitational field equation (75) is free of factor γ_B or γ_F , i.e., it is equally applicable to bosons and conjugated fermions in $(3+1)D$ curvilinear quantum spacetime. Similarly, its first order equation is also applicable to fermions in $(3+1)D$ curvilinear quantum spacetime (see Section 5 below for more details).*

Note it that, unlike the dimension of Einsteinian gravitational field equation in General Relativity [19], the quantum gravitational field equation (75) has the dimension as $[M^2 L^2 T^{-2}]$.

For $G \mathcal{T}_{\mu\nu} = (i\hbar)^2 G T_{\mu\nu}$ in (75), the mass dimension of gravitational constant $G \equiv [M^{-1} L^3 T^{-2}]$ vanishes. So, not owing to the mass dimension of G , the perturbative version of the quantum gravitational field equation (75) is renormalizable in $(3+1)D$ curvilinear quantum spacetime. Though, we do not explore this idea in this article, instead we are going to use a different pathway in Section 10 below to prove that the present theory of Quantum Gravity is actually ‘multiplicatively renormalizable’.

5. Duality of $(3+1)D$ Quantum Field Theory and $(3+1)D$ Quantum Differential Geometry in Fermionic Scenarios

Theorem 8. *Let M be a spin manifold. If $\mathcal{P}$ is the quantum scalar curvature of M then, the d’Alembertian operator $\square$ on a spinor (see page 308 of [20], for example),*

$$\begin{aligned} \square \psi &= \left(i\hbar \nabla_{\mu}^{\mu} \vec{\nabla}_{\mu} \right) \left(i\hbar \nabla_{\nu}^{\nu} \vec{\nabla}_{\nu} \right) \psi + \frac{1}{4} \mathcal{P} \psi \\ &= (i\hbar)^2 \nabla_{\mu}^{\mu} \vec{\nabla}_{\mu} \nabla_{\nu}^{\nu} \vec{\nabla}_{\nu} \psi + \frac{1}{4} \mathcal{P} \psi, \end{aligned} \quad (76)$$

where,

$$\square \psi = (i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} \psi \doteq \frac{\partial^2}{\partial \hat{s}^2} \psi \quad [\text{for (19)}], \quad (77)$$

yields,

$$\left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} + \frac{1}{12} \mathcal{P} \right] \psi = 0,$$

for the construction $\psi = (\varphi, \varphi_{\mu}, \varphi_{\mu\nu})$ when the spin of the particle(s) (i.e., scalars, gauge bosons, gravitons, etc.) is $(s = 0, s = 1, s = 2)$ with the conditions $\partial^{\mu} \varphi_{\mu} = 0$, $\partial^{\mu} \varphi_{\mu\nu} = 0$, $\mathcal{V}^{\mu\nu} \varphi_{\mu\nu} = 0$ and the total symmetrization of the indices as $\varphi_{\mu\nu}(x) = \varphi_{(\mu\nu)}$.

Proof. The d’Alembertian operator $\square$ on a spinor is yielded by combining (76) and (77) as,

$$\begin{aligned} \square \psi &= \left(i\hbar \nabla_{\mu}^{\mu} \vec{\nabla}_{\mu} \right) \left(i\hbar \nabla_{\nu}^{\nu} \vec{\nabla}_{\nu} \right) \psi + \frac{1}{4} \mathcal{P} \psi \\ &= (i\hbar)^2 \nabla_{\mu}^{\mu} \vec{\nabla}_{\mu} \nabla_{\nu}^{\nu} \vec{\nabla}_{\nu} \psi + \frac{1}{4} \mathcal{P} \psi \\ &\doteq (i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} \psi = \frac{\partial^2}{\partial \hat{s}^2} \psi, \end{aligned} \quad (78)$$

which yields the following relation after multiplying the second and the last line terms in (78) by $\mathcal{V}_{\mu\nu}$ while using $\mathcal{V}_{\mu\nu} \bar{\nabla}^\nu = \bar{\nabla}_\mu$ and $\bar{\nabla}^\mu \bar{\nabla}_\mu = 4\mathcal{K}_4$,

$$\begin{aligned} (i\hbar)^2 \mathcal{V}_{\mu\nu} \mathcal{V}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \psi &= (i\hbar)^2 \mathcal{V}_{\mu\nu} \bar{\nabla}^\mu \bar{\nabla}_\mu \bar{\nabla}_\nu \psi + \frac{1}{4} \mathcal{V}_{\mu\nu} \mathcal{P} \psi \\ &= (i\hbar)^2 \bar{\nabla}^\mu \bar{\nabla}_\mu \bar{\nabla}_\nu \psi + \frac{1}{4} \mathcal{V}_{\mu\nu} \mathcal{P} \psi, \\ \therefore (i\hbar)^2 \mathcal{V}_{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}^\mu \psi &= 4(i\hbar)^2 \bar{\nabla}_\mu \bar{\nabla}_\nu \psi + \frac{1}{4} \mathcal{V}_{\mu\nu} \mathcal{P} \psi. \end{aligned}$$

Since $\bar{\nabla}_\nu = \mathcal{V}_{\mu\nu} \bar{\nabla}^\mu$, then we have,

$$\begin{aligned} &\left[4(i\hbar)^2 \mathcal{V}_{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}^\mu + \frac{1}{4} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi - (i\hbar)^2 \mathcal{V}_{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}^\mu \psi = 0, \\ \Rightarrow &\left[3(i\hbar)^2 \mathcal{V}_{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}^\mu + \frac{1}{4} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi = 0, \\ \text{hence,} &\left[(i\hbar)^2 \mathcal{V}_{\mu\nu} \bar{\nabla}_\mu \mathcal{V}^{\mu\nu} \bar{\nabla}_\nu + \frac{1}{12} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi = 0 \\ &\quad \quad \quad [\text{for } \bar{\nabla}^\mu = \mathcal{V}^{\mu\nu} \bar{\nabla}_\nu], \\ \therefore &\left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu + \frac{1}{12} \mathcal{P} \right] \psi = 0. \end{aligned} \quad (79)$$

This completes the proof. $\square$ $\square$

Theorem 9 (Duality of Quantum Gravitational Field and Conjugated Fermionic Field). *A non-vacuum quantum gravitational field equation in $(3+1)D$ curvilinear quantum spacetime can metamorphose into the Kline-Gordon-like equation (63) for conjugated fermions.*

Proof. For (79), we can introduce a new kind of line element as,

$$\frac{\partial^2}{\partial \hat{s}^2} \psi = (i\hbar)^2 \mathcal{V}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \psi \doteq -\frac{1}{12} \mathcal{P} \psi. \quad (80)$$

Combination of $[\partial^2/\partial \hat{s}^2] \psi = (i\hbar \bar{\nabla}^\mu \bar{\nabla}_\mu)(i\hbar \bar{\nabla}^\nu \bar{\nabla}_\nu) \psi + \frac{1}{4} \mathcal{P} \psi$ of (78) with (80) yields,

$$\begin{aligned} &(i\hbar \bar{\nabla}^\mu \bar{\nabla}_\mu)(i\hbar \bar{\nabla}^\nu \bar{\nabla}_\nu) \psi + \frac{1}{4} \mathcal{P} \psi = -\frac{1}{12} \mathcal{P} \psi, \\ \therefore &(i\hbar \bar{\nabla}^\mu \bar{\nabla}_\mu)(i\hbar \bar{\nabla}^\nu \bar{\nabla}_\nu) \psi = -\frac{1}{3} \mathcal{P} \psi. \end{aligned} \quad (81)$$

By introducing $\mathcal{P} = \mathcal{V}^{\mu\nu} \mathcal{P}_{\mu\nu}$ in (81), where $\mathcal{P}_{\mu\nu}$ is the trace of quantum curvature tensor, we have a tensor calculus-free definition of $\mathcal{P}_{\mu\nu}$ as,

$$\mathcal{V}_{\mu\nu} (i\hbar \bar{\nabla}^\mu \bar{\nabla}_\mu)(i\hbar \bar{\nabla}^\nu \bar{\nabla}_\nu) \psi = -\frac{4}{3} \mathcal{P}_{\mu\nu} \psi \quad [\text{for (18)}]. \quad (82)$$

By the virtue of the universality of (75) in Remark 6 that the non-vacuum quantum gravitational field equation (75) is equally applicable to bosons as well as conjugated fermions, too, in $(3+1)D$ curvilinear quantum spacetime, then we can modify (75) by using (82) and (81) as follows,

$$\begin{aligned}
& -3 \left[\frac{1}{4} \mathcal{V}_{\mu\nu} (i\hbar \nabla^\mu \vec{\nabla}_\mu) (i\hbar \nabla^\nu \vec{\nabla}_\nu) - \frac{1}{2} \mathcal{V}_{\mu\nu} (i\hbar \nabla^\mu \vec{\nabla}_\mu) (i\hbar \nabla^\nu \vec{\nabla}_\nu) \right] \psi + \\
& \quad + \mathcal{V}_{\mu\nu} \Lambda' \psi = \kappa \mathcal{T}_{\mu\nu} \psi, \\
& \therefore \mathcal{V}_{\mu\nu} (i\hbar \nabla^\mu \vec{\nabla}_\mu) (i\hbar \nabla^\nu \vec{\nabla}_\nu) \psi + \frac{4}{3} \mathcal{V}_{\mu\nu} \Lambda' \psi = \frac{4\kappa}{3} \mathcal{T}_{\mu\nu} \psi, \\
& \Rightarrow (i\hbar \nabla^\mu \vec{\nabla}_\mu) (i\hbar \nabla^\nu \vec{\nabla}_\nu) \psi + \frac{4}{3} \Lambda' \psi = \frac{\kappa}{3} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi \quad [\text{for (18)}],
\end{aligned}$$

i.e., by using the first line and the last line of (78) it yields,

$$(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi - \frac{1}{4} \mathcal{P} \psi + \frac{4}{3} \Lambda' \psi = \frac{\kappa}{3} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi,$$

hence, by using (80), we have,

$$\begin{aligned}
& (i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi + 3(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi + \frac{4}{3} \Lambda' \psi = \frac{\kappa}{3} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi, \\
& \therefore (i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi + \frac{1}{3} \Lambda' \psi - \frac{\kappa}{12} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi = 0,
\end{aligned} \tag{83}$$

which is a non-vacuum quantum gravitational field equation for conjugated fermions accompanied by Dark Energy, while the 'naïve' cosmological constant yields as,

$$\begin{aligned}
& \Lambda' \psi = -3(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi + \frac{\kappa}{4} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi, \\
& \therefore \Lambda \psi = -3\mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi + \frac{\kappa}{4} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi,
\end{aligned} \tag{84}$$

since $\Lambda' = (i\hbar)^2 \Lambda$ and $\mathcal{T}_{\mu\nu} = (i\hbar)^2 T_{\mu\nu}$. Inserting (84) into (75), we can get for (18) as,

$$\mathcal{V}^{\mu\nu} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi - 12(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi = 0,$$

which returns us to (79), since $\mathcal{V}^{\mu\nu} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P} \right] = -\mathcal{P}$ again for (18).

The comparison between (83), (80) and (65) yields,

$$(i\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi - \frac{1}{3} \Lambda' \psi + \frac{\kappa}{12} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi = 0. \tag{85}$$

Then (85) indicates us that (70) must be rewritten as,

$$\frac{\partial^2}{\partial \hat{s}^2} \psi = -(i\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi = \frac{\kappa}{12} \mathcal{V}^{\mu\nu} \mathcal{T}_{\mu\nu} \psi - \frac{1}{3} \Lambda' \psi. \tag{86}$$

Hence, (85) and/or (86) modifies (83) as,

$$\left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi = 0. \tag{87}$$

which is nothing but the Kline-Gordon-like equation (63) for conjugated fermions. This completes the proof. $\square \square$

Remark 7 (Origination of Dark Energy). *It is clear from (84) that Λ is directly related to the conjugated fermionic fields (or bosonic fields if we consider (62) instead of (63), which we are going to develop in Theorem 11, see below) and the trace of energy-momentum fields. Thus,*

Claim 4. *Dark Energy may originate from matter fields.*

Definition 13 ((Quantum) Ricci-like Flow, or Quantum Metric Flow in Conjugated Fermionic Fields). Combination of (A26) from Appendix A.2 with (A25) to have $\Theta_F m \psi = i \hbar \leq^\mu \vec{\nabla}_\mu \psi = i \hbar \gamma_F \partial_0 \psi \equiv \vec{\nabla}_{0(\tau)} \psi$, so as we may take for (A29) as $\Theta_F^2 m^2 \psi = (i \hbar)^2 \mathcal{A} \psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi = (i \hbar)^2 \gamma_F^2 \partial_0^2 \psi \equiv \vec{\nabla}_{0(\tau)}^2 \psi$, which can be used to modify (80) in such a way that,

$$\frac{\partial^2}{\partial \hat{s}^2} \psi \equiv \vec{\nabla}_{0(\tau)}^2 \psi \doteq -\frac{1}{12} \mathcal{P} \psi. \quad (88)$$

Since $\vec{\nabla}_{0(\tau)} = [\partial/\partial(c\tau)]$ from (A25), thus, after introducing $\mathcal{P} = \mathcal{A} \psi^{\mu\nu} \mathcal{P}_{\mu\nu}$ into (88), the RHS two relations in (88) transform into,

$$\begin{aligned} \frac{\partial^2}{\partial \tau^2} \mathcal{A} \psi_{\mu\nu} \psi &= -\frac{c^2}{3} \mathcal{P}_{\mu\nu} \psi \quad [\text{for (18)}], \\ \therefore \frac{\partial}{\partial \tau} \mathcal{A} \psi_{\mu\nu} \psi &= -\frac{c^2 \tau}{3} \mathcal{P}_{\mu\nu} \psi, \end{aligned}$$

or alternatively,

$$i \hbar \frac{\partial}{\partial t} \mathcal{A} \psi_{\mu\nu} \psi = -(i \hbar)^{-1} \frac{c^2 t}{3 \gamma_F^2} \mathcal{P}_{\mu\nu} \psi \quad [\text{for } \tau = (i \hbar)^{-1} \gamma_F^{-1} t],$$

or since $(i \hbar)^{-1} x^0 = (i \hbar)^{-1} (c t)$, let $(i \hbar)^{-1} \gamma_F^{-1} x^0 = (i \hbar)^{-1} x'^0$, then,

$$\begin{aligned} i \hbar \frac{\partial}{\partial t} \mathcal{A} \psi_{\mu\nu} \psi &= \frac{i}{\hbar} \frac{c x'^0}{3 \gamma_F} \mathcal{P}_{\mu\nu} \psi, \\ \text{or, } i \hbar \frac{\partial}{\partial x'^0} \mathcal{A} \psi_{\mu\nu} \psi &= \frac{i}{\hbar} \frac{x'^0}{3} \mathcal{P}_{\mu\nu} \psi. \end{aligned} \quad (89)$$

Suppose, the inverse operator of the momentum operator in $(3+1)D$ curvilinear quantum spacetime is [21],

$$\frac{1}{\hat{p}'_0} \psi(x'^0) = \frac{i}{\hbar} \int_{-\infty}^{x'^0} dy'^0 \psi(y'^0),$$

hence,

$$\frac{i}{\hbar} x'^0 \psi = \frac{i}{\hbar} \int_{-\infty}^{x'^0} dy'^0 \psi(y'^0) = \frac{1}{\hat{p}'_0} \psi(x'^0),$$

provided ψ vanishes at $-\infty$. In such a case,

$$\partial_{x'^0} \int_{-\infty}^{x'^0} dy'^0 = \int_{-\infty}^{x'^0} dy'^0 \partial_{x'^0} = 1.$$

Then (89) yields,

$$i \hbar \frac{\partial}{\partial x'^0} \mathcal{A} \psi_{\mu\nu} \psi = \frac{1}{3 \hat{p}'_0} \mathcal{P}_{\mu\nu} \psi' \quad [\text{for } \psi' = \psi(x'^0)]. \quad (90)$$

Thus, the evolution equation (90) is the flow of the metric $\mathcal{A} \psi_{\mu\nu}$ by the quantum curvature $\mathcal{P}_{\mu\nu}$ somehow analogous to the unnormalized Hamilton's Ricci Flow [18], but since $(3 \hat{p}'_0)^{-1} \neq 2$, the flow equation (90) is not exactly Hamilton's Ricci Flow but a new kind of (Quantum) Ricci-like Flow. Let us simply name it as a Quantum Metric Flow in conjugated fermionic fields.

Theorem 10 (Duality of Quantum Gravitational Field and Fermionic Field). A non-vacuum quantum gravitational field equation in $(3+1)D$ curvilinear quantum spacetime can metamorphose into the Dirac-like equation (68) for free fermions.

Proof. Let us assume a simplest system of linear homogeneous differential equations of first order which meets the conditions of (80) as,

$$\frac{\partial}{\partial \hat{s}} \psi = i \hbar \hat{\epsilon}^{\mu} \vec{\nabla}_{\mu} \psi \doteq \frac{i}{2\sqrt{3}} \sqrt{\mathcal{P}} \psi, \quad (91)$$

then it yields a spinor field representation with quantum scalar curvature as,

$$\left[i \hbar \hat{\epsilon}^{\mu} \vec{\nabla}_{\mu} - \frac{i}{2\sqrt{3}} \sqrt{\mathcal{P}} \right] \psi = 0, \quad (92)$$

for the construction $\psi = (\varphi, \varphi_{\nu})$ when the spin of the particle is $(s = 1/2, s = 3/2)$ with $\partial^{\nu} \varphi_{\nu} = 0$ and $\hat{\epsilon}^{\nu} \varphi_{\nu} = 0$. By comparing (92) with (68), we can get,

$$\begin{aligned} \mathcal{A}\psi^{\mu\nu} \gamma_{\mathbb{F}} m c \psi &= \frac{i}{2\sqrt{3}} \sqrt{\mathcal{P}} \psi, \\ \Rightarrow \quad \mathcal{A}\psi^{\mu\nu} \Theta_{\mathbb{F}} m \psi &= \frac{i}{2\sqrt{3}} \sqrt{\mathcal{P}} \psi \quad [\text{for (A23)}], \\ \therefore \quad m_0 \psi &= \frac{i}{2\sqrt{3}} \frac{1}{\mathcal{A}\psi^{\mu\nu} \Theta_{\mathbb{F}} \gamma_{\mathbb{F}}} \sqrt{\mathcal{P}} \psi \quad [\text{for (A20)}]. \end{aligned} \quad (93)$$

Comparing $m_0 = i \hbar \Theta_{\mathbb{F}}^{-1} \partial_0$ from (A24) with (93), we have,

$$\sqrt{\mathcal{P}} \psi = -i 2 \sqrt{3} \mathcal{A}\psi^{\mu\nu} i \hbar \gamma_{\mathbb{F}} \partial_0 \psi,$$

and with the proper time $\tau = (i \hbar)^{-1} \gamma_{\mathbb{F}}^{-1} t$ it becomes for (A25) as,

$$\sqrt{\mathcal{P}} \psi = -i 2 \sqrt{3} \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)} \psi, \quad (94)$$

which appears like as a quantum ‘operator’.

Squaring up (94), we get,

$$\begin{aligned} \mathcal{P} \psi &= -12 (\mathcal{A}\psi^{\mu\nu})^2 \vec{\nabla}_{0(\tau)}^2 \psi, \\ \therefore \quad \mathcal{A}\psi_{\mu\nu} \mathcal{P} \psi &= -48 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)}^2 \psi \quad [\text{for (18)}], \end{aligned} \quad (95)$$

so, by using $\mathcal{P} = \mathcal{A}\psi^{\mu\nu} \mathcal{P}_{\mu\nu}$, we can get (95) as,

$$\mathcal{P}_{\mu\nu} \psi = -12 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)}^2 \psi \quad [\text{for (18)}], \quad (96)$$

then we can have a modification of the non-vacuum quantum gravitational field equation (75) in $(3+1)D$ curvilinear quantum spacetime by using (96) and (95) as,

$$\begin{aligned} -12 \left[\mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)}^2 - 2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)}^2 \right] \psi + \mathcal{A}\psi_{\mu\nu} \Lambda' \psi &= \kappa \mathcal{T}_{\mu\nu} \psi \quad [\text{for (18)}], \\ \therefore \quad \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)}^2 \psi + \frac{1}{12} \mathcal{A}\psi_{\mu\nu} \Lambda' \psi &= \frac{\kappa}{12} \mathcal{T}_{\mu\nu} \psi, \end{aligned} \quad (97)$$

hence, by multiplying (97) with $\mathcal{A}\psi^{\mu\nu}$ and by reusing the proper time $\tau = (i \hbar)^{-1} \gamma_{\mathbb{F}}^{-1} t$, we can get,

$$(i \hbar)^2 (\mathcal{A}\psi^{\mu\nu})^2 \gamma_{\mathbb{F}}^2 \partial_0^2 \psi + \frac{1}{3} \Lambda' \psi = \frac{\kappa}{12} \mathcal{A}\psi^{\mu\nu} \mathcal{T}_{\mu\nu} \psi \quad [\text{for (18)}]. \quad (98)$$

The simplest system of linear homogeneous differential equations of first order which meets the conditions of (98) reads for $\mathcal{T}_{\mu\nu} = (i\hbar)^2 T_{\mu\nu}$ as,

$$i\hbar \leq^{\mu} \mathcal{V}^{\mu\nu} \gamma_F \partial_0 \psi + \left(\frac{1}{3} \Lambda'\right)^{1/2} \psi = i\hbar \leq^{\mu} \frac{\sqrt{\kappa}}{2\sqrt{3}} T_{\mu\nu}^{1/2} \psi. \quad (99)$$

By the way, another non-vacuum quantum gravitational field equation for spinor in $(3+1)D$ curvilinear quantum spacetime is possible to be derived by the simplest system of linear homogeneous differential equations of first order which meets the conditions of (83) reads for $\mathcal{T}_{\mu\nu} = (i\hbar)^2 T_{\mu\nu}$ as,

$$i\hbar \leq^{\mu} \vec{\nabla}_{\mu} \psi + \left(\frac{1}{3} \Lambda'\right)^{1/2} \psi = i\hbar \leq^{\mu} \frac{\sqrt{\kappa}}{2\sqrt{3}} T_{\mu\nu}^{1/2} \psi. \quad (100)$$

Similarly, the simplest system of linear homogeneous differential equations of first order which meets the conditions of (85) reads for $\mathcal{T}_{\mu\nu} = (i\hbar)^2 T_{\mu\nu}$ as,

$$\mathcal{V}^{\mu\nu} \gamma_F m c \psi + \left(\frac{1}{3} \Lambda'\right)^{1/2} \psi = i\hbar \leq^{\mu} \frac{\sqrt{\kappa}}{2\sqrt{3}} T_{\mu\nu}^{1/2} \psi. \quad (101)$$

Comparing (100) with (99) and (101), we have,

$$i\hbar \leq^{\mu} \vec{\nabla}_{\mu} \psi \doteq i\hbar \leq^{\mu} \mathcal{V}^{\mu\nu} \gamma_F \partial_0 \psi \doteq \mathcal{V}^{\mu\nu} \gamma_F m c \psi, \quad (102)$$

then, we can recover a first order field equation from (102) as,

$$\left[i\hbar \leq^{\mu} \vec{\nabla}_{\mu} - \mathcal{V}^{\mu\nu} \gamma_F m c \right] \psi = 0, \quad (103)$$

which is nothing but the Dirac-like equation (68) for free fermions. This completes the proof. $\square \square$

Remark 8. Other two first order field equations from (102) are possible to write as,

$$\begin{aligned} \left[i\hbar \leq^{\mu} \vec{\nabla}_{\mu} - i\hbar \leq^{\mu} \mathcal{V}^{\mu\nu} \gamma_F \partial_0 \right] \psi &= 0, \\ \left[i\hbar \leq^{\mu} \mathcal{V}^{\mu\nu} \gamma_F \partial_0 - \mathcal{V}^{\mu\nu} \gamma_F m c \right] \psi &= 0. \end{aligned}$$

At present, we do not know what are their purposes actually.

Since, (68), more precisely (69), is nothing but a gravitational field equation for free fermions as we have seen in Theorem 10 and on the other hand, (68) is nothing but a ‘Special’-within-Special Relativistic QFT equation because of γ_F , then, with a little bravery, we can say that: *a ‘Special’-within-Special frame of relativity makes a General frame of relativity emerge.*

But (68) (so as (A28) below) is **not** technically a Dirac equation since it has a direct relation with the gravitational field equation of General (Quantum) Relativity for fermions. Though, for practical purposes, we shall **never** use any of these ‘dummy’ Dirac-like equations, instead we shall use a properly developed, local non-vacuum quantum gravitational field equation (117), see below, for fermions, which is effectively applicable to various particle interactions in $(3+1)D$ curvilinear quantum spacetime (see Section 9 below).

Definition 14 ((Quantum) Ricci-like Flow, or Quantum Metric Flow in Fermionic Fields). *Combination of (A26) from Appendix A.2 with (A25) to have $\Theta_F m \psi = i\hbar \leq^{\mu} \vec{\nabla}_{\mu} \psi = i\hbar \gamma_F \partial_0 \psi \equiv \vec{\nabla}_{0(\tau)} \psi$ can be used to modify (91) in such a way that,*

$$\frac{\partial}{\partial \hat{s}} \psi = i\hbar \leq^{\mu} \vec{\nabla}_{\mu} \psi \equiv i\hbar \gamma_F \partial_0 \psi \equiv \vec{\nabla}_{0(\tau)} \psi \doteq \frac{i}{2\sqrt{3}} \sqrt{\mathcal{P}} \psi, \quad (104)$$

thus, after introducing $\mathcal{P} = \mathcal{A}^{\mu\nu} \mathcal{P}_{\mu\nu}$ into (104), the RHS last two terms in (104) transform into the following as,

$$\begin{aligned} \frac{\partial}{\partial \tau} \sqrt{\mathcal{A}^{\mu\nu}} \psi &= i \frac{c}{\sqrt{3}} \sqrt{\mathcal{P}_{\mu\nu}} \psi \quad [\text{for (18)}], \\ \therefore i \hbar \frac{\partial}{\partial t} \sqrt{\mathcal{A}^{\mu\nu}} \psi &= i \frac{c}{\sqrt{3} \gamma_F} \sqrt{\mathcal{P}_{\mu\nu}} \psi \quad [\text{for } \tau = (i \hbar)^{-1} \gamma_F^{-1} t], \end{aligned}$$

or since $(i \hbar)^{-1} x^0 = (i \hbar)^{-1} (c t)$, let $(i \hbar)^{-1} \gamma_F^{-1} x^0 = (i \hbar)^{-1} x'^0$, then,

$$i \hbar \frac{\partial}{\partial x'^0} \sqrt{\mathcal{A}^{\mu\nu}} \psi = \frac{i}{\sqrt{3}} \sqrt{\mathcal{P}_{\mu\nu}} \psi,$$

which is the flow of the metric $\mathcal{A}^{\mu\nu}_{1/2}$ by the quantum curvature $\mathcal{P}_{\mu\nu}^{1/2}$. Let us name it as a Quantum Metric Flow in fermionic fields.

Definition 15 (Dark Energy and Its Quantum Factor $\gamma_{F\Lambda}$). The cosmological constant is $\Lambda = [(G M_{\text{PL}})/(\ell_{\text{PL}}^2 c^2)]^2$ at the Planck scale for the Planck length ℓ_{PL} and the Planck mass M_{PL} [22]. Since the Planck length ℓ_{PL} is a unit of length introduced in the system of Planck units and the Planck length can acquire a physical meaning in the framework of quantum gravity [23], let us assume a line element as follows for the temporal axis $(i \hbar)^{-1} \ell_{\text{PL}}^0 = (i \hbar)^{-1} c t$ if t is not corresponding to Planck time t_{PL} , i.e., $t \neq t_{\text{PL}}$, but t satisfies the transformation $\{t_{\text{initial}}|_{t \rightarrow 0} \rightarrow t_{\text{final}}|_{t=1}\}$ (see Subsection 9.2 below for more details) as,

$$\frac{\partial^2}{\partial \hat{s}_\Lambda^2} \psi = (i \hbar)^2 \mathcal{A}^{\mu\nu} \frac{(v_{\text{PL}}^\mu v_{\text{PL}}^\nu)^2}{c^4} \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \frac{\partial}{\partial \ell_{\text{PL}}^\nu} \psi \equiv (i \hbar)^2 \Lambda \psi \doteq \Lambda' \psi, \quad (105)$$

where, let us consider that,

$$\frac{(v_{\text{PL}}^\mu v_{\text{PL}}^\nu)}{c^2} = \eta_\mu, \quad \text{while} \quad v_{\text{PL}}^\mu = \frac{d \ell_{\text{PL}}^\mu}{d t}, \quad (106)$$

for the four-velocity v_{PL}^μ . Admitting Claim 4 that Dark Energy may originate from matter fields, let us state that the simplest system of linear homogeneous differential equations of first order which meets the conditions of (105) as,

$$\frac{\partial}{\partial \hat{s}_\Lambda} \psi = i \hbar \mathbb{E}^\mu \eta_\mu \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \psi \equiv i \hbar \Lambda^{1/2} \psi \doteq (\Lambda')^{1/2} \psi. \quad (107)$$

Note that $i \hbar \mathbb{E}^\mu \eta_\mu (\partial/\partial \ell_{\text{PL}}^\mu) \psi$ has the dimension of momentum as $[M L T^{-1}]$. From (107), we can also write,

$$\begin{aligned} \frac{\partial}{\partial \hat{s}_\Lambda} \psi &= i \hbar \mathbb{E}^\mu \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \frac{(v_{\text{PL}}^\mu v_{\text{PL}}^\nu)}{c^2} \psi \\ &\equiv i \hbar \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \left[\mathbb{E}^0 \frac{(v_{\text{PL}}^0 v_{\text{PL}}^0)}{c^2} + \mathbb{E}^i \frac{(v_{\text{PL}}^i v_{\text{PL}}^j)}{c^2} \right] \psi \\ &= i \hbar \frac{\partial}{\partial \ell_{\text{PL}}^\mu} [\mathbb{E}^0 \eta_0 + \mathbb{E}^i \eta_i] \psi \equiv i \hbar \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \gamma_{F\Lambda}^{-1} \psi, \end{aligned}$$

for the temporal axis $(i \hbar)^{-1} L_{\text{PL}}^0 = (i \hbar)^{-1} c t$, when $\gamma_{F\Lambda} \neq \gamma_F$, thus, t must satisfy the proper time $\tau_\Lambda = (i \hbar)^{-1} \gamma_{F\Lambda}^{-1} t$. It must be remembered that,

Claim 5. Dark Energy is strongly sterile and never reacts with other physical particles anyway, even it never shows any self-interactions either.

Now, the insertion of (107) into (100) yields,

$$\left[i \hbar \leq^{\mu} \vec{\nabla}_{\mu} + \frac{1}{\sqrt{3}} i \hbar \leq^{\mu} \eta_{\mu} \frac{\partial}{\partial \ell_{\text{PL}}^{\mu}} - i \hbar \leq^{\mu} \frac{\sqrt{\kappa}}{2\sqrt{3}} T_{\mu\nu}^{1/2} \right] \psi = 0. \quad (108)$$

Though (108) looks quite promising, but it does not contain any $\leq^{\mu}$ -free mass term $m c$ as like as the Dirac-like equation (68). Thus, the structure of (108) is not complete yet, so we need some more crucial discussions as follows.

6. Generalized Quantum Gravitational Field Theory in Fermionic Scenarios

Definition 16 (Generalized Quantum Gravitational Conjugated Fermionic Fields). *To complete the General Relativity as a logical structure, we used to set the equation of motion for the gravitational field as, $R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$, in practice (see [20,24,25], for example). Likewise, from (75), it implies the equation of motion for the quantum gravitational field as,*

$$- \mathcal{P} \psi = [\kappa \mathcal{T} - 4 \Lambda'] \psi \quad [\text{for (18)}], \quad (109)$$

$$\text{thus, } \mathcal{P}_{\mu\nu} \psi = \left[\kappa \left(\mathcal{T}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{T} \right) + \mathcal{V}_{\mu\nu} \Lambda' \right] \psi \quad [\text{for (75)}]. \quad (110)$$

Insertion of (82) and then use of the first line with the last line of (78) along with (85) and (18) into (110) yields,

$$\begin{aligned} & - \frac{3}{4} \left[(i \hbar)^2 \mathcal{V}_{\mu\nu} \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} - \frac{1}{4} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi \\ & = \left[3 \mathcal{V}_{\mu\nu} (\mathcal{V}^{\mu\nu} \gamma_F m c)^2 - 6 \mathcal{V}_{\mu\nu} (\mathcal{V}^{\mu\nu} \gamma_F m c)^2 + \right. \\ & \quad \left. + \mathcal{V}_{\mu\nu} \Lambda' - 2 \mathcal{V}_{\mu\nu} \Lambda' \right] \psi + \mathcal{V}_{\mu\nu} \Lambda' \psi, \\ \therefore \quad (i \hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} \psi - \frac{1}{4} \mathcal{P} \psi & = 4 (\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi, \end{aligned} \quad (111)$$

hence, for (109), it yields,

$$\left[(i \hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} + \frac{\kappa}{4} \mathcal{T} - \Lambda' \right] \psi + 4 (i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi = 0, \quad (112)$$

thus, by inserting (105) and $\mathcal{T}_{\mu\nu} = (i \hbar)^2 T_{\mu\nu}$, we have,

$$\begin{aligned} & \left[(i \hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} + (i \hbar)^2 \frac{\kappa}{4} \mathcal{V}^{\mu\nu} T_{\mu\nu} - (i \hbar)^2 \eta_{\mu}^2 \mathcal{V}^{\mu\nu} \frac{\partial}{\partial \ell_{\text{PL}}^{\mu}} \frac{\partial}{\partial \ell_{\text{PL}}^{\nu}} \right] \psi + \\ & + (2 i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi = 0, \end{aligned} \quad (113)$$

which is a non-vacuum quantum gravitational field equation for conjugated fermions. Thus, (113) actually gives,

$$\begin{aligned} & \left[(i \hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} + (i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi + \\ & + \left[(i \hbar)^2 \frac{\kappa}{4} \mathcal{V}^{\mu\nu} T_{\mu\nu} + (i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi - \\ & - \left[(i \hbar)^2 \eta_{\mu}^2 \mathcal{V}^{\mu\nu} \frac{\partial}{\partial \ell_{\text{PL}}^{\mu}} \frac{\partial}{\partial \ell_{\text{PL}}^{\nu}} - (i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi + \\ & + (i \mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi = 0. \end{aligned} \quad (114)$$

The positive energy-momentum tensor $(i \hbar)^2 (\kappa/4) \mathcal{V}^{\mu\nu} T_{\mu\nu}$ in (114) guarantees that gravity obtained from this conjugated fermionic field is always attractive, on the other hand, negative value of $-(i \hbar)^2 \eta_{\mu}^2 \mathcal{V}^{\mu\nu} (\partial/\partial \ell_{\text{PL}}^{\mu})$

$(\partial/\partial \ell_{\text{PL}}^\nu)$ therein guarantees that Dark Energy obtained from the same conjugated fermionic field is always repulsive.

Importantly, (113) is exactly equivalent to (110) for the geometry coupled to matter. Instead of (75) and (83) or (85), it would be appropriate if we consider (114) as the non-vacuum quantum gravitational field equation for conjugated fermions in $(3+1)D$ curvilinear quantum spacetime.

Definition 17 (Equations of Gravity and Dark Energy Exclusivity). *Comparison between (83) and (113) directly yields,*

$$\left[(i\hbar)^2 \frac{\kappa}{12} \mathcal{V}^{\mu\nu} T_{\mu\nu} - \frac{1}{3} (i\hbar)^2 \eta_\mu^2 \mathcal{V}^{\mu\nu} \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \frac{\partial}{\partial \ell_{\text{PL}}^\nu} + (i\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi = 0, \quad (115)$$

which is an excellent equation of conjugated fermionic fields exclusively for gravity and Dark Energy. By the way, we may also have a simplest system of linear homogeneous differential equation of first order which meets the conditions of (115) as,

$$\left[i\hbar \not{\nabla}^\mu \frac{\sqrt{\kappa}}{2\sqrt{3}} T_{\mu\nu}^{1/2} - \frac{1}{\sqrt{3}} i\hbar \not{\nabla}^\mu \eta_\mu \frac{\partial}{\partial \ell_{\text{PL}}^\mu} - \mathcal{V}^{\mu\nu} \gamma_F m c \right] \psi = 0,$$

which is equivalent to (101).

Remark 9. Since $\mathcal{V}^{\mu\nu} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P} \right] = -\mathcal{P}$, we can also transform (111) into,

$$\begin{aligned} (i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi + \frac{1}{4} \mathcal{V}^{\mu\nu} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi + (2i\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \psi &= 0, \\ \therefore \left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + \frac{1}{4} \mathcal{V}^{\mu\nu} G_{\mu\nu} + (2i\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi &= 0, \end{aligned} \quad (116)$$

which is a direct relation between Einstein-like tensor $G_{\mu\nu} = \mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P}$ and Kline-Gordon-like equation in $(3+1)D$ curvilinear quantum spacetime. More precisely, (112) or (116) yields,

$$\begin{aligned} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{V}_{\mu\nu} \mathcal{P} \right] \psi &= \kappa T_{\mu\nu} \psi - \mathcal{V}_{\mu\nu} \Lambda' \psi \\ &= -\mathcal{V}_{\mu\nu} \left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (2i\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi \\ &\quad \text{[for (18)]} \\ &\equiv \mathcal{V}_{\mu\nu} \left[\hbar^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (2\mathcal{V}^{\mu\nu} \gamma_F m c)^2 \right] \psi. \end{aligned}$$

Thus, we have an exact correspondence between Quantum Differential Geometry and conjugated fermionic Quantum Field Theory.

Definition 18 (Generalized Quantum Gravitational Fermionic Fields). *The simplest system of linear homogeneous differential equation of first order, which meets the conditions of (113) while satisfying fermionic fields causing QCD, gravity (GRV) and Dark Energy (DE), is as,*

$$\begin{aligned} &\left[i\hbar \not{\nabla}^\mu \vec{\nabla}_\mu + i\hbar \not{\nabla}^\mu \frac{\sqrt{\kappa}}{2} T_{\mu\nu}^{1/2} - i\hbar \not{\nabla}^\mu \eta_\mu \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \right] \psi - 2\mathcal{V}^{\mu\nu} \gamma_F m c \psi = 0, \\ \text{or, } &\left[i\hbar \not{\nabla}^\mu \vec{\nabla}_\mu - \mathcal{V}^{\mu\nu} \gamma_F m c \right]_{\text{causing QCD}} \psi + \\ &+ \left[i\hbar \not{\nabla}^\mu \frac{\sqrt{\kappa}}{2} T_{\mu\nu}^{1/2} - i\hbar \not{\nabla}^\mu \eta_\mu \frac{\partial}{\partial \ell_{\text{PL}}^\mu} - \mathcal{V}^{\mu\nu} \gamma_F m c \right]_{\text{causing GRV, DE}} \psi = 0. \end{aligned} \quad (117)$$

The structure of (117) is an excellent evidence why we should consider (117) instead of (108). It is the best local non-vacuum quantum gravitational field equation for fermions, which will be effectively applicable to various particle interactions in $(3+1)D$ curvilinear quantum spacetime (see Section 9 below).

7. Generalized Quantum Gravitational Field Theory in Bosonic Scenarios

Since the Kline-Gordon-like equation (63) for conjugated fermions arises in Theorem 9 from (79), which is actually a distorted form of (75) then the universality of (75) assures us that no matter how (79) has been emerged from conjugated fermions, it can be always valid for all bosons (i.e., not just for conjugated fermions exclusively). Since $\mathcal{A}\psi^{\mu\nu} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{A}\psi_{\mu\nu} \mathcal{P} \right] \psi = -\mathcal{P} \psi$ for (18), let us consider the bosonic version analogous to (79) as,

$$-\mathcal{P} \psi - 12 (i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi = 0, \quad (118)$$

$$\therefore \mathcal{A}\psi^{\mu\nu} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{A}\psi_{\mu\nu} \mathcal{P} \right] \psi - 12 (i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi = 0. \quad (119)$$

By comparing (118) with (62) and for the proper time $\tau = (i\hbar)^{-1} \gamma_B^{-1} t$, we can get,

$$\begin{aligned} \mathcal{P} \psi &= 12 (i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \psi, \\ \Rightarrow \mathcal{P} \psi &= 12 (i\mathcal{A}\psi^{\mu\nu} \Theta_B m)^2 \psi \quad [\text{for (A9)}], \\ \therefore \mathcal{P} \psi &= 12 \left(i\mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)} \right)^2 \psi \quad [\text{for (A13)}], \\ \text{hence, } \sqrt{\mathcal{P}} \psi &= i 2 \sqrt{3} \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_{0(\tau)} \psi, \end{aligned} \quad (120)$$

which appears like as a quantum ‘operator’.

Theorem 11 (Duality of Quantum Gravitational Field and Bosonic Field). *A non-vacuum quantum gravitational field equation in $(3+1)D$ curvilinear quantum spacetime can metamorphose into the Kline-Gordon-like equation (62) for bosons.*

Proof. Comparison between (119), (75) and (62) yields,

$$\begin{aligned} 12(i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \psi - 4\Lambda' \psi + \kappa \mathcal{A}\psi^{\mu\nu} \mathcal{T}_{\mu\nu} \psi &= 0 \quad [\text{for (18)}], \\ \text{i.e., } 3(i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \psi - \Lambda' \psi + \frac{\kappa}{4} \mathcal{A}\psi^{\mu\nu} \mathcal{T}_{\mu\nu} \psi &= 0. \end{aligned} \quad (121)$$

Addition of (119) with the first equation of (121) implies that,

$$\begin{aligned} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{A}\psi_{\mu\nu} \mathcal{P} \right] \psi + \mathcal{A}\psi_{\mu\nu} \Lambda' \psi - \kappa \mathcal{T}_{\mu\nu} \psi \\ = 3\mathcal{A}\psi_{\mu\nu} \left[(i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \right] \psi, \end{aligned} \quad (122)$$

or alternatively for (18),

$$\begin{aligned} -\frac{1}{12} \mathcal{P} \psi + \frac{1}{3} \Lambda' \psi - \frac{\kappa}{12} \mathcal{A}\psi^{\mu\nu} \mathcal{T}_{\mu\nu} \psi \\ = \left[(i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \right] \psi. \end{aligned}$$

This completes the proof. $\square \square$

Corollary 1. *The bosonic version analogous to (79) in $(3+1)D$ curvilinear quantum spacetime yields the Kline-Gordon-like equation (62) for bosons.*

So, we have seen that a quantum gravitational field equation is easily deductible into a Kline-Gordon-like equation (63), or (62), in Theorem 9, or Theorem 11, respectively, establishing that a Kline-Gordon-like equation (63), or (62), is itself nothing but a quantum gravitational field equation analogous to Einstein's field equation in General Relativity. Thus, (62) (so as (A12), or better to say that conjugated spinorial (63) along with (A31) for example) is **not** technically a Kline-Gordon equation since it has a direct relation with the gravitational field equation of General (Quantum) Relativity (as we have seen in Theorem 11, or Theorem 9, above). Though, for practical purposes, we shall **never** use any of these 'dummy' Kline-Gordon-like equations, instead we shall use a properly developed, local non-vacuum quantum gravitational field equation (124), see below, for bosons, which is effectively applicable to various particle interactions in $(3+1)D$ curvilinear quantum spacetime (see Section 9 below).

Furthermore, by simply adding (62) with the second equation of (121), we can get,

$$\left[(i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + \frac{\kappa}{4} \mathcal{A}\psi^{\mu\nu} \mathcal{T}_{\mu\nu} - \Lambda' \right] \psi + (2i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \psi = 0, \quad (123)$$

and the insertion of (105) and $\mathcal{T}_{\mu\nu} = (i\hbar)^2 T_{\mu\nu}$ into it yields,

$$\begin{aligned} & \left[(i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i\hbar)^2 \frac{\kappa}{4} \mathcal{A}\psi^{\mu\nu} T_{\mu\nu} - (i\hbar)^2 \eta_\mu^2 \mathcal{A}\psi^{\mu\nu} \frac{\partial}{\partial \ell_{PL}^\mu} \frac{\partial}{\partial \ell_{PL}^\nu} \right] \psi + \\ & \quad + (2i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \psi = 0, \\ \text{or, } & \left[(i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \right]_{\text{for QCD}} \psi + \\ & \quad + \left[(i\hbar)^2 \frac{\kappa}{4} \mathcal{A}\psi^{\mu\nu} T_{\mu\nu} + (i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \right]_{\text{for GRV}} \psi - \\ & \quad - \left[(i\hbar)^2 \eta_\mu^2 \mathcal{A}\psi^{\mu\nu} \frac{\partial}{\partial \ell_{PL}^\mu} \frac{\partial}{\partial \ell_{PL}^\nu} - (i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \right]_{\text{for DE}} \psi + \\ & \quad + (i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2_{\text{BAC}} \psi = 0, \end{aligned} \quad (124)$$

which is a non-vacuum quantum gravitational field equation for bosons, whereas (113) is the non-vacuum quantum gravitational field equation for conjugated fermions. The structure of (124) insists us to determine,

Testable Prediction 4. *Color force, gravity and Dark Energy interaction are simultaneous but not inter-dependent, while they all require an additional background mass field $(i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2_{\text{BAC}}$ to perform their roles.*

Additionally, (123) also yields,

$$\begin{aligned} \left[\mathcal{P}_{\mu\nu} - \frac{1}{2} \mathcal{A}\psi_{\mu\nu} \mathcal{P} \right] \psi &= \kappa \mathcal{T}_{\mu\nu} \psi - \mathcal{A}\psi_{\mu\nu} \Lambda' \psi \\ &= -\mathcal{A}\psi_{\mu\nu} \left[(i\hbar)^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (2i\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \right] \psi \\ & \quad \text{[for (18)]} \\ &\equiv \mathcal{A}\psi_{\mu\nu} \left[\hbar^2 \mathcal{A}\psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (2\mathcal{A}\psi^{\mu\nu} \gamma_B m c)^2 \right] \psi. \end{aligned}$$

Thus, we have an exact correspondence between Quantum Differential Geometry and bosonic Quantum Field Theory.

8. Fermionic and Bosonic Mass-Spacetime Uncertainties and Geometries of Their Masses

8.1. Fermionic Scenarios

Let us assume that the quantum relativistic fermion mass $m = \gamma_F m_0$ for (A20), where $m_0 = \Theta_F^{-1} i \hbar \partial_0$ for (A24) and $\Theta_F = \gamma_F c$ for (A22), then it yields the following relation due to $(i \hbar)^{-1} x^0 = (i \hbar)^{-1} (c t)$ as,

$$m = \Theta_F^{-1} i \hbar \gamma_F \partial_0 \doteq \frac{i \hbar}{c^2} \frac{\partial}{\partial t}, \quad (125)$$

which appears like as a quantum 'operator'. Then, we can say that,

1. (quantum) relativistic mass and time 'operators' m and t in (125) are unbounded operators, similarly,
2. quantum scalar curvature and the proper time 'operators' $\sqrt{\mathcal{P}}$ and τ in (94), are also unbounded operators.

Let they define on suitable dense subspaces, suppose, $\text{Dom}(m)$, $\text{Dom}(t)$ and $\text{Dom}((2\sqrt{3})^{-1} \sqrt{\mathcal{P}})$, $\text{Dom}(\tau)$ of $L^2(\mathbb{R}^4)$.

Proposition 3. *The (quantum) relativistic mass and time operators m and t , similarly, quantum scalar curvature and the proper time operators $\sqrt{\mathcal{P}}$ and τ , commute canonically for $c = 1$, i.e.,*

$$(m t - t m) = i \hbar I, \quad \text{and} \quad \frac{1}{2\sqrt{3}} [\sqrt{\mathcal{P}} \tau - \tau \sqrt{\mathcal{P}}] = -i \mathcal{V}^{\mu\nu} I.$$

Using [11]. Using the product rule we may calculate that,

$$\begin{aligned} m t \psi &= \frac{i \hbar}{c^2} \frac{\partial}{\partial t} (t \psi(t)) \\ &= \frac{i \hbar}{c^2} \psi(t) + \frac{i \hbar}{c^2} t \frac{\partial \psi}{\partial t} \\ &= \frac{i \hbar}{c^2} \psi(t) + t m \psi, \end{aligned}$$

and for the proper time $\tau = (i \hbar)^{-1} \gamma_F^{-1} t$,

$$\begin{aligned} \sqrt{\mathcal{P}} \tau \psi &= -i 2 \sqrt{3} \mathcal{V}^{\mu\nu} \frac{\partial}{\partial (c \tau)} (\tau \psi(\tau)) \\ &= -i \frac{2 \sqrt{3}}{c} \mathcal{V}^{\mu\nu} \psi(\tau) - i \frac{2 \sqrt{3}}{c} \mathcal{V}^{\mu\nu} \tau \frac{\partial \psi}{\partial \tau} \\ &= -i \frac{2 \sqrt{3}}{c} \mathcal{V}^{\mu\nu} \psi(t) + \tau \sqrt{\mathcal{P}} \psi. \end{aligned}$$

Let $c = 1$, then the canonical commutation relations are $[m, t] = i \hbar I$ and $(2\sqrt{3})^{-1} [\sqrt{\mathcal{P}}, \tau] = -i \mathcal{V}^{\mu\nu} I$. $\square \square$

Definition 19 (From [11]). *If A is a symmetric operator on $\mathcal{H}$, then for all unit vectors ψ in $\text{Dom}(A)$, the uncertainty $\Delta_\psi A$ of A in the state ψ is given by,*

$$(\Delta_\psi A)^2 = \langle (A - \langle A \rangle_\psi I) \psi, (A - \langle A \rangle_\psi I) \psi \rangle. \quad (126)$$

Theorem 12 (Uncertainty Relations for Fermions). *Suppose ψ is a unit vector in $L^2(\mathbb{R}^4)$ belonging to $\text{Dom}(m_0) \cap \text{Dom}(t)$, or belonging to,*

$$\text{Dom}((2\sqrt{3})^{-1} \sqrt{\mathcal{P}}) \cap \text{Dom}(\tau),$$

then,

$$(\Delta_\psi m) (\Delta_\psi t) \geq \frac{\hbar}{2}, \quad \text{and} \quad \frac{1}{\sqrt{3}} (\Delta_\psi \sqrt{\mathcal{P}}) (\Delta_\psi \tau) \geq -\mathcal{V}^{\mu\nu}, \quad (127)$$

where $\Delta_\psi m$ and $\Delta_\psi t$, or $\Delta_\psi \sqrt{\mathcal{P}}$ and $\Delta_\psi \tau$, are given by the Definition 19.

Using [11]. For a particle moving in $\mathbb{R}^{(3+1)}$, let the quantum Hilbert space be $L^2(\mathbb{R}^4)$ and define the relativistic mass and time operators M and T , similarly quantum scalar curvature and the proper time operators P and $\mathcal{T}$, by

$$\begin{aligned} M\psi(t) &= \frac{i\hbar}{c^2} \frac{d\psi}{dt}, & T\psi(t) &= t\psi(t), \\ P\psi(\tau) &= -i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \frac{d\psi}{d\tau}, & \mathcal{T}\psi(\tau) &= \tau\psi(\tau). \end{aligned}$$

According to Stone's theorem, $\forall \psi \in \text{Dom}(m)$, or $\psi \in \text{Dom}((2\sqrt{3})^{-1}\sqrt{\mathcal{P}})$, we have,

$$\begin{aligned} (M\psi)(t) &= \frac{i\hbar}{c^2} \lim_{a \rightarrow 0} \frac{\psi(t+a) - \psi(t)}{a}, \\ (P\psi)(\tau) &= -i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \lim_{b \rightarrow 0} \frac{\psi(\tau+b) - \psi(\tau)}{b}. \end{aligned}$$

Thus,

$$\begin{aligned} \langle T\psi, M\psi \rangle &= \lim_{a \rightarrow 0} \left\langle T\psi, \frac{i\hbar}{c^2} \left(\frac{\psi(t+a) - \psi(t)}{a} \right) \right\rangle \\ &= \lim_{a \rightarrow 0} \left(\frac{1}{a} \left\langle \frac{i\hbar}{c^2} \psi(t+a), -t\psi(t) \right\rangle - \frac{i\hbar}{ac^2} \langle T\psi, \psi \rangle \right) \\ &= \lim_{a \rightarrow 0} \left(\frac{1}{a} \left\langle \psi(x), -\frac{i\hbar}{c^2} (x-a)\psi(x-a) \right\rangle - \frac{i\hbar}{ac^2} \langle T\psi, \psi \rangle \right), \\ \langle \mathcal{T}\psi, P\psi \rangle &= \lim_{b \rightarrow 0} \left\langle \mathcal{T}\psi, -i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \left(\frac{\psi(\tau+b) - \psi(\tau)}{b} \right) \right\rangle \\ &= \lim_{b \rightarrow 0} \left(\frac{1}{b} \left\langle -i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \psi(\tau+b), -\tau\psi(\tau) \right\rangle + i \frac{2\sqrt{3}}{bc} \mathcal{V}^{\mu\nu} \langle \mathcal{T}\psi, \psi \rangle \right) \\ &= \lim_{b \rightarrow 0} \left(\frac{1}{b} \left\langle \psi(y), i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} (y-b)\psi(y-b) \right\rangle + i \frac{2\sqrt{3}}{bc} \mathcal{V}^{\mu\nu} \langle \mathcal{T}\psi, \psi \rangle \right), \end{aligned}$$

if $x = t + a$ and $y = \tau + b$. Renaming the variable of integration back to t , or τ , we get,

$$\begin{aligned} \langle T\psi, M\psi \rangle &= \lim_{a \rightarrow 0} \left(\left\langle \frac{i\hbar}{c^2} T \left(\frac{\psi(t-a) - \psi(t)}{a} \right), \psi(t) \right\rangle - \frac{i\hbar}{c^2} \langle \psi(t-a), \psi(t) \rangle \right) \\ &= \lim_{a \rightarrow 0} \left(\left\langle \frac{i\hbar}{c^2} \left(\frac{\psi(t-a) - \psi(t)}{a} \right), T\psi(t) \right\rangle - \frac{i\hbar}{c^2} \langle \psi(t-a), \psi(t) \rangle \right) \\ &= \langle M\psi, T\psi \rangle - \frac{i\hbar}{c^2} \langle \psi, \psi \rangle, \\ \langle \mathcal{T}\psi, P\psi \rangle &= \lim_{b \rightarrow 0} \left(\left\langle -i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \mathcal{T} \left(\frac{\psi(\tau-b) - \psi(\tau)}{b} \right), \psi(\tau) \right\rangle + \right. \\ &\quad \left. + i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \langle \psi(\tau-b) - \psi(\tau) \rangle \right) \\ &= \lim_{b \rightarrow 0} \left(\left\langle -i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \left(\frac{\psi(\tau-b) - \psi(\tau)}{b} \right), \mathcal{T}\psi(\tau) \right\rangle + \right. \\ &\quad \left. + i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \langle \psi(\tau-b), \psi(\tau) \rangle \right) \\ &= \langle P\psi, \mathcal{T}\psi \rangle + i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \langle \psi, \psi \rangle, \end{aligned}$$

if $\psi \in \text{Dom}(t)$, then $\psi(t - a) \in \text{Dom}(t)$ for each fixed a , similarly, if $\psi \in \text{Dom}(\tau)$, then $\psi(\tau - b) \in \text{Dom}(\tau)$ for each fixed b . For any real constants α and β , similarly γ and δ , we have,

$$\begin{aligned} \langle (T - \alpha I) \psi, (M - \beta I) \psi \rangle &= \langle (M - \beta I) \psi, (T - \alpha I) \psi \rangle - \frac{i\hbar}{c^2} \langle \psi, \psi \rangle, \\ \langle (T - \gamma I) \psi, (P - \delta I) \psi \rangle &= \langle (P - \delta I) \psi, (T - \gamma I) \psi \rangle + i \frac{2\sqrt{3}}{c} \mathcal{V}^{\mu\nu} \langle \psi, \psi \rangle. \end{aligned} \quad (128)$$

Solving (128) for $\langle \psi, \psi \rangle$ gives,

$$\begin{aligned} \langle \psi, \psi \rangle &= \frac{c^2}{i\hbar} (\langle (M - \beta I) \psi, (T - \alpha I) \psi \rangle - \langle (T - \alpha I) \psi, (M - \beta I) \psi \rangle) \\ &= \frac{2}{\hbar} \text{Im} \langle (M - \beta I) \psi, (T - \alpha I) \psi \rangle \quad [\text{if } c = 1] \\ &\leq \frac{2}{\hbar} \| (M - \beta I) \psi \| \| (T - \alpha I) \psi \|, \\ \langle \psi, \psi \rangle &= -\frac{1}{i\mathcal{V}^{\mu\nu}} \frac{c}{2\sqrt{3}} (\langle (P - \delta I) \psi, (T - \gamma I) \psi \rangle - \langle (T - \gamma I) \psi, (P - \delta I) \psi \rangle) \\ &= -\frac{2}{\mathcal{V}^{\mu\nu}} \frac{1}{2\sqrt{3}} \text{Im} \langle (P - \delta I) \psi, (T - \gamma I) \psi \rangle \quad [\text{if } c = 1] \\ &\leq -\frac{2}{\mathcal{V}^{\mu\nu}} \frac{1}{2\sqrt{3}} \| (P - \delta I) \psi \| \| (T - \gamma I) \psi \|, \end{aligned}$$

by the Cauchy–Schwarz inequality. If ψ is a unit vector and we take $\alpha = \langle T \rangle_\psi$ and $\beta = \langle M \rangle_\psi$, then $\| (T - \alpha I) \psi \|^2 = (\Delta_\psi T)^2$ and $\| (M - \beta I) \psi \|^2 = (\Delta_\psi M)^2$, similarly we take $\gamma = \langle T \rangle_\psi$ and $\delta = \langle P \rangle_\psi$, then $\| (T - \gamma I) \psi \|^2 = (\Delta_\psi T)^2$ and $\| (P - \delta I) \psi \|^2 = (\Delta_\psi P)^2$. Thus, we get,

$$1 \leq \frac{2}{\hbar} (\Delta_\psi M) (\Delta_\psi T), \quad \text{and} \quad 1 \leq -\frac{1}{\mathcal{V}^{\mu\nu}} \frac{1}{\sqrt{3}} (\Delta_\psi P) (\Delta_\psi T).$$

This completes the proof. $\square \quad \square$

Definition 20 (Geometry of Fermionic Mass). *Insertion of (54) into (125) yields,*

$$\begin{aligned} m &= \frac{i\hbar}{c} \left(\frac{1}{(\gamma_F^2 - \mathbb{K}_4)} \frac{1}{2} \left\{ \begin{smallmatrix} i \\ \mp \end{smallmatrix} \right\} \partial_0^{-1} \partial_i \partial_j \right) \quad \forall i, j \in \{1, 2, 3\} \\ &\doteq \frac{i\hbar}{c} \left(\frac{c}{(\gamma_F^2 - \mathbb{K}_4) v^i} \frac{1}{2} \left\{ \begin{smallmatrix} i \\ \mp \end{smallmatrix} \right\} \partial_j \right) \quad [\text{for } \partial_0^{-1} \partial_i = (v^i)^{-1} c] \\ &= \frac{i\hbar}{c^2} \Phi_i^F \frac{1}{2} \left\{ \begin{smallmatrix} i \\ \mp \end{smallmatrix} \right\} \partial_j, \end{aligned} \quad (129)$$

where $\Phi_i^F = [(\gamma_F^2 - \mathbb{K}_4) v^i]^{-1} c^2$ is the stretching/shrinking parameters. Combining (125) with (129) we can get a first order spinor field representation as,

$$\begin{aligned} m \psi &= \frac{i\hbar}{c^2} \left(\Phi_0^F \frac{\partial}{\partial(ct)} + \Phi_i^F \frac{1}{2} \left\{ \begin{smallmatrix} i \\ \mp \end{smallmatrix} \right\} \partial_j \right) \psi \\ &\doteq \frac{i\hbar}{c^2} (\Phi_0^F \partial_0 - \Phi_i^F \partial_j) \psi = \frac{i\hbar}{c^2} \Phi_\mu^F \mathcal{V}^{\mu\nu} \vec{\nabla}_\nu \psi, \end{aligned} \quad (130)$$

if a stretching/shrinking parameters $\Phi_0^F = c$ and $i\hbar \Phi_\mu^F \mathcal{V}^{\mu\nu} \vec{\nabla}_\nu$ is the $(3+1)D$ stretched/shrank spacetime due to stretching/shrinking parameters Φ_μ^F , where,

$$\Phi_i^F = [(\gamma_F^2 - \mathbb{K}_4) v^i]^{-1} c^2 = (\gamma_F^2 - \mathbb{K}_4)^{-1} \frac{\partial t}{\partial x^i} c^2 = \varphi_i^F \frac{\partial t}{\partial x^i} c^2, \quad (131)$$

for $\varphi_i^F = (\gamma_F^2 - \mathbb{K}_4)^{-1}$. Considering $\varphi_0^F = 1$, more precisely, (130) may regenerate alternatively as,

$$\begin{aligned} m\psi &= \frac{i\hbar}{c^2} \left(c^2 t \varphi_0^F \frac{\partial^2}{\partial (ct)^2} + c^2 t \varphi_i^F \frac{1}{2} \{ \not{\epsilon}^i, \not{\epsilon}^j \} \partial_i \partial_j \right) \psi \\ &\doteq i\hbar t \varphi_\mu^F \not{\epsilon}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi, \end{aligned} \quad (132)$$

while satisfying the Clifford algebra as $\{ \not{\epsilon}^\mu, \not{\epsilon}^\nu \} = \not{\epsilon}^\mu \not{\epsilon}^\nu + \not{\epsilon}^\nu \not{\epsilon}^\mu = 2 \not{\epsilon}^{\mu\nu} \mathbb{K}_4$. Thus, (130) yields a first order equation as,

$$\begin{aligned} (i\hbar \Phi_\mu^F \not{\epsilon}^{\mu\nu} \vec{\nabla}_\nu - mc^2) \psi &= 0, \\ \therefore (i\hbar \Phi_\mu^F \not{\epsilon}^{\mu\nu} \vec{\nabla}_\nu - \hat{E}) \psi &= 0 \quad [\text{for (4)}], \end{aligned} \quad (133)$$

where the total energy is,

$$\hat{E} \psi \doteq mc^2 \psi = i\hbar \Phi_\mu^F \not{\epsilon}^{\mu\nu} \vec{\nabla}_\nu \psi, \quad (134)$$

whereas, (132) yields a second order equation for (131) as,

$$\begin{aligned} \left(i\hbar \varphi_\mu^F \not{\epsilon}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu - \frac{1}{t} m \right) \psi &= 0, \\ \text{or, } \left[(i\hbar)^2 \varphi_\mu^F \not{\epsilon}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu - \frac{i\hbar}{t} m \right] \psi &= 0. \end{aligned} \quad (135)$$

Suppose, $(i\hbar m)/t = (m_h^F c)^2$ for a hypothetical mass $m_h^F < m$, then (135) gives,

$$\left[(i\hbar)^2 \varphi_\mu^F \not{\epsilon}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i m_h^F c)^2 \right] \psi = 0. \quad (136)$$

Thus, fermionic matter is represented by a tensor of the second rank [26].

Note it that the equation (134) is the direct and exclusively straightforward relation between Quantum Mechanics and the Einsteinian mass-energy relationship of Relativity for a fermion.

Definition 21 (Mass-Spacetime Uncertainty for Fermions). *Following Theorem 13 but without any further repetition, we can develop a new uncertainty relation for (130) as,*

$$(\Phi_\mu^F)^{-1} (\Delta_\psi m) (\Delta_\psi \not{\epsilon}_{\mu\nu} dx^\nu) \geq \frac{\hbar}{2}. \quad (137)$$

Thus, the mass m of a fermionic particle and its inversely stretched/shrank $(3+1)D$ spacetime $(\Phi_\mu^F)^{-1} \not{\epsilon}_{\mu\nu} dx^\nu$, which conjugated in the uncertainty principle (137), cannot have definite and constant values at the same time.

Testable Prediction 5. *It is quite clear if a quark satisfying (133) somehow starts behaving ‘bosonic’ as (136), i.e., where (virtual) quark-antiquark pair condenses in the same energy level, then this quark can start ‘oscillating’ between the matter-antimatter states. Furthermore, since $m_h^F < m$ from its definition then a visible mass difference between the observed matter-antimatter eigenstates must be occurred (see the LHCb measurement of mixing and CP violation in neutral charm mesons [27], for example).*

Testable Prediction 5 is always true for D^0 meson, K^0 meson and two types of B mesons. Of course, in the whole scenario, the mass-spacetime uncertainty (137) must play a crucial role.

8.2. Bosonic Scenarios

Let us assume that the quantum relativistic boson mass $m = \gamma_B m_0$ for (A6), where $m_0 = \Theta_B^{-1} i \hbar \partial_0$ for (A10) and $\Theta_B = \gamma_B c$ for (A8), then it yields the following relation due to $(i \hbar)^{-1} x^0 = (i \hbar)^{-1} (c t)$ as,

$$m = \Theta_B^{-1} i \hbar \gamma_B \partial_0 \doteq \frac{i \hbar}{c^2} \frac{\partial}{\partial t}, \quad (138)$$

which appears like as a quantum ‘operator’. Then, we can say that,

1. (quantum) relativistic mass and time ‘operators’ m and t in (138) are unbounded operators, similarly,
2. quantum scalar curvature and the proper time ‘operators’ $\sqrt{\mathcal{P}}$ and τ in (120), are also unbounded operators.

Let they define on suitable dense subspaces, suppose, $\text{Dom}(m)$, $\text{Dom}(t)$ and $\text{Dom}((2\sqrt{3})^{-1} \mathcal{P})$, $\text{Dom}(\tau)$ of $L^2(\mathbb{R}^4)$, then we can easily reach:

Theorem 13 (Uncertainty Relations for Bosons). *Suppose ψ is a unit vector in $L^2(\mathbb{R}^4)$ belonging to $\text{Dom}(m_0) \cap \text{Dom}(t)$, or belonging to,*

$$\text{Dom}((2\sqrt{3})^{-1} \mathcal{P}) \cap \text{Dom}(\tau),$$

then,

$$(\Delta_\psi m) (\Delta_\psi t) \geq \frac{\hbar}{2}, \quad \text{and} \quad \frac{1}{\sqrt{3}} (\Delta_\psi \sqrt{\mathcal{P}}) (\Delta_\psi \tau) \geq \mathcal{V}^{\mu\nu},$$

where $\Delta_\psi m$ and $\Delta_\psi t$, or $\Delta_\psi \sqrt{\mathcal{P}}$ and $\Delta_\psi \tau$, are given by the Definition 19.

Now, it is not very hard to show that the geometry of bosonic mass and the mass-spacetime uncertainty for bosons are exactly analogous with the fermionic scenarios. We conclude here with minimum repetitions.

Definition 22 (Geometry of Φ_μ^B -dependent Bosonic Mass). *A first order bosonic field representation by combining (138) with (52) yields,*

$$m \psi \doteq \frac{i \hbar}{c^2} (\Phi_0^B \partial_0 - \Phi_i^B \partial_i) \psi = \frac{i \hbar}{c^2} \Phi_\mu^B \mathcal{V}^{\mu\nu} \vec{\nabla}_\nu \psi, \quad (139)$$

if a stretching/shrinking parameters $\Phi_0^B = c$ and $i \hbar \Phi_\mu^B \mathcal{V}^{\mu\nu} \vec{\nabla}_\nu$ is the $(3+1)D$ stretched/shrank spacetime due to stretching/shrinking parameters Φ_μ^B , where,

$$\Phi_i^B = \left[(\gamma_B^2 - 1) v^i \right]^{-1} c^2 = (\gamma_B^2 - 1)^{-1} \frac{\partial t}{\partial x^i} c^2 = \varphi_i^B \frac{\partial t}{\partial x^i} c^2, \quad (140)$$

for $\varphi_i^B = (\gamma_B^2 - 1)^{-1}$. Considering $\varphi_0^B = 1$, more precisely, (139) may regenerate alternatively as,

$$\begin{aligned} m \psi &= \frac{i \hbar}{c^2} \left(c^2 t \varphi_0^B \frac{\partial^2}{\partial (c t)^2} - c^2 t \varphi_i^B \partial_i \partial_j \right) \psi \\ &\doteq i \hbar t \varphi_\mu^B \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi. \end{aligned} \quad (141)$$

Thus, (139) yields a first order equation as,

$$\begin{aligned} \left(i \hbar \Phi_{\mu}^B \mathcal{A}^{\mu\nu} \vec{\nabla}_{\nu} - m c^2 \right) \psi &= 0, \\ \therefore \left(i \hbar \Phi_{\mu}^B \mathcal{A}^{\mu\nu} \vec{\nabla}_{\nu} - \hat{E} \right) \psi &= 0 \quad [\text{for (4)}], \end{aligned} \quad (142)$$

where the total energy is,

$$\hat{E} \psi \doteq m c^2 \psi = i \hbar \Phi_{\mu}^B \mathcal{A}^{\mu\nu} \vec{\nabla}_{\nu} \psi, \quad (143)$$

whereas, (141) yields a second order equation for (140) as,

$$\begin{aligned} \left(i \hbar \Phi_{\mu}^B \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} - \frac{1}{t} m \right) \psi &= 0, \\ \text{or, } \left[(i \hbar)^2 \Phi_{\mu}^B \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} - \frac{i \hbar}{t} m \right] \psi &= 0. \end{aligned} \quad (144)$$

Suppose, $(i \hbar m)/t = (m_h^B c)^2$ for a hypothetical mass $m_h^B < m$, then (144) gives,

$$\left[(i \hbar)^2 \Phi_{\mu}^B \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} + (i m_h^B c)^2 \right] \psi = 0. \quad (145)$$

Thus, bosonic matter is represented by a tensor of the second rank [26].

The equation (143) is also the direct and exclusively straightforward relation between Quantum Mechanics and the Einsteinian mass-energy relationship of Relativity for a boson.

Additionally, we can develop the mass-spacetime uncertainty for bosons using (139) as,

$$(\Phi_{\mu}^B)^{-1} (\Delta_{\psi} m) (\Delta_{\psi} \mathcal{A}_{\mu\nu} dx^{\nu}) \geq \frac{\hbar}{2}, \quad (146)$$

where the mass m of a bosonic particle and its inversely stretched/shrank $(3+1)D$ spacetime $(\Phi_{\mu}^B)^{-1} \mathcal{A}_{\mu\nu} dx^{\nu}$, which conjugated in the uncertainty principle (146), cannot have definite and constant values at the same time.

Definition 23 (Geometry of Φ_{μ}^B -independent Bosonic Mass). A second order bosonic field representation from the mass operator m in (41) yields,

$$\begin{aligned} \left(i \hbar \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} - \frac{1}{t} m \right) \psi &= 0, \\ \text{or, } \left[(i \hbar)^2 \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} - \frac{i \hbar}{t} m \right] \psi &= 0. \end{aligned} \quad (147)$$

If we assume a hypothetical mass $m_h^B < m$ such as $(i \hbar m)/t = (m_h^B c)^2$, then (147) gives,

$$\left[(i \hbar)^2 \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} + (i m_h^B c)^2 \right] \psi = 0. \quad (148)$$

Thus, bosonic matter is again represented by a tensor of the second rank [26].

Testable Prediction 6. For all derivations of the quantum time-reversal symmetry of open quantum systems for microscopic dynamics, descending from the increasing entanglement between a system and its environment, when the Markov approximation is performed (for example [28]),

1. the transformation of stretching/shrinking parameters from Φ_i^B to $(\Phi_i^B)^{\dagger}$ (or Φ_i^F to $(\Phi_i^F)^{\dagger}$) as well as
 2. the mass-hypothetical mass relation for $m > m_h^B$ (or $m > m_h^F$)
- have to be satisfied.

9. General Unified Theory

Let us accept a partial field equation for gravitation (where Dark Energy sector is neglected) by splitting it from (117) as follows,

$$\begin{aligned} & \left[i \hbar \leq^{\mu} \vec{\nabla}_{\mu} - \mathcal{V}^{\mu\nu} \gamma_F m c \right]_{\text{causing QCD}} \psi + \\ & + \left[i \hbar \leq^{\mu} \frac{\sqrt{\kappa}}{2} T_{\mu\nu}^{1/2} - \mathcal{V}^{\mu\nu} \gamma_F m c \right]_{\text{causing GRV}} \psi = 0, \end{aligned} \quad (149)$$

where m is the mass of fermion causing gravity, but not the mass of a graviton itself.

9.1. $SL(2, \mathbb{C})$ Gravity Sector [29–31]

The gauge field theory approach to gravitation was extended by Carmeli, who formulated Einstein's theory as a local gauge theory with the gauge group $SL(2, \mathbb{C})$. The $SL(2, \mathbb{C})$ gauge theory of gravity seems to be the best existing theory for the exploration of gravitation, which is renormalizable to all orders. The $SL(2, \mathbb{C})$ gauge theory of gravitation is based on the fact that any tensorial field defined on the pseudo-Riemannian manifold of spacetime has an underlying spinor structure [29–31].

Let there are four 2×2 Hermitian matrices, denoted by $\sigma_{AB'}^{\mu}$, where Roman capital indices are the spinor indices taking the values $\{0, 1\}$. Let the spinor equivalent of the quantum operator forms of the (quantum) metric tensors (44) and (46), respectively, are given by,

$$\begin{aligned} \mathcal{V}_{AB'CD'} &= \sigma_{AB'}^{\mu} \sigma_{CD'}^{\nu} \mathcal{V}_{\mu\nu} = \varepsilon_{AB'} \varepsilon_{CD'}, \\ \mathcal{V}^{AB'CD'} &= \sigma_{\mu}^{AB'} \sigma_{\nu}^{CD'} \mathcal{V}^{\mu\nu} = \varepsilon^{AB'} \varepsilon^{CD'}, \end{aligned}$$

where $\varepsilon_{AB'}$ and $\varepsilon^{AB'}$ are the skew-symmetric Levi-Civita metric spinors. In addition, the Hermitian matrices σ_{μ} and σ^{μ} also satisfy (44) and (46), respectively, as follows,

$$\begin{aligned} \sigma_{\mu AB'} \sigma_{\nu}^{AB'} &= \mathcal{V}_{\mu\nu} \rightarrow \hat{\mathcal{P}}_{\mu} \cdot \partial_{\hat{\mathcal{P}}_{\nu}}, \\ \sigma_{AB'}^{\mu} \sigma^{\nu AB'} &= \mathcal{V}^{\mu\nu} \rightarrow \partial_{\hat{\mathcal{P}}_{\mu}} \cdot \hat{\mathcal{P}}_{\nu}, \\ \sigma_{AB'}^{\mu} \sigma_{\nu}^{AB'} &= \delta_{\nu}^{\mu}, \\ \sigma_{AB'}^{\mu} \sigma_{\mu}^{CD'} &= \delta_A^C \delta_{B'}^{D'}. \end{aligned} \quad (150)$$

Thus, the contravariant vector $\varepsilon_{AB'}$, considered as the tangent vector, defines the directional derivative,

$$\varepsilon_{AB'} = \sigma_{AB'}^{\mu} \hat{\mathcal{P}}_{\mu}^{1/2} \cdot \partial_{\hat{\mathcal{P}}_{\nu}}^{1/2}.$$

The above spinors are now having the form,

$$\mathcal{V}^{AB'CD'} = \mathcal{V}_{AB'CD'} = \varepsilon_{AC} \varepsilon_{B'D'} = \begin{pmatrix} 0 & 0 & 1 \\ & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let η_a^A be a normalized spinor basis, i.e., a pair of spinors subject to the normalization condition $\varepsilon_{AB} \eta_a^A \eta_b^B = \varepsilon_{ab}$, where a and b are dyad indices. Possessing spinorial character with respect to the index A and vectorial character with respect to the index μ , the covariant derivatives $\eta_{a;\mu}^A$ may have the representation, $\eta_{a;\mu}^A = B_{a\mu}^b \eta_b^A$. The coefficients $B_{a\mu}^b$, which form a set of four 2×2 complex matrices,

will be taken as the gauge potentials of the theory. With the aid of the gauge potential $\mathcal{G}_{a\mu}^b$, the gauge field of the $SL(2, \mathbb{C})$ theory which have the dyad components is possible to define by [29–31],

$$\mathcal{G}_{a\mu\nu}^b = \partial_\nu \mathcal{G}_{a\mu}^b - \partial_\mu \mathcal{G}_{a\nu}^b + \mathcal{G}_{a\mu}^c \mathcal{G}_{c\nu}^b - \mathcal{G}_{a\nu}^c \mathcal{G}_{c\mu}^b. \quad (151)$$

Then the consideration of the simplest Lagrangian density, which is invariant under both general coordinate transformation and spin frame transformation, is,

$$\mathcal{L}_G = -\frac{1}{4} \sqrt{-\mathcal{V}} \text{Tr}(\mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu}). \quad (152)$$

From the $\mathcal{G}_\mu$ and $\mathcal{G}_{\mu\nu}$, we can obtain two new representations of the $\mathcal{G}$ and $\mathcal{G}$ matrices as,

$$\mathcal{G}_{AB'} = \sigma_{AB'}^\mu \mathcal{G}_\mu, \quad \mathcal{G}_{AB'CD'} = \sigma_{AB'}^\mu \sigma_{CD'}^\nu \mathcal{G}_{\mu\nu}.$$

But there is a number of things remain unanswered in this scenario. (we shall discuss them in the end of the following Subsection 9.2)

9.2. Dark Energy Sector

Let us accept a partial field equation for Dark Energy (where gravity sector is neglected) by splitting it from (117) as follows,

$$\left[i \hbar \not{\nabla}_\mu - \mathcal{V}^{\mu\nu} \gamma_F m c \right]_{\text{causing QCD}} \psi - \left[i \hbar \not{\nabla}_\mu \eta_\mu \frac{\partial}{\partial \ell_{\text{PL}}^\mu} + \mathcal{V}^{\mu\nu} \gamma_F m c \right]_{\text{causing DE}} \psi = 0, \quad (153)$$

where m is the mass of fermion causing Dark Energy, but not the mass of a Dark Energy gauge boson itself.

Definition 24 (Evolving Dark Energy). In (153), the dimensionless factor η_μ according to (106) ranges

1. from $\eta_\mu \equiv 1$ when $t \approx 5.38752 \times 10^{-44}$ s (around Planck time), i.e., ‘naïve’ cosmological constant Λ' was almost $42.5728 \text{ J}^2 \text{ s}^2 \text{ m}^{-2}$ during the Big Bang
2. to $\eta_\mu \approx 2.90253 \times 10^{-87}$ when $t \equiv 1$ s, i.e., ‘naïve’ cosmological constant Λ' will reach almost $3.58663 \times 10^{-172} \text{ J}^2 \text{ s}^2 \text{ m}^{-2}$ at the time of Big Freeze, thus, the cosmological constant Λ will reach almost $3.22504 \times 10^{-104} \text{ m}^{-2}$ at the time of Big Freeze, because the value $\Lambda = 3.22504 \times 10^{-104} \text{ m}^{-2}$ will be sufficient to help the Universe to achieve its absolute zero temperature state.

The present-day value of cosmological constant is $\Lambda = 1.088 \times 10^{-52} \text{ m}^{-2}$ [32] thus, the density of Dark Energy is decreasing as t is growing continuously.

Hence, the *inconstant* ‘naïve’ cosmological constant Λ' , which agrees with the best-fit of the current result [33], technically provides us a unit as the square of ‘energy-per-velocity’, i.e., $\text{J}^2 \text{ s}^2 \text{ m}^{-2}$.

Testable Prediction 7. For near-future surveys (DESI-extension and the Vera C. Rubin Observatory LSST), we can show the exact rate of overall Dark Energy evolution by using the factor η_μ and time t , while additionally accepting the forecast given in [34]. We can also calculate the possible lifespan of the Universe from the evolving value of the cosmological constant Λ due to Definition 24.

(The author of the present article does not willing to accept a different, ultra-light version of the axion as like as [34] *a priori* at this very early stage of his work because he is not sure whether our ‘massive’ Dark Energy candidate, which will be discussed below in (167), is actually a ultra-light version of the axion or not and neither he exactly targets to show the current best-fit w_ϕ CDM model to make it distinguishable from Λ CDM at over 9σ .)

It is clear that the mutual field of (149) and (153) mediated by the same fermion of mass m in (117) is a bilocal field. Let the binary system with graviton a at $x_\mu = X_\mu + r_\mu/2$ and Dark Energy gauge boson b at $x'_\mu = X_\mu - r_\mu/2$ be described by a bilocal field $\Phi_a^b(x|x') = \Phi_a^b(X, r)$. Let the propagator of bilocal field is [35,36],

$$\mathfrak{D}(X; r, r') = \frac{1}{(2\pi)^4} \int \tilde{\mathfrak{D}}(k; r, r') \exp[-ikX] dk,$$

as,

$$\tilde{\mathfrak{D}}(k; r, r') = -i\alpha_1 \int \mathbf{G}(k; r, r'; \exp[-i\tau]) \exp[i\alpha(k^2)\tau - \varepsilon\tau] d\tau,$$

where ε being a positive infinitesimal and,

$$\mathbf{G}(k; r, r'; \zeta) = \pi^{-2} \rho^{-4} (1 - \zeta^{2\lambda})^{-1/2} (1 - \zeta^2)^{-3/2} \exp[-\eta_{\parallel}^{\mu\nu} f_{\mu\nu}(\zeta^\lambda) + \eta_{\perp}^{\mu\nu} f_{\mu\nu}(\zeta)],$$

with,

$$f_{\mu\nu}(\zeta) = -\frac{1}{4\rho^2} \left[(r - r')_\mu (r - r')_\nu \frac{1 + \zeta}{1 - \zeta} + (r + r')_\mu (r + r')_\nu \frac{1 - \zeta}{1 + \zeta} \right],$$

and $\alpha(k^2) = \alpha_0 + \alpha_1 k^2$ is the liner trajectory with coefficients $\alpha_0 = -n^2$ for an arbitrary natural number n and $\alpha_1 = 1/m^2$ for the single fermion of (117) with mass m . Thus, for the propagation of binary system from, let, (y, y') to (x, x') , we have the factor,

$$i\mathfrak{D}\left(\frac{x+x'}{2} - \frac{y+y'}{2}; x-x', y-y'\right).$$

So, by considering $\hbar = c = 1$ and $i\hbar \leq^\mu = \gamma$ while,

$$i\hbar \leq^\mu \frac{\sqrt{\kappa}}{2} T_{\mu\nu}^{1/2} \rightarrow \gamma \cdot \partial_1 \quad \text{and} \quad i\hbar \leq^\mu \eta_\mu \frac{\partial}{\partial \ell_{\text{PL}}^\mu} \rightarrow \gamma \cdot \partial_2,$$

let the Bethe-Salpeter equation be given by using (149) and (153) as,

$$\begin{aligned} & \left[(\gamma \cdot \vec{\nabla}_\mu + \gamma \cdot \partial_1) - 2\mathcal{A}\psi^{\mu\nu} \gamma_\mu m \right] \psi(\vec{\nabla}; \partial_1, \partial_2) \left[(\gamma \cdot \vec{\nabla}_\mu - \gamma \cdot \partial_2) - 2\mathcal{A}\psi^{\mu\nu} \gamma_\mu m \right] \\ & = \lambda G(\partial_1, \partial_2) \psi(\vec{\nabla}; \partial_1, \partial_2), \end{aligned} \quad (154)$$

where the linear operator $G(\partial_1, \partial_2)$ describes the propagation of graviton and Dark Energy gauge boson. Then the matrix element of the decay process bilocal fields $\rightarrow \{\text{graviton, Dark Energy gauge boson}\}$ with momenta p_1 and p_2 can be found from the reduction formula [37],

$$\begin{aligned} \langle \text{pair} | b \rangle &= \int d^4x d^4x' \bar{f}^{p_1}(x) \left[(\gamma \cdot \vec{\nabla}_\mu + \gamma \cdot \partial_1) - 2\mathcal{A}\psi^{\mu\nu} \gamma_\mu m \right] \cdot \\ & \langle 0 | T[\psi(x), \bar{\psi}(x')] | b \rangle \cdot \left[(\gamma \cdot \vec{\nabla}_\mu - \gamma \cdot \partial_2) - 2\mathcal{A}\psi^{\mu\nu} \gamma_\mu m \right] \cdot g^{p_2}(x'), \end{aligned}$$

where f and g contain the same mutual spinor u of the same fermion,

$$\begin{aligned} f^{p_1}(x) &= [(2\pi)^3 2\omega_1]^{-1/2} \exp[-ip_1 x] u_{p_1}, \\ g^{p_2}(x) &= [(2\pi)^3 2\omega_2]^{-1/2} \exp[ip_2 x] u_{p_2}. \end{aligned}$$

Thus, we can conclude that Dark Energy gauge boson is definitely satisfying the gauge field of the antifundamental representation $\text{SL}^*(2, \mathbb{C})$. Additionally, the Bethe-Salpeter equation (154) of bilocal fields for graviton and Dark Energy gauge boson helps us to realize that the spin of Dark Energy gauge boson from $\text{SL}^*(2, \mathbb{C})$ would be either 0 or 2. But ℓ_{PL}^μ is not a tensor, thus the Dark Energy

gauge boson from $SL^*(2, \mathbb{C})$ must be a strongly sterile, non-self-interacting (due to Claim 5 above), ‘massive’ (during gauge symmetry breaking, see (167) below), spin-0 particle. Note it here that Dark Energy is not considered ‘massive’ in the traditional sense of elementary particles (see Remark 11 in Subsection 9.3 below).

But neither one of the References [29–31] in Subsection 9.1 and not our above discussion able to show us a tiny possibility for an antifundamental representation $SL^*(2, \mathbb{C})$ for ‘massive’ Dark Energy gauge boson. Due to this reason, we do not focus either on [29–31] or our above discussion anymore, instead, we shall try to develop something conventional here as follows.

9.3. Gravitational-Weak Symmetry Group $SU(2)_L \oplus iSU(2)_G$ (Using [38])

Let us formulate the kinematics of the $SU(2)$ gauge fields in terms of the $SL(2, \mathbb{C})$ spinors. Any element of $SU(2)$ can be written in the form $U = \exp(-i \alpha^k \tau^k)$ where the α^k are three real numbers and the τ^k are the three generators of the group $SU(2)$. For the global $SU(2)$ symmetry to be made into a local $SU(2)$ symmetry, with $U = U(x)$ dependent on space and time coordinates, let us introduce a vector gauge field $W_\mu^k(x)$ for each generator τ^k , then the transformation law for the matrices $\mathcal{W}_\mu(x) = W_\mu^k(x) \tau^k$ is,

$$\mathcal{W}_\mu(x) \rightarrow \mathcal{W}'_\mu(x) = U(x) \mathcal{W}_\mu(x) U^\dagger(x) + \sum_{n=0}^2 \frac{2i}{g^n} (\partial_\mu U(x)) U^\dagger(x),$$

where for (117),

$$\partial_\mu = \partial_\mu + \frac{\sqrt{\kappa}}{2} T_{\mu\nu}^{1/2} - \eta_\mu \frac{\partial}{\partial \ell_{PL}^\mu}, \quad (155)$$

and the matrices,

$$\begin{aligned} \mathcal{W}_\mu(x) &= \begin{bmatrix} \begin{pmatrix} W_\mu^3 & W_\mu^1 - i W_\mu^2 \\ W_\mu^1 + i W_\mu^2 & -W_\mu^3 \end{pmatrix} & 0 \\ 0 & i \begin{pmatrix} W_\mu^0 & W_\mu^1 - i W_\mu^3 \\ W_\mu^1 + i W_\mu^3 & -W_\mu^0 \end{pmatrix} \end{bmatrix}, \\ &= \begin{bmatrix} \mathcal{W}_\mu(x) & 0 \\ 0 & \bar{\mathcal{W}}_\mu(x) \end{bmatrix}, \end{aligned} \quad (156)$$

or,

$$\begin{aligned} \mathcal{W}_\mu(x) &= \begin{bmatrix} \begin{pmatrix} W_\mu^3 & W_\mu^1 - i W_\mu^2 \\ W_\mu^1 + i W_\mu^2 & -W_\mu^3 \end{pmatrix} & 0 \\ 0 & i \begin{pmatrix} W_\mu^0 & W_\mu^2 - i W_\mu^3 \\ W_\mu^2 + i W_\mu^3 & -W_\mu^0 \end{pmatrix} \end{bmatrix}, \\ &= \begin{bmatrix} \mathcal{W}_\mu(x) & 0 \\ 0 & \bar{\mathcal{W}}_\mu(x) \end{bmatrix}, \end{aligned} \quad (157)$$

are Hermitian and have zero trace. Let us define,

$$D_\mu \Phi = \left[\partial_\mu + i \frac{g_0}{2} \bar{\mathcal{W}}_\mu + i \frac{g_1}{2} W_\mu^h + i \frac{g_2}{2} \mathcal{W}_\mu \right] \Phi, \quad (158)$$

straightforwardly,

$$\begin{aligned} D'_\mu \Phi' &= \left[\partial_\mu + i \frac{g_0}{2} \bar{\mathcal{W}}'_\mu + i \frac{g_1}{2} W_\mu'^h + i \frac{g_2}{2} \mathcal{W}'_\mu \right] \Phi' \\ &= \exp(-i\theta) U D_\mu \Phi, \end{aligned}$$

where $\Phi' = \exp(-i\theta) \mathbf{U} \Phi$. The field strength tensors are taken to be,

$$\begin{aligned}\bar{W}_{\mu\nu} &= \left[\partial_\mu + i \frac{g_0}{2} \bar{W}_\mu \right] \bar{W}_\nu - \left[\partial_\nu + i \frac{g_0}{2} \bar{W}_\nu \right] \bar{W}_\mu, \\ W_{\mu\nu} &= \left[\partial_\mu + i \frac{g_2}{2} W_\mu \right] W_\nu - \left[\partial_\nu + i \frac{g_2}{2} W_\nu \right] W_\mu.\end{aligned}$$

Using the results $[\tau^2, \tau^3] = 2i\tau^1$, etc., the matrices $\bar{W}_{\mu\nu}$ and $W_{\mu\nu}$ may be written as, $\bar{W}_{\mu\nu} = W_{\mu\nu}^i \tau^i$ and $W_{\mu\nu} = W_{\mu\nu}^i \tau^i$, respectively, where,

$$\begin{aligned}W_{\mu\nu}^0 &= i\partial_\mu W_\nu^0 - i\partial_\nu W_\mu^0 - g_0(W_\mu^1 W_\nu^2 - W_\nu^1 W_\mu^2), \\ &\simeq i\partial_\mu W_\nu^0 - i\partial_\nu W_\mu^0 - g_0(W_\mu^3 W_\nu^1 - W_\nu^3 W_\mu^1), \\ &\simeq i\partial_\mu W_\nu^0 - i\partial_\nu W_\mu^0 - g_0(W_\mu^2 W_\nu^3 - W_\nu^2 W_\mu^3), \\ W_{\mu\nu}^1 &= \partial_\mu W_\nu^1 - \partial_\nu W_\mu^1 - g_2(W_\mu^2 W_\nu^3 - W_\nu^2 W_\mu^3), \\ W_{\mu\nu}^2 &= \partial_\mu W_\nu^2 - \partial_\nu W_\mu^2 - g_2(W_\mu^3 W_\nu^1 - W_\nu^3 W_\mu^1), \\ W_{\mu\nu}^3 &= \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3 - g_2(W_\mu^1 W_\nu^2 - W_\nu^1 W_\mu^2),\end{aligned}$$

and $\text{Tr}(\tau^i)^2 = 2$, while $\text{Tr}(\tau^i \tau^j) = 0, i \neq j$, so the Lagrangian density expresses as,

$$\mathcal{L}_{\text{dyn}} = -\frac{1}{4} W_{\mu\nu}^g W^{g\mu\nu} - \frac{1}{4} W_{\mu\nu}^h W^{h\mu\nu} - \sum_{i=0}^3 \frac{1}{4} W_{\mu\nu}^i W^{i\mu\nu}. \quad (159)$$

The fields W_μ^1 and W_μ^2 are electrically charged, but W_μ^0 is chargeless. It is convenient to define here the complex combinations,

$$\begin{aligned}W_\mu^+ &= (W_\mu^1 - iW_\mu^2)/\sqrt{2}, \\ W_\mu^- &= (W_\mu^1 + iW_\mu^2)/\sqrt{2}, \\ W_\mu^g &= i(W_\mu^1 - iW_\mu^3)/\sqrt{2} \simeq i(W_\mu^2 - iW_\mu^3)/\sqrt{2}, \\ W_\mu^h &= i(W_\mu^1 + iW_\mu^3)/\sqrt{2} \simeq i(W_\mu^2 + iW_\mu^3)/\sqrt{2},\end{aligned}$$

where the field W_μ^- is the complex conjugate of the field W_μ^+ , so as the field W_μ^h is the complex conjugate of the field W_μ^g , thus, we can define,

$$\begin{aligned}W_{\mu\nu}^g &= i(W_{\mu\nu}^1 - iW_{\mu\nu}^3)/\sqrt{2} \simeq i(W_{\mu\nu}^2 - iW_{\mu\nu}^3)/\sqrt{2} \\ &= (\partial_\mu + i g_0 W_\mu^0) W_\nu^g - (\partial_\nu + i g_0 W_\nu^0) W_\mu^g, \\ W_{\mu\nu}^h &= i(W_{\mu\nu}^1 + iW_{\mu\nu}^3)/\sqrt{2} \simeq i(W_{\mu\nu}^2 + iW_{\mu\nu}^3)/\sqrt{2} \\ &= (\partial_\mu + i g_1 W_\mu^0) W_\nu^h + (\partial_\nu + i g_1 W_\nu^0) W_\mu^h, \\ W_{\mu\nu}^+ &= (W_{\mu\nu}^1 - iW_{\mu\nu}^2)/\sqrt{2} \\ &= (\partial_\mu + i g_2 W_\mu^3) W_\nu^+ - (\partial_\nu + i g_2 W_\nu^3) W_\mu^+, \\ W_{\mu\nu}^- &= (W_{\mu\nu}^1 + iW_{\mu\nu}^2)/\sqrt{2} \\ &= (\partial_\mu + i g_2 W_\mu^3) W_\nu^- + (\partial_\nu + i g_2 W_\nu^3) W_\mu^-.\end{aligned}$$

We can also write,

$$\begin{aligned}W_{\mu\nu}^0 &= i\partial_\mu W_\nu^0 - i\partial_\nu W_\mu^0 - i g_0(W_\mu^g W_\nu^g - W_\nu^g W_\mu^g), \\ W_{\mu\nu}^3 &= \partial_\mu W_\nu^3 - \partial_\nu W_\mu^3 - i g_2(W_\mu^- W_\nu^+ - W_\nu^- W_\mu^+),\end{aligned}$$

hence, (159) becomes,

$$\begin{aligned}\mathcal{L}_{\text{dyn}} = & -\frac{1}{4} W_{\mu\nu}^0 W^{0\mu\nu} - \frac{1}{4} W_{\mu\nu}^g W^{g\mu\nu} - \frac{1}{4} W_{\mu\nu}^h W^{h\mu\nu} - \frac{1}{4} W_{\mu\nu}^3 W^{3\mu\nu} - \\ & - \frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} .\end{aligned}\quad (160)$$

By breaking the $SU(2) \oplus iSU(2)$ symmetry, we have from (158) and (156) or (157) in terms of the scalar Higgs field $H(x)$ and the vacuum state expectation value of the Higgs field ϕ_0 as,

$$\begin{aligned}D^\mu \Phi = & \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial^\mu H \end{pmatrix} + i \frac{g_1}{2\sqrt{2}} \begin{pmatrix} 0 \\ \sqrt{2} W^{h\mu} (\phi_0 + H) \end{pmatrix} + \\ & + i \frac{g_0}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} W^{g\mu} (\phi_0 + H) \\ -W^{0\mu} (\phi_0 + H) \end{pmatrix} + i \frac{g_2}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} W^{+\mu} (\phi_0 + H) \\ -W^{3\mu} (\phi_0 + H) \end{pmatrix},\end{aligned}$$

where g_0 is now,

$$g_0 = \frac{\alpha_0}{\frac{1}{\sqrt{2}} (\phi_0 + H)}, \quad (161)$$

then multiplying $(D_\mu \Phi)^\dagger$ by $D^\mu \Phi$, we find for (161) as,

$$\begin{aligned}\mathcal{L}_\Phi = & \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{\alpha_0^2}{2} W_\mu^g W^{g\mu} + \frac{g_2^2}{4} W_\mu^- W^{+\mu} (\phi_0 + H)^2 + \\ & + \frac{1}{2} \left[\frac{g_0^2}{4} W_\mu^0 W^{0\mu} - \frac{g_1 g_0}{2} W_\mu^0 W^{h\mu} + \frac{g_1^2}{4} W_\mu^h W^{h\mu} \right] (\phi_0 + H)^2 + \\ & + \frac{1}{2} \left[\frac{g_2^2}{4} W_\mu^3 W^{3\mu} - \frac{g_1 g_2}{2} W_\mu^3 W^{h\mu} + \frac{g_1^2}{4} W_\mu^h W^{h\mu} \right] (\phi_0 + H)^2 - \\ & - V(H) \\ = & \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{\alpha_0^2}{2} W_\mu^g W^{g\mu} + \frac{g_2^2}{4} W_\mu^- W^{+\mu} (\phi_0 + H)^2 + \\ & + \frac{1}{8} (g_1^2 + g_0^2) Q_\mu Q^\mu (\phi_0 + H)^2 + \frac{1}{8} (g_1^2 + g_2^2) Z_\mu Z^\mu (\phi_0 + H)^2 - \\ & - V(H),\end{aligned}$$

for,

$$\begin{aligned}Q_\mu = & W_\mu^0 \cos \theta_D - W_\mu^h \sin \theta_D, \\ Z_\mu = & W_\mu^3 \cos \theta_W - W_\mu^h \sin \theta_W,\end{aligned}\quad (162)$$

where,

$$\begin{aligned}\cos \theta_D = & \frac{g_0}{(g_1^2 + g_0^2)^{1/2}}, & \sin \theta_D = & \frac{g_1}{(g_1^2 + g_0^2)^{1/2}}, \\ \cos \theta_W = & \frac{g_2}{(g_1^2 + g_2^2)^{1/2}}, & \sin \theta_W = & \frac{g_1}{(g_1^2 + g_2^2)^{1/2}},\end{aligned}$$

whereas θ_D is the Dark angle and θ_W is the Weinberg angle. Along with the fields Q_μ and Z_μ , respectively, if we define the orthogonal combinations,

$$A_\mu = W_\mu^0 \sin \theta_D + W_\mu^h \cos \theta_D = W_\mu^3 \sin \theta_W + W_\mu^h \cos \theta_W, \quad (163)$$

then (162) and (163) correspond to the rotations of axes in (W_μ^h, W_μ^0) and (W_μ^h, W_μ^3) spaces, respectively, so the rotations can be inverted to give,

$$\begin{aligned} W_\mu^h &= A_\mu \cos \theta_D - Q_\mu \sin \theta_D = A_\mu \cos \theta_W - Z_\mu \sin \theta_W, \\ W_\mu^0 &= A_\mu \sin \theta_D + Q_\mu \cos \theta_D, \\ W_\mu^3 &= A_\mu \sin \theta_W + Z_\mu \cos \theta_W, \end{aligned} \quad (164)$$

which yield,

$$\begin{aligned} W_{\mu\nu}^h &= A_{\mu\nu} \cos \theta_D - Q_{\mu\nu} \sin \theta_D = A_{\mu\nu} \cos \theta_W - Z_{\mu\nu} \sin \theta_W, \\ W_{\mu\nu}^0 &= A_{\mu\nu} \sin \theta_D + Q_{\mu\nu} \cos \theta_D - i g_0 (W_\mu^g W_\nu^g - W_\nu^g W_\mu^g), \\ W_{\mu\nu}^3 &= A_{\mu\nu} \sin \theta_W + Z_{\mu\nu} \cos \theta_W - i g_2 (W_\mu^- W_\nu^+ - W_\nu^- W_\mu^+), \end{aligned}$$

where,

$$\begin{aligned} A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ Q_{\mu\nu} &= \partial_\mu Q_\nu - \partial_\nu Q_\mu, \\ Z_{\mu\nu} &= \partial_\mu Z_\nu - \partial_\nu Z_\mu. \end{aligned} \quad (165)$$

Using (164) and (165) in (160), to express the field W_μ^h and W_μ^0 in terms of the fields A_μ and Q_μ , similarly, to express the field W_μ^h and W_μ^3 in terms of the fields A_μ and Z_μ , we may write the full Lagrangian density,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu H \partial^\mu H - (\bar{\psi} \gamma^\mu \gamma_5 m)^2 H^2 - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} + \frac{1}{2} \alpha_0^2 W_\mu^g W^{g\mu} - \\ &\quad - \frac{1}{4} Q_{\mu\nu} Q^{\mu\nu} + \frac{1}{4} \phi_0^2 (g_1^2 + g_0^2) Q_\mu Q^\mu - \\ &\quad - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{1}{4} \phi_0^2 (g_1^2 + g_2^2) Z_\mu Z^\mu - \\ &\quad - \frac{1}{2} [(D_\mu W_\nu^+)^* - (D_\nu W_\mu^+)^*] [D^\mu W^{+\nu} - D^\nu W^{+\mu}] + \\ &\quad + \frac{1}{2} \phi_0^2 g_2^2 W_\mu^- W^{+\mu}, \end{aligned} \quad (166)$$

and,

$$D_\mu W_\nu^+ = (\partial_\mu + i g_2 \sin \theta_W A_\mu) W_\nu^+.$$

We can identify the three vector fields, W_μ^+ , W_μ^- , Z_μ , with the mediators of the weak interaction, the W^+ , W^- , Z particles. Similarly, the three vector fields, W_μ^g , W_μ^h , Q_μ , have the mediators of the gravito-dark interaction, the W^g , A , Q particles. We can also identify that,

$$\begin{aligned} \frac{1}{\sqrt{2}} \phi_0 (g_1^2 + g_0^2)^{1/2} &= \frac{1}{\sqrt{2}} \phi_0 \left(g_1^2 + \frac{2\alpha_0^2}{(\phi_0 + H)^2} \right)^{1/2} = M_Q \quad [\text{for (161)}], \\ \frac{1}{\sqrt{2}} \phi_0 (g_1^2 + g_2^2)^{1/2} &= M_Z = 91.1880 \pm 0.0020 \text{ GeV} \quad (\text{from [32]}), \\ \frac{1}{\sqrt{2}} \phi_0 g_2 &= M_W = 80.366 \pm 0.012 \text{ GeV} \quad (\text{from [32]}). \end{aligned} \quad (167)$$

On the other hand, the bosons W^g and A are massless in (166) and none of them does carry any color, weak or electromagnetic charges, however the former one uniquely interacts with the energy momentum tensor due to gravity sector in Subsection 9.1. To make a difference between them, let us consider that the former one is a graviton whereas the later one is a photon (γ).

Testable Prediction 8. By neglecting quantum corrections to the mass ratio, we have,

$$\begin{aligned}\cos \theta_D &= \frac{1}{\sqrt{2}} \frac{\phi_0 g_0}{M_Q} = \left(1 + \frac{g_1^2}{g_0^2}\right)^{-1/2}, \\ \cos \theta_W &= \frac{M_W}{M_Z} = 0.8810 \pm 0.0016.\end{aligned}\quad (168)$$

In other words, if we consider $\cos \theta_D = \cos \theta' / M_Q = a_D / F_D$, then it depends upon the acceleration (a_D) in the direction of motion caused by per unit force (F_D) of Dark Energy applied at an angle θ' , i.e., then $\cos \theta_D$ depends upon the Dark Energy density, thus, for the evolving Dark Energy model [34], $\cos \theta_D$ will change as Dark Energy density change with time while M_Q in (168) still remains constant.

Remark 10 (Dilemma about α_0). Definitely, (168) assures us that g_0 , which is used to achieve $\cos \theta_D$, must have a dimensionless value. We do not sure whether α_0 in (161) is actually dimensionless or not, though, the current observations strongly favor the massless graviton [32]. But to assume $\cos \theta_D$ as dimensionless in (168), we have to take α_0 as a minuscule mass with incredibly tiny upper limit that does not significantly alter large-scale gravity [39,40], so as it would appear harmlessly in (168) resulting $\cos \theta_D$ as dimensionless. Possibly, the best way to express a 'massive' graviton to be appeared as a massless one is to consider a universal quantum 'supertunneling' through spacetime, which is currently unknown to us.

Definition 25 (Resolving Hierarchy Problem without Fine-tuning). Feebleness of gravity in Definition 8 guarantees us that gravity must be quantized to resolve the Hierarchy Problem without fine-tuning, i.e., the problem concerning the large discrepancy between aspects of the weak force and gravity (see [41], for example). According to (166), α_0^2 is a companion to graviton. Thus, a minuscule mass of graviton with incredibly tiny upper limit due to α_0^2 makes gravity $\sim 10^{-24}$ times weaker than the weak force in (166) since $\alpha_0^2 \ll \phi_0^2$.

Testable Prediction 9 (Mass of Q Boson). If α_0 is a minuscule mass with incredibly tiny upper limit, then it leads to $2\alpha_0^2 / (\phi_0 + H)^2 \approx 0$ for M_Q in (167). Thus, we may have $M_Q \lesssim 10$ GeV.

Remark 11 (Sterile Nature of Q Boson). In (167), Q boson is pseudo-massive due to its peculiar mass-construction for $\alpha_0^2 / (\phi_0 + H)^2$, which ought to be dimensionless by taking α_0 as a minuscule mass with incredibly tiny upper limit. If we consider Remark 10, then $\alpha_0^2 / (\phi_0 + H)^2$ would switch between dimensional and dimensionless states in a mutually exclusive way depending upon the observational conditions. Possibly that is why Q boson is naturally undetectable by a particle detector and this peculiarity may be the reason that it is a highly sterile particle in nature. Additionally, this peculiarity possibly ensures Q to be a scalar particle rather than a vector particle. Without any doubt, Q boson is the particle resulting Dark Energy.

Theorem 14 (Confirmation of Chargeless W^8 and A bosons). Both W^8 and A bosons are chargeless.

Proof. Let the hypercharge current is $J_\mu^Y = 2J_\mu^{EM} - 2J_\mu^3$. To introduce gravito-weak isospin and gravito-weak hypercharge interactions, let us consider [42],

$$\begin{aligned}&-i g_0 J_\mu^i W^{i\mu} - i \frac{g_1}{2} J_\mu^Y W^{h\mu}, \\ &-i g_2 J_\mu^i W^{i\mu} - i \frac{g_1}{2} J_\mu^Y W^{h\mu}.\end{aligned}$$

Expressions for the neutral currents are,

$$\begin{aligned}
& -i \left[g_0 \sin \theta_D J_\mu^3 + \frac{g_1}{2} \cos \theta_D J_\mu^Y \right] A^\mu, \\
& -i \left[g_2 \sin \theta_W J_\mu^3 + \frac{g_1}{2} \cos \theta_W J_\mu^Y \right] A^\mu, \\
& -i \left[g_0 \cos \theta_D J_\mu^3 - \frac{g_1}{2} \sin \theta_D J_\mu^Y \right] Q^\mu, \\
& -i \left[g_2 \cos \theta_W J_\mu^3 - \frac{g_1}{2} \sin \theta_W J_\mu^Y \right] Z^\mu.
\end{aligned}$$

Let us represent the electric current,

$$\begin{aligned}
e J^{EM} &= g_0 \sin \theta_D J^3 + \frac{g_1}{2} \cos \theta_D J^Y \\
&= g_2 \sin \theta_W J^3 + \frac{g_1}{2} \cos \theta_W J^Y,
\end{aligned}$$

then we get,

$$e J^{EM} = e J^3 + \frac{e}{2} J^Y,$$

hence,

$$\begin{aligned}
e &= g_0 \sin \theta_D = g_1 \cos \theta_D \\
&= g_2 \sin \theta_W = g_1 \cos \theta_W.
\end{aligned}$$

Suppose, W^8 and A bosons are carrying charge $\pm e$, then we can identify,

$$\begin{aligned}
e = g_0 \sin \theta_D &= g_1 \cos \theta_D = \frac{1}{\sqrt{2}} \frac{\phi_0 g_0 g_1}{M_Q} \quad [\text{for (168)}] \\
&\doteq \frac{1}{\sqrt{2}} \frac{\phi_0 g_0^2}{M_Q} \tan \theta_D \quad [\text{as } g_1 = g_0 \tan \theta_D]. \quad (169)
\end{aligned}$$

Hence, e have to change as Dark Energy density change with time while M_Q is constant. But none of W^8 and A bosons depend upon Dark Energy density. Thus, it confirms that W^8 and A bosons (i.e., graviton and photon) are chargeless in nature. This completes the proof. $\square$ $\square$

Testable Prediction 10. *If a graviton couples with a field where the Dark Energy density vanishes (i.e., where acceleration (a_D) in the direction of motion caused by per unit force (F_D) of Dark Energy is no longer applicable), then it can show surface charge density without satisfying the Weinberg-Witten theorem [43] (which tells us that a non-zero $U(1)$ charge may make sense if the graviton is massive) since graviton is massless due to (166) as the dimension of mass vanishes due to (161) (or at least its mass does not significantly alter large-scale gravity [39,40], for example). For this purpose, we can focus some findings like Reference [44]. This surface charge density may observe non-classically near the event horizon of a Black Hole.*

9.4. General Unified Theory

Suppose, the scenario of Standard Model in elementary particle physics is concerned the QCD, Electroweak, and Electromagnetism along with gravity and Dark Energy interactions. Let us call this scenario as the ‘General Unified Theory’. The equations of QCD and Electroweak along with gravity and Dark Energy interactions obeyed by the fermions would be easily achievable by using (117) in companion to the bosonic fields (124) (without repeating here the well-known formalism of the Standard Model again). For the symmetry part, let us consider that the $SU(5)$ generators are T_a and

the covariant derivative is (by analogy with the Gravito-weak symmetry group $SU(2)_L \oplus iSU(2)_G$ and the QCD symmetry group $SU(3)_C$ namely),

$$\partial_\mu - i g_5 T_a V_\mu^a = \partial_\mu - i g_5 (T_0 W_\mu^0 + T_1 W_\mu^h + \dots),$$

where V_μ^a are the $SU(5)$ gauge bosons and ∂_μ satisfies (155), whereas only one coupling (g_5) for all interactions. So, the General Unified Theory is coupled with gauge theory to describe the interactions as,

$$SU(5) = SU(3)_C \times (SU(2)_L \oplus iSU(2)_G), \quad (170)$$

where the gauge fields are comprising eight gluons (g) form an octet of $SU(3)_C$, three massive weak bosons (W^+ , W^- , Z^0) from the weak interaction part, i.e., $SU(2)_L$, of Gravito-weak symmetry group $SU(2)_L \oplus iSU(2)_G$, while they all are spin-1 vector bosons, in association with one 'massless' spin-2 graviton (W^g), one massless spin-1 photon (γ) and one 'massive' spin-0 Dark Energy gauge boson (Q) from the gravitational interaction part, i.e., $iSU(2)_G$, of Gravito-weak symmetry group $SU(2)_L \oplus iSU(2)_G$. This completes the generalized form of all particle interactions in baryonic matters. Definitely, non-baryonic matter, i.e., Dark Matter, does not satisfy a gauge group like (170). We must need a different scenario for this purpose.

10. Renormalization [45] of General Quantum Gravity

Let $\phi(x)$ is a field quantity or field operator or simply the field, which is the linear operator depending on a point $(i\hbar)^{-1}x$ in four-dimensional $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^4)$ space. Let there exist a non-singular matrix ε such that hermitian conjugation $[\varepsilon \Lambda(\partial)]^+ = \varepsilon \Lambda(-\partial)$ with $\Lambda(\partial)$ is of the form,

$$\Lambda(\partial) = \sum_{\ell=0}^3 (i\hbar)^{1,\dots,\ell} \Lambda_{\mu_1\dots\mu_\ell} \partial_{\mu_1} \dots \partial_{\mu_\ell},$$

where $\Lambda_{\mu_1\dots\mu_\ell}$ is symmetric in all pairs of indices and independent of $(i\hbar)^{-1}x$. Then (124) divisor exists such that,

$$\begin{aligned} \Lambda(\partial) d(\partial) &= d(\partial) \Lambda(\partial) \\ &= \left[(i\hbar)^2 \mathcal{A}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i\hbar)^2 \frac{\kappa}{4} \mathcal{A}^{\mu\nu} T_{\mu\nu} - (i\hbar)^2 \eta_\mu^2 \mathcal{A}^{\mu\nu} \frac{\partial}{\partial \ell_{PL}^\mu} \frac{\partial}{\partial \ell_{PL}^\nu} \right] + \\ &\quad + (2i\mathcal{A}^{\mu\nu} \gamma_B m c)^2. \end{aligned}$$

The field equation is quantum Lorentz invariant and $[\ell_{\mu\nu}, \Lambda(\partial)] = 0$, where $\ell_{\mu\nu}$ is: $\ell_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + i S_{\mu\nu}$, and $S_{\mu\nu}$ is defined by the transformation of $\phi(x)$ under an infinitesimal Lorentz-like transformation, i.e.,

$$\phi(x) \rightarrow \phi'(x') = \left[1 + \frac{i}{2} S_{\mu\nu} \delta \omega_{\mu\nu} \right] \phi(x),$$

when,

$$(i\hbar)^{-1} x_\mu \rightarrow (i\hbar)^{-1} x'_\mu = (i\hbar)^{-1} (\delta_{\mu\nu} + \delta \omega_{\mu\nu}) x_\nu,$$

and $\delta \omega_{\mu\nu} + \delta \omega_{\nu\mu} = 0$.

Now, with this sufficient background along with Section 2 of Chapter V in page 112 of [45], we can easily show that the operator $\hat{\mathcal{P}}_\mu$ and the commutation relation of $\phi_\alpha(x)$ are suitable for acceptance, so that,

$$i\hbar \vec{\nabla}_\mu \phi_\alpha(x) = [\phi_\alpha(x), \hat{\mathcal{P}}_\mu] \equiv \{[\phi_\alpha(x), p_0], [\phi_\alpha(x), \hat{p}_i]\},$$

for $i \in \{1, 2, 3\}$, is consistent with the equation $\Lambda(\partial) \phi(x) = 0$ or equivalently $\Lambda_{\alpha\beta}(\partial) \phi_\beta(x) = 0$.

Let the (quantum) metric tensor $\mathcal{V}^{\mu\nu} = \mathcal{V}^{\alpha\beta} \left(\frac{i\hbar \vec{\nabla}_\alpha}{i\hbar \vec{\nabla}_\mu} \frac{i\hbar \vec{\nabla}_\beta}{i\hbar \vec{\nabla}_\nu} \right)$ of (17) may rearrange as,

$$(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu = (i\hbar)^2 \mathcal{V}^{\alpha\beta} \vec{\nabla}_\alpha \vec{\nabla}_\beta \equiv \frac{\partial^2}{\partial \hat{s}^2} \doteq \partial_{\hat{s}}^2 \quad [\text{for (19)}].$$

Then $i\hbar \vec{\nabla}_\mu \phi(x) = [\phi(x), \hat{\mathcal{P}}_\mu]$ may yield:

$$\begin{aligned} \sum_{\mu, \nu=0}^3 (i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \phi(x)^2 &\equiv \sum_{\mu, \nu=0}^3 \mathcal{V}^{\mu\nu} \hat{\mathcal{P}}_\mu \hat{\mathcal{P}}_\nu \phi(x)^2 \\ &= \sum_{\mu, \nu=0}^3 \mathcal{V}^{\mu\nu} [\phi(x)^2, \hat{\mathcal{P}}_\mu \hat{\mathcal{P}}_\nu] \equiv [\phi(x)^2, \partial_{\hat{s}}^2]. \end{aligned}$$

Thus (with the summation convention),

$$\mathcal{V}^{\mu\nu} [\phi(x)^2, \hat{\mathcal{P}}_\mu \hat{\mathcal{P}}_\nu] \equiv \mathcal{V}^{\alpha\beta} [\phi(x)^2, \hat{\mathcal{P}}_\alpha \hat{\mathcal{P}}_\beta], \quad (171)$$

$$\text{i.e., } \mathcal{V}^{\mu\nu} = \mathcal{V}^{\alpha\beta} \frac{[\phi(x)^2, \hat{\mathcal{P}}_\alpha \hat{\mathcal{P}}_\beta]}{[\phi(x)^2, \hat{\mathcal{P}}_\mu \hat{\mathcal{P}}_\nu]}. \quad (172)$$

Hence, the metric (17) is now an object of quantization. So, both the (quantum) Christoffel symbol (71) and the trace of $\mathfrak{P}_{(3,1)}$ curvature tensor, i.e., $\mathcal{P}_{\mu\nu}$, in (72) are possible to become the objects of quantization for (172) in the following ways,

$$\begin{aligned} \Gamma_{\mu\nu}^k \phi(x) &= \frac{1}{2} \mathcal{V}^{k\ell} \left[[\phi(x), \hat{\mathcal{P}}_\mu] \mathcal{V}_{\nu\ell} + [\phi(x), \hat{\mathcal{P}}_\nu] \mathcal{V}_{\mu\ell} - [\phi(x), \hat{\mathcal{P}}_\ell] \mathcal{V}_{\mu\nu} \right], \\ \mathcal{P}_{\mu\nu} \phi(x)^2 &= \left[[\phi(x), \hat{\mathcal{P}}_\ell] \Gamma_{\mu\nu}^\ell \phi(x) - [\phi(x), \hat{\mathcal{P}}_\nu] \Gamma_{\mu\ell}^\ell \phi(x) + \Gamma_{k\ell}^\ell \Gamma_{\mu\nu}^k \phi(x)^2 - \right. \\ &\quad \left. - \Gamma_{k\nu}^\ell \Gamma_{\mu\ell}^k \phi(x)^2 \right]. \end{aligned}$$

Likewise, the curvature (3,1)-tensor $\mathfrak{P}_{(3,1)}$, i.e., $\mathcal{P}_{\mu\nu\alpha\beta}$, and the scalar curvature $\mathfrak{P}$, i.e., $\mathcal{P}$, also reserve the similar theory of quantization.

For a real scalar field with a $\lambda \phi^4$ interaction along with the acceptance of the map $\phi \mapsto \varphi$ (though, the following formalism remains true without considering this map) and for (79), the renormalized Lagrangian is,

$$\begin{aligned} \mathcal{L}_R &= \frac{1}{2} (1 + \delta z_1) \mathcal{V}^{\alpha\beta} [\varphi_R(x)^2, \hat{\mathcal{P}}_\alpha \hat{\mathcal{P}}_\beta] - \frac{1}{24} (1 + \delta z_2) \mathcal{P} \varphi_R^2 - \\ &\quad - (1 + \delta z_4) \lambda_R \varphi_R^4. \end{aligned}$$

By denoting $z_1 = 1 + \delta z_1$, $z_1 z_{\mathcal{P}} = 1 + \delta z_2$ and $z_1^2 z_\lambda = 1 + \delta z_4$, we get,

$$\mathcal{L}_R = \frac{1}{2} z_1 \mathcal{V}^{\alpha\beta} [\varphi_R(x)^2, \hat{\mathcal{P}}_\alpha \hat{\mathcal{P}}_\beta] - \frac{1}{24} z_1 z_{\mathcal{P}} \mathcal{P} \varphi_R^2 - z_1^2 z_\lambda \lambda_R \varphi_R^4.$$

It is clear that the theory is ‘multiplicatively renormalizable’, and the renormalized Lagrangian $\mathcal{L}_R$ is related to the bare Lagrangian by the renormalization transformation $\varphi = z_1^{1/2} \varphi_R$, $\mathcal{P} = z_{\mathcal{P}} \mathcal{P}_R$ and

$\lambda = z_\lambda \lambda_R$, where the renormalization constants $z_{1,\varphi,\lambda}$ depend on the coupling constant λ_R and on regularization parameters. Since (172) is considered as true for (171), then, $\mathcal{L}_R$ could be written as,

$$\begin{aligned}\mathcal{L}'_R &= \frac{1}{2} z_1 (z'_1)^{-1} \mathcal{U}^{\alpha\beta} \frac{[\varphi_R(x)^2, \hat{\mathbf{P}}_\alpha \hat{\mathbf{P}}_\beta]}{[\varphi_R(x)^2, \hat{\mathbf{P}}_\mu \hat{\mathbf{P}}_\nu]} - \frac{1}{24} z_1 (z'_1)^{-1} z_\varphi \frac{\mathcal{P}_R}{[\varphi_R(x)^2, \hat{\mathbf{P}}_\mu \hat{\mathbf{P}}_\nu]} \varphi_R^2 - \\ &\quad - z_1^2 (z'_1)^{-1} z_\lambda \frac{\lambda_R}{[\varphi_R(x)^2, \hat{\mathbf{P}}_\mu \hat{\mathbf{P}}_\nu]} \varphi_R^4 \\ &= \frac{1}{2} z_1 (z'_1)^{-1} \varphi_R \mathcal{U}^{\mu\nu} \varphi_R - \frac{1}{24} z_1 (z'_1)^{-1} z_\varphi [\varphi_R(x)^2, \hat{\mathbf{P}}_\mu \hat{\mathbf{P}}_\nu]^{-1} \mathcal{P}_R \varphi_R^2 - \\ &\quad - z_1^2 (z'_1)^{-1} z_\lambda [\varphi_R(x)^2, \hat{\mathbf{P}}_\mu \hat{\mathbf{P}}_\nu]^{-1} \lambda_R \varphi_R^4,\end{aligned}\quad (173)$$

where $\mathcal{L}'_R = [\varphi_R(x)^2, \hat{\mathbf{P}}_\mu \hat{\mathbf{P}}_\nu]^{-1} \mathcal{L}_R$ and $z'_1 = 1 + \delta z'_1$. Thus, (173) is multiplicatively renormalizable. So as the (quantum) metric tensor (172) and quantum scalar curvature $\mathcal{P}$ are multiplicatively renormalizable, too. Differently, we can show that the (quantum) Christoffel symbol $\Gamma_{\mu\nu}^k$ is actually multiplicatively renormalizable in such a way that,

$$\begin{aligned}\mathcal{L}_R &= \frac{1}{4} z_1 (z'_1)^{-1} \varphi_R \mathcal{U}^{k\ell} \varphi_R \left[(z'_1)^{1/2} \varphi_R [\varphi_R(x), \hat{\mathbf{P}}_\mu] \left(z_1 (z'_1)^{-1} \varphi_R \mathcal{U}_{\nu\ell} \varphi_R \right) + \right. \\ &\quad + (z'_1)^{1/2} \varphi_R [\varphi_R(x), \hat{\mathbf{P}}_\nu] \left(z_1 (z'_1)^{-1} \varphi_R \mathcal{U}_{\mu\ell} \varphi_R \right) - \\ &\quad \left. - (z'_1)^{1/2} \varphi_R [\varphi_R(x), \hat{\mathbf{P}}_\ell] \left(z_1 (z'_1)^{-1} \varphi_R \mathcal{U}_{\mu\nu} \varphi_R \right) \right].\end{aligned}$$

Thus, by performing a little calculation, we can show that all quantum differential geometric curvatures, i.e., $\mathcal{P}_{\mu\nu\alpha\beta}$, $\mathcal{P}_{\mu\nu}$ and $\mathcal{P}$, are evidently become multiplicatively renormalizable.

Although, separately, by using [20,46–49] and omitting any further repetitions, we can easily show that (117), as well as (108), are naturally renormalizable. Consequently, all kind of non-vacuum quantum gravitational field equations, for example the universal form (75), bosonic form (83), fermionic forms (99), (100) and (101), even including bosonic and fermionic particle field equations, for example (87) and (103), respectively, are all renormalizable by the definitions.

PART III: Discussion and Conclusion

The insuperable obstacle between gravity and quantum mechanics is their spacetimes [50–54]. But John Donoghue was correct when he stated about the claims that, “general relativity clashes with quantum mechanics are wrong or at least misleadingly simplistic” [55]. By accepting the quantum temporal axis $(i\hbar)^{-1} x^0 = (i\hbar)^{-1} (ct)$ as our ‘new’ conception of time throughout this article, we have melted down this insuperable obstacle between gravity and Quantum Mechanics in Theorem 3 by developing a $(3+1)D$ curvilinear Schrödinger equation representing total energy $\hat{E}$ and dynamic time t in a $(3+1)D$ curvilinear quantum spacetime.

Speaking philosophically, time in General Quantum Gravity is nothing but an *essential illusion* necessary for mechanics concerned with the bodies in motions or forces in equilibrium. That is to say that all events are in time, but time is not a separate, independent reality, outside and strongly apart from all-that-is-in-time, i.e., the contingent, as like as Nyāya-Vaiśeṣika and the Sāṅkhya-Yoga schools. Time is phenomenal like as the system of Advaita Vedānta but not extraordinary. Time is conceived as instant, a series of discontinuous instants, because of the quantized spatial dimensions in (52) or (57), for example, but not like as William James, Husserl and Bergson or neither as one instant being superseded by the Buddhist doctrine of momentariness. Time as instant is real and not as unitary, but these instants are present in a linear order as like as the Jaina tradition, however, time is neither an absolute dimension as like as Jaina, Nyāya, Newton, Bergson or Aristotle and neither it is unchanging due to Definition 10. The relativistic and quantized time in (52) or (57) is not transcendental like Kant or Husserl. It is not like as Leibnitz, too, where time is a system of relations, or not even like as Einstein, where time is simply relativistic but not quantized. Time (as well as space) must be finite and has a beginning when $v^i \neq 0$ but again not quite like Einsteinian due to the quantized spatial dimensions

in (52) and (57). Time is only ‘relativistically’ reversible at $v^i = c$, otherwise it has an irreversible direction. (See Refs. [56] and [57] for a brief account of time.)

In this article, we have strongly avoided to focus on any parts of the orthodox Quantum Mechanics including orthodox Kline-Gordon and Dirac equations. On the other hand, we have also avoided to focus on the definitions of Special and/or General Relativity. Instead, we have employed a new trick here, we have named it as ‘**Metric Mechanics**’, by manipulating various kind of line elements (also known as, metric functions) for both bosonic and fermionic fields throughout this article to achieve all sorts of required QFT equations without adopting the orthodox Kline-Gordon or Dirac formalisms. As a result, we have developed here the unorthodox versions of Kline-Gordon-like and Dirac-like equations suitable for $(3+1)D$ curvilinear Quantum Mechanics so as, from the very beginning, these Kline-Gordon-like and Dirac-like equations would make themselves appropriate for the General (Quantum) Relativity, rather than Special Relativity. For this reason, both Kline-Gordon-like and Dirac-like equations have appeared quite different from their orthodox counterparts in QFT while carrying an innocent-looking but quite straightforward **General (Quantum) Relativistic signature** within their mass fields, e.g., $i\mathcal{V}^{\mu\nu}\gamma_B mc$ for bosons and $\mathcal{V}^{\mu\nu}\gamma_F mc$ for fermions, instead of the orthodox mass term ‘ mc ’, we commonly use in QFT. Speaking more precisely, while in orthodox QFT, $\hat{\mathcal{P}}^2\psi = (mc)^2\psi$ obeys the Klein-Gordon equation and $\gamma^\mu\hat{\mathcal{P}}_\mu\psi = mc\psi$ obeys the Dirac equation, we have shown in this article that there always exists a possibility of an unorthodox ‘(General) Relativistic Quantum Mechanics’ alongside of the fundamental QFT (quite similar to Remark 3 and Note 4, where we have already established a deep relationship between this article and the fundamental QFT). To satisfy this, we have established in Section 4 that $\hat{\mathcal{P}}^2\psi = (i\mathcal{V}^{\mu\nu}\gamma_B mc)^2\psi$ should obey the Klein-Gordon-like equation whereas $\hat{\mathcal{P}}_\mu\psi = \mathcal{V}^{\mu\nu}\gamma_F mc\psi$ should obey the Dirac-like equation, which are actually proved in Section 5 as the modified versions of Einstein-like gravitational field equations by themselves in $(3+1)D$ curvilinear quantum spacetime. On the other hand, their orthodox Quantum Mechanical counterparts, i.e., $\hat{\mathcal{P}}^2\psi = (mc)^2\psi$ and $\gamma^\mu\hat{\mathcal{P}}_\mu\psi = mc\psi$, are solely Special Relativistic and they are unable to develop any gravitational field equations like Einsteinian.

Since (33) as well as (39) satisfy (36) and (42), respectively, thus, *Quantum Gravity and Quantum Mechanics are able to express each other simultaneous in a $(3+1)D$ curvilinear quantum spacetime*, while this $(3+1)D$ curvilinear quantum spacetime is locally Quantum Mechanical in $3D$ space for its global $(3+1)D$ curvilinear spacetime covering, i.e., it holds $3D \subset (3+1)D$ in this curvilinear quantum spacetime. Quantum gravitational wave packets (33) or (39) in $(3+1)D$ curvilinear spacetime are universally existed as the covering companions to the three-dimensional wave packets $\Psi(\vec{r}, t)$ in (1).

Usually, when gravity itself is treated in QFT, there arouses a problematic part that the metric itself is not a well-defined quantum concept [55]. By considering a (quantum) metric tensor $\mathcal{V}_{\mu\nu}$, which is multiplicatively renormalizable by nature, we have overthrown such a dilemma in our present article.

From the second and the fourth terms of (A11) from the Appendix A.1, it becomes, $(i\hbar)^2\gamma_B^2\partial_0^2\psi \doteq (i\hbar)^2\mathcal{V}^{\mu\nu}\vec{\nabla}_\mu\vec{\nabla}_\nu\psi$, then the rearrangement yields,

$$\begin{aligned}\mathcal{V}_{\mu\nu} &= \left(\frac{i\hbar\vec{\nabla}_\mu}{i\hbar\gamma_B\partial_0} \frac{i\hbar\vec{\nabla}_\nu}{i\hbar\gamma_B\partial_0} \right) \equiv \left(\frac{i\hbar\vec{\nabla}_\mu}{\partial/\partial\hat{s}} \frac{i\hbar\vec{\nabla}_\nu}{\partial/\partial\hat{s}} \right) \quad [\text{for (A11)}] \\ &\doteq \mathcal{V}_{\alpha\beta} \left(\frac{i\hbar\vec{\nabla}_\mu}{i\hbar\vec{\nabla}_\alpha} \frac{i\hbar\vec{\nabla}_\nu}{i\hbar\vec{\nabla}_\beta} \right) \quad [\text{for (19)}],\end{aligned}\quad (174)$$

hence it is a (quantum) metric tensor as like as (17). So, a non-vacuum quantum gravitational field equation (75) exists for (174), better to say, for (A11). In other words, it is also true for the fields of bosonic (62), or (63), and fermionic (68) since (174) is originated from the same (19). Then, for example, the first order quantum equations, i.e., bosonic (52) and fermionic (56) are undoubtedly the pseudo-gravity equations for (174) ready to satisfy the non-vacuum quantum gravitational field equation (75) in $(3+1)D$ curvilinear quantum spacetime through bosonic (62) or (63), and fermionic (68), since (52) and (56) are also originated from (19). Undoubtedly, (52) and (56) are equally eligible to

stand along with some other alternative quantum theories, like the Newton-Schrödinger equation (as we know, for example [3]), or some modified Schrödinger equations (as we know, for example [58]), or the Gravitational Schrödinger equations (as we know, for example [59]), etc.

In General Theory of Relativity, matter means everything that is not the gravitational field (metric), but in (79) as well as in the free fermionic field representation (92), we have replaced matter with quantum scalar curvature $\mathcal{P}$. General Relativistic Ricci scalar R never depicts such dynamical possibilities like (79) or (92). Similarly, neither General Relativity and not the orthodox Quantum Mechanics can predict something like a metamorphosis of bosonic fields from (83) or (85) to (87), or a metamorphosis of fermionic fields from (99) or (100) or (101) to (103), or their vice versa.

The fraction $1/12$ in (79) is the nonminimal parameter. But (79) is not the conformally invariant wave equation (as we know, for example, [60–63]), on the other hand, (79) is neither even a free classical Hermitian Klein-Gordon field on a curved spacetime when $m = 0$ (as we know, for example [15,64]). Thus, (79) is a Quantum Gravitational Klein-Gordon-like equation of its ‘own kind’ for conjugated fermions. Although, the factor $\frac{1}{12} \mathcal{P} = 2 \Lambda'$ is being of relevance when we consider curved space-time [65] for the ‘naïve’ cosmological constant $\Lambda' = (i\hbar)^2 \Lambda$. Thus, from (79), we have a bosonic field representation of the cosmological constant as,

$$[\mathcal{U}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + 2\Lambda] \psi = 0. \quad (175)$$

Then, the simplest system of linear homogeneous differential equations of first order which meets the conditions of (175) would be as,

$$[\leq^\mu \vec{\nabla}_\mu - (2\Lambda)^{1/2}] \psi = 0. \quad (176)$$

In association with (175) and (176), consideration of $m = \sum_{i=1}^N m_i$ in (115), where N is an even number, may help us to explore the possibilities for opening a study on cosmological bodies solely based on gravity and/or Dark Energy.

In Section 8, the inversely stretched/shrank $(3+1)D$ spacetimes for fermions, i.e., $(\Phi_\mu^F)^{-1} \mathcal{U}_{\mu\nu} dx^\nu$, in (137) and for bosons, i.e., $(\Phi_\mu^B)^{-1} \mathcal{U}_{\mu\nu} dx^\nu$, in (146) become quite complicated to presume. The situation is not altered in their second order field representations with the hypothetical mass m_h^F or m_h^B . Even the transformation of (148) from the mass operator m in (41) makes it equally complicated. Definitely, both the first order and the second order quantum geometries of mass of fermionic or bosonic particle is not as simple as we can usually expect. We need an elaborative study on it elsewhere.

By introducing a direct transformation from (133) to (136) for fermions, so as (142) to (145) for bosons, where (136) mirroring bosonic fields while (142) mirroring fermionic fields, we have demonstrated a new kind of boson-fermion ‘Supersymmetry’ without ever considering Supersymmetry and its troupe of surprisingly heavy superparticles, since $m_h^{F,B} < m$.

Since the orthodox Quantum Mechanics is typically set up in constant velocity frameworks and since both Kline-Gordon and Dirac equations are developed from Special Relativity, whereas on the other hand, Newton-Schrödinger equations cannot be derived from General Relativity plus Quantum Field Theory [3], then we can say that no experimental formalism has designed yet to understand the interaction of quantum matter with classical gravity in accelerating frames at today’s technical platform to determine whether the generality (i.e., universally validation) of quantum gravity, unless otherwise we consider some well-known classical findings ranging from Schrödinger equation to Einstein’s mass-energy relationship, or something like which we have established in this article by using different aspects of General Quantum Gravity. To justify the General Quantum Gravity, there are some good references, as we know, for example [66–72], explaining the proposals and highlighting the important ways, which can help us to predict in such a way the interpretation of General Quantum Gravity may experimentally access. Though, such an experiment might be feasible in the near future. Despite the lack of experimental evidences for quantum gravity, Testable Prediction 9 is a unique and exclusive way to probe General Quantum Gravity quite straightforwardly apart from some other

near-future experimental possibilities to probe quantum gravity, such as [2,73,74], for example. If Testable Prediction 9 is provable then overall scenario of General Quantum Gravity will be proved automatically. Perhaps, a careful anomaly detection search for narrow-width resonances beyond the Standard Model based on the dijet final state using proton-proton collisions at $\sqrt{s} = 13$ TeV recorded during 2015-2018 period in the ATLAS detector at the Large Hadron Collider (LHC) at CERN may probe $M_Q \lesssim 10$ GeV by revealing a deficit in the m_{JJ} potentially signal-enriched region (using something like [75], for example).

Appendix A. (3 + 1)D Curvilinear Quantum Field Theory

Appendix A.1. Bosons

Let (22), whose temporal direction t is dynamic, yield a quantum factor γ_B^{-2} for (21) as,

$$\begin{aligned} (i\hbar)^4 \left(\frac{d\hat{s}}{dx^0} \right)^2 \Psi &= (i\hbar)^2 \mathcal{V}_{\mu\nu} \frac{dx^\mu}{d(ct)} \frac{dx^\nu}{d(ct)} \Psi \\ &= (i\hbar)^2 \left[\mathcal{V}_{00} \frac{dx^0}{d(ct)} \frac{dx^0}{d(ct)} + \mathcal{V}_{ij} \frac{dx^i}{d(ct)} \frac{dx^j}{d(ct)} \right] \Psi \\ &= (i\hbar)^2 \left(\mathcal{V}_{00} + \mathcal{V}_{ij} \frac{(v^i v^j)}{c^2} \right) \Psi \equiv (i\hbar)^2 \gamma_B^{-2} \Psi, \end{aligned} \quad (A1)$$

where the subscript B is intending bosons and $v^i = (dx^i/dt)$ is the three-dimensional velocity vector, i.e., (using the summation convention),

$$\begin{aligned} \gamma_B^2 &= \left(\mathcal{V}_{00} + \mathcal{V}_{ij} \frac{(v^i v^j)}{c^2} \right)^{-1} \equiv \mathcal{V}^{\mu\nu} \left(\frac{dx^\mu}{d(ct)} \frac{dx^\nu}{d(ct)} \right)^{-1}, \\ \therefore \quad \gamma_B^2 &\neq \gamma^2 \quad \text{for the Lorentz factor } \gamma^2 = \left(1 - \frac{v^2}{c^2} \right)^{-1}. \end{aligned} \quad (A2)$$

A (quantum) Lorentz-like transformation originated from (22) is possible to obtain in a (3 + 1)D curvilinear quantum spacetime on contrary to the Special Relativistic Lorentz transformation as follows.

Definition A1 ((Quantum) Lorentz-like transformation for bosons or composites). *Let us consider,*

$$i\hbar \partial'_\mu \psi = i\hbar \Lambda_\mu^\nu \partial_\nu \psi + a_\mu \psi,$$

where $\Lambda \equiv (\Lambda_\mu^\nu)$ is a matrix with constant elements, and a_μ is a constant four-vector. If we consider Λ and Λ^{-1} matrices as,

$$\Lambda = \begin{pmatrix} \gamma_B^{-1} & -\gamma_B^{-1} \frac{c}{v} & 0 & 0 \\ -\gamma_B^{-1} \frac{c}{v} & \gamma_B^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^{-1} = \begin{pmatrix} \gamma_B^{-1} & \gamma_B^{-1} \frac{c}{v} & 0 & 0 \\ \gamma_B^{-1} \frac{c}{v} & \gamma_B^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then let a standard form would be for Λ ,

$$\begin{aligned} i\hbar \mathcal{V}^{00} \partial'_0 \psi &= i\hbar \gamma_B^{-1} \left(\mathcal{V}^{00} \partial_0 + \mathcal{V}^{11} \frac{c}{v} \partial_1 \right) \psi, \\ i\hbar \mathcal{V}^{11} \partial'_1 \psi &= i\hbar \gamma_B^{-1} \left(\mathcal{V}^{11} \partial_1 + \mathcal{V}^{00} \frac{c}{v} \partial_0 \right) \psi, \\ i\hbar \mathcal{V}^{22} \partial'_2 \psi &= i\hbar \mathcal{V}^{22} \partial_2 \psi, \\ i\hbar \mathcal{V}^{33} \partial'_3 \psi &= i\hbar \mathcal{V}^{33} \partial_3 \psi, \end{aligned} \quad (A3)$$

i.e., the linear (Quantum) Lorentz-like (homogeneous) transformation of the form would be,

$$i \hbar \mathcal{L}^{\mu\nu} \partial'_\mu \psi = i \hbar \mathcal{L}^{\mu\nu} \Lambda^\nu_\mu \partial_\nu \psi, \quad (\text{A4})$$

which is nothing but the Special Relativistic Lorentz transformation (SRLT) written reciprocally in a generalized $(3+1)D$ curvilinear quantum spacetime, thus, (A3) always satisfies the inverse map,

$$f^{-1}(i \hbar \mathcal{L}^{\mu\nu} \partial'_\mu) \longmapsto (i \hbar)^{-1} \text{SRLT},$$

or vice versa. Additionally, the inhomogeneous transformations would be,

$$i \hbar \mathcal{L}^{\mu\nu} \partial'_\mu \psi = i \hbar \mathcal{L}^{\mu\nu} \Lambda^\nu_\mu \partial_\nu \psi + i \hbar \mathcal{L}^{\mu\nu} \partial_\mu^0 \psi,$$

where $i \hbar \mathcal{L}^{\mu\nu} \partial_\mu^0$ describes translations.

Remark A1 (Bridging Link between Simple (Quantum) Relativity and General (Quantum) Relativity). For the ‘four-momentum’ operator $\hat{\mathcal{P}}_\mu$ of (13), the linear (Quantum) Lorentz-like transformation in (A4) would become as,

$$\mathcal{L}^{\mu\nu} \hat{\mathcal{P}}'_\mu \psi = \mathcal{L}^{\mu\nu} \Lambda^\nu_\mu \hat{\mathcal{P}}_\nu \psi,$$

which can inject a (Quantum) Lorentz-like signature in the (quantum) scalar curvature $\mathcal{P} = \mathcal{L}^{\mu\nu} \mathcal{P}_{\mu\nu}$ while considering (72), see later in the main text, as,

$$\begin{aligned} \mathcal{P} \psi &= \mathcal{L}^{\mu\nu} \mathcal{P}_{\mu\nu} \psi \\ &\doteq \mathcal{L}^{\mu\nu} \left[\hat{\mathcal{P}}'_\ell \Gamma^\ell_{\mu\nu} - \hat{\mathcal{P}}'_\nu \Gamma^\ell_{\mu\ell} + \Gamma^\ell_{k\ell} \Gamma^k_{\mu\nu} - \Gamma^\ell_{kv} \Gamma^k_{\mu\ell} \right] \psi \\ &= \mathcal{L}^{\mu\nu} \left[\Lambda^\nu_\ell \hat{\mathcal{P}}_\nu \Gamma^\ell_{\mu\nu} - \Lambda^\mu_\nu \hat{\mathcal{P}}_\mu \Gamma^\ell_{\mu\ell} + \Gamma^\ell_{k\ell} \Gamma^k_{\mu\nu} - \Gamma^\ell_{kv} \Gamma^k_{\mu\ell} \right] \psi. \end{aligned}$$

Definitely, it is a strong bridge between Simple (Quantum) Relativity and General (Quantum) Relativity. No such bridges are possible to exist in Einsteinian relativity.

By the way, we can also establish the fact that elements of the proper, orthochronous, homogeneous (Quantum) Lorentz-like group ‘qL’ can be described by means of elements of $\text{SL}(2, \mathbb{C})$, i.e., the group of all 2×2 complex matrices where a, b, c , and d are complex numbers satisfying,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2,2}, \quad \text{with} \quad \det g = ad - bc = 1,$$

associates with each four-vector $(i \hbar)^{-1} x^\mu$ a Hermitian matrix,

$$Q = \begin{pmatrix} i \hbar \partial_0 + i \hbar \partial_3 & i \hbar \partial_1 + i \hbar \partial_2 \\ i \hbar \partial_1 - i \hbar \partial_2 & i \hbar \partial_0 - i \hbar \partial_3 \end{pmatrix},$$

which can also be written as $Q = i \hbar \partial_\mu \sigma^\mu \doteq \hat{\mathcal{P}}_\mu \sigma^\mu$ due to (13) for σ^0 and the three Pauli matrices $\sigma^k|_{k=1,2,3}$.

Putting it another way, let the metric be derived from the $\mathbb{C}^{2,2}$ -norm as,

$$\begin{aligned}
||\mathcal{X}|| &\equiv ||(i\hbar)^{-1}X|| \\
&= \left(|(i\hbar)^{-1}x^0|^2 - |(i\hbar)^{-1}x^1|^2 - |(i\hbar)^{-1}x^2|^2 - |(i\hbar)^{-1}x^3|^2 \right)^{1/2} \\
&= \left(|(i\hbar)^{-1}(ct)|^2 - |(i\hbar)^{-1}x^1|^2 - |(i\hbar)^{-1}x^2|^2 - |(i\hbar)^{-1}x^3|^2 \right)^{1/2} \\
&\doteq \left(|a|^2 - |b|^2 - |c|^2 - |d|^2 \right)^{1/2}, \\
\text{and } \mathcal{X} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2,2}, \quad \text{with } \det g = ad - bc = 1,
\end{aligned}$$

then this metric space can also establish a $SL(2, \mathbb{C})$ space, as the transformation (A4) preserves the scalar product since,

$$\begin{aligned}
&\left((i\hbar)^{-1}x'^0 \right)^2 - \left((i\hbar)^{-1}x'^1 \right)^2 - \left((i\hbar)^{-1}x'^2 \right)^2 - \left((i\hbar)^{-1}x'^3 \right)^2 = \det Q' \\
&= \det Q = \left((i\hbar)^{-1}x^0 \right)^2 - \left((i\hbar)^{-1}x^1 \right)^2 - \left((i\hbar)^{-1}x^2 \right)^2 - \left((i\hbar)^{-1}x^3 \right)^2.
\end{aligned}$$

As a result, the description of the representations of the (Quantum) Lorentz-like group qL is equivalent to that of the group $SL(2, \mathbb{C})$.

In $\mathcal{V} \subset \mathcal{H} \otimes \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^4)$ space, $i\hbar \vec{\nabla}_\mu = \frac{1}{2} \text{Tr} \mathcal{D} \sigma_\mu$, where σ_0 is the 2×2 identity matrix and $\sigma_{1,2,3}$ are the Pauli spin matrices. So, the inner product would be given by $(i\hbar)^2 (\vec{\nabla}, \vec{\nabla}) = \det \mathcal{D}$. By considering the transformation $\mathcal{D} \rightarrow A\mathcal{D}A^*$ with A in $SL(2, \mathbb{C})$ and since $\det A = 1$, this transformation preserves $(i\hbar)^2 (\vec{\nabla}, \vec{\nabla})$. Writing $\mathcal{D}' = A\mathcal{D}A^* = i\hbar \vec{\nabla}'_\mu \sigma_\mu$, we can see that there is a Lorentz-like transformation Λ_A such that $i\hbar \vec{\nabla}' = i\hbar \Lambda_A \vec{\nabla}$. It is clear that in this identification we must have $\Lambda_A = \Lambda_{-A}$.

Definition A2 (Some Crucial Properties). Let us consider the operators P_μ and $J_{\alpha\beta}$ by the rules $P_\mu \psi = i(i\hbar) \partial_\mu \psi$ and $J_{\alpha\beta} \psi = (i\hbar)^{-1} (\eta_{\alpha\nu} x^\nu P_\beta - \eta_{\beta\nu} x^\nu P_\alpha) \psi$, where P_μ and $J_{\alpha\beta}$ are the generators of spacetime translations and the Lorentz rotations, respectively, then P_μ and $J_{\alpha\beta}$ define the Lie algebra of the Poincaré group [20]. Then there is a unitary representation $U(R, \alpha)$ of the Poincaré group on $\mathcal{H}$, where α ranges over all spacetime translations and R ranges over all Lorentz boosts and spatial rotations, which fulfills $P_0 \geq 0$ and $(P_0)^2 - P_i P_i \geq 0$ as well as a unique state ψ_0 , called a vacuum, represented by a ray in $\mathcal{H}$, which is invariant under the action of the Poincaré group. For each test function f , defined on spacetime, there exists a set of operators $\varphi_1(f), \dots, \varphi_n(f)$ (tempered distributions regarded as a functional of f) which, together with their adjoints defined on a domain D , are defined on a dense subset of $\mathcal{H}$, containing the vacuum $\psi_0 \in D$. The Hilbert space $\mathcal{H}$ is spanned by the field polynomials acting on the vacuum (cyclicity condition). The relation $U(R, \alpha) \varphi_i(f) U(R, \alpha)^{-1} = \sum S_{ij}(R^{-1}) \varphi_j(\{R, \alpha\}f)$ is valid when each side is applied to any vector in domain D . Finally, if the supports of two fields are space-like separated, then the fields either commute or anticommute.

Over again, we have (A1) as,

$$\begin{aligned}
&(i\hbar)^{-4} \left(\frac{\partial^2}{\partial (x^0)^2} \right)^{-1} \frac{\partial^2}{\partial \hat{s}^2} \psi = (i\hbar)^{-2} \gamma_B^2 \psi, \\
\therefore \quad &\frac{\partial^2}{\partial \hat{s}^2} \psi = (i\hbar)^2 \gamma_B^2 \left(\frac{\partial^2}{\partial (x^0)^2} \right) \psi \equiv (i\hbar)^2 \gamma_B^2 \partial_0^2 \psi, \\
\text{i.e.,} \quad &\gamma_B^{-2} c^2 \frac{\partial^2}{\partial \hat{s}^2} \psi = (i\hbar)^2 \frac{\partial^2}{\partial t^2} \psi = \hat{E}^2 \psi \quad [\text{for (34)}]. \tag{A5}
\end{aligned}$$

Thus, (A5) implies that the total energy operator $\hat{E} \rightarrow i\hbar \partial_t$ is also possible to define as $\hat{E} \rightarrow \gamma_B^{-1} c (\partial/\partial \hat{s})$ in $(3+1)D$ curvilinear quantum spacetime.

Remark A2 (Parameter Θ_B and Its Relation with Bosonic Field). *Imitating the Special Relativistic mass $m = \gamma m_0$ for the Lorentz factor $\gamma = (1 - [(v^i v^j)/c^2])^{-1/2}$, let us assume analogously that the (quantum) relativistic mass of a boson in quantum spacetime is,*

$$m = \gamma_B m_0 \quad [\text{for (A2)}]. \quad (\text{A6})$$

Let us insert (A5) 'twice' into (A6) one after another as follows. At first by inserting the value of γ_B we get,

$$m = (i\hbar)^{-1} m_0 \left(\frac{\partial}{\partial x^0} \right)^{-1} \left(\frac{\partial}{\partial \hat{s}} \right),$$

and then by inserting the value of $(\partial/\partial \hat{s})$ it yields,

$$m = (i\hbar)^{-1} m_0 \left(\frac{\partial}{\partial x^0} \right)^{-1} i\hbar \gamma_B \partial_0, \quad (\text{A7})$$

and by using $\hat{E} \rightarrow i\hbar \partial_t$ and $m = \gamma_B m_0$ for (A6), if a parameter Θ_B yields,

$$\Theta_B = i\hbar \frac{1}{m_0} \left(\frac{\partial}{\partial x^0} \right) \equiv \frac{\hat{E}}{m_0 c} = \frac{m c}{m_0} \div \gamma_B c \quad [\text{for } E = m c^2 \text{ and (A6)}], \quad (\text{A8})$$

then (A7) gives,

$$\Theta_B m = i\hbar \gamma_B \partial_0, \quad \text{i.e.,} \quad \Theta_B m \div \frac{\hat{E} m}{m_0 c} = \frac{m^2 c}{m_0} \div \gamma_B m c. \quad (\text{A9})$$

Thus, insertion of $m = \gamma_B m_0$ from (A6) into $\Theta_B m = i\hbar \gamma_B \partial_0$ of (A9) yields,

$$\Theta_B m_0 = i\hbar \partial_0, \quad (\text{A10})$$

i.e., the rest mass m_0 of boson in association with Θ_B is now able to behave like as an expression of the energy operator $c^{-1} \hat{E}$ of (34). This is a bridging relation between (Quantum) Relativity and Quantum Mechanics.

Definition A3 (Kline-Gordon-like Equation). *Insertion of (20) or (19) in (A5) and then using (A9) it yields,*

$$\begin{aligned} \frac{\partial^2}{\partial \hat{s}^2} \psi &= (i\hbar)^2 \mathcal{A} \psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu \psi \div \left[(i\hbar)^2 \partial_0^2 - (i\hbar)^2 \partial_i \partial_j \right] \psi \\ &\equiv (i\hbar)^2 \gamma_B^2 \partial_0^2 \psi = (\Theta_B m)^2 \psi \div (\gamma_B m c)^2 \psi. \end{aligned} \quad (\text{A11})$$

Equalizing the second and last terms of (A11) yields a second order wave equation as,

$$\left[(i\hbar)^2 \mathcal{A} \psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + (i\gamma_B m c)^2 \right] \psi = 0, \quad (\text{A12})$$

and by using $E = m c^2$ and (A12) yields,

$$\left[(i\hbar)^2 \mathcal{A} \psi^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu + \left(i\gamma_B \frac{\hat{E}}{c} \right)^2 \right] \psi = 0,$$

for the construction $\psi = (\varphi, \varphi_\mu, \varphi_{\mu\nu})$ when the spin of the particle is ($s = 0, s = 1, s = 2$) with the conditions $\partial^\mu \varphi_\mu = 0, \partial^\mu \varphi_{\mu\nu} = 0, \mathcal{A} \psi^{\mu\nu} \varphi_{\mu\nu} = 0$ and the total symmetrization of the indices as $\varphi_{\mu\nu}(x) = \varphi_{(\mu\nu)}$. So, (A12) is nothing but an analogy to Kline-Gordon equation, though, $\Theta_B m = \gamma_B m c$ makes it clear that (A12) is not an orthodox Quantum Relativistic Kline-Gordon equation, though, it is actually a hidden (quantum) gravitational field equation (see (62), or more precisely, see (63) or the origin of (87), in the main text for more details) which have to satisfy the $(3+1)D$ curvilinear quantum coordinate system $((i\hbar)^{-1}x^0, \dots, (i\hbar)^{-1}x^3)$ for (19). Thus,

unlike orthodox Quantum Relativistic Kline-Gordon equation, here (A12) (or (62), in other words, (63) or (87), for example) is technically applicable in accelerating frames for Definition 6.

If we consider $\tau = (i\hbar)^{-1} \gamma_B^{-1} t$ as the proper time, then (A9) yields,

$$\Theta_B m = i\hbar \gamma_B \partial_0 = \frac{\partial}{\partial (c (i\hbar)^{-1} \gamma_B^{-1} t)} \doteq \frac{\partial}{\partial (c \tau)} \equiv \vec{\nabla}_{0(\tau)} \doteq \partial_{0(\tau)}, \quad (\text{A13})$$

thus, once again we can show that, in reality, (A12) is not a Kline-Gordon equation by inserting (A13) in (A9) to have $\Theta_B m = \gamma_B m c \doteq \partial_{0(\tau)}$ in such a way that,

$$\left[(i\hbar)^2 \mathcal{V}^{\mu\nu} \vec{\nabla}_\mu \vec{\nabla}_\nu - \partial_{0(\tau)}^2 \right] \psi = 0,$$

which is mass-independent.

Appendix A.2. Fermions

Let us write a fermionic field representation by using a set of (quantum) line elements, i.e., $d\hat{s}$ and $(\partial/\partial\hat{s})$, using the gamma matrices $\hat{\mathbb{E}}^\mu$ when,

$$\hat{\mathbb{E}}^\mu = \begin{pmatrix} \gamma^\mu & 0 & 0 & 0 \\ 0 & \gamma^\mu & 0 & 0 \\ 0 & 0 & -\tilde{\gamma}^\mu & 0 \\ 0 & 0 & 0 & -\tilde{\gamma}^\mu \end{pmatrix}, \quad (\text{A14})$$

while satisfying the Clifford algebra as $\{\hat{\mathbb{E}}^\mu, \hat{\mathbb{E}}^\nu\} = \hat{\mathbb{E}}^\mu \hat{\mathbb{E}}^\nu + \hat{\mathbb{E}}^\nu \hat{\mathbb{E}}^\mu = 2\mathcal{V}^{\mu\nu} \mathbb{K}_4$, where γ^μ are the Dirac gamma-matrices and $\tilde{\gamma}^\mu$ are the Majorana gamma-matrices, as follows,

$$d\hat{s} \Psi = (i\hbar)^{-1} \hat{\mathbb{E}}_\mu dx^\mu \Psi, \quad (\text{A15})$$

$$\text{and } \frac{\partial}{\partial \hat{s}} \psi = i\hbar \left(\hat{\mathbb{E}}^0 \frac{\partial}{\partial x^0} - \hat{\mathbb{E}}^i \frac{\partial}{\partial x^i} \right) \psi = i\hbar \hat{\mathbb{E}}^\mu \vec{\nabla} \psi, \quad (\text{A16})$$

while the temporal axis is $(i\hbar)^{-1} x^0 = (i\hbar)^{-1} (ct)$.

Let (A15) yield a quantum factor γ_F for fermionic fields as,

$$\begin{aligned} (i\hbar)^2 \left(\frac{d\hat{s}}{dx^0} \right) \Psi &= i\hbar \hat{\mathbb{E}}_\mu \frac{dx^\mu}{d(ct)} \Psi = i\hbar \left[\hat{\mathbb{E}}^0 \frac{dx^0}{d(ct)} - \hat{\mathbb{E}}^i \frac{dx^i}{d(ct)} \right] \Psi \\ &= i\hbar \left(\hat{\mathbb{E}}^0 - \hat{\mathbb{E}}^i \frac{v^i}{c} \right) \Psi \equiv i\hbar \gamma_F^{-1} \Psi, \end{aligned} \quad (\text{A17})$$

where the subscript F means fermions, definitely $\gamma_F \neq \gamma_B$, thus for (A17), the quantum factor for spinor is,

$$(i\hbar)^{-1} \gamma_F = (i\hbar)^{-1} \left(\hat{\mathbb{E}}^0 - \hat{\mathbb{E}}^i \frac{v^i}{c} \right)^{-1} \equiv (i\hbar)^{-1} \hat{\mathbb{E}}^\mu \frac{d(ct)}{dx^\mu} \doteq (i\hbar)^{-1} \frac{dx^0}{\hat{\mathbb{E}}_\mu dx^\mu}. \quad (\text{A18})$$

But (A17) also yields,

$$\begin{aligned} (i\hbar)^{-2} \left(\frac{\partial}{\partial x^0} \right)^{-1} \frac{\partial}{\partial \hat{s}} \psi &= (i\hbar)^{-1} \gamma_F \psi, \\ \therefore \frac{\partial}{\partial \hat{s}} \psi &= i\hbar \gamma_F \left(\frac{\partial}{\partial x^0} \right) \psi = i\hbar \gamma_F \partial_0 \psi, \\ \text{i.e., } \gamma_F^{-1} c \frac{\partial}{\partial \hat{s}} \psi &= i\hbar \frac{\partial}{\partial t} \psi = \hat{E} \psi. \end{aligned} \quad (\text{A19})$$

Thus, (A19) implies that the energy operator $\hat{E} \rightarrow i \hbar \partial_t$ is now for fermions, which is also possible to be written as $\hat{E} \rightarrow \gamma_F^{-1} c (\partial/\partial \hat{s})$ in $(3+1)D$ curvilinear quantum spacetime.

Definition A4 ((Quantum) Lorentz-like transformation for Fermions). *It is possible to develop a standard form of (Quantum) Lorentz-like transformation for fermions using the gamma matrices γ^μ as,*

$$\begin{aligned} i \hbar \gamma_F^0 \partial'_0 \psi &= i \hbar \gamma_F^{-1} \left(\gamma_F^0 \partial_0 - \gamma_F^1 \frac{c}{v} \partial_1 \right) \psi, \\ i \hbar \gamma_F^1 \partial'_1 \psi &= i \hbar \gamma_F^{-1} \left(\gamma_F^1 \partial_1 - \gamma_F^0 \frac{c}{v} \partial_0 \right) \psi, \\ i \hbar \gamma_F^2 \partial'_2 \psi &= i \hbar \gamma_F^2 \partial_2 \psi, \\ i \hbar \gamma_F^3 \partial'_3 \psi &= i \hbar \gamma_F^3 \partial_3 \psi. \end{aligned}$$

Interestingly, this transformation does not have any Special Relativistic counterparts.

Remark A3 (Parameter Θ_F and Its Relation with Fermionic Field). *Let us assume that the quantum relativistic fermion mass is,*

$$m = \gamma_F m_0 \quad [\text{for (A18)}]. \quad (\text{A20})$$

Let us insert (A19) 'twice' into (A20) one after another as follows, at first by inserting the value of γ_F we get,

$$m = (i \hbar)^{-1} m_0 \left(\frac{\partial}{\partial x^0} \right)^{-1} \left(\frac{\partial}{\partial \hat{s}} \right),$$

and then by inserting the value of $(\partial/\partial \hat{s})$ it yields,

$$m = (i \hbar)^{-1} m_0 \left(\frac{\partial}{\partial x^0} \right)^{-1} i \hbar \gamma_F \partial_0, \quad (\text{A21})$$

and by using $\hat{E} \rightarrow i \hbar \partial_t$ and $m = \gamma_F m_0$ for (A20), if a parameter Θ_F yields,

$$\Theta_F = i \hbar \frac{1}{m_0} \left(\frac{\partial}{\partial x^0} \right) \equiv \frac{\hat{E}}{m_0 c} = \frac{m c}{m_0} \doteq \gamma_F c, \quad (\text{A22})$$

then (A21) gives,

$$\Theta_F m = i \hbar \gamma_F \partial_0, \quad \text{i.e.,} \quad \Theta_F m \doteq \frac{\hat{E} m}{m_0 c} = \frac{m^2 c}{m_0} \doteq \gamma_F m c. \quad (\text{A23})$$

Thus, insertion of $m = \gamma_F m_0$ from (A20) into $\Theta_F m = i \hbar \gamma_F \partial_0$ of (A23) yields,

$$\Theta_F m_0 = i \hbar \partial_0, \quad (\text{A24})$$

i.e., the rest mass m_0 of fermion in association with Θ_F is now able to behave like as an expression of the energy operator $c^{-1} \hat{E}$ of (34). This is again a bridging relation between (Quantum) Relativity and Quantum Mechanics.

Due to $\Theta_F m$ of (A23), if we consider $\tau = (i \hbar)^{-1} \gamma_F^{-1} t$ as the proper time, then (A23) yields,

$$\Theta_F m = i \hbar \gamma_F \partial_0 = \frac{\partial}{\partial (c (i \hbar)^{-1} \gamma_F^{-1} t)} \doteq \frac{\partial}{\partial (c \tau)} \equiv \vec{\nabla}_{0(\tau)} \doteq \partial_{0(\tau)}. \quad (\text{A25})$$

Note it that fermions that satisfy $\tau = (i \hbar)^{-1} \gamma_F^{-1} t$ are not tachyons, the property of τ is a special characteristic of fermions as long as (A15) is satisfied.

Definition A5 (Dirac-like Equation). By inserting (A19) in (A16) and then using (A23) it yields,

$$\begin{aligned}\frac{\partial}{\partial \hat{s}} \psi &= i \hbar \not{\epsilon}^{\mu} \vec{\nabla}_{\mu} \psi \doteq \left[i \hbar \not{\epsilon}^0 \partial_0 + i \hbar \not{\epsilon}^i \partial_i \right] \psi \\ &\equiv i \hbar \gamma_F \partial_0 \psi = \Theta_F m \psi \doteq \gamma_F m c \psi.\end{aligned}\quad (\text{A26})$$

So, we are able to yield a first order spinor field representation using the third and the fourth terms and/or the second and the last terms of (A26) as,

$$\left[i \hbar \not{\epsilon}^0 \partial_0 + i \hbar \not{\epsilon}^i \partial_i \right] \psi = i \hbar \gamma_F \partial_0 \psi, \quad (\text{A27})$$

$$\text{and/or} \quad \left[i \hbar \not{\epsilon}^{\mu} \vec{\nabla}_{\mu} - \gamma_F m c \right] \psi = 0, \quad (\text{A28})$$

and by using $E = m c^2$ and (A28) yields,

$$\left[i \hbar \not{\epsilon}^{\mu} \vec{\nabla}_{\mu} - \gamma_F \frac{\hat{E}}{c} \right] \psi = 0,$$

in terms of the Dirac tensor spinors $\psi = \psi_{\mu_1 \dots \mu_n}$ for the construction $\psi = (\varphi, \varphi_v)$ when the spin of the particle is ($s = 1/2, s = 3/2$) with $\partial^{\nu} \varphi_v = 0$ and $\not{\epsilon}^{\nu} \varphi_v = 0$. Though, (A28) is analogous to Dirac equation, but $\Theta_F m = \gamma_F m c$ of (A23) make it clear that (A28) is not an orthodox Quantum Relativistic Dirac equation, though, it is actually a hidden (quantum) gravitational field equation (for example, see (68), or more precisely, see the origin of (103), in the main text for more details) which have to satisfy the $(3+1)D$ curvilinear quantum coordinate system $((i \hbar)^{-1} x^0, \dots, (i \hbar)^{-1} x^3)$ for (19). Thus, unlike orthodox Quantum Relativistic Dirac equation, (A28) (or (68), in other words, (103) for example) is technically only applicable in accelerating frames for Definition 6.

Let us consider that an anticommutative line operator for $q\bar{q}$ mesons made of two-quarks is possible to be written by squaring the second and the fourth terms of (A26) as,

$$(i \hbar)^2 \gamma_F^2 \partial_0^2 \psi = \frac{1}{2} (i \hbar)^2 \{ \not{\epsilon}^{\mu}, \not{\epsilon}^{\nu} \} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} \psi, \quad (\text{A29})$$

$$\text{or alternatively, } \frac{\partial^2}{\partial \hat{s}_{q\bar{q}}^2} \psi = \frac{1}{2} (i \hbar)^2 \{ \not{\epsilon}^{\mu}, \not{\epsilon}^{\nu} \} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} \psi,$$

$$\text{thus, by comparing, } \frac{\partial^2}{\partial \hat{s}_{q\bar{q}}^2} \psi \doteq (i \hbar)^2 \gamma_F^2 \partial_0^2 \psi, \quad (\text{A30})$$

if here and hereafter $\{, \}$ represents the anticommutator, which is satisfying the Clifford algebra as $\{ \not{\epsilon}^{\mu}, \not{\epsilon}^{\nu} \} = \not{\epsilon}^{\mu} \not{\epsilon}^{\nu} + \not{\epsilon}^{\nu} \not{\epsilon}^{\mu} = 2 \mathcal{A}^{\mu\nu} \mathbb{K}_4$. Moreover, (A29) yields as,

$$\frac{1}{2} (i \hbar)^2 \{ \not{\epsilon}^{\mu}, \not{\epsilon}^{\nu} \} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} \psi - (i \hbar)^2 \gamma_F^2 \partial_0^2 \psi = 0,$$

$$\therefore (i \hbar)^2 \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} \psi - (i \hbar)^2 \gamma_F^2 \partial_0^2 \psi = 0,$$

$$\text{or alternatively, } \left[(i \hbar)^2 \mathcal{A}^{\mu\nu} \vec{\nabla}_{\mu} \vec{\nabla}_{\nu} + (i \gamma_F m c)^2 \right] \psi = 0 \quad [\text{for (A26)}]. \quad (\text{A31})$$

This second order wave equation is analogous to Kline-Gordon-like equation (A12), but now with γ_F .

References

1. I. J. R. Aitchison, A. J. G. Hey, *Gauge Theories in Particle Physics: A Practical Introduction*, Vol. 2 (CRC Press, 2024).
2. G. Tobar et al., *Nature Communications* **15**, 7229 (2024).
3. B.-L. B. Hu, E. Verdaguer, *Semiclassical and Stochastic Gravity: Quantum Field Effects on Curved Spacetime* (Cambridge University Press, 2020).

4. M. J. D. Hamilton, *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics* (Springer International, 2017).
5. F. Müller, "On the Paradigms of Quantum Gravity 2016" in *Experimental Search for Quantum Gravity*, ed. by S. Hossenfelder (Springer, 2018), pp. 21-27.
6. M. Blagojevic, *Gravitation and Gauge Symmetries* (Institute of Physics Publishing, 2002).
7. P. M. Alsing, J. C. Evans, K. K. Nandi, *Gen. Rel. Grav.* **33**, 1459 (2001).
8. P. Collas, D. Klein, *The Dirac Equation in Curved Spacetime* (Springer Nature, 2019).
9. J. L. Powell, B. Crasemann, *Quantum Mechanics* (Addison-Wesley Publishing Company, Inc. 1961).
10. M. Jammer, *Concepts of Mass in Contemporary Physics and Philosophy*, (Princeton University Press, 2000), pp. 69-70.
11. B. C. Hall, *Quantum Theory for Mathematicians* (Springer, 2013).
12. E. Gaztañaga, K. S. Kumar, J. Marto, *Class. Quantum Grav.* **43**, 015023 (2026).
13. H. Moussa et al., *Nat. Phys.* **19**, 863 (2023).
14. L. E. Parker, D. J. Toms, *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* (Cambridge University Press, 2009).
15. S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, 1989).
16. A. Aste, *Symmetry* **2**, 1776 (2010).
17. M. Gasperini, *Theory of Gravitational Interactions* (Springer, 2017).
18. B. Chow, P. Lu, L. Ni, *Hamilton's Ricci Flow* (American Mathematical Society and Science Press, 2010).
19. N. Olivi-Tran, *Adv. Studies Theor. Phys.* **3**, 9 (2009).
20. I. L. Buchbinder, I. L. Shapiro, *Introduction to Quantum Field Theory with Applications to Quantum Gravity* (Oxford University Press, 2021).
21. J. Matehkolaei Mehdi, *Pramana-J. Phys.* **93**, 84 (2019).
22. B. M. Novakovic, D. B. Novakovic, A. B. Novakovic, *AIP Conf. Proc.* **839**, 144 (2006).
23. C. Cremaschini, *Eur. Phys. J. C* **83**, 729 (2023).
24. E. Zeidler, *Quantum Field Theory III: Gauge Theory* (Springer-Verlag, 2011).
25. H. Friedrich, A. Rendall, "The Cauchy Problem for the Einstein Equations" in *Einstein's Field Equations and Their Physical Implications*, ed. by B. G. Schmidt (Springer, 2000).
26. E. Schrödinger, *Space-time Structure* (Cambridge University Press, 1997), p. 6l.
27. R. Aaij et al., *Phys. Rev. Lett.* **127**, 111801 (2021).
28. T. Guff, C. U. Shastry, A. Rocco, *Sci Rep* **15**, 3658 (2025).
29. M. Carmeli, *Group Theory and General Relativity* (McGraw-Hill, 1977).
30. M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (John Wiley, 1982).
31. M. Camneli, E. Leibowitz, N. Nissani, *Gravitation: $SL(2,C)$ Gauge Theory and Conservation Laws* (World Scientific, 1990).
32. S. Navas et al. (Particle Data Group), *Phys. Rev. D* **110**, 030001 (2024).
33. Editorial, *Nature Astronomy* **9**, 471 (2025).
34. A. J. Shajib, J. A. Frieman, *Phys. Rev. D* **112**, 063508 (2025).
35. I. Sogami, S. Nishimoto, *Il Nuovo Cimento* **52**, 531 (1979).
36. I. Sogami, S. Nishimoto, *Prog. Theor. Phys.* **62**, 1768 (1979).
37. W. Kummer, *Il Nuovo Cimento* **31**, 219 (1964).
38. W. N. Cottingham, D. A. Greenwood, *An Introduction to the Standard Model of Particle Physics* (Cambridge University Press, 2007).
39. A. Piórkowska-Kurpas, S. Cao, M. Biesiada, *Journal of High Energy Astrophysics* **33**, 37 (2022).
40. A. Piórkowska-Kurpas, *Universe* **8**, 83 (2022).
41. B. Carr, *Universe or Multiverse?* (Cambridge University Press, 2007).
42. R. Demina, A. Garcia-Bellido, *Electroweak Symmetry and its Breaking* (World Scientific, 2023).
43. S. Weinberg, E. Witten, *Phys. Lett.* **96B**, 59 (1980).
44. S. Nair, A. Vijaykumar, S. Sarkar, *JCAP* **11**, 004 (2024).
45. Y. Takahashi, *An Introduction to Field Quantization* (Pergamon Press, 1969).
46. J. C. Collins, *Renormalization: An introduction to renormalization, the renormalization group, and the operator-product expansion* (Cambridge University Press, 2023).
47. W. D. McComb, *Renormalization Methods: A Guide for Beginners* (Oxford University Press, 2007).
48. E. B. Manoukian, *Renormalization* (Academic Press, 1983).
49. K. Costello, *Renormalization and Effective Field Theory* (American Mathematical Society, 2011).

50. E. Anderson, *The Problem of Time: Quantum Mechanics Versus General Relativity* (Springer, 2017).
51. M. Bojowald, "A Momentous Arrow of Time" in *The Arrows of Time: A Debate in Cosmology*, eds. by L. Mersini-Houghton, R. Vaas (Springer, 2012), pp. 169-189.
52. T. P. Singh, "Classical and Quantum: A Conflict of Interest" in *Gravity and the Quantum*, eds. by J. S. Bagla, S. Engineer (Springer, 2017), pp. 411-425.
53. C. Kiefer, "Quantum Gravity: Whence, Whither?" in *Quantum Field Theory and Gravity: Conceptual and Mathematical Advances in the Search for a Unified Framework*, eds. by F. Finster et al. (Springer, 2012), pp. 1-13.
54. C. K. Raju, *Time: Towards a Consistent Theory* (Springer, 1994).
55. J. F. Donoghue, "Quantum General Relativity and Effective Field Theory" in *Handbook of Quantum Gravity*, eds. by C. Bambi et al. (Springer Nature, 2024), pp. 3-26.
56. A. N. Balslev, *A Study of Time in Indian Philosophy* (Munshiram Manoharlal Publishers Pvt. Ltd., 1999).
57. A. N. Balslev, J. N. Mohanty (eds.), *Religion And Time* (E. J. Brill, 1993).
58. M. Maziashvili, Z. K. Silagadze, *Gen. Relativ. Gravit.* **53**, 67 (2021).
59. D. H. Bernstein, E. Giladi, K. R. W. Jones, *Mod. Phys. Lett. A* **13**, 2327 (1998).
60. J. Natario, A. Sasane, *Ann. Henri Poincaré* **23** 2345 (2022),.
61. Y. Choquet-Bruhat, C. De Witt-Morette, M. Dillard-Bleick, *Analysis, Manifolds, and Physics, Part I: Basics* (Elsevier, 1996).
62. R. Penrose, W. Rindler, *Spinors and Space-Time*, Vol 1 (Cambridge University Press, Cambridge, 2003).
63. T.-P. Hack, *Cosmological Applications of Algebraic Quantum Field Theory in Curved Spacetimes* (Springer, 2016).
64. S. Hollands, W. Ruan, *Ann. Henri Poincaré* **3**, 635 (2002).
65. R. Penrose, W. Rindler, *Spinors and Space-Time*, Vol. 2 (Cambridge University Press, Cambridge, 1999).
66. N. Huggett, N. Linnemann, M. D. Schneider, *Quantum Gravity in a Laboratory?* (Cambridge University Press, 2023).
67. S. Hossenfelder (ed.), *Experimental Search for Quantum Gravity* (Springer, 2018).
68. C. Pfeifer, C. Lämmerzahl (eds.), *Modified and Quantum Gravity: From Theory to Experimental Searches on All Scales* (Springer Nature, 2023).
69. G. Z. Adunas, E. Rodriguez-Milla, D. V. Ahluwalia, *Phys. Lett. B* **485**, 215 (2000).
70. A. R. Brown et al., *PRX Quantum* **4**, 010320 (2023).
71. S. Nezami et al., *PRX Quantum* **4**, 010321 (2023).
72. A. Garcia-Soto, D. Garg, M. H. Reno, C. A. Argüelles, *Phys. Rev. D* **107**, 033009 (2023).
73. R. Schützhold, *Phys. Rev. Lett.* **135**, 171501 (2025).
74. S. M. Vermeulen et al., *Phys. Rev. X* **15**, 011034 (2025).
75. G. Aad et al., *Phys. Rev. D* **112**, 072009 (2025).

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