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Article

Circular and Hyperbolic Symmetry Unified in Hyper-Spacetime

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Abstract

We present a unified geometric framework for the full Lorentz group, based on a bounded angle parametrization of spacetime transformations within a hyper-spherical geometry. By mapping the unbounded hyperbolic angle φ to a bounded angle β using the Gudermannian function, hyperbolic, causal, and Euclidean three-spheres are brought together into a single structure: **hyper-spacetime**. This structure unifies the Euclidean $\mathbb{R}^4$ and Minkowski $\mathbb{R}^{1,3}$ domains, incorporates discrete symmetries in a continuous way, and removes discontinuities at the lightlike boundary. Each three-sphere carries a natural spinor set, encoding symmetry, and acting as eigen-spinors of corresponding observables. These reproduce the Dirac spectrum while confining singular behavior to a scalar factor. The bounded angle parametrization therefore provides a continuous, closed representation of the full Lorentz group and a transparent geometric basis for spacetime symmetry.

Keywords: full Lorentz group; spacetime spinors; three-sphere; hyper-spacetime; geometric algebra; Dirac equation; null spinors; geometric duality

1. Introduction

The full Lorentz group $O(1, 3)$ [1–6] encodes the fundamental symmetries of spacetime. It includes both continuous transformations - boosts and spatial rotations - as well as discrete reflections. Conventionally, Lorentz boosts are parameterized by the unbounded angle $\varphi \in (-\infty, +\infty)$ (rapidity). While effective, this parametrization leads to discontinuities at the lightlike boundary, and leaves the discrete symmetries disconnected from identity.

In this paper, we introduce a bounded representation of the full Lorentz group by embedding its transformations into a closed hyper-spherical geometry, which we term **hyper-spacetime**. The main step is to map the unbounded angle φ to a bounded angle β using the Gudermannian function [7,8]. This mapping enables smooth passage across infinity, incorporates discrete transformations in a continuous manner, and eliminates algebraic singularities at the lightlike boundary.

The core of the paper (Sections 2–6) develops the mathematical framework. We first reformulate Lorentz boosts using a bounded angle parametrization that unifies circular and hyperbolic symmetry and applies naturally to special relativity, yielding harmonic relativistic proportionality factors [9]. Extending to four dimensions, we introduce a triplet of three-spheres that define hyper-spacetime and show how the full Lorentz group, with both continuous and discrete symmetries, emerges within its hyper-spherical coordinates. Spinor sets are then constructed and analyzed as eigen-spinors of geometric observables, reproducing the Dirac spectrum while confining singularities to scalar factors. The final part of the paper (Sections 7–8) turns to interpretation and outlook, discussing the physical implications of hyper-spacetime, and concludes with a summary of its unifying features.

The mathematical formalism throughout is based on the geometric algebra (GA) of spacetime (STA) as developed by David Hestenes [10,11] (Appendix A). Foundations of geometric algebra where laid by Grassmann [12] and Clifford [13] in the 19th century, and the decisive motivation for its use lies in the generalization of rotations via rotors [14–18]. In STA [10], a spacetime inertial frame

$\{t, x, y, z\}$ is represented by four orthogonal basis vectors $\gamma_\mu = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, which satisfy the algebra of the Dirac gamma matrices [4].

2. Hyperbolic Rotation

Lorentz boosts are conventionally expressed as hyperbolic rotations, parameterized by the unbounded angle $\varphi \in (-\infty, +\infty)$ (rapidity) (**Figure 1a**).

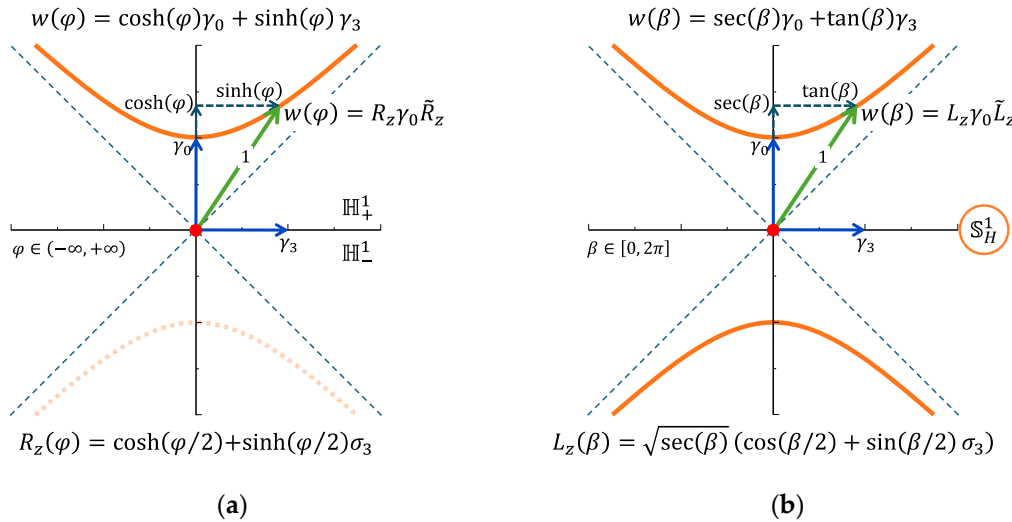


Figure 1. Comparison of hyperbolic rotation. (a) In the unbounded parametrization, the rotor $R_z(\varphi)$ covers only one branch $\mathbb{H}_+^1$, with a discontinuity at lightlike boundaries. (b) Using the bounded angle β , the rotor $L_z(\beta)$ spans the full symmetry $\mathbb{S}_H^1 = \mathbb{H}_+^1 \oplus \mathbb{H}_-^1$. The Gudermannian mapping ensures smooth continuity through infinity, closing the hyperbola into a one-sphere $\mathbb{S}_H^1$.

A boost along the γ_3 -axis (z -axis) is generated by the hyperbolic rotor $R_z(\varphi)$:

$$R_z(\varphi) = \exp(\varphi \sigma_3/2) = \cosh(\varphi/2) + \sinh(\varphi/2)\sigma_3, \quad \varphi \in (-\infty, +\infty), \quad R_z \tilde{R}_z = 1 \quad (1)$$

Where bivector $\sigma_3 = \gamma_3\gamma_0$ represents the temporal plane of rotation. Acting on the temporal basis vector γ_0 , the rotor generates the boosted spacetime vector $w(\varphi)$:

$$w(\varphi) = R_z \gamma_0 \tilde{R}_z = \cosh(\varphi)\gamma_0 + \sinh(\varphi)\gamma_3, \quad \|w(\varphi)\|^2 = 1 \quad (2)$$

However, rotor $R_z(\varphi)$ only covers the positive branch $\mathbb{H}_+^1$ of the hyperbola. The negative branch of the hyperbola $\mathbb{H}_-^1$ is not present (**Figure 1a**).

To cover the full hyperbolic symmetry in a single parameter domain, we map the unbounded angle $\varphi \in (-\infty, +\infty)$ onto the bounded angle $\beta \in [0, 2\pi]$ using the Gudermannian function $\varphi = \tanh^{-1}(\sin(\beta))$ (Appendix B) [7,8]. Substituting of this function into $R_z(\varphi)$ yields the rotor $L_z(\beta)$:

$$\begin{aligned} L_z(\beta) &= \exp(\tanh^{-1}(\sin(\beta))\sigma_3/2) = \sqrt{\sec(\beta)} L_{u1}(\beta), & L_z \tilde{L}_z &= 1 \\ L_{u1}(\beta) &= \cos(\beta/2) + \sin(\beta/2)\sigma_3, & L_{u1} \tilde{L}_{u1} &= \cos(\beta) \end{aligned} \quad (3)$$

The scalar factor $\sqrt{\sec(\beta)}$ ensures normalization, while $L_{u1}(\beta)$ acts as a **temporal spinor** (Appendix C). Applying rotor $L_z(\beta)$ to γ_0 generates the boosted spacetime vector $w(\beta)$:

$$w(\beta) = L_z \gamma_0 \tilde{L}_z = \sec(\beta)\gamma_0 + \tan(\beta)\gamma_3, \quad \beta \in [0, 2\pi], \quad \|w(\beta)\|^2 = 1 \quad (4)$$

Where the full hyperbolic symmetry is covered continuously (**Figure 1b**).

In summary, the bounded angle parametrization provides several advantages. It removes singularities by isolating them in the scalar density factor $\sqrt{\sec(\beta)}$, while the associated spinor remains regular. At $\beta = \pm\pi/2$, the mapping passes smoothly through infinity, ensuring continuity across the lightlike boundary where crossing corresponds to a time reversal. The two disconnected

hyperbolic branches $\mathbb{H}_+^1$ and $\mathbb{H}_-^1$ are thereby closed into a single hyperbolic one-sphere $\mathbb{S}_H^1$ (**Figure 1b**). In this way, hyperbolic symmetry is unified within a single angular cycle.

3. Harmonic Relativistic Proportionality Factors

In special relativity, the relative speed v/c is conventionally mapped to the unbounded angle $\varphi \in (-\infty, +\infty)$ (rapidity). Using instead the bounded angle $\beta \in [0, 2\pi]$ introduced in Section 2, this mapping provides a harmonic set of relativistic proportionality factors:

$$\begin{aligned} \sin(\beta) &= \pm v/c, & \cos(\beta) &= \pm \sqrt{1 - (v/c)^2} \\ \sec(\beta) &= \frac{1}{\pm \sqrt{1 - (v/c)^2}}, & \tan(\beta) &= \frac{\pm v/c}{\pm \sqrt{1 - (v/c)^2}} \end{aligned} \quad (5)$$

These factors form a bridge between circular and hyperbolic symmetry:

$$\begin{aligned} \sin^2(\beta) + \cos^2(\beta) &= 1, & \sec^2(\beta) - \tan^2(\beta) &= 1 \\ \mathbb{S}^1 \subset \mathbb{R}^2 & & \leftrightarrow & \mathbb{S}_H^1 \subset \mathbb{R}^{1,1} \end{aligned} \quad (6)$$

Consequently, the bounded angle mapping unifies circular $\mathbb{S}^1 \subset \mathbb{R}^2$ with hyperbolic $\mathbb{S}_H^1 \subset \mathbb{R}^{1,1}$ symmetry (**Figure 1b**). A complete cycle on one-sphere $\mathbb{S}^1 \subset \mathbb{R}^2$ corresponds one-to-one with a complete cycle on hyperbolic one-sphere $\mathbb{S}_H^1 \subset \mathbb{R}^{1,1}$. This includes a smooth passage through infinity at the lightlike boundaries $\beta = \pm\pi/2$.

To incorporate all the symmetries of the full Lorentz group $O(1, 3)$, the framework must extend from the two-dimensional (2D) relation $\mathbb{R}^2 \leftrightarrow \mathbb{R}^{1,1}$ to a four-dimensional (4D) relation that unifies the Euclidean domain $\mathbb{R}^4$ with the Minkowski domain $\mathbb{R}^{1,3}$.

4. One Unified Geometry Bridging $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$

The generalization from two to four dimensions is realized by the introduction of a triplet of three-spheres embedded in a Euclidean $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ and a Minkowski $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ domain (Appendix A). We define three distinct but related three-spheres:

$$\begin{aligned} \mathbb{S}_H^3 &= \{p = a_0\gamma_0 + a_k\gamma_k \in \mathbb{R}^{1,3} \mid a_\mu \in \mathbb{R}, \quad \|p\|^2 = 1\}, & \text{metric: } \{+1, -1, -1, -1\} \\ \mathbb{S}_C^3 &= \{q = \gamma_0 + a_k\gamma_k \in \mathbb{R}^{1,3} \mid a_k \in \mathbb{R}, \quad \|q\|^2 = \cos^2(\beta)\}, & \text{metric: } \{+1, -1, -1, -1\} \\ \mathbb{S}^3 &= \{u_r = a_0\gamma_0 + a_k\sigma_k \in \mathbb{R}^4 \mid a_\mu \in \mathbb{R}, \sigma_k = \gamma_k\gamma_0, \|u_r\|^2 = 1\}, & \text{metric: } \{+1, +1, +1, +1\} \end{aligned} \quad (7)$$

Here:

- The hyperbolic three-sphere $\mathbb{S}_H^3$ is preserved by spacetime rotor Λ_1 (**Figure 2b**).
- The causal three-sphere $\mathbb{S}_C^3$ is preserved by spacetime spinor U_1 (**Figure 3a**).
- The Euclidean three-sphere $\mathbb{S}^3$ is preserved by Euclidean spinor V_1 (**Figure 3b**).

Each of these three-spheres is preserved by a specific spinor structure (as defined in section 4.1), which ensures that the geometry remains invariant under both Lorentzian and Euclidean transformations. In this way, every three-sphere is not only defined by its metric but also carries a natural spinor representation that encodes its symmetry.

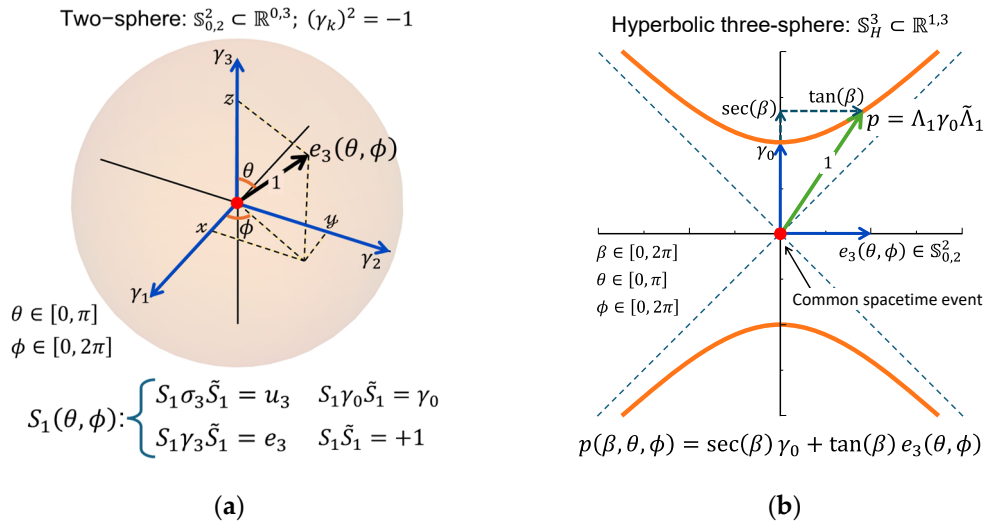


Figure 2. (a) The spatial two-sphere $\mathbb{S}^2_{0,2} \subset \mathbb{R}^{0,3}$, spanned by vectors $e_3(\theta, \phi)$, is preserved by the spatial spinor S_1 . (b) The hyperbolic three-sphere $\mathbb{S}^3_H \subset \mathbb{R}^{1,3}$, spanned by vectors p , is preserved by the spacetime rotor Λ_1 . The spatial vectors $e_3 \in \mathbb{S}^2_{0,2}$ are all orthogonal to the temporal axis γ_0 .

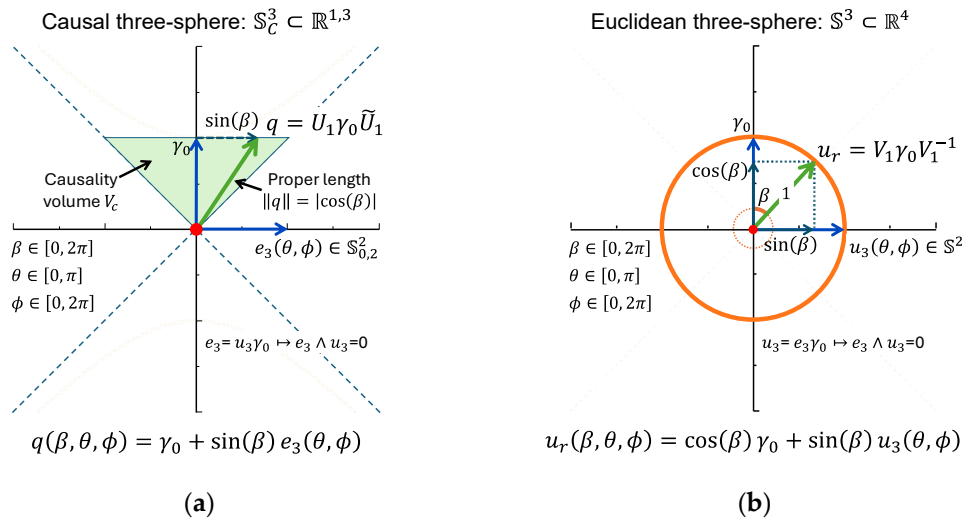


Figure 3. (a) The causal three-sphere $\mathbb{S}^3_C \subset \mathbb{R}^{1,3}$, spanned by vectors q , is preserved by the spacetime spinor U_1 . (b) The Euclidean three-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, spanned by vectors u_r , is preserved by the Euclidean spinor V_1 . In this structure, each of the aligned vectors $e_3 \in \mathbb{S}^2_{0,2}$ and $u_3 \in \mathbb{S}^2$ is orthogonal to temporal axis γ_0 in its respective domain, thereby fixing a common temporal bivector plane..

All three-spheres share: a common temporal axis γ_0 , a common origin (spacetime event), and a common temporal bivector plane. They differ in their metric and spatial subspaces $\{\gamma_k\} \subset \mathbb{R}^{1,3}$ and $\{\sigma_k\} \subset \mathbb{R}^4$ (Eq. 7), which define two complementary two-spheres:

$$\begin{aligned} \mathbb{S}^2_{0,2} &= \{e_3 = a_k \gamma_k \in \mathbb{R}^{0,3} \mid a_k \in \mathbb{R}, (e_3)^2 = -1\} \\ \mathbb{S}^2 &= \{u_3 = a_k \sigma_k \in \mathbb{R}^{3,0} \mid a_k \in \mathbb{R}, (u_3)^2 = +1\} \end{aligned} \quad (8)$$

Both two-spheres are preserved by spatial spinor S_1 (Figure 2a):

$$S_1(\theta, \phi) = \exp(-i\sigma_3\phi/2)\exp(-i\sigma_2\theta/2), \quad S_1\tilde{S}_1 = 1, \quad S_1\gamma_0\tilde{S}_1 = \gamma_0 \quad (9)$$

Applying spinor S_1 generates the spatial unit vectors e_3 and u_3 (Figure 2 and 3):

$$\begin{aligned} e_3(\theta, \phi) &= S_1\gamma_3\tilde{S}_1 = \sin(\theta)\cos(\phi)\gamma_1 + \sin(\theta)\sin(\phi)\gamma_2 + \cos(\theta)\gamma_3, & (e_3)^2 &= -1, & e_3 &= u_3\gamma_0 \\ u_3(\theta, \phi) &= S_1\sigma_3\tilde{S}_1 = \sin(\theta)\cos(\phi)\sigma_1 + \sin(\theta)\sin(\phi)\sigma_2 + \cos(\theta)\sigma_3, & (u_3)^2 &= +1, & u_3 &= e_3\gamma_0 \end{aligned} \quad (10)$$

They are orthogonal to γ_0 , aligned ($e_3 \wedge u_3 = 0$), and define spatial direction. Moreover, they span a common temporal bivector plane ($e_3 = u_3 \gamma_0 \leftrightarrow u_3 = e_3 \gamma_0$), which is a cornerstone of the unified geometry. The Minkowski vectors $e_3 \in \mathbb{R}^{1,3}$ thus reappear in the Euclidean domain as temporal bivectors, while the Euclidean vectors $u_3 \in \mathbb{R}^4$ play the same role in the Minkowski domain. This geometrical duality: specifies spatial direction while simultaneously encoding for temporal orientation in the opposite domain. Their associated dual spatial bivector planes $\mathbb{I}u_3$ belongs to the common three-dimensional even subalgebra (Appendix A), ensuring domain-independence of spatial orientation.

4.1. Definition of the Spinors $\{U_1, V_1\}$, Rotor $\Lambda_1 = \rho U_1$, and Vectors $\{p, q, u_r\}$

The two spinors $\{U_1, V_1\}$, and associated rotor $\Lambda_1 = \rho U_1$, form a spinor structure that preserves the triplet of three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ (Figure 2 and 3). Thus, every three-sphere carries a natural spinor representation that encodes its symmetry. The Euclidean spinor V_1 (preserving $\mathbb{S}^3$) and spacetime spinor U_1 (preserving $\mathbb{S}_C^3$ and $\mathbb{S}_H^3$) are defined as:

$$\begin{aligned} V_1(\beta, \theta, \phi) &= S_1(\theta, \phi) L_{v1}(\beta), & L_{v1}(\beta) &= \cos(\beta/2) + \sin(\beta/2) \gamma_3, & V_1 V_1^{-1} &= 1 \\ U_1(\beta, \theta, \phi) &= S_1(\theta, \phi) L_{u1}(\beta), & L_{u1}(\beta) &= \cos(\beta/2) + \sin(\beta/2) \sigma_3, & U_1 \tilde{U}_1 &= \cos(\beta) \\ \Lambda_1(\beta, \theta, \phi) &= \sqrt{\sec(\beta)} U_1(\beta, \theta, \phi), & & & \Lambda_1 \tilde{\Lambda}_1 &= 1 \end{aligned} \quad (11)$$

Here S_1 acts as a spatial spinor (Eq. 9), L_{v1} and L_{u1} act as temporal spinors (Appendix C), and the scalar factor $\sqrt{\sec(\beta)}$ ensures the unitarity of rotor Λ_1 , the normalized form of U_1 . Applying the spinors U_1 and V_1 yields the vectors spanning the three-spheres:

$$\begin{aligned} p \in \mathbb{S}_H^3 : \quad p(\beta, \theta, \phi) &= \Lambda_1 \gamma_0 \tilde{\Lambda}_1 = \sec(\beta) \gamma_0 + \tan(\beta) e_3(\theta, \phi), \quad \|p\|^2 = 1 \\ q \in \mathbb{S}_C^3 : \quad q(\beta, \theta, \phi) &= U_1 \gamma_0 \tilde{U}_1 = \gamma_0 + \sin(\beta) e_3(\theta, \phi), \quad \|q\|^2 = \cos^2(\beta) \\ u_r \in \mathbb{S}^3 : \quad u_r(\beta, \theta, \phi) &= V_1 \gamma_0 V_1^{-1} = \cos(\beta) \gamma_0 + \sin(\beta) u_3(\theta, \phi), \quad \|u_r\|^2 = 1 \end{aligned} \quad (12)$$

Here, $p \in \mathbb{S}_H^3$ is a hyperbolic unit vector, $q \in \mathbb{S}_C^3$ is a causal vector with variable norm, and $u_r \in \mathbb{S}^3$ is a Euclidean unit vector. The causal three-sphere $\mathbb{S}_C^3$ covers the spacetime region bounded by the light cone of a past event (Figure 3a). Each vector $q \in \mathbb{S}_C^3$ is future pointing and its norm represents proper length, vanishing at the lightlike boundaries. In this way, $\mathbb{S}_C^3$ integrates a natural form of causality in the triplet of three-spheres.

In summary, the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ (Eq. 7) exhibit the following features:

- A common temporal axis γ_0 with a shared origin (spacetime event).
- Unit vectors $e_3 \in \mathbb{R}^{1,3}$ and $u_3 \in \mathbb{R}^4$ that encode spatial direction and temporal orientation through the dual relation $\{e_3 = u_3 \gamma_0 \leftrightarrow u_3 = e_3 \gamma_0\}$, expressing a geometric duality between domains.
- Domain-independent spatial bivector planes $\mathbb{I}u_3 \subset \mathbb{R}^3$, common to both domains.

A complete cycle on the Euclidean three-sphere $\mathbb{S}^3$ corresponds one-to-one with a complete cycle on hyperbolic three-sphere $\mathbb{S}_H^3$, thereby unifying $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$. The apparent boundary at infinity ($\beta = \pm\pi/2$) is absorbed smoothly, making the entire structure closed and traversable. We therefore define the unified geometry constructed from the triplet of three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ as **hyper-spacetime** $\mathcal{B}$:

$$\mathcal{B} = \{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\} \quad (13)$$

Its coordinates are given by the hyper-spherical set (β, θ, ϕ) , where the spatial angles (θ, ϕ) encodes direction and the temporal angle β encodes relative speed (Eq. 5). Hence, the familiar relativistic parameters of direction and speed are reformulated as angular variables in a closed domain, and continuity at the lightlike boundaries.

5. Full Lorentz Group

The full Lorentz group $O(1,3)$ [19–21] contains the symmetries of spacetime that preserve the Minkowski metric. It includes: 1) continuous transformations as boosts and spatial rotations, and 2) spacetime event reflections $\{I, P, T, PT\}$: identity (I), parity (P), time-reversal (T), and spacetime-reversal (PT). Thus, the full Lorentz group is defined as:

$$O(1,3) = \{I, P, T, PT\} \cdot SO^+(1,3) \quad (14)$$

where $SO^+(1,3)$ is the proper orthochronous Lorentz subgroup connected to identity.

In the conventional formalism: boosts in $SO^+(1,3)$ are parameterized by the unbounded angle $\varphi \in (-\infty, +\infty)$, leading to singularities at the lightlike boundaries. Moreover, the discrete reflections $\{P, T, PT\}$ remain disconnected from identity. The standard way to extend the full Lorentz group $O(1,3)$ is through its double cover $\text{Pin}(1,3)$, as it includes both proper and improper transformations [21]. However, even within $\text{Pin}(1,3)$, the discrete transformations remain disconnected from identity, leaving a gap between reflections and continuous transformations. This gap is closed by hyper-spacetime, in which continuous and discrete symmetries arise naturally within a unified geometry.

5.1. A Unified Geometry

In the hyper-spherical coordinates (β, θ, ϕ) of hyper-spacetime (defined in Section 4), the spacetime event reflections $\{I, P, T, PT\}$ manifest naturally as smooth angular transformations:

$$\begin{aligned} \text{Identity} & \quad I : (\pm\beta, \theta, \phi) \\ \text{Parity} & \quad P : (\mp\beta, \theta \pm \pi, \phi) \\ \text{Time-reversal} & \quad T : (\mp\beta \pm \pi, \theta, \phi) \\ \text{Spacetime-reversal} & \quad PT : (\pm\beta \pm \pi, \theta \pm \pi, \phi) \end{aligned} \quad (15)$$

These transformations correspond to changes in temporal orientation (sign changes or $\pm\pi$ shifts in β) and spatial orientation ($\pm\pi$ shifts in θ). Crucially, all four reflections are smoothly connected to identity, eliminating discontinuities.

5.2. Spinor Representations

The spinor structure of hyper-spacetime provides explicit representations of the $\{I, P, T, PT\}$ transformations. Starting from the spacetime spinor $U_1(\beta, \theta, \phi)$, variations in the hyper-spherical coordinates (β, θ, ϕ) generate the full set of spacetime spinors $\{U_j\}$, where each of the spinors represents one of the $\{I, P, T, PT\}$ transformations:

$$\begin{aligned} I: \quad U_1(\beta, \theta, \phi) &= U_1(+\beta, \theta, \phi) = S_1(\theta, \phi)(+\cos(\beta/2) + \sin(\beta/2)\sigma_3) \\ P: \quad U_2(\beta, \theta, \phi) &= U_1(-\beta, \theta + \pi, \phi) = S_2(\theta, \phi)(+\cos(\beta/2) - \sin(\beta/2)\sigma_3) \\ T: \quad U_3(\beta, \theta, \phi) &= U_1(-\beta + \pi, \theta, \phi) = S_1(\theta, \phi)(-\sin(\beta/2) - \cos(\beta/2)\sigma_3) \\ PT: \quad U_4(\beta, \theta, \phi) &= U_1(+\beta + \pi, \theta + \pi, \phi) = S_2(\theta, \phi)(-\sin(\beta/2) + \cos(\beta/2)\sigma_3) \end{aligned} \quad (16)$$

With $U_j \tilde{U}_j = \cos(\beta)\{+1, +1, -1, -1\}$. Similarly, changes in $V_1(\beta, \theta, \phi)$ generate the full set of Euclidean spinors $\{V_j\}$:

$$\begin{aligned} I: \quad V_1(\beta, \theta, \phi) &= V_1(+\beta, \theta, \phi) = S_1(\theta, \phi)(+\cos(\beta/2) + \sin(\beta/2)\gamma_3) \\ P: \quad V_2(\beta, \theta, \phi) &= V_1(-\beta, \theta + \pi, \phi) = S_2(\theta, \phi)(+\cos(\beta/2) - \sin(\beta/2)\gamma_3) \\ T: \quad V_3(\beta, \theta, \phi) &= V_1(-\beta + \pi, \theta, \phi) = S_1(\theta, \phi)(-\sin(\beta/2) - \cos(\beta/2)\gamma_3) \\ PT: \quad V_4(\beta, \theta, \phi) &= V_1(+\beta + \pi, \theta + \pi, \phi) = S_2(\theta, \phi)(-\sin(\beta/2) + \cos(\beta/2)\gamma_3) \end{aligned} \quad (17)$$

With $V_j V_j^{-1} = \{+1, +1, +1, +1\}$. Note the algebraic closure: each of the spinors U_j and V_j can be a basis to generate the full set of transformations. The spatial spinors $\{S_1, S_2\}$ in U_j and V_j are generated from spatial spinor S_1 (**Figure 2a**) via a parity transformation:

$$P: \quad S_1(\theta + \pi, \phi) = S_2(\theta, \phi), \quad S_k \tilde{S}_i = \delta_{ki} \quad (18)$$

This orthonormal set $\{S_k\}$ corresponds to the spatial Pauli spinors [22].

Each set $\{U_j\}$ and $\{V_j\}$ forms an orthonormal basis within its respective metric, and the discrete transformations remain continuously connected to identity. The components of both sets are identical, differing only in their temporal bivector plane: $(\sigma_3 = \gamma_3 \gamma_0) \in \mathbb{R}^{1,3}$ in the Minkowski domain versus $(\gamma_3 = \sigma_3 \gamma_0) \in \mathbb{R}^4$ in the Euclidean domain.

In summary, hyper-spacetime provides:

- A smooth, closed, continuous angular domain for the full Lorentz group $O(1,3)$.
- Discrete reflections $\{P, T, PT\}$, continuously connected to identity.
- Spinor sets $\{U_j, V_j\}$ forming orthonormal bases in their respective metrics, unifying the Euclidean domain $\mathbb{R}^4$ with the Minkowski domain $\mathbb{R}^{1,3}$.
- A closed realization of the double cover $\text{Pin}(1,3)$ through the spinor sets $\{U_j, V_j\}$.
- Isolation of infinities at the lightlike boundaries $\beta = \pm \pi/2$ into the scalar density factor $\sqrt{\sec(\beta)}$ (Eq. 11), while the spinors themselves remain regular.

This prepares the ground for Section 6, where the eigen-spinor and eigenvalue structure of physical observables in hyper-spacetime is developed.

6. Eigen-Spinors and Eigenvalues in the Geometry of Hyper-Spacetime

Analogous to how the observables of $O(2)$ lie on the surface of the one-sphere $\mathbb{S}^1$, the observables of hyper-spacetime reside on the surfaces of the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$, each preserved by the spinor sets $\{U_j, V_j\}$. These spinor sets act not only as symmetry generators, but also as **eigen-spinors** of the corresponding observables $\{p, q, u_r\}$ (Eq. 12). This follows directly from the geometric algebra eigen-spinor eigenvalue relation (Appendix D):

$$\mathcal{O} \Omega_k = \lambda_k \Omega_k e_1 \mapsto \mathcal{O} = \frac{\lambda_k}{\Omega_k \tilde{\Omega}_k} \Omega_k e_1 \tilde{\Omega}_k \quad (19)$$

Here, $\mathcal{O}$ is an observable, e_1 is a basis vector (reference observable, e.g., γ_0, σ_3 , or γ_3), Ω_k are the eigen-spinors of $\mathcal{O}$, and λ_k the corresponding eigenvalues. In hyper-spacetime, however, the spinor sets $\{U_j, V_j\}$ are generated geometrically by variations in the hyper-spherical coordinates (β, θ, ϕ) (Eq. 15), rather than being obtained by solving an eigen-spinor eigenvalue equation explicitly (Appendix D).

6.1. Spatial Eigen-Spinors

The spatial spinor set S_k (Eq. 18), generated from a parity transformation, act as eigen-spinors of the spatial unit vectors $e_3 \in \mathbb{S}_{0,2}^2 \subset \mathbb{R}^{1,3}$ (**Figure 2a**) and $u_3 \in \mathbb{S}^2 \subset \mathbb{R}^4$ (Eq. 10). They satisfy the GA eigen-spinor eigenvalue relations (Appendix D):

$$\begin{aligned} e_3 \in \mathbb{S}_{0,2}^2 &\mapsto e_3 S_k = \lambda_k S_k \gamma_3, & \lambda_k \in \{\pm 1\} \\ u_3 \in \mathbb{S}^2 &\mapsto u_3 S_k = \lambda_k S_k \sigma_3, & \lambda_k \in \{\pm 1\} \end{aligned} \quad (20)$$

So, e_3 and u_3 are observables with eigen-spinors S_k and eigenvalues $\lambda_k = \pm 1$. Physically, these correspond to spin-up or spin-down measurements along the spatial directions of $u_3(\theta, \phi)$ and $e_3(\theta, \phi)$.

6.2. Spacetime Eigen-Spinors

The spacetime spinor set $\{U_j\}$ (Eq. 16) and Euclidean spinor set $\{V_j\}$ (Eq. 17) act as eigen-spinors of the hyper-spacetime observables $\{p, q, u_r\}$ (Eq. 12):

$$\begin{aligned}
p \in \mathbb{S}_H^3 &\mapsto pU_j = \eta_j U_j \gamma_0, & \eta_j \in \{+1, +1, -1, -1\} \\
q \in \mathbb{S}_C^3 &\mapsto qU_j = \kappa_j U_j \gamma_0, & \kappa_j \in \cos(\beta) \{+1, +1, -1, -1\} \\
u_r \in \mathbb{S}^3 &\mapsto u_r V_j = \eta_j V_j \gamma_0, & \eta_j \in \{+1, +1, -1, -1\}
\end{aligned} \tag{21}$$

Here, the eigenvalues η_j encodes two constant spin states combined with positive/negative temporal orientation in $\mathbb{S}_H^3$ and $\mathbb{S}^3$, while the eigenvalues κ_j encodes two variable ($\pm \cos(\beta)$) spin states combined with positive/negative temporal orientation in $\mathbb{S}_C^3$. Unlike η_j , the values of κ_j depend on relative speed via angle β (Eq. 5). Hence, a spin state in $\mathbb{S}_C^3$ vanishes at the lightlike boundaries $\beta = \pm \pi/2$.

6.3. Null Spinors

At the lightlike boundaries $\beta = \pm \pi/2$ ($v = \pm c$), the spacetime spinors U_j reduce to null spinors:

$$U_j(\beta, \theta, \phi) = S_k(\theta, \phi) L_{uj}(\beta) \mapsto U_j\left(\pm \frac{\pi}{2}, \theta, \phi\right) = \frac{1}{\sqrt{2}} S_k(\theta, \phi) (\pm 1 \pm \sigma_3), \quad U_j \tilde{U}_j = 0 \tag{22}$$

These spinors have zero intensity but preserve direction, lying on the light cone of $\mathbb{S}_C^3$ and $\mathbb{S}_H^3$. For the Euclidean spinors V_j , the corresponding boundary form is:

$$V_j(\beta, \theta, \phi) = S_k(\theta, \phi) L_{vj}(\beta) \mapsto V_j\left(\pm \frac{\pi}{2}, \theta, \phi\right) = \frac{1}{\sqrt{2}} S_k(\theta, \phi) (\pm 1 \pm \gamma_3), \quad V_j V_1^{-1} = 1 \tag{23}$$

Thus, the Euclidean spinors V_j keep their unit intensity, while the intensities of spacetime spinors U_j become zero. Each spinor set pass smoothly through the lightlike boundary.

6.4. Correspondence with the Dirac Equation

The eigen-spinors U_j (Eq. 21) and eigenvalues η_j of the observable $p \in \mathbb{S}_H^3$ reproduce those of the Dirac equation [14] (two spin states combined with positive/negative temporal orientation, i.e., matter/antimatter) (Appendix E). This agreement is remarkable, because in hyper-spacetime these eigen-spinors are generated geometrically (Eq. 16), rather than being obtained by solving an eigen-spinor eigenvalue equation (Appendix D). This establishes a direct geometric origin for the Dirac spectrum, demonstrating that the fundamental structure of spacetime eigen-spinors emerges naturally within the hyper-spacetime framework.

In summary, the eigen-spinor structure of hyper-spacetime reproduces the Dirac spectrum, isolates singularities into a scalar density, and reveals null spinors as natural lightlike states.

7. Discussion

The unified hyper-spacetime framework $\mathcal{B} = \{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$ extends the full Lorentz group $O(1,3)$ into a continuous, bounded, and closed representation that reproduces the Dirac eigen-spinor spectrum while exposing structural features obscured in conventional formulations.

The agreement between Dirac eigen-spinors and the spinors of hyper-spacetime is striking: whereas Dirac spinors encode intrinsic particle properties, in hyper-spacetime they emerge directly from geometry through variations in the hyper-spherical coordinates (β, θ, ϕ) (Eq. 15). This correspondence shows that the algebraic structure of the Dirac equation can be understood as a geometric property.

In hyper-spacetime, boosts and spatial rotations combine seamlessly with the discrete symmetries $\{I, P, T, PT\}$, which appear as smooth angular transformations in the closed domain of (β, θ, ϕ) . The replacement of the unbounded rapidity φ by the bounded angle β closes the hyperbola $\mathbb{S}_H^1$, incorporating the lightlike boundaries as regular points. This generalizes naturally to the four-dimensional (4D) hyperbolic three-sphere $\mathbb{S}_H^3$, where infinities are absorbed into a smooth boundary.

The causal three-sphere $\mathbb{S}_C^3$ covers the spacetime region bounded by the light cone of a past event, a four-dimensional causality volume containing all information accessible to observation (Figure 3a). It is spanned by future-pointing vectors q of variable norm $\cos^2(\beta)$. The Euclidean three-sphere $\mathbb{S}^3$, with its positive-definite metric, ensures completeness. Together, the three-spheres share a common temporal axis γ_0 , origin (spacetime event), and temporal bivector plane ($e_3 = u_3\gamma_0 \leftrightarrow u_3 = e_3\gamma_0$) (Eq. 10). This bivector plane expresses the geometric duality of the aligned spatial vectors, as e_3 and u_3 act as vector $\leftrightarrow$ bivector counterparts in opposite domains. They are linking spatial direction with temporal orientation and thereby unifying the Euclidean $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ and Minkowski $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ domains. Hyper-spacetime thus provides a natural setting for Wick rotations and periodic time in path integrals [23,24], without the need to introduce imaginary time.

A direct manifestation of this duality is the emergence of temporal spinors (Appendix C). Both spacetime spinors U_j and the Euclidean spinors V_j decompose into spatial and temporal spinors (Eq. 16, 17). The spatial spinors encode orientation in the spatial bivector plane $\mathbb{I}u_3(\theta, \phi) \subset \mathbb{R}^3$, which lies in the common three-dimensional even subalgebra and is domain-independent, while the temporal spinors encode orientation in the shared temporal bivector plane. In this way, the spinor decomposition mirrors the dual geometric structure of hyper-spacetime.

The physical status of hyper-spacetime must ultimately be tested experimentally. We therefore challenge experimentalists [25,26] to search for phenomena that could reveal the predicted link between particle properties and spacetime geometry. Such confirmation would show that hyper-spacetime is not only mathematically consistent but also offers a new perspective on nature. This article has focused on establishing hyper-spacetime as a consistent framework: a closed representation of the full Lorentz group enriched by new geometric elements - temporal spinors, null spinors, and a shared bivector plane - that provide a foundation for future work.

8. Conclusion

The hyper-spacetime framework provides a bounded, continuous, and closed representation of the full Lorentz group $O(1,3)$. Through a unified three-sphere geometry and hyper-spherical coordinates (β, θ, ϕ) , it removes discontinuities, incorporates both continuous and discrete transformations in a single smooth domain, and reproduces the Dirac eigen-spinor spectrum with singularities isolated in scalar factors. Beyond these established results, hyper-spacetime introduces new elements - temporal spinors, null spinors, and a shared bivector plane - that offer a fresh perspective on spacetime symmetry. Together, these features define a consistent framework and a foundation for future exploration of geometric structure in nature.

Appendix A: Spacetime Algebra

Table 1A. Spacetime algebra bridging $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$

#	Spacetime algebra [10,14,27,28]	Description
1	$\{t, x, y, z\} = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} \subset \mathbb{R}^{1,3} \quad \gamma_k = \sigma_k \gamma_0$ $\{t, x, y, z\} = \{\gamma_0, \sigma_1, \sigma_2, \sigma_3\} \subset \mathbb{R}^4 \quad \sigma_k = \gamma_k \gamma_0$ Metric diagonal for $\mathbb{R}^{1,3} \rightarrow \{+1, -1, -1, -1\}$ Metric diagonal for $\mathbb{R}^4 \rightarrow \{+1, +1, +1, +1\}$ $(\gamma_0)^2 = +1 \quad (\gamma_k)^2 = -1 \quad (\sigma_k)^2 = +1$	The four orthogonal basis vectors $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ and $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ span the same four-dimensional (4D) spacetime axes $\{t, x, y, z\}$. The common temporal basis vector $\gamma_0 \in (\mathbb{R}^{1,3} \cap \mathbb{R}^4)$ squares to plus one (+1). The spatial basis vectors $\{\gamma_k\} \subset \mathbb{R}^{1,3}$ square to minus one (-1). The spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ square to plus one (+1).
2	$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$ $\sigma_k \cdot \sigma_j + \sigma_j \cdot \sigma_k = 2\delta_{kj}$	The basis vectors $\{\gamma_\mu\} \subset \mathbb{R}^{1,3}$ satisfy the algebra of the Dirac gamma matrices, while the spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ satisfy the algebra of three-dimensional space.

3	Grade (#) $\langle M \rangle_j$	Spacetime $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$	Spacetime $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$	The algebra of four-dimensional (4D) spacetime, unifying the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ with the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$, is a graded $\langle M \rangle_j$ linear space with dimension $2^4 = 16$. Multi-vectors M are “geometric numbers” being scalars $\langle M \rangle$, vectors $\langle M \rangle_1$, bivectors $\langle M \rangle_2$, trivectors $\langle M \rangle_3$, and pseudoscalars $\langle M \rangle_4$. Both domains, $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ and $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$, share: scalar 1, temporal basis vector γ_0, three spatial bivectors $\mathbb{I}\sigma_k$, four trivectors $\mathbb{I}\gamma_\mu$, and pseudoscalar $\mathbb{I}$. The spatial basis vectors $\{\gamma_k\} \subset \mathbb{R}^{1,3}$ correspond to temporal bivectors $\sigma_k \gamma_0 \subset \mathbb{R}^4$ in the Euclidean domain, while the spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ correspond to temporal bivectors $\gamma_k \gamma_0 \subset \mathbb{R}^{1,3}$ in the Minkowski domain. This geometric duality (spatial vector $\leftrightarrow$ temporal bivector) unifies the two domains at the level of temporal bivector planes.
	$\langle M \rangle$ (1)	1		
	$\langle M \rangle_1$ (4)	$\{\gamma_0, \gamma_k\}$	$\{\gamma_0, \sigma_k\}$	
	$\langle M \rangle_2$ (6)	$\sigma_k = \gamma_k \gamma_0$	$\gamma_k = \sigma_k \gamma_0$	
		$(\mathbb{I}\sigma_k = \mathbb{I}\gamma_k \gamma_0) \in \mathbb{R}^3$		
	$\langle M \rangle_3$ (4)	$\mathbb{I}\gamma_\mu = \{\mathbb{I}\gamma_0, \mathbb{I}\sigma_k \gamma_0\}$		
	$\langle M \rangle_4$ (1)	$\mathbb{I} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$		
4	$\mathbb{I} = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ $\mathbb{I} = \sigma_1 \sigma_2 \sigma_3 = \gamma_1 \gamma_0 \wedge \gamma_2 \gamma_0 \wedge \gamma_3 \gamma_0 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ $\mathbb{I} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$ $\mathbb{I}^2 = -1$			Pseudoscalar $\mathbb{I}$ in the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$. Pseudoscalar $\mathbb{I}$ in 3D space $\mathbb{R}^3$ of the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$. These pseudoscalars are equal.
5	$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3 + \langle M \rangle_4$ $\tilde{M} = \langle M \rangle + \langle M \rangle_1 - \langle M \rangle_2 - \langle M \rangle_3 + \langle M \rangle_4$			The form of a general spacetime multi-vector M. Revision operator $\sim$ on top of a multi-vector is an invariant type of conjugation.
6	$M_+ = \frac{1}{2}(M - \mathbb{I}M\mathbb{I}) = \langle M \rangle + \langle M \rangle_2 + \langle M \rangle_4$ $M_+ M_+ = M_+$ $M_+ \tilde{M}_+ = \langle M \rangle + \langle M \rangle_4$			The even subalgebra M_+ (see A7). The even subalgebra M_+ is closed under a geometric product (GP). The GP of $M_+ \tilde{M}_+$ gives a scalar $\langle M \rangle$ plus a pseudoscalar $\langle M \rangle_4$.
7	Grade (#)	3D even subalgebra M_+		The three orthogonal basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$ span the spatial axes $\{x, y, z\}$ of 3D space. Together with scalar 1, the spatial bivectors $\mathbb{I}\sigma_k$, and the pseudoscalar $\mathbb{I}$, they provide an eight-dimensional common 3D even subalgebra M_+. The spatial basis vectors $\{\sigma_k\} \subset \mathbb{R}^4$, orthogonal to the temporal basis vector γ_0, correspond to the temporal bivectors $\{\gamma_k \gamma_0\} \subset \mathbb{R}^{1,3}$ in the Minkowski domain. Both domains $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$ therefore share the three spatial bivectors $\mathbb{I}\sigma_k \in \mathbb{R}^3$, ensuring domain-independence of spatial orientation.
	$\langle M \rangle$ (1)	1		
	$\langle M \rangle_2$ (6)	$\sigma_k = \gamma_k \gamma_0$ $\mathbb{I}\sigma_k = \{\sigma_2 \sigma_3, \sigma_3 \sigma_1, \sigma_1 \sigma_2\}$		
	$\langle M \rangle_4$ (1)	$\mathbb{I} = \sigma_1 \sigma_2 \sigma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$		

Appendix B: Gudermannian Function

The Gudermannian* function [7,8]:

$$\tanh(\varphi) = \sin(\beta) \leftrightarrow \varphi = \tanh^{-1}(\sin(\beta)) \quad (\text{B1})$$

relates the unbounded hyperbolic angle $\varphi \in (\mp\infty, \pm\infty)$ to the bounded and periodic angle $\beta \in [0, 2\pi]$ (**Figure B1**).

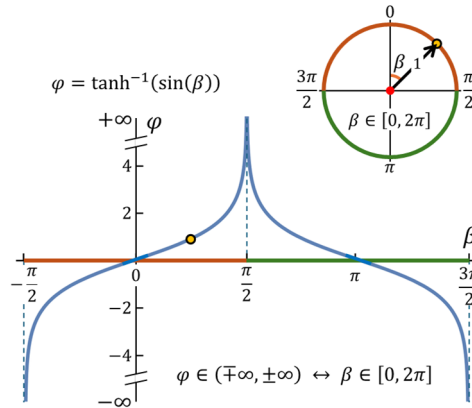


Figure B1. The Gudermannian function $\varphi = \tanh^{-1}(\sin(\beta))$ provides a smooth mapping between the unbounded angle $\varphi \in (\mp\infty, \pm\infty)$ and the bounded angle $\beta \in [0, 2\pi]$.

A hyperbolic unit vector $\pm w(\varphi)$ spans the two separate branches of a hyperbola.

$$\pm w(\varphi) = \pm \cosh(\varphi)\gamma_0 \pm \sinh(\varphi)\gamma_3, \quad \varphi \in (\mp\infty, \pm\infty), \quad \|w(\varphi)\|^2 = 1 \quad (\text{B2})$$

Substitution of the Gudermannian function $\varphi = \tanh^{-1}(\sin(\beta))$ into $\pm w(\varphi)$ eliminates the need to treat the two branches separately, yielding a single smooth parametrization:

$$w(\beta) = \sec(\beta)\gamma_0 + \tan(\beta)\gamma_3, \quad \beta \in [0, 2\pi], \quad \|w(\beta)\|^2 = 1 \quad (\text{B3})$$

Consequently, the full hyperbola is covered in one continuous domain, including the points at infinity ($\beta = \pm\pi/2$), which appear as regular boundary points.

The choice of the Gudermannian function is not arbitrary. It establishes a direct one-to-one correspondence between trigonometric and hyperbolic functions without introducing complex arguments. The Gudermannian function in its primary form maps unbounded angle φ to the bounded angle interval $\beta \in [-\pi/2, \pi/2]$, where the hyperbolic functions are positive. To extend this mapping to the full periodic domain $\beta \in [0, 2\pi]$, one must also include the negative hyperbolic functions, which correspond to a $\pm\pi$ shift of the trigonometric functions. This yields the following Gudermannian functions:

$$+\tanh(\varphi) = \sin(\beta), \quad +\text{sech}(\varphi) = \cos(\beta), \quad +\sinh(\varphi) = \tan(\beta) \quad \mapsto \beta \in [-\pi/2, \pi/2] \quad (\text{B4})$$

$$-\tanh(\varphi) = \sin(\beta \pm \pi), \quad -\text{sech}(\varphi) = \cos(\beta \pm \pi), \quad -\sinh(\varphi) = \tan(\beta \pm \pi) \quad \mapsto \beta \pm \pi \in [+ \pi/2, 3\pi/2]$$

Together, the positive and negative branches cover the full periodic domain $\beta \in [0, 2\pi]$ (**Figure B1**). So, the Gudermannian function $\varphi = \tanh^{-1}(\sin(\beta))$ unifies circular $\mathbb{S}^1$ and hyperbolic $\mathbb{S}_H^1$ symmetry, ensuring that both boosts and rotations are treated within a single angular domain.

* Christoph Gudermann (25 March 1798 – 25 September 1852) introduced this function and the concept of uniform convergence. See Wikipedia.

Appendix C: Spinors and Rotors

In geometrical algebra (GA) the rotation of a multivector M by a rotor R is performed by a double-sided geometric product (GP):

$$M' = RM\tilde{R} \quad (\text{C1})$$

This is a powerful generalization of rotation, as it applies in any dimension and to any multivector. A rotor R can be written as a scalar density ρ times a spinor S_R :

$$R = \rho S_R, \quad R\tilde{R} = 1 \mapsto \rho = (S_R \tilde{S}_R)^{-1/2} \quad (C2)$$

Here the unitarity of the rotor determines the scalar density $\rho \in \mathbb{R}$. In general, for a spinor to define a proper rotor, its intensity must be a scalar:

$$(S_R \tilde{S}_R \vee S_R S_R^{-1}) \in \mathbb{R} \quad (C3)$$

Any nonzero additional grade contribution to the intensity would prevent normalization with a scalar density ρ (Eq. C2).

A spinor is characterized by its bivector, i.e., the plane of rotation. In hyper-spacetime the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ and the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ share the three spatial bivectors $\mathbb{I}\sigma_k \in \mathbb{R}^3$ (Appendix A). Hence a general spatial spinor is:

$$S_{xyz} = \cos(\alpha/2) + \mathbb{I}\sigma_k \sin(\alpha/2), \quad S_{xyz} \tilde{S}_{xyz} = 1 \mapsto \rho = 1, \quad (\mathbb{I}\sigma_k)^2 = -1 \quad (C4)$$

The three temporal bivectors in the Minkowski domain $\{\gamma_0, \gamma_k\} \subset \mathbb{R}^{1,3}$ are $\sigma_k = \gamma_k \gamma_0$. Hence, a general Minkowski temporal spinor is:

$$S_t = \cos(\alpha/2) + \sigma_k \sin(\alpha/2), \quad S_t \tilde{S}_t = \cos(\alpha) \mapsto \rho = \sqrt{\sec(\alpha)}, \quad (\sigma_k)^2 = +1 \quad (C5)$$

The three temporal bivectors in the Euclidean domain $\{\gamma_0, \sigma_k\} \subset \mathbb{R}^4$ are $\gamma_k = \sigma_k \gamma_0$. Hence, a general Euclidean temporal spinor is:

$$S_t = \cos(\alpha/2) + \gamma_k \sin(\alpha/2), \quad S_t S_t^{-1} = 1 \mapsto \rho = 1, \quad (\gamma_k)^2 = -1 \quad (C6)$$

Both the spatial S_{xyz} and the two temporal spinors S_t are pure spinors, since their intensities reduce to scalar values.

The pure spinors $\{U_i, V_j\}$ (Eq. 16, 17) preserving hyper-spacetime and acting as eigen-spinors, are geometrically generated from the spinors U_1 and V_1 (Eq. 11). Both are composed of a spatial spinor $S_1(\theta, \phi)$ (Eq. 9) and the temporal spinors $L_{v1}(\beta)$ and $L_{u1}(\beta)$:

$$\begin{aligned} V_1(\beta, \theta, \phi) &= S_1(\theta, \phi) L_{v1}(\beta), & L_{v1}(\beta) &= \cos(\beta/2) + \sin(\beta/2) \gamma_3, & V_1 V_1^{-1} &= 1 \\ U_1(\beta, \theta, \phi) &= S_1(\theta, \phi) L_{u1}(\beta), & L_{u1}(\beta) &= \cos(\beta/2) + \sin(\beta/2) \sigma_3, & U_1 \tilde{U}_1 &= \cos(\beta) \end{aligned} \quad (C7)$$

The associated proper rotor Λ_1 , the normalized form of U_1 , is:

$$\Lambda_1(\beta, \theta, \phi) = \sqrt{\sec(\beta)} U_1(\beta, \theta, \phi), \quad \Lambda_1 \tilde{\Lambda}_1 = 1 \quad (C8)$$

The rotor associated with spinor V_1 is equal to V_1 , since its intensity satisfies $V_1 V_1^{-1} = 1$. The same holds for the spatial spinor S_1 , which also has unit intensity.

The temporal spinors L_{v1} and L_{u1} (Eq. C7), characterized by the temporal bivectors ($\gamma_3 = \sigma_3 \gamma_0$) $\subset \mathbb{R}^4$ and ($\sigma_3 = \gamma_3 \gamma_0$) $\subset \mathbb{R}^{1,3}$, reveal the dual role of these temporal bivector planes across hyper-spacetime $\mathcal{B} = \{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$. In this dual relation, the temporal bivectors ($\gamma_k = \sigma_k \gamma_0$) of the Euclidean domain appear as the spatial basis vectors of the Minkowski domain, while the temporal bivectors ($\sigma_k = \gamma_k \gamma_0$) of the Minkowski domain play the same role in the Euclidean domain. This geometric duality unifies the two domains at the level of their temporal bivector planes. Moreover, the two domains share the three spatial bivectors $\mathbb{I}\sigma_k \subset \mathbb{R}^3$ (Eq. C4), which belong to a common three-dimensional even subalgebra, ensuring that the spatial spinors remain domain-independent.

Appendix D: Eigen-Spinors and Eigenvalues

The hyper-spacetime spinor sets $\{U_i, V_j\}$ (Eq. 16, 17), which preserve the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$, act also as eigen-spinors of the corresponding observables $\{p, q, u_r\}$. Their connection is described by the geometric algebra eigen-spinor eigenvalue relation:

$$\mathcal{O} \Omega_k = \lambda_k \Omega_k e_1 \mapsto \mathcal{O} = \frac{\lambda_k}{\Omega_k \tilde{\Omega}_k} \Omega_k e_1 \tilde{\Omega}_k \quad (D1)$$

Here, $\mathcal{O}$ is a observable, e_1 is a basis vector (reference observable, e.g., γ_0, σ_3 , or γ_3), Ω_k is the eigen-spinor set of $\mathcal{O}$, and λ_k the eigenvalues of $\mathcal{O}$. However, in hyper-spacetime these spinor sets $\{U_i, V_j\}$

are geometrically generated (Eq. 16, 17), rather than being obtained by solving an eigen-spinor eigenvalue equation.

To calculate eigen-spinors and eigenvalues in practice, one uses the linear algebra (LA) eigen-spinor eigenvalue equation:

$$[\mathcal{O}][\Omega_k] = \lambda_k[\Omega_k] \quad (\text{D2})$$

where $[\mathcal{O}]$ is the matrix representation of the observable, $[\Omega_k]$ the column vector representation of the eigen-spinors, and λ_k the eigenvalues. A one-to-one conversion between GA and LA is obtained via the standard matrix representations of the basis vectors:

$$\begin{aligned} \sigma_\mu &\leftrightarrow [\sigma_\mu] = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \quad i = \sqrt{-1} \\ \gamma_\mu &\leftrightarrow [\gamma_\mu] = \left\{ \begin{pmatrix} [\sigma_0] & 0 \\ 0 & -[\sigma_0] \end{pmatrix}, \begin{pmatrix} 0 & -[\sigma_k] \\ [\sigma_k] & 0 \end{pmatrix} \right\} \end{aligned} \quad (\text{D3})$$

Using these, the observables $\{p, q, u_r\}$ become 4 by 4 matrices:

$$\begin{aligned} p \in \mathbb{S}_H^3: \quad p &= \sec(\beta) \gamma_0 + \tan(\beta) e_3(\theta, \phi) \leftrightarrow [p] = \begin{pmatrix} +\sec(\beta) [\sigma_0] & -\tan(\beta) [\mathcal{S}] \\ +\tan(\beta) [\mathcal{S}] & -\sec(\beta) [\sigma_0] \end{pmatrix} \\ q \in \mathbb{S}_C^3: \quad q &= \gamma_0 + \sin(\beta) e_3(\theta, \phi) \leftrightarrow [q] = \begin{pmatrix} +[\sigma_0] & -\sin(\beta) [\mathcal{S}] \\ +\sin(\beta) [\mathcal{S}] & -[\sigma_0] \end{pmatrix} \\ u_r \in \mathbb{S}^3: \quad u_r &= \cos(\beta) \gamma_0 + \sin(\beta) u_3(\theta, \phi) \leftrightarrow [u_r] = \begin{pmatrix} +\cos(\beta) [\sigma_0] & +\sin(\beta) [\mathcal{S}] \\ +\sin(\beta) [\mathcal{S}] & -\cos(\beta) [\sigma_0] \end{pmatrix} \end{aligned} \quad (\text{D4})$$

where $[\mathcal{S}]$ is the 2 by 2 spatial spin observable matrix associated with the spatial unit vectors e_3 and u_3 (Eq. 10):

$$[\mathcal{S}] = \begin{pmatrix} +\cos(\theta) & +\sin(\theta) e^{-i\phi} \\ +\sin(\theta) e^{+i\phi} & -\cos(\theta) \end{pmatrix}, \quad i = \sqrt{-1} \quad (\text{D5})$$

Solving the LA eigen-spinor eigenvalue equation (D2) for $[\mathcal{S}]$ gives the spatial eigen-spinors S_k with eigenvalues $\lambda_k = \pm 1$:

$$[\mathcal{S}][S_k] = \lambda_k[S_k] \rightarrow [S_k] = \left\{ \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{+i\phi/2} \end{pmatrix}, \begin{pmatrix} -\sin(\theta/2) e^{-i\phi/2} \\ +\cos(\theta/2) e^{+i\phi/2} \end{pmatrix} \right\} \quad (\text{D6})$$

Here, the set $[S_k]$ are the orthonormal spatial Pauli spinors [22]. Similarly, solving equation (D2) for the observables $\{p, q, u_r\}$ yields the column vector eigen-spinor sets:

$$\begin{aligned} [p][U_j] &= \eta_j[U_j] \rightarrow [U_j] = \left\{ \begin{pmatrix} +\cos(\beta/2)[S_1] \\ +\sin(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} +\cos(\beta/2)[S_2] \\ -\sin(\beta/2)[S_2] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_1] \\ -\cos(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_2] \\ +\cos(\beta/2)[S_2] \end{pmatrix} \right\} \\ [q][U_j] &= \kappa_j[U_j] \\ [u_r][V_j] &= \eta_j[V_j] \rightarrow [V_j] = \left\{ \begin{pmatrix} +\cos(\beta/2)[S_1] \\ +\sin(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} +\cos(\beta/2)[S_2] \\ -\sin(\beta/2)[S_2] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_1] \\ -\cos(\beta/2)[S_1] \end{pmatrix}, \begin{pmatrix} -\sin(\beta/2)[S_2] \\ +\cos(\beta/2)[S_2] \end{pmatrix} \right\} \end{aligned} \quad (\text{D7})$$

with eigenvalues $\kappa_j \in \cos(\beta) \{+1, +1, -1, -1\}$ and $\eta_j \in \{+1, +1, -1, -1\}$.

In summary, the eigen-spinors U_j and V_j (D7) obtained by solving the LA eigen-spinor eigenvalue equation are equal to the geometrically generated (Eq. 15) spacetime spinor sets $\{U_j, V_j\}$ (Eq. 16, 17).

Appendix E: Dirac Equation

This appendix provides the explicit correspondence between 1) the spacetime spinors U_j obtained as solutions of the Dirac equation and 2) the geometrically generated spacetime spinors U_j (Eq. 16).

The observables $\{p, q, u_r\}$, as elements of the three-spheres $\{\mathbb{S}_H^3, \mathbb{S}_C^3, \mathbb{S}^3\}$, are normalized vectors and can directly be linked to relativistic dynamics. The hyperbolic three-sphere $\mathbb{S}_H^3$ vector p corresponds to normalized momentum:

$$p = \frac{p_m}{m_0 c} \rightarrow p_m = p_t \gamma_0 + p_x \gamma_1 + p_y \gamma_2 + p_z \gamma_3 = m_0 c (\sec(\beta) \gamma_0 + \tan(\beta) e_3(\theta, \phi)), \quad (p_m)^2 = m_0^2 c^2 \quad (E1)$$

While the causal three-sphere $\mathbb{S}_c^3$ vector q corresponds to normalized distance:

$$q = \frac{q_t}{c \Delta t} \rightarrow q_t = t \gamma_0 + x \gamma_1 + y \gamma_2 + z \gamma_3 = c \Delta t (\gamma_0 + \sin(\beta) e_3(\theta, \phi)), \quad (q_t)^2 = c^2 \Delta t^2 \cos^2(\beta) \quad (E2)$$

The geometric product of p_m and q_t yields relativistic action S [29]:

$$S = p_m q_t = m_0 c^2 \Delta t \cos(\beta) \mapsto S = \pm m_0 c^2 \Delta t \sqrt{1 - (v/c)^2} \quad (E3)$$

Treating action as a phase $S/\hbar$, a free particle plane-wave state [14] is defined as:

$$\psi_1(c \Delta t, \beta, \theta, \phi) = U_1(\beta, \theta, \phi) e^{-i \sigma_3 p_m q_t / \hbar} \quad (E4)$$

Where, $p_m q_t / \hbar$ is the normalized action, and $U_1(\beta, \theta, \phi)$ is a constant spacetime spinor. This plane-wave state ψ_1 is a solution of the Dirac equation in geometric algebra [14]:

$$\hbar \nabla \psi_1(x) \mathbb{I} \sigma_3 - m_0 c^2 \psi_1(x) \gamma_0 = 0, \quad \nabla = \gamma^\mu \partial_\mu = \gamma^0 \frac{\partial}{c \partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z}, \quad \gamma^\mu \cdot \gamma_\nu = \delta_\nu^\mu \quad (E5)$$

which is equivalent to the linear algebra (LA) form [4]:

$$(i \hbar [\gamma^\mu] \partial_\mu - m_0 c^2 [\mathbb{I}])[\psi_1(x)] = 0 \quad (E6)$$

Substitution of the plane wave state ψ_1 in (Eq. E5) shows that spinor U_1 satisfies the GA eigen-spinor eigenvalue relation:

$$p U_1 = U_1 \gamma_0 \quad (E7)$$

Multiplying from the right with $\tilde{U}_1$ gives:

$$p U_1 \tilde{U}_1 = U_1 \gamma_0 \tilde{U}_1 \mapsto U_1 \tilde{U}_1 = \cos(\beta) \mapsto p \cos(\beta) = q \mapsto \gamma_0 + \sin(\beta) e_3(\theta, \phi) = \gamma_0 + \sin(\beta) e_3(\theta, \phi) \quad (E8)$$

Hence, plane wave state $\psi_1(c \Delta t, \beta, \theta, \phi)$ (Eq. E4) is a solution of the Dirac equation.

Finally, variations in the hyper-spherical coordinates (β, θ, ϕ) of ψ_1 , encoded by the spacetime event reflections $\{I, P, T, PT\}$ (Eq. 15), generate the four Dirac plane-wave solutions:

$$\begin{aligned} I: \quad \psi_1(c \Delta t, \beta, \theta, \phi) &= \psi_1(c \Delta t, +\beta, \theta, \phi) = U_1(\beta, \theta, \phi) e^{-i \sigma_3 p_m q_t / \hbar} \mapsto S = +m_0 c^2 \Delta t \cos(\beta) \\ P: \quad \psi_2(c \Delta t, \beta, \theta, \phi) &= \psi_1(c \Delta t, -\beta, \theta + \pi, \phi) = U_2(\beta, \theta, \phi) e^{-i \sigma_3 p_m q_t / \hbar} \mapsto S = +m_0 c^2 \Delta t \cos(\beta) \\ T: \quad \psi_3(c \Delta t, \beta, \theta, \phi) &= \psi_1(c \Delta t, -\beta + \pi, \theta, \phi) = U_3(\beta, \theta, \phi) e^{+i \sigma_3 p_m q_t / \hbar} \mapsto S = -m_0 c^2 \Delta t \cos(\beta) \\ PT: \quad \psi_4(c \Delta t, \beta, \theta, \phi) &= \psi_1(c \Delta t, +\beta + \pi, \theta + \pi, \phi) = U_4(\beta, \theta, \phi) e^{+i \sigma_3 p_m q_t / \hbar} \mapsto S = -m_0 c^2 \Delta t \cos(\beta) \end{aligned} \quad (E9)$$

The orientation of action $S = p_m q_t$ depends on the temporal angle β , such that matter and antimatter solutions follow naturally from the spacetime event reflections $\{I, P, T, PT\}$. Each of these plane-waves ψ_j is a valid solution of the Dirac equation, demonstrating the remarkable correspondence between the Dirac equation and the spacetime event reflections $\{I, P, T, PT\}$.

In addition to the four fermionic Dirac plane-wave solutions ψ_j , these solutions also yield a lightlike boundary solution at $\beta = \pm \pi/2$:

$$\psi_j(c \Delta t, \beta = \pm \pi/2, \theta, \phi) = \frac{1}{\sqrt{2}} S_k(\theta, \phi) (\pm 1 \pm \sigma_3) \quad (E10)$$

This corresponds to a photon-like state with vanishing mass term. At the lightlike boundary, the fermionic plane-wave solutions reduce to null spinors with zero intensity while retaining directionality. Thus, the hyper-spacetime framework produces both fermionic (massive) and photonic (massless) solutions as limiting cases of a unified spinor structure.

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