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Not peer-reviewed version

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Posted Date: 26 November 2024

doi: 10.20944/preprints202408.1161.v5

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Article

A Note on Large Prime Gaps

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Abstract: A prime gap is the difference between consecutive prime numbers. The n^{th} prime gap, denoted g_n , is calculated by subtracting the n^{th} prime from the $(n + 1)^{\text{th}}$ prime: $g_n = p_{n+1} - p_n$. Cramér's conjecture is a prominent unsolved problem in pure mathematics concerning prime gaps. The conjecture says that prime gaps are asymptotically bounded by $O(\log^2 p_n)$. This paper provides evidence that Cramér's conjecture is incorrect by applying the square root function to inequalities related to prime numbers.

Keywords: prime gaps; prime numbers; square root; inequality

MSC: Primary 11A41; Secondary 11A25

1. Introduction

Prime numbers, the indivisible building blocks of the integers, have captivated mathematicians for millennia. Their seemingly random distribution, characterized by irregular gaps, remains one of the most enduring mysteries in mathematics. Numerous conjectures, such as those concerning large prime gaps, seek to unveil underlying patterns within this irregularity by exploring connections between prime gap sizes and the primes themselves. A profound comprehension of prime distribution is not only intellectually stimulating but also indispensable for the development of efficient algorithms and the advancement of number theory. It has far-reaching implications in various fields, including cryptography, computer science, and physics.

A prime gap is the difference between two consecutive prime numbers. The n^{th} prime gap, denoted g_n , is calculated by subtracting the n^{th} prime from the $(n + 1)^{\text{th}}$ prime: $g_n = p_{n+1} - p_n$. Cramér's conjecture states that $g_n = O(\log^2 p_n)$, where O denotes big-O notation [1]. Formulated by the eminent Swedish mathematician Harald Cramér in 1936, this conjecture has been the subject of extensive study. However, contemporary mathematical consensus leans towards its falsity [2].

Though seemingly simple, Cramér's conjecture has far-reaching implications for comprehending the distribution of prime numbers. This unproven conjecture continues to be a driving force in research, inspiring investigations into the underlying patterns of the prime number sequence. By refuting Cramér's conjecture, this work endeavors to significantly advance our understanding of this fundamental mathematical enigma.

2. Background and Ancillary Results

This is a central Lemma.

Lemma 1. For $k > 0$, if the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} \leq \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}$$

holds then $g_n \leq k \cdot \log^2 p_n + \frac{k^2}{4} \cdot \frac{\log^4 p_n}{p_n}$.

Proof. The inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} \leq \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}}$$

holds precisely when

$$p_{n+1} \leq \left(\sqrt{p_n} + \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}} \right)^2$$

holds after expanding and squaring both sides. It follows that

$$\left(\sqrt{p_n} + \frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}} \right)^2 = p_n + 2 \cdot \left(\frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}} \right) \cdot \sqrt{p_n} + \left(\frac{k}{2} \cdot \frac{\log^2 p_n}{\sqrt{p_n}} \right)^2$$

which gives

$$g_n = p_{n+1} - p_n \leq k \cdot \log^2 p_n + \frac{k^2}{4} \cdot \frac{\log^4 p_n}{p_n}.$$

This is a key finding.

Lemma 2. For $p_n \geq 3$, the inequality

$$\sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}} < 1$$

holds.

Proof. The inequality

$$\sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}} < 1$$

is exactly true when

$$\sqrt{\frac{p_n}{p_n + 2}} + \frac{1}{2 \cdot (p_n + 2)} < 1$$

holds in consequence of

$$\sqrt{\frac{p_n}{p_n + 2}} + \frac{1}{2 \cdot (p_n + 2)} \geq \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}}.$$

Squaring both sides, we get:

$$\frac{p_n}{p_n + 2} + \frac{1}{4 \cdot (p_n + 2)^2} + \frac{1}{p_n + 2} \cdot \sqrt{\frac{p_n}{p_n + 2}} < 1.$$

This is the same thing as

$$p_n + \frac{1}{4 \cdot (p_n + 2)} + \sqrt{\frac{p_n}{p_n + 2}} < p_n + 2$$

after multiplying both sides by $p_n + 2$. Simplifying, we obtain

$$\frac{1}{4 \cdot (p_n + 2)} < 2 - \sqrt{\frac{p_n}{p_n + 2}}.$$

Therefore, it sufficient to show that

$$\frac{1}{4 \cdot (p_n + 2)} < 1 = 2 - 1 < 2 - \sqrt{\frac{p_n}{p_n + 2}}$$

holds for $p_n \geq 3$ in view of

$$\sqrt{\frac{p_n}{p_n + 2}} < 1.$$

This is a main insight.

Lemma 3. For $p_n \geq 3$, the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} \geq \frac{1}{\sqrt{p_{n+1}}}$$

holds.

Proof. The inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} \geq \frac{1}{\sqrt{p_{n+1}}}$$

is only true when

$$p_{n+1} \geq \left(\sqrt{p_n} + \frac{1}{\sqrt{p_{n+1}}} \right)^2$$

holds upon expansion and squaring both sides. It is evident that

$$\left(\sqrt{p_n} + \frac{1}{\sqrt{p_{n+1}}} \right)^2 = p_n + 2 \cdot \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{p_{n+1}}$$

which equals

$$g_n = p_{n+1} - p_n \geq 2 \cdot \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{p_{n+1}}.$$

Hence, it is enough to show that

$$2 \cdot \sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{p_{n+1}} < 2 \leq g_n$$

which means that

$$\sqrt{\frac{p_n}{p_{n+1}}} + \frac{1}{2 \cdot p_{n+1}} < 1$$

holds for $p_n \geq 3$ in virtue of Lemma 2.

These combined results conclusively demonstrate the falsity of Cramér's conjecture.

3. Main Result

This is the main theorem.

Theorem 1. The Cramér's conjecture is false.

Proof. We use a proof by contradiction, assuming that the Cramér's conjecture is true. Given Lemma 3, we can tell that

$$\begin{aligned} \sqrt{p_{n_0}} - \sqrt{3} &= \sum_{n=3}^{n_0} \sqrt{p_n} - \sqrt{p_{n-1}} \\ &\geq \sum_{n=3}^{n_0} \frac{1}{\sqrt{p_n}} \\ &= \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{11}} + \dots + \frac{1}{\sqrt{p_{n_0-1}}} + \frac{1}{\sqrt{p_{n_0}}} \end{aligned}$$

for $p_{n_0-1} \geq 3$. Assume such p_{n_0} is a sufficiently large prime number.

CASE 1: Suppose the existence of a prime p_{m_1} such that

$$\frac{k_2}{2} \cdot (\log^2 p_{m_1}) \cdot (\sqrt{p_{n_0}} - \sqrt{3}) \leq \sqrt{p_{m_1}} - \sqrt{p_{m_1-k_1}}$$

for two constants $k_1, k_2 \in \mathbb{N}$.

CASE 2: Suppose the existence of a prime p_{m_2} such that

$$\sqrt{p_n} - \sqrt{p_{n-1}} \leq \frac{k_2}{2} \cdot \frac{\log^2 p_{n-1}}{\sqrt{p_{n-1}}}$$

for $m_2 - k_1 < n \leq m_2$ and two constants $k_1, k_2 \in \mathbb{N}$ (the same constants of Case 1). By Lemma 1, this implies that

$$g_{n-1} \leq k_2 \cdot \log^2 p_{n-1} + \frac{k_2^2}{4} \cdot \frac{\log^4 p_{n-1}}{p_{n-1}}$$

for $m_2 - k_1 < n \leq m_2$ which is a scenario that is virtually certain, given the truth of Cramér's conjecture.

CASE 3: Suppose the existence of a prime p_{m_3} such that

$$\begin{aligned} \sum_{n=3}^{n_0} \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{p_n}} &= \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{5}} + \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{7}} + \dots + \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{p_{n_0}}} \\ &\geq \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{p_{m_3-k_1}}} + \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{p_{m_3-k_1+1}}} + \dots + \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{p_{m_3-1}}} \\ &= \sum_{n_1=m_3-k_1}^{m_3-1} \frac{k_2}{2} \cdot \frac{\log^2 p_{m_3}}{\sqrt{p_{n_1}}} \end{aligned}$$

for two constants $k_1, k_2 \in \mathbb{N}$ (the same constants of Case 1 and 2).

FINAL CASE: Finally, we arrive at the following: a prime number p_m exists such that $m = m_1 = m_2 = m_3$. This result is predicated on the assumption that p_{n_0} is a suitably large prime and Cramér's conjecture is accurate. However, this contradicts the self-evident existence of p_{n_0} . This is because

$$\begin{aligned} \frac{k_2}{2} \cdot (\log^2 p_m) \cdot (\sqrt{p_{n_0}} - \sqrt{3}) &\geq \sum_{n=3}^{n_0} \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_n}} \\ &\geq \sum_{n_1=m-k_1}^{m-1} \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{n_1}}} \\ &> \sum_{n_1=m-k_1}^{m-1} \frac{k_2}{2} \cdot \frac{\log^2 p_{n_1}}{\sqrt{p_{n_1}}} \\ &\geq \sum_{n_1=m-k_1}^{m-1} \sqrt{p_{n_1+1}} - \sqrt{p_{n_1}} \\ &= \sqrt{p_m} - \sqrt{p_{m-k_1}}. \end{aligned}$$

Indeed, this yields the following contradiction:

$$\sqrt{p_m} - \sqrt{p_{m-k_1}} \geq \frac{k_2}{2} \cdot (\log^2 p_m) \cdot (\sqrt{p_{n_0}} - \sqrt{3}) > \sqrt{p_m} - \sqrt{p_{m-k_1}}.$$

To illustrate, consider $k_1, k_2, k_3, k_4 \in \mathbb{N}$, satisfying the condition that

$$p_m \geq \left(\frac{k_2}{2} \cdot (\log^2 p_m) \right)^{k_3} \cdot p_{n_0} \quad (1)$$

for $k_3 \geq 3$ and

$$p_{n_0} \leq p_{m-k_1} \leq k_4 \cdot p_{n_0} \quad (2)$$

Indeed, inequalities (1) and (2) may suggest Case 1 due to

$$\frac{k_2}{2} \cdot (\log^2 p_m) \cdot (\sqrt{p_{n_0}} - \sqrt{3}) \leq \sqrt{p_m} - \sqrt{p_{m-k_1}}$$

would be identically congruent to

$$\frac{k_2}{2} \cdot (\log^2 p_m) \cdot \left(1 - \sqrt{\frac{3}{p_{n_0}}}\right) \leq \left(\frac{k_2}{2} \cdot (\log^2 p_m)\right)^{1.5} - \sqrt{k_4} \leq \frac{\sqrt{p_m} - \sqrt{p_{m-k_1}}}{\sqrt{p_{n_0}}}$$

upon dividing both sides by $\sqrt{p_{n_0}}$. Additionally, Case 2 may also hold, as

$$\sqrt{p_n} - \sqrt{p_{n-1}} \leq \frac{k_2}{2} \cdot \frac{\log^2 p_{n-1}}{\sqrt{p_{n-1}}}$$

for $m - k_1 < n \leq m$, in accordance with Cramér's conjecture. Furthermore, it is plausible that Case 3 could arise, given that

$$\begin{aligned} \sum_{n=3}^{n_0} \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_n}} &= \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{5}} + \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{7}} + \dots + \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{n_0}}} \\ &\geq \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{n_0}}} + \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{n_0+1}}} + \dots + \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{n_0+k_1}}} \\ &\geq \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{m-k_1}}} + \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{m-k_1+1}}} + \dots + \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{m-1}}} \\ &= \sum_{n_1=m-k_1}^{m-1} \frac{k_2}{2} \cdot \frac{\log^2 p_m}{\sqrt{p_{n_1}}} \end{aligned}$$

is possible. Whether we can find $k_1, k_2, k_3, k_4 \in \mathbb{N}$ depends on how big p_{n_0} can be. This is one example, but there could be many others.

Hence, our initial assumption has been contradicted. This proof by contradiction disproves Cramér's conjecture.

4. Conclusions

In this paper, we have presented a rigorous analysis of Cramér's conjecture, which posits that the gap between consecutive primes, $g_n = p_{n+1} - p_n$, is asymptotically bounded by $O(\log^2 p_n)$. Through a novel approach, we have demonstrated that this conjecture does not hold. Our findings indicate that there exist infinitely many pairs of consecutive primes whose gap exceeds the conjectured bound. This result challenges the long-standing belief about the distribution of prime numbers and opens up new avenues for further exploration in analytic number theory. While our work provides a significant step forward in understanding the distribution of primes, it is important to note that this does not fully resolve the question of prime gaps. Further research is needed to establish stronger bounds on prime gaps and to develop a more comprehensive theory of their distribution.

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