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Article

Laplacian Spectrum and Vertex Connectivity of the Unit Graph of the Ring $\mathbb{Z}_{p^r q^s}$

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Abstract: In this paper, we examine the interplay between the structural and spectral properties of the unit graph $G(\mathbb{Z}_n)$ for $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where $p_1, p_2, ..., p_k$ are distinct primes and $k, r_1, r_2, ..., r_k$ are positive integers such that at least one of the r_i must be greater than 1. We first analyze the structure of the unit graph of $\mathbb{Z}_n$ as a generalized join graph under these conditions. We then determine the Laplacian spectrum of $G(\mathbb{Z}_n)$ and deduce that it is integral for all n. Consequently, we obtain Laplacian spectral radius and algebraic connectivity of $G(\mathbb{Z}_n)$. We also prove that the vertex connectivity of $G(\mathbb{Z}_{pq})$ is $(p-2)q$, where $2 \neq p < q$. We deduce the vertex connectivity of $G(\mathbb{Z}_n)$ when $n = p^r q^s$, where $2 \neq p < q$ are primes and r, s are positive integers. Finally, we present conjectures about the vertex connectivity of $G(\mathbb{Z}_n)$ when $n = p_1 p_2 ... p_k$ and $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where p_i are distinct primes, r_i are positive integers, and $1 \leq i \leq k$.

Keywords: unit graph; laplacian spectrum; laplacian spectral radius; algebraic connectivity; vertex connectivity

MSC: 05C25; 05C50; 05C75

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1. Introduction

For a positive integer n, $\mathbb{Z}_n$ denotes the ring of integers modulo n. In 1990, the unit graph was first introduced by Grimaldi [1] for the ring $\mathbb{Z}_n$ as follows: the unit graph $G(\mathbb{Z}_n)$ is the graph obtained by setting all the elements of $\mathbb{Z}_n$ to be vertices and defining distinct vertices x and y to be adjacent if and only if $x + y \in U(\mathbb{Z}_n)$. He discussed certain basic properties of the structure of the unit graph $G(\mathbb{Z}_n)$ and studied the degree of a vertex, covering number, independence number, Hamilton cycles, and chromatic polynomial of the graph $G(\mathbb{Z}_n)$. More about the unit graph $G(\mathbb{Z}_n)$ can be seen in [2–4]. Later, Ashrafi et al. [5] generalized the unit graph from $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R. They studied the chromatic index, diameter, girth, and planarity of $G(R)$. Some of the work associated with the unit graph on the rings can be found in [6–9].

In recent years, many researchers have studied the Laplacian spectrum and vertex connectivity of graphs associated with algebraic structures. In 2020, Chattopadhyay et al. [10] studied the Laplacian spectrum of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ of the ring $\mathbb{Z}_n$. They discussed the Laplacian integrality, algebraic connectivity, vertex connectivity, and Laplacian spectral radius of $\Gamma(\mathbb{Z}_n)$. For other related works on the Laplacian spectrum and vertex connectivity of graphs associated to the ring $\mathbb{Z}_n$, one may refer to [11,12]. Shen et al. [3] determined the Laplacian spectrum of the unit graphs of the ring $\mathbb{Z}_n$ for $n = p^m$, where p is an odd prime and m is a positive integer. They proved that the algebraic connectivity and vertex connectivity of $G(\mathbb{Z}_n)$ coincide if and only if $n = p^m$.

In this paper, we investigate the structure of $G(\mathbb{Z}_n)$. Based on this structure, we study the Laplacian spectrum and vertex connectivity of $G(\mathbb{Z}_n)$ for various n. The paper is arranged as follows: In Section 2, we provide the preliminary concepts and results that are used throughout the paper. In Section 3, we examine the structure of $G(\mathbb{Z}_n)$ for $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where $p_1, p_2, ..., p_k$ are distinct primes and $k, r_1, r_2, ..., r_k$ are positive integers such that at least one of the r_i must be greater than 1. We prove that the graph $G(\mathbb{Z}_n)$ is a generalized join of certain complete graphs and null graphs. In Section 4, we

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study the Laplacian spectrum of $G(\mathbb{Z}_n)$, we prove that $G(\mathbb{Z}_n)$ is Laplacian integral, and we deduce the algebraic connectivity and Laplacian spectral radius of $G(\mathbb{Z}_n)$. In Section 5, we investigate the vertex connectivity of $G(\mathbb{Z}_{pq})$ and $G(\mathbb{Z}_{p^r q^s})$, where $2 \neq p < q$ are primes and r and s are positive integers, based on their structure and Menger's theorem. Moreover, we present the following conjectures:

Conjecture I: Let $n = p_1 p_2 ... p_k$, where $2 \neq p_1 < p_2 < ... < p_k$ are primes. Then, the vertex connectivity of $G(\mathbb{Z}_n)$ is $(p_1 - 2) \prod_{2 \leq i \leq k} p_i$.

Conjecture II: Let $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where $2 \neq p_1 < p_2 < ... < p_k$ are primes and $k, r_1, r_2, ..., r_k$ are positive integers such that at least one of the r_i must be greater than 1. Then, the vertex connectivity of $G(\mathbb{Z}_n)$ is $(p_1 - 2)p_1^{(r_1-1)} \prod_{2 \leq i \leq k} p_i^{r_i}$.

2. Preliminaries

In this section, we will present preliminary definitions and theorems that will be necessary for the following sections. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. For $1 \leq i, j \leq n$, two vertices v_i and v_j in G are adjacent (or neighbors) in G if v_i and v_j are endpoints of an edge e of G, and we write $v_i \sim v_j$ if v_i is adjacent to v_j in G. For $v \in V(G)$, we denote by $N_G(v)$ the set of all neighbors of v in G. The degree of a vertex v in G, denoted by $deg(v)$, is the number of edges incident with it. A path in a graph is a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the next vertex of it. The graph G is said to be connected if G contains a path between every pair of vertices. A complete graph is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. The complement of K_n is a null graph and is denoted by $\bar{K}_n$. A clique of a graph G is a complete subgraph of G. A coclique in a graph G is a set of pairwise nonadjacent vertices. An isomorphism of graphs G_1 and G_2, $G_1 \cong G_2$, is a bijection ϕ between the vertex sets of G_1 and G_2 such that for any two vertices x and y of G_1, x and y are adjacent in G_1 if and only if $\phi(x)$ and $\phi(y)$ are adjacent in G_2. For two graphs G_1 and G_2 with disjoint vertex sets, the join $G_1 \vee G_2$ of G_1 and G_2 is the graph obtained from the union of G_1 and G_2 by adding new edges from each vertex of G_1 to every vertex of G_2.

Let R be a ring with unity and $U(R)$ be the set of units of R. The unit graph $G(R)$ of R is the graph whose vertices are all the elements of R, defining distinct vertices x and y to be adjacent if and only if $x + y$ is a unit in R. Let R be a commutative ring with unity. An element $r \in R$ is nilpotent if there exists an integer $k > 1$ such that $r^k = 0$. The nilradical of R, denoted $nil(R)$, is the set of all nilpotent elements of R. An ideal $N \neq R$ in R is a prime ideal if $ab \in N$ implies that either $a \in N$ or $b \in N$ for $a, b \in R$. The nilradical of R is the intersection of prime ideals of R. A maximal ideal of R is an ideal m different from R such that there is no proper ideal I of R properly containing m. The Jacobson radical $J(R)$ of R is the intersection of maximal ideals of R. Every maximal ideal in R is a prime ideal. So, $J(R) = nil(R)$. Recall that, the sum of a unit and a nilpotent element is a unit. The following two results give some properties of the unit graph $G(R)$.

Lemma 1. *[5] Let R be a commutative ring and suppose that $J(R)$ denotes the Jacobson radical of R. If $x, y \in R$, then the following statements hold:*

1. *If $x + J(R)$ and $y + J(R)$ are adjacent in $G(R/J(R))$, then every element of $x + J(R)$ is adjacent to every element of $y + J(R)$.*
2. *If $2x \notin U(R)$, then $x + J(R)$ is a coclique in $G(R)$.*
3. *If $2x \in U(R)$, then $x + J(R)$ is a clique in $G(R)$.*

Proposition 1. *[5] Let R be a finite ring. Then, the following statements hold for the unit graph of R:*

1. *If $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph.*

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2. *If* $2 \in U(R)$*, then for every* $x \in U(R)$ *we have* $deg(x) = |U(R)| - 1$*, and for every* $x \in R \backslash U(R)$ *we have* $deg(x) = |U(R)|$.

For a finite simple undirected graph G, the adjacency matrix $A(G)$ is defined as the $n \times n$ matrix whose $(i,j)th$ entry is 1 if $v_i \sim v_j$ and 0 otherwise. The Laplacian matrix $L(G)$ of G is defined by $L(G) := D(G) - A(G)$, where $D(G) = Diag(d_1, d_2, ..., d_n)$ is the diagonal matrix such that d_i are degrees of vertices of G. The matrix $L(G)$ is a real, symmetric, and positive semidefinite so that its eigenvalues are real and nonnegative. Since the sum of each row of $L(G)$ is zero, the smallest eigenvalue of $L(G)$ is 0. The largest eigenvalue of $L(G)$ is known as the Laplacian spectral radius $\lambda(G)$ of G, and the second smallest eigenvalue of $L(G)$ is known as the algebraic connectivity $\mu(G)$ of G and $\mu(G) > 0$ if and only if G is connected. A graph G is called Laplacian integral if all the eigenvalues of $L(G)$ are integers. More literature about the Laplacian matrix of graphs can be seen in [13,14].

The spectrum of a square matrix C, denoted by $\sigma(C)$, is the multiset of all the eigenvalues of C. If $\eta_1, \eta_2, ..., \eta_c$ are distinct eigenvalues of B with respective multiplicities $\gamma_1, \gamma_2, ..., \gamma_c$, then we shall denote the spectrum of C by

$$\sigma(C) = \left\{ \begin{array}{cccc} \eta_1 & \eta_2 & \cdots & \eta_c \\ \gamma_1 & \gamma_2 & \cdots & \gamma_c \end{array} \right\}.$$

For a graph G, the Laplacian spectrum of G is the spectrum of $L(G)$, we write $\sigma(L(G))$ as $\sigma_L(G)$. For example,

$$\sigma_L(K_n) = \left\{ \begin{array}{cc} 0 & n \\ 1 & n-1 \end{array} \right\} \text{ and } \sigma_L(\bar{K}_n) = \left\{ \begin{array}{c} 0 \\ n \end{array} \right\}. \tag{1}$$

Let G be a graph on k vertices with $V(G) = \{v_1, v_2, ..., v_k\}$ and let $H_1, H_2, ..., H_k$ be k pairwise disjoint graphs. The G-generalized join graph $G[H_1, H_2, ..., H_k]$ of $H_1, H_2, ..., H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_i to every vertex of H_j whenever $v_i \sim v_j$ in G [15]. The following result is useful in the sequel.

Theorem 1. *[16] Let G be a graph on k vertices with* $V(G) = \{v_1, v_2, ..., v_k\}$ *and let* $H_1, H_2, ..., H_k$ *be k pairwise disjoint graphs on* $n_1, n_2, ..., n_k$ *vertices, respectively. Then, the Laplacian spectrum of* $G[H_1, H_2, ..., H_k]$ *is given by*

$$\sigma_L(G[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k} (M_j + (\sigma_L(H_j)) \backslash \{0\}) \right) \bigcup \sigma(\boldsymbol{L}(G)), \tag{2}$$

where

$$M_j = \begin{cases} \sum_{v_i \sim v_j} n_i & if\ N_G(v_j) \neq \emptyset; \\ 0 & otherwise, \end{cases}$$

$$\boldsymbol{L}(G) = \begin{bmatrix} M_1 & -s_{1,2} & \cdots & -s_{1,k} \\ -s_{1,2} & M_2 & \cdots & -s_{2,k} \\ \cdots & \cdots & \cdots & \cdots \\ -s_{1,k} & -s_{2,k} & \cdots & M_k \end{bmatrix},$$

and

$$s_{i,j} = \begin{cases} \sqrt{n_i n_j} & if\ v_i \sim v_j\ in\ G; \\ 0 & otherwise. \end{cases}$$

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In (2), $\sigma_L(H_j)\backslash\{0\}$ means that one copy of the eigenvalue 0 is removed from the multiset $\sigma_L(H_j)$, and $(M_j + (\sigma_L(H_j))\backslash\{0\})$ means that M_j is added to each element of $\sigma_L(H_j)\backslash\{0\}$.

Let n be a positive integer. Euler's totient function, denoted by $\varphi(n)$, is the number of positive integers less than or equal to n that are relatively prime to n. Let $n = p_1p_2...p_k$ and p_i be distinct primes for $i = 1, 2, ..., k$. Note that, $\varphi(p_i) = p_i - 1$ and $\varphi(p_1p_2...p_k) = \varphi(p_1)\varphi(p_2)...\varphi(p_k)$, so $\varphi(n) = \prod_{i=1}^{k} \varphi(p_i)$. Fakieh et al. studied the Laplacian spectrum of the unit graphs associated to the ring $\mathbb{Z}_{p_1p_2...p_k}$, where p_i are distinct primes and $i = 1, 2, ..., k$ [4]. This result is the main tool to prove Theorem 4.

Theorem 2. *[4] Let $p_i \neq 2$ be distinct primes and k be a positive integer, $1 \leq i, j \leq k$. Then:*

1. *If $n = p_1p_2...p_k$, then the Laplacian spectrum of $G(\mathbb{Z}_n)$ is*

$$\sigma_L(G(\mathbb{Z}_n)) = \begin{Bmatrix} 0 & [\varphi(p_i) \pm 1]\prod_{i\neq j}\varphi(p_j) & [\varphi(p_i)\varphi(p_j) \pm 1]\prod_{h\neq i,j}\varphi(p_h) & \cdots \\ 1 & \frac{\varphi(p_i)}{2} & \frac{\varphi(p_i)\varphi(p_j)}{2} & \cdots \end{Bmatrix.$$

$$\begin{matrix} [\varphi(p_1)\varphi(p_2)...\varphi(p_{k-1}) \pm 1]\varphi(p_k) & [\prod_{1\leq i\leq k}\varphi(p_i)] \pm 1 \\ \frac{\prod_{1\leq i\leq k-1}\varphi(p_i)}{2} & \frac{\prod_{1\leq i\leq k}\varphi(p_i)}{2} \end{matrix}\Bigg\}.$$

2. *If $n = 2p_1p_2...p_k$, then the Laplacian spectrum of $G(\mathbb{Z}_n)$ is*

$$\sigma_L(G(\mathbb{Z}_n)) = \begin{Bmatrix} 0 & [\varphi(p_i) \pm 1]\prod_{i\neq j}\varphi(p_j) & [\varphi(p_i)\varphi(p_j) \pm 1]\prod_{h\neq i,j}\varphi(p_h) & \cdots \\ 1 & \varphi(p_i) & \varphi(p_i)\varphi(p_j) & \cdots \end{Bmatrix.$$

$$\begin{matrix} [\varphi(p_1)\varphi(p_2)...\varphi(p_{k-1}) \pm 1]\,\varphi(p_k) & [\prod_{1\leq i\leq k}\varphi(p_i)] \pm 1 & 2\prod_{1\leq i\leq k}\varphi(p_i) \\ \prod_{1\leq i\leq k-1}\varphi(p_i) & \prod_{1\leq i\leq k}\varphi(p_i) & 1 \end{matrix}\Bigg\}.$$

The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G leaves a disconnected or trivial graph. A family of two or more paths in a graph G is said to be internally disjoint if no vertex of G is an internal vertex of more than one path in the family. There is a strong result for the vertex connectivity $\kappa(G)$ by Menger's theorem. Menger's theorem says that the maximum number of internally disjoint uv-paths in G is equal to the minimum number of vertices whose deletion destroys all uv-paths, where u and v are nonadjacent in G [17]. So that

$$\kappa(G) = \min\{\rho(u,v) : \rho(u,v) \text{ is the maximum number of internally disjoint } uv\text{-paths in } G, \text{ where } u \nsim v\}.$$

This paper uses Menger's theorem to examine the vertex connectivity of $G(\mathbb{Z}_n)$.

3. $G(\mathbb{Z}_n)$ as a Generalized Join Graph

For a positive integer n, $\mathbb{Z}_n$ denotes the ring of integers modulo n. The elements of the ring $\mathbb{Z}_n$ are referred to as $0, 1, 2,$ and $n - 1$. A nonzero element $x \in \mathbb{Z}_n$ is a unit in $\mathbb{Z}_n$ if x is relatively prime with n, $(x, n) = 1$. Through this section, we use p_i as a prime number. Also, an integer n can be written in the

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form $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where $p_1, p_2, ..., p_k$ are distinct primes and $k, r_1, r_2, ..., r_k$ are positive integers such that at least one of the r_i must be greater than 1. In this section, we prove that $G(\mathbb{Z}_n)$ is a generalized join graph of some complete graphs and null graphs. To this end, first, we study the structure of $G(\mathbb{Z}_n)$. Denote the maximal ideal of $\mathbb{Z}_n$ by $\mathfrak{m}_i$, $\mathfrak{m}_i$ is an ideal generated by the prime divisors p_i of n, that is, $\mathfrak{m}_i = \langle p_i \rangle$. So, $J(\mathbb{Z}_n) = \cap\, \mathfrak{m}_i = \left\langle \prod_{i=1}^{k} p_i \right\rangle \neq \{0\}$. Put $\mathbf{p} = \prod_{i=1}^{k} p_i$, then

$$\mathbb{Z}_n / J(\mathbb{Z}_n) = \{i + J(\mathbb{Z}_n) : i \in \{0, 1, ..., \mathbf{p} - 1\}\} \text{ and } \mathbb{Z}_n / J(\mathbb{Z}_n) \cong \mathbb{Z}_{\mathbf{p}}.$$

Let S be the set of distinct representatives of $\mathbb{Z}_n / J(\mathbb{Z}_n)$. For $i \in S$, we denote

$$B_i = i + J(\mathbb{Z}_n).$$

Note that the sets $B_0, B_1, ..., B_{\mathbf{p}-1}$ form a partition of the vertex set of $G(\mathbb{Z}_n)$. Thus,

$$V(G(\mathbb{Z}_n)) = \bigcup_{i=0}^{\mathbf{p}-1} B_i. \tag{3}$$

Let $B_0 = J(\mathbb{Z}_n)$. Then, for $j \in S$

$$|B_j| = |B_0| = |J(\mathbb{Z}_n)| = p_1^{r_1 - 1} p_2^{r_2 - 1} ... p_k^{r_k - 1}.$$

The following result describes the adjacency criterion of vertices in $G(\mathbb{Z}_n)$, where $V(G(\mathbb{Z}_n))$ is described in Equation (3).

Lemma 2. *For $i, j \in S$, every vertex of B_i is adjacent to every vertex of B_j in $G(\mathbb{Z}_n)$ if and only if $(i + j, n) = 1$.*

Proof. ($\Rightarrow$) Clearly.
($\Leftarrow$) Suppose that $(i + j, n) = 1$. Let $a \in B_i$ and $b \in B_j$, which can be written as $a = i + a_i$ and $b = j + b_j$, where $a_i, b_j \in J(\mathbb{Z}_n)$. Now, $a + b = (i + j) + (a_i + b_j)$. Here $i + j$ is a unit by assumption, and $a_i + b_j$ is nilpotent. So, $a + b$ is a unit, and hence every vertex of B_i is adjacent to every vertex of B_j in $G(\mathbb{Z}_n)$. □

By using Lemmas 1 [(2),(3)] and 2, the following is evident.

Corollary 1. *The following statements hold:*

1. *For $i \in S$, the induced subgraph $G(B_i)$ of $G(\mathbb{Z}_n)$ on the vertex set B_i is either the complete graph $K_{|J(\mathbb{Z}_n)|}$ or its complement graph $\bar{K}_{|J(\mathbb{Z}_n)|}$. Indeed, $G(B_i)$ is $K_{|J(\mathbb{Z}_n)|}$ if and only if $2i \in U(\mathbb{Z}_n)$.*
2. *For $i, j \in S$ with $i \neq j$, a vertex of $G(B_i)$ is adjacent to either all or none of the vertices of $G(B_j)$ in $G(\mathbb{Z}_n)$.*

The above corollary implies that the partition $B_0 \cup B_1 \cup ... \cup B_{\mathbf{p}-1}$ of the vertex set $V(G(\mathbb{Z}_n))$ of $G(\mathbb{Z}_n)$ is an equitable partition in such a way that every vertex of the B_i has equal number of neighbors in B_j for all $i, j \in S$.

We define G_n by the simple graph whose vertices are the distinct representatives of $\mathbb{Z}_n / J(\mathbb{Z}_n)$, that is, the set of vertices is S, and in which two distinct vertices i and j are adjacent if and only if $(i + j, n) = 1$. The graph G_n will play an important role in the rest of the paper.

Lemma 3. $G(\mathbb{Z}_{\mathbf{p}}) \cong G_n$.

Proof. We define a map $\phi : V(G(\mathbb{Z}_{\mathbf{p}})) \to V(G_n)$ such that $\phi(B_i) = i$. Clearly, ϕ is well-defined and bijection. From Lemma 2, the adjacency relationships are preserved by ϕ. Hence, the result follows. □

The following lemma states that $G(\mathbb{Z}_n)$ can be expressed as a generalized join of certain complete graphs and null graphs.

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Lemma 4. *Let $G(B_i)$ be the induced subgraph of $G(\mathbb{Z}_n)$ on the vertex set B_i for $0 \le i \le \mathbf{p} - 1$. Then,*

$$G(\mathbb{Z}_n) = G_n[G(B_0), G(B_1), \ldots, G(B_{\mathbf{p}-1})].$$

Proof. Replace the vertex i of G_n by $G(B_i)$ for $0 \le i \le \mathbf{p} - 1$. Thus, the result follows from Lemma 2 and Corollary 1. □

Example 1. *The unit graph $G(\mathbb{Z}_{25})$ is shown in Figure 1. By Lemma 4, we have*

$$G(\mathbb{Z}_{25}) = G_{25}[G(B_0), G(B_1), G(B_2), G(B_3), G(B_4)],$$

where G_{25} is shown in Figure 2, $G(B_0) = \bar{K}_5$, and $G(B_i) = K_5$ for $1 \le i \le 4$. In Figure 1, the lines between two squares mean that each vertex in one square is adjacent to every vertex in the other square.

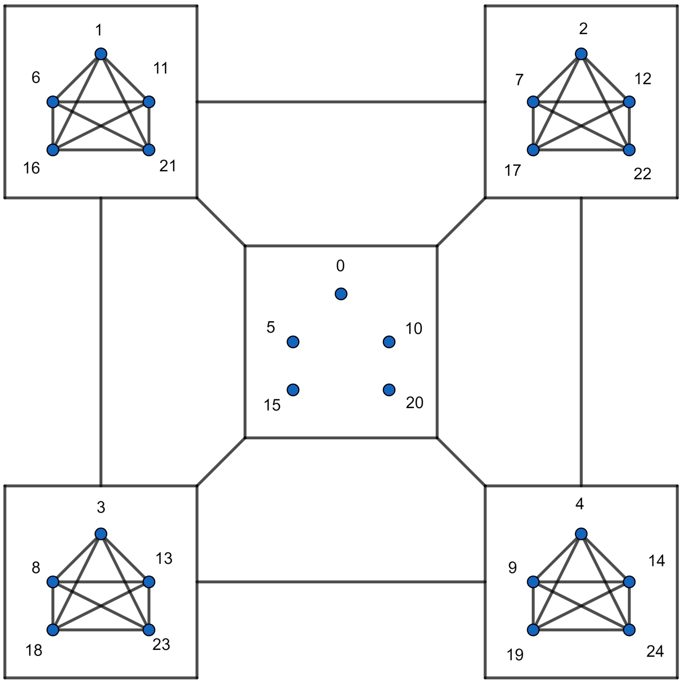

Figure 1. The graph $G(\mathbb{Z}_{25})$.

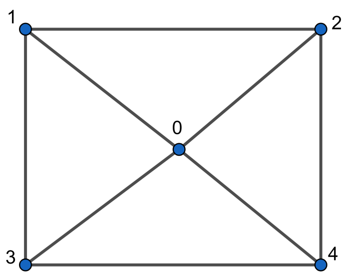

Figure 2. The graph G_{25}.

4. Laplacian Spectrum of $G(\mathbb{Z}_n)$

In this section, we investigate the Laplacian spectrum of $G(\mathbb{Z}_n)$ for $n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$, where $2 \neq p_1 < p_2 < \ldots < p_k$ are primes. For $0 \le i \le \mathbf{p} - 1$, we give the weight $|J(\mathbb{Z}_n)| = t$ to the vertex i of the graph G_n. Define the integer

$$M_j = tb_j, \text{ where } b_j = deg(j) \text{ in } G_n$$

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for $0 \leq j \leq \mathbf{p}-1$. The $\mathbf{p} \times \mathbf{p}$ vertex weighted Laplacian matrix $\mathbf{L}(G_n)$ of G_n defined in Theorem 1 is given by

$$\mathbf{L}(G_n) = \begin{bmatrix} M_0 & -h_{0,1} & \dots & -h_{0,\mathbf{p}-1} \\ -h_{0,1} & M_1 & \dots & -h_{1,\mathbf{p}-1} \\ \dots & \dots & \dots & \dots \\ -h_{0,\mathbf{p}-1} & -h_{1,\mathbf{p}-1} & \dots & M_{\mathbf{p}-1} \end{bmatrix}, \tag{4}$$

where

$$h_{i,j} = \begin{cases} t & if\ i \sim j\ in\ G_n; \\ 0 & otherwise \end{cases}$$

for $0 \leq i \neq j \leq \mathbf{p}-1$.

The following remark is an immediate result of Proposition 1.

Remark 1. *The following statements hold:*

1. *If* $2 \notin U(\mathbb{Z}_\mathbf{p})$, *then* $M_j = t\varphi(\mathbf{p})$.
2. *If* $2 \in U(\mathbb{Z}_\mathbf{p})$, *then*

$$M_j = \begin{cases} t(\varphi(\mathbf{p})-1) & if\ j \in U(\mathbb{Z}_\mathbf{p}); \\ \\ t\varphi(\mathbf{p}) & if\ j \notin U(\mathbb{Z}_\mathbf{p}). \end{cases}$$

Lemma 5. $\boldsymbol{L}(G_n) = t(L(G_n))$

Proof. The proof is direct from definition of $\mathbf{L}(G_n)$ in Equation (4). □

The following theorem describes the Laplacian spectrum of $G(\mathbb{Z}_n)$.

Theorem 3. *Let* $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, *where* $p_1 < p_2 < \dots < p_k$ *are primes,* r_i *and* k *are positive integers, and* $1 \leq i \leq k$. *The Laplacian spectrum of* $G(\mathbb{Z}_n)$ *is given by*

$$\sigma_L(G(\mathbb{Z}_n)) = \left(\bigcup_{j=0}^{\mathbf{p}-1} (M_j + (\sigma_L(G(B_j)) \backslash \{0\})) \right) \bigcup t\sigma_L(G_n),$$

where $j \in S$ *and* $M_j + (\sigma_L(G(B_j) \backslash \{0\})$ *means that* M_j *is added to each element of the multiset* $\sigma_L(G(B_j) \backslash \{0\}$.

Proof. By Lemma 4, we have

$$G(\mathbb{Z}_n) = G_n[G(B_0), G(B_1), \dots, G(B_{\mathbf{p}-1})].$$

Consequently, the result can be obtained by using Theorem 1 and Lemma 5. □

By Corollary 1, $G(B_i)$ is either K_t or $\bar{K}_t$ for $1 \leq i \leq k$. By Theorem 3, out of the n number of Laplacian eigenvalues of $G(\mathbb{Z}_n)$, $n-\mathbf{p}$ of them are known to be nonzero integer values. The remaining $\mathbf{p}$ Laplacian eigenvalues of $G(\mathbb{Z}_n)$ will come from the Laplacian eigenvalues of $G(\mathbb{Z}_\mathbf{p})$.

Corollary 2. *Let* $2 \neq p_1 < p_2 < \dots < p_k$ *be distinct primes and* r, r_i, s_i, k *be positive integers, where* $1 \leq i \leq k$. *Then, the Laplacian spectrum of* $G(\mathbb{Z}_n)$ *is given by*

1. *If* $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, *then*

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$$\sigma_L(G(\mathbb{Z}_n)) = \underbrace{(t(\varphi(\mathbf{p})-1)+(\sigma_L(K_t)\backslash\{0\}))}_{\varphi(\mathbf{p})\text{-times}} \bigcup \underbrace{(t(\varphi(\mathbf{p})+(\sigma_L(\bar{K}_t)\backslash\{0\}))}_{[\mathbf{p}-\varphi(\mathbf{p})]\text{-times}} \bigcup t\sigma_L(G_n).$$

2. *If* $n = 2^r p_1^{s_1} p_2^{s_2} ... p_k^{s_k}$, *then*

$$\sigma_L(G(\mathbb{Z}_n)) = \underbrace{(t(\varphi(\mathbf{p})+(\sigma_L(\bar{K}_t)\backslash\{0\}))}_{2\mathbf{p}\text{-times}} \bigcup t\sigma_L(G_n).$$

Proof. By the above argument and Remark 1, the result holds. □

The following result gives the Laplacian spectrum of $G(\mathbb{Z}_n)$ for $n = p^r q^s$, where $p < q$ are primes and r, s are positive integers.

Theorem 4. *Let* $2 \neq p < q$ *be primes and* r, s *be positive integers. Then:*

1. *If* $n = p^r q^s$, *then the Laplacian spectrum of* $G(\mathbb{Z}_n)$ *consists of*

$$\begin{Bmatrix} 0 & p^{r-1}q^{s-1}[\varphi(p)-1]\varphi(q) & p^{r-1}q^{s-1}[\varphi(q)-1]\varphi(p) & p^{r-1}q^{s-1}[\varphi(p)\varphi(q)-1] \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(q)}{2} & \frac{\varphi(p)\varphi(q)}{2} \end{Bmatrix.$$
$$\begin{matrix} p^{r-1}q^{s-1}[\varphi(p)\varphi(q)+1] & p^{r-1}q^{s}\varphi(p) & p^{r}q^{s-1}\varphi(q) & p^{r-1}q^{s-1}\varphi(p)\varphi(q) \\ \frac{\varphi(p)\varphi(q)}{2} & \frac{\varphi(q)}{2} & \frac{\varphi(p)}{2} & p^r q^s - pq \end{matrix}\Bigg\}. \quad (5)$$

2. *If* $n = 2^r q^s$, *then the Laplacian spectrum of* $G(\mathbb{Z}_n)$ *is*

$$\sigma_L(G(\mathbb{Z}_n)) = \begin{Bmatrix} 0 & 2^{r-1}p^{s-1}[\varphi(p)-1] & 2^{r-1}p^s & 2^r p^{s-1}\varphi(p) & 2^{r-1}p^{s-1}\varphi(p) \\ 1 & \varphi(p) & \varphi(p) & 1 & 2^r p^s - 2p \end{Bmatrix}.$$

Proof. 1. Let $n = p^r q^s$, where $2 < p < q$ are primes and r, s are positive integers. So, the Jacobson radical of $\mathbb{Z}_n$ is $\langle pq \rangle$ and the set of distinct representatives of $\mathbb{Z}_n / \langle pq \rangle$ is $\{0, 1, ..., pq-1\}$. Thus,

$$G(\mathbb{Z}_{p^r q^s}) = G_{p^r q^s}[\underbrace{K_{p^{r-1}q^{s-1}}, \dots, K_{p^{r-1}q^{s-1}}}_{\varphi(pq)\text{-times}}, \underbrace{\bar{K}_{p^{r-1}q^{s-1}}, \dots, \bar{K}_{p^{r-1}q^{s-1}}}_{[pq-\varphi(pq)]\text{-times}}].$$

By Corollary 2, the Laplacian spectrum of $G(\mathbb{Z}_n)$ is given by

$$\sigma_L(G(\mathbb{Z}_n)) = \underbrace{(t(\varphi(pq)-1)+(\sigma_L(K_{p^{r-1}q^{s-1}})\backslash\{0\}))}_{\varphi(pq)\text{-times}} \bigcup \underbrace{(t(\varphi(pq)+(\sigma_L(\bar{K}_t)\backslash\{0\}))}_{[pq-\varphi(pq)]\text{-times}}$$
$$\bigcup p^{r-1}q^{s-1}\sigma_L(G_{p^r q^s}).$$

By Eq (1), the Laplacian spectrum of $K_{p^{r-1}q^{s-1}}$ and $\bar{K}_{p^{r-1}q^{s-1}}$ are

$$\sigma_L(K_{p^{r-1}q^{s-1}}) = \begin{Bmatrix} 0 & p^{r-1}q^{s-1} \\ 1 & p^{r-1}q^{s-1}-1 \end{Bmatrix} \text{ and } \sigma_L(\bar{K}_{p^{r-1}q^{s-1}}) = \begin{Bmatrix} 0 \\ p^{r-1}q^{s-1} \end{Bmatrix}.$$

Then,

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$$\sigma_L(G(\mathbb{Z}_n)) = \left\{ \begin{array}{c} p^{r-1}q^{s-1}[\varphi(pq)-1] + p^{r-1}q^{s-1} \\ \\ \varphi(pq)(p^{r-1}q^{s-1}-1) \end{array} \right\} \bigcup \left\{ \begin{array}{c} p^{r-1}q^{s-1}\varphi(pq) \\ \\ pq - \varphi(pq)(p^{r-1}q^{s-1}-1) \end{array} \right\}$$

$$\bigcup \left\{ p^{r-1}q^{s-1}\,\sigma_L(G_{p^r q^s}) \right\}.$$

By using Lemma 3, $G_{p^r q^s}$ is isomorphic to $G(\mathbb{Z}_{pq})$, and hence $\sigma_L(G_{p^r q^s}) = \sigma_L(G(\mathbb{Z}_{pq}))$. So, by Theorem 2, we have

$$\sigma_L(G_{p^r q^s}) = \begin{Bmatrix} 0 & [\varphi(p)-1]\varphi(q) & [\varphi(q)-1]\varphi(p) & \varphi(p)\varphi(q)-1 & \varphi(p)\varphi(q)+1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(q)}{2} & \frac{\varphi(p)\varphi(q)}{2} & \frac{\varphi(p)\varphi(q)}{2} \end{Bmatrix.$$

$$\left. \begin{matrix} \varphi(p)[\varphi(q)+1] & \varphi(q)[\varphi(p)+1] \\ \frac{\varphi(q)}{2} & \frac{\varphi(p)}{2} \end{matrix} \right\}.$$

So,

$$\sigma_L(G(\mathbb{Z}_{p^r q^s})) = \left\{ \begin{array}{c} p^{r-1}q^{s-1}\varphi(pq) \\ \\ p^r q^s - pq \end{array} \right\} \bigcup \left\{ \begin{matrix} 0 & p^{r-1}q^{s-1}[\varphi(p)-1]\varphi(q) & p^{r-1}q^{s-1}[\varphi(q)-1]\varphi(p) \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(q)}{2} \end{matrix} \right.$$

$$\left. \begin{matrix} p^{r-1}q^{s-1}[\varphi(p)\varphi(q)-1] & p^{r-1}q^{s-1}[\varphi(p)\varphi(q)+1] & p^{r-1}q^{s}\varphi(p) & p^{r}q^{s-1}\varphi(q) \\ \frac{\varphi(p)\varphi(q)}{2} & \frac{\varphi(p)\varphi(q)}{2} & \frac{\varphi(q)}{2} & \frac{\varphi(p)}{2} \end{matrix} \right\}.$$

Hence, the Laplacian spectrum of $G(\mathbb{Z}_{p^r q^s})$ is as in Eq (5).

2. Let $n = 2^r p^s$, where $p \neq 2$ is a prime and r, s are positive integers. Note that, $S = \{0, 1, ..., 2p - 1\}$ is the vertex set of the graph $G_{2^r p^s}$. Thus,

$$G(\mathbb{Z}_{2^r p^s}) = G_{2^r p^s}[\underbrace{\bar{K}_{2^{r-1}p^{s-1}}, \bar{K}_{2^{r-1}p^{s-1}}, \ldots, \bar{K}_{2^{r-1}p^{s-1}}}_{2p\text{-times}}].$$

By Corollary 2, the Laplacian spectrum of $G(\mathbb{Z}_n)$ is given by

$$\sigma_L(G(\mathbb{Z}_{2^r p^s})) = \underbrace{(2^{r-1}p^{s-1}(\varphi(p) + (\sigma_L(\bar{K}_{2^{r-1}p^{s-1}})\backslash\{0\}))}_{2p\text{-times}} \bigcup 2^{r-1}p^{s-1}\sigma_L(G_{2^r p^s})$$

$$= \left\{ \begin{array}{c} 2^{r-1}p^{s-1}\varphi(p) \\ \\ 2p(2^{r-1}p^{s-1}-1) \end{array} \right\} \bigcup \left\{ 2^{r-1}p^{s-1}\sigma_L(G_{2^r p^s}) \right\}.$$

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By using Lemma 3, $G_{2^r p^s}$ is isomorphic to $G(\mathbb{Z}_{2p})$, and hence $\sigma_L(G_{2^r p^s}) = \sigma_L(G(\mathbb{Z}_{2p}))$. So, by Theorem 2, we have

$$\sigma_L(G_{2^r p^s}) = \left\{ \begin{array}{cccc} 0 & \varphi(p)-1 & \varphi(p)+1 & 2\varphi(p) \\ 1 & \varphi(p) & \varphi(p) & 1 \end{array} \right\}.$$

Hence, the Laplacian spectrum of $G(\mathbb{Z}_n)$ is

$$\sigma_L(G(\mathbb{Z}_n)) = \left\{ \begin{array}{cccccc} 0 & 2^{r-1}p^{s-1}[\varphi(p)-1] & 2^{r-1}p^s & 2^r p^{s-1}\varphi(p) & 2^{r-1}p^{s-1}\varphi(p) \\ 1 & \varphi(p) & \varphi(p) & 1 & 2^r p^s - 2p \end{array} \right\}. \quad \square$$

Now, we find the Laplacian spectrum of $G(\mathbb{Z}_n)$ for $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, where $p_1 < p_2 < \dots < p_k$ are primes, r_i, k are positive integers, and $1 \le i \le k$. The following theorem can be obtained by arguments similar to those used in the proof of Theorem 4, and therefore the proof is omitted.

Theorem 5. *Let $2 \neq p_1 < p_2 < \dots < p_k$ be primes and r, r_i, s_i, k be positive integers, $1 \le i \le k$. Then:*

1. *If $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, then the Laplacian spectrum of $G(\mathbb{Z}_n)$ consists of*

$$\left\{ \begin{array}{cccc} 0 & [\varphi(p_i) \pm 1] \prod\limits_{1 \le i \le k} p_i^{(r_i-1)} \prod\limits_{i \neq j} \varphi(p_j) & [\varphi(p_i)\varphi(p_j) \pm 1] \prod\limits_{1 \le i \le k} p_i^{(r_i-1)} \prod\limits_{h \neq i,j} \varphi(p_h) & \dots \\ 1 & \dfrac{\varphi(p_i)}{2} & \dfrac{\varphi(p_i)\varphi(p_j)}{2} & \dots \end{array} \right.$$

$$\begin{array}{cc} [\varphi(p_1)\varphi(p_2)\dots\varphi(p_{k-1}) \pm 1]\varphi(p_k) \prod\limits_{1 \le i \le k} p_i^{(r_i-1)} & \left[\left[\prod\limits_{1 \le i \le k} \varphi(p_i)\right] \pm 1\right] \prod\limits_{1 \le i \le k} p_i^{(r_i-1)} \\ \dfrac{\prod\limits_{1 \le i \le k-1} \varphi(p_i)}{2} & \dfrac{\prod\limits_{1 \le i \le k} \varphi(p_i)}{2} \end{array}$$

$$\left. \begin{array}{c} \prod\limits_{1 \le i \le k} p_i^{(r_i-1)} \varphi(p_i) \\ n - p_1 p_2 \dots p_k \end{array} \right\}.$$

2. *If $n = 2^r p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$, then the Laplacian spectrum of $G(\mathbb{Z}_n)$ consists of*

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$$\begin{cases} 0 & 2^{r-1}[\varphi(p_i)\pm 1]\prod_{1\le i\le k} p_i^{(s_i-1)}\prod_{i\ne j}\varphi(p_j) & 2^{r-1}[\varphi(p_i)\varphi(p_j)\pm 1]\prod_{1\le i\le k} p_i^{(s_i-1)}\prod_{h\ne i,j}\varphi(p_h) & \ldots \\ 1 & \varphi(p_i) & \varphi(p_i)\varphi(p_j) & \ldots \end{cases}$$

$$\begin{matrix} 2^{r-1}[\varphi(p_1)\varphi(p_2)\ldots\varphi(p_{k-1})\pm 1]\varphi(p_k)\prod_{1\le i\le k} p_i^{(s_i-1)} & 2^{r-1}\left[\left[\prod_{1\le i\le k}\varphi(p_i)\right]\pm 1\right]\prod_{1\le i\le k} p_i^{(s_i-1)} \\ \prod_{1\le i\le k-1}\varphi(p_i) & \prod_{1\le i\le k}\varphi(p_i) \end{matrix}$$

$$\left.\begin{matrix} 2^r\prod_{1\le i\le k} p_i^{(s_i-1)}\varphi(p_i) & 2^{(r-1)}\prod_{1\le i\le k} p_i^{(s_i-1)}\varphi(p_i) \\ 1 & n-2p_1p_2\ldots p_k \end{matrix}\right\}.$$

As an immediate consequence of Theorem 5, we have the following results.

Corollary 3. *$G(\mathbb{Z}_n)$ is Laplacian integral for all n.*

Corollary 4. *The Laplacian spectral radius of $G(\mathbb{Z}_n)$ is*

$$\lambda(G(\mathbb{Z}_n)) = \begin{cases} [\varphi(p_1)+1]\prod_{1\le i\le k} p_i^{(r_i-1)}\prod_{j\ne 1}\varphi(p_j) & \text{if } n = p_1^{r_1}p_2^{r_2}\ldots p_k^{r_k}; \\ 2^r\prod_{1\le i\le k} p_i^{(s_i-1)}\varphi(p_i) & \text{if } n = 2^r p_1^{s_1}p_2^{s_2}\ldots p_k^{s_k}, \end{cases}$$

where $2 \ne p_1 < p_2 < \ldots < p_k$ are primes, r, r_i, s_i, k are positive integers, and $1 \le i \le k$.

Corollary 5. *The algebraic connectivity of $G(\mathbb{Z}_n)$ is*

$$\mu(G(\mathbb{Z}_n)) = \begin{cases} [\varphi(p_1)-1]\prod_{1\le i\le k} p_i^{(r_i-1)}\prod_{j\ne 1}\varphi(p_j) & \text{if } n = p_1^{r_1}p_2^{r_2}\ldots p_k^{r_k}; \\ 2^{r-1}[\varphi(p_1)-1]\prod_{1\le i\le k} p_i^{(s_i-1)}\prod_{j\ne 1}\varphi(p_j) & \text{if } n = 2^r p_1^{s_1}p_2^{s_2}\ldots p_k^{s_k}, \end{cases}$$

where $2 \ne p_1 < p_2 < \ldots < p_k$ are primes, r, r_i, s_i, k are positive integers, and $1 \le i \le k$.

5. Vertex Connectivity of $G(\mathbb{Z}_n)$

In this section, we obtain the vertex connectivity of $G(\mathbb{Z}_n)$ when $n = pq$ and $n = p^r q^s$, where $2 \ne p < q$ are primes and r and s are positive integers. To achieve this goal, we calculate the number of internally disjoint paths between any two nonadjacent vertices in $G(\mathbb{Z}_{pq})$, which allow us to be ready to explore the vertex connectivity of $G(\mathbb{Z}_{pq})$ by using Menger's theorem. We end this section by presenting conjectures about the vertex connectivity of $G(\mathbb{Z}_n)$ when $n = p_1p_2\ldots p_k$ and $n = p_1^{r_1}p_2^{r_2}\ldots p_k^{r_k}$, where p_i are distinct primes, r_i are positive integers, and $1 \le i \le k$.

5.1. Structure of $G(\mathbb{Z}_{pq})$

The following result will be used in the sequel.

Lemma 6. *[18] Let $n = pq$, where p and q are distinct odd primes. Then the following statements hold:*

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1. *Let $i \in \{1, 2, ..., p-1\}$. The induced subgraph $G(i + \langle p \rangle)$ of $G(\mathbb{Z}_n)$ is isomorphic to $K_1 \vee CP(q-1)$*[1]. *If $i = 0$, then $G(i + \langle p \rangle)$ is $\bar{K}_q$.*
2. *Let $i, j \in \{0, 1, ..., p-1\}$ and $i \neq j$. If $(i + j, p) = 1$, then every vertex of $G(i + \langle p \rangle)$ is adjacent to $(q-1)$ vertices of $G(j + \langle p \rangle)$.*
3. *Let $i, j \in \{1, 2, ..., p-1\}$ and $i \neq j$. If $(i + j, p) \neq 1$, then every vertex of $G(i + \langle p \rangle)$ is nonadjacent to any vertex of $G(j + \langle p \rangle)$.*

Now, we study the structure of $G(\mathbb{Z}_{pq})$, where $2 \neq p < q$ are primes, analogously to Section 3. In this case, we choose the maximal ideal $\langle p \rangle$. Let S' be the set of distinct representatives of $\mathbb{Z}_{pq} / \langle p \rangle$. For $i \in S'$, we denote

$$B'_i = i + \langle p \rangle .$$

Note that the sets $B'_0, B'_1, ..., B'_{p-1}$ form a partition of the vertex set of $G(\mathbb{Z}_{pq})$. Thus,

$$V(G(\mathbb{Z}_n)) = \bigcup_{i=0}^{p-1} B'_i.$$

This implies that any two vertices x and y that belong to the above union are adjacent if and only if $(x + y, pq) = 1$. Let $B'_0 = \langle p \rangle$. Then, $|B'_i| = |B'_0| = |\langle p \rangle| = q$ for $i \in S'$.

Note that, Lemma 6 implies that the partition $\cup_{i=0}^{p-1} G(B'_i)$ of $V(G(\mathbb{Z}_{pq}))$ is an almost equitable partition in such a way that every vertex of $G(B'_i)$ has an equal number of neighbors in $G(B'_j)$ where $i \neq j$ and $i, j \in S'$. Also, $G(B'_0)$ is isomorphic to $\bar{K}_q$ and $G(B'_i)$ is isomorphic to $K_1 \vee CP(q-1)$, where $i \in S' - \{0\}$.

Let G'_{pq} be defined as the simple graph whose vertices are $G(B'_i)$, where $i \in S'$, so that $|V(G'_{pq})| = p$. Two distinct vertices $G(B'_i)$ and $G(B'_j)$ in G'_{pq} are adjacent if and only if $(i + j, p) = 1$, which is equivalent to each vertex in $G(B'_i)$ being adjacent to $(q-1)$ vertices in $G(B'_j)$.

Example 2. *The vertex set of $G(\mathbb{Z}_{15})$, denoted as $V(G'_{15})$, can be expressed as the union $V(G'_{15}) = G(B'_0) \cup G(B'_1) \cup G(B'_2)$, where $G(B'_0) = G(0 + \langle 3 \rangle)$, $G(B'_1) = G(1 + \langle 3 \rangle)$, and $G(B'_2) = G(2 + \langle 3 \rangle)$. From Figure 3 bellow, we observe the following:*

1. *$G(B'_1)$ is nonadjacent to $G(B'_2)$ since $(1 + 2, 3) \neq 1$.*
2. *Each vertex in $G(B'_1)$ and $G(B'_2)$ is adjacent to 4 vertices in $G(B'_0)$.*
3. *$G(B'_0)$ is isomorphic to $\bar{K}_5$ and $G(B'_1)$ and $G(B'_2)$ are isomorphic to $K_1 \vee CP(4)$. The red vertices 0, 5, and 10 represent K_1 in $G(B'_0)$, $G(B'_2)$, and $G(B'_1)$, respectively. Note that these vertices are multiples of 5.*

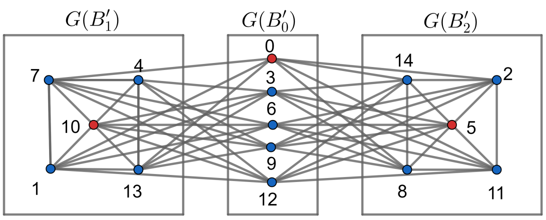

Figure 3. The graph $G(\mathbb{Z}_{15})$.

The following two results determine the neighbors and the number of common neighbors of the vertices in G'_{pq}, which help us to calculate the number of internally disjoint paths between any two nonadjacent vertices in $\cup_{i=0}^{p-1} G(B'_i)$.

[1] $CP(q-1)$ is the cocktail party graph, which is obtained from the complete graph K_{2s}, $2s = q-1$, by deleting a perfect matching, where a perfect matching of graph G is a 1-regular spanning subgraph H of G.

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Lemma 7. *Let $i \in S'$. If $i \neq 0$, then there are $(p-2)$ neighbors of $G(B'_i)$ in G'_{pq}. On the other hand, $G(B'_0)$ has $(p-1)$ neighbors in G'_{pq}.*

Proof. Let $i \neq 0$. For $j \neq i$, $(i+j,p) = 1$ for all $j \in S'$ except $j = p-i$. Consequently, $G(B'_i)$ is adjacent to $G(B'_j)$ in G'_{pq} when $j \neq i, p-i$. Then, there are $(p-2)$ neighbors of $G(B_i)$ in G'_{pq}. If $i = 0$, then $(0+j,p) = 1$ for all $j \neq i$. Thus, $G(B'_0)$ is adjacent to all vertices in G'_{pq}. Therefore, there are $(p-1)$ neighbors of $G(B'_0)$ in G'_{pq}. □

Lemma 8. *If $i,j \in S'$, then the following statements hold:*

1. *If $G(B'_i)$ and $G(B'_j)$ are nonadjacent in G'_{pq}, then the number of common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq} is $(p-2)$.*
2. *If $G(B'_i)$ and $G(B'_j)$ are adjacent in G'_{pq}, where $i,j \neq 0$, then the number of common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq} is $(p-4)$.*
3. *The number of common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq} is $(p-3)$.*

Proof. Let $i,j \in S'$.

1. Let $G(B'_i)$ and $G(B'_j)$ be nonadjacent in G'_{pq}. By Lemma 7, $j = p-i$ and $i = p-j$. Hence, $G(B'_i)$ is adjacent to all vertices in G'_{pq} except $G(B'_{p-i})$. Similarly, $G(B'_j)$ is adjacent to all vertices in G'_{pq} except $G(B'_{p-j})$. So, $G(B'_i)$ and $G(B'_j)$ are adjacent to all vertices in G'_{pq} except $G(B'_j)$ and $G(B'_i)$. Therefore, there are $(p-2)$ common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}.
2. Let $G(B'_i)$ and $G(B'_j)$ be adjacent in G'_{pq}, where $i,j \neq 0$. According to Lemma 7, $G(B'_i)$ and $G(B'_j)$ are nonadjacent to $G(B'_{p-i})$ and $G(B'_{p-j})$ in G'_{pq}, respectively. So, $G(B'_i)$ and $G(B'_j)$ are adjacent to all vertices in G'_{pq} except $G(B'_{p-i})$ and $G(B'_{p-j})$, respectively. Then, the set of common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq} is

$$V(G'_{pq}) - \{G(B'_i), G(B'_j), G(B'_{p-i}), G(B'_{p-j})\}.$$

   Thus, there are $(p-4)$ common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}.
3. By Lemma 7, $G(B'_0)$ is adjacent to all vertices in G'_{pq}. Also, $G(B'_i)$ is adjacent to all vertices in G'_{pq} except $G(B'_{p-i})$. So, the set of common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq} is

$$V(G'_{pq}) - \{G(B'_0), G(B'_i), G(B'_{p-i})\}.$$

   Then, there are $(p-3)$ common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq}. □

The following result determines the number of common neighbors between nonadjacent vertices $x, y \in G(B'_i)$ through $G(B'_z)$, where $G(B'_z)$ is a neighbor of $G(B'_i)$ in G'_{pq}.

Lemma 9. *Let $x, y \in G(B'_i)$ be nonadjacent and $G(B'_z)$ be a neighbor of $G(B'_i)$ in G'_{pq}. Then, x and y have $(q-2)$ common neighbors in $G(B'_z)$.*

Proof. By Part 2 of Lemma 6, both x and y are adjacent to $(q-1)$ vertices in $G(B'_z)$. Suppose that x and y have the same neighbors in $G(B'_z)$. Then, x and y are adjacent to all vertices in $G(B'_z)$ except z'. This implies that z' is adjacent to $(q-2)$ vertices in $G(B'_i)$, a contradiction with Part 2 of Lemma 6. Then, x and y are adjacent to all vertices in $G(B'_z)$ except a_z and b_z, respectively. So, the number of common neighbors between x and y in $G(B'_z)$ is $(q-2)$. □

The following result determines the number of common neighbors between nonadjacent vertices $x \in G(B'_i)$ and $y \in G(B'_j)$ through $G(B'_z)$, where $G(B'_z)$ is a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}.

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Lemma 10. *Let $x \in G(B'_i)$ and $y \in G(B'_j)$ be nonadjacent and $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Then:*

1. *If x and y have the same neighbors in $G(B'_z)$, then the number of common neighbors in $G(B'_z)$ between x and y is $(q-1)$.*
2. *If x and y do not have the same neighbors in $G(B'_z)$, then the number of common neighbors in $G(B'_z)$ between x and y is $(q-2)$.*

Proof. Let $x \in G(B'_i)$ and $y \in G(B'_j)$ be nonadjacent and $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}.

1. The proof is direct from Part 2 of Lemma 6.
2. Let x and y do not have the same neighbors in $G(B'_z)$. By Part 2 of Lemma 6, both x and y have $(q-1)$ neighbors in $G(B'_z)$. That is, x and y are adjacent to all vertices in $G(B'_z)$ except a_z and b_z, respectively. So, the number of the common neighbors in $G(B'_z)$ between x and y is $(q-2)$. □

From now to the rest of this section, we denote x, which is a multiple of q in $G(B'_i)$, by x^* (see Example 2). The following proposition characterizes the nonadjacent vertices of $G(\mathbb{Z}_{pq})$ for which the relation in Part 1 of Lemma 10 holds when $G(B'_i)$ is adjacent to $G(B'j)$ in $G'pq$.

Proposition 2. *Let $x \in G(B'_i)$ and $y \in G(B'_j)$ be nonadjacent, $G(B'_i)$ be adjacent to $G(B'_j)$ in G'_{pq}, and $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Then, x and y have the same neighbors in $G(B'_z)$ if and only if $x = x^*$ and $y = y^*$.*

Proof. ($\Rightarrow$) Suppose that x and y have the same neighbors in $G(B'_z)$. Then, x and y are adjacent to all vertices in $G(B'_z)$ except z' by Part 2 of Lemma 6. We assume that $x \neq x^*$ and $y \neq y^*$. Suppose that $x = i + a'p$, $y = j + b'p$, and $z' = z + c'p$, where $i, j, z \in S'$ and $a', b', c' \in \{0, 1, ..., q-1\}$. Since $G(B'_i)$ is adjacent to $G(B'_j)$ in G'_{pq} and x is nonadjacent to y, then $(i+j, p) = 1$ and $((i+j) + (a'+b')p, pq) \neq 1$, this implies that $((i+j) + (a'+b')p, q) \neq 1$. Similarly, since $G(B'_z)$ is adjacent to $G(B'_i)$ and $G(B'_j)$ in G'_{pq}, then $((i+z') + (a'+c')p, q) \neq 1$ and $((j+z') + (b'+c')p, q) \neq 1$. So,

$$x + y = (i+j) + (a'+b')p = a_1 q,$$
$$x + z' = (i+z) + (a'+c')p = a_2 q,$$
$$y + z' = (j+z) + (b'+c')p = a_3 q.$$

So, $2z' = (a_2 + a_3 - a_1)q$; this implies that 2 divide $(a_2 + a_3 - a_1)$, and hence z' is multiple of q, in this case $z' = z^*$. But z^* is adjacent to all vertices in $G(B'_i)$ and $G(B'_j)$ except $x = x^*$ and $y = y^*$, respectively. This is a contradiction with $x \neq x^*$ and $y \neq y^*$. So, $x = x^*$ and $y = y^*$.
($\Leftarrow$) Assume that $x = x^*$ and $y = y^*$. Then, x is adjacent to $(q-1)$ vertices of $G(B'_z)$ by Part 2 of Lemma 6, that is x is adjacent to all vertices of $G(B'_z)$ except z^*, where $z^* = cq$ and $c \in \{0, 1, ..., q-1\}$. Similarly, y is adjacent to all vertices of $G(B'_z)$ except z^*. Then, the result is obtained. □

5.2. Number of Internally Disjoint Paths between Nonadjacent Vertices in $G(\mathbb{Z}_{pq})$

The following two lemmas calculate the number of internally disjoint paths between nonadjacent vertices x and y in $G(B'_i)$.

Lemma 11. *Let $x, y \in G(B'_i)$ be nonadjacent. Then, there are $(p-1)(q-2)$ internally disjoint paths of length 2 between x and y.*

Proof. If $i \neq 0$, then $G(B'_i)$ is isomorphic to $K_1 \vee CP(q-1)$ by Part 1 of Lemma 6. Since x^* represents K_1, then x^* adjacent to all vertices in $G(B'_i)$. Since x is nonadjacent to y, then both x and y are not x^* in $G(B'_i)$. So, there are $(q-2)$ common neighbors between x and y, and hence there are $(q-2)$ internally disjoint paths of length 2 between x and y in $G(B'_i)$. Let $G(B'_z)$ be a neighbor of $G(B'_i)$ in G'_{pq}.

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By Lemma 9, there are $(q-2)$ common neighbors between x and y in $G(B'_z)$. Hence, there are $(q-2)$ internally disjoint paths of length 2 through $G(B'_z)$. By Lemma 7, there are $(p-2)$ neighbors of $G(B'_i)$ in G'_{pq}, and hence there are $(p-2)(q-2)$ internally disjoint paths of length 2 between x and y through all neighbors of $G(B'_i)$ in G'_{pq}. Thus, the total number of internally disjoint paths of length 2 between x and y is

$$(q-2)+(p-2)(q-2)=(p-1)(q-2).$$

If $i=0$, then $G(B'_0)$ is isomorphic to $\bar{K}_q$, by Part 1 of Lemma 6. So, there is no path between x and y in $G(B'_0)$. By Lemma 9, there are $(q-2)$ common neighbors between x and y in $G(B'_z)$, where $G(B'_z)$ is a neighbor of $G(B'_0)$ in G'_{pq}. Hence, there are $(q-2)$ internally disjoint paths of length 2 through $G(B'_z)$. By Lemma 7, there are $(p-1)$ neighbors of $G(B'_0)$ in G'_{pq}, and hence there are $(p-1)(q-2)$ internally disjoint paths of length 2 between x and y. □

Lemma 12. *Let $x, y \in G(B'_i)$ be nonadjacent. Then:*

1. *If $i \neq 0$, then there are $(p-2)$ internally disjoint paths of length 4 between x and y.*
2. *If $i = 0$, then there are $(p-1)$ internally disjoint paths of length 4 between x and y.*

Proof. Let $x, y \in G(B'_i)$ be nonadjacent and $G(B'_z)$ be a neighbor of $G(B'_i)$ in G'_{pq}. By proof of Lemma 9, x and y are adjacent to all vertices of $G(B'_z)$ except a_z and b_z, respectively.

1. Let $i \neq 0$. By Part 2 of Lemma 6, a_z and b_z in $G(B'_z)$ are adjacent to $(q-1)$ vertices of $G(B'_{p-i})$. Since we investigate the internally disjoint paths between x and y through a_z and b_z, we can choose a vertex w from $G(B'_{p-i})$ that is adjacent to both a_z and b_z. This path will be of length 4, as illustrated in Figure 4 (a). Similarly, for each neighbor $G(B'_z)$ of $G(B'_i)$ in G'_{pq} there is one internally disjoint path of length 4 between x and y. By Lemma 7, there are $(p-2)$ neighbors of $G(B'_i)$ in G'_{pq}. Therefore, there are $(p-2)$ internally disjoint paths of length 4 between x and y through all neighbors of $G(B'_i)$.
2. Let $i=0$. By Part 2 of Lemma 6, a_z and b_z in $G(B'_z)$ are adjacent to $(q-1)$ vertices of $G(B'_0)$. Approaching the proof in a similar manner as with Part 1, there is a path of length 4, as shown in Figure 4 (b). By Lemma 7, the number of neighbors of $G(B'_0)$ in G'_{pq} is $(p-1)$. So, there are $(p-1)$ internally disjoint paths of length 4 between x and y through all neighbors of $G(B'_0)$. □

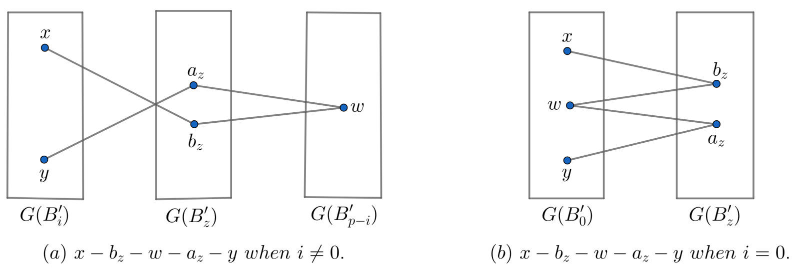

Figure 4. (a) $x - b_z - w - a_z - y$ when $i \neq 0$. **(b)** $x - b_z - w - a_z - y$ when $i = 0$.

The following two lemmas determine the number of internally disjoint paths between nonadjacent vertices $x \in G(B'_i)$ and $y \in G(B'_j)$, where $G(B'_i)$ is nonadjacent to $G(B'_j)$ in G'_{pq}.

Lemma 13. *Let $x \in G(B'_i)$ and $y \in G(B'_j)$ be nonadjacent, $G(B'_i)$ be nonadjacent to $G(B'_j)$ in G'_{pq}, and $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Then:*

1. *If x and y have the same neighbors in $G(B'_z)$, then there are $(p-2)(q-1)$ internally disjoint paths of length 2 between x and y.*
2. *If x and y do not have the same neighbors in $G(B'_z)$, then there are $(p-2)(q-2)$ internally disjoint paths of length 2 between x and y.*

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Proof. Let $G(B'_i)$ be nonadjacent to $G(B'_j)$ in G'_{pq}. By Part 1 of Lemma 8, there are $(p-2)$ common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Then:

1. If x and y have the same neighbors in $G(B'_z)$, then there are $(q-1)$ common neighbors between x and y in $G(B'_z)$ by Lemma 10. Thus, there are $(q-1)$ internally disjoint paths of length 2 between x and y through $G(B'_z)$. Hence, there are $(p-2)(q-1)$ internally disjoint paths of length 2 between x and y through all common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}.
2. If x and y do not have the same neighbors in $G(B'_z)$, then there are $(q-2)$ common neighbors between x and y in $G(B'_z)$ by Lemma 10. So, there are $(q-2)$ internally disjoint paths of length 2 between x and y through $G(B'_z)$. Therefore, there are $(p-2)(q-2)$ internally disjoint paths of length 2 between x and y through all common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. □

Lemma 14. *Let $x \in G(B'_i)$ and $y \in G(B'_j)$ be nonadjacent, $G(B'_i)$ be nonadjacent to $G(B'_j)$ in G'_{pq}, and $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Then:*

1. *If x and y have the same neighbors in $G(B'_z)$, then there are $(p-2)$ internally disjoint paths of length 4 between x and y.*
2. *If x and y do not have the same neighbors in $G(B'_z)$, then there are $2(p-2)$ internally disjoint paths of length 3 between x and y.*

Proof. Let $G(B'_i)$ be nonadjacent to $G(B'_j)$ in G'_{pq}. By Part 1 of Lemma 8, there are $(p-2)$ common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}.

1. Let x and y have the same neighbors in $G(B'_z)$. By Lemma 6, $x \in G(B'_i)$ has at least $(q-2)$ neighbors in $G(B'_i)$ and each of them is adjacent to $(q-1)$ vertices of $G(B'_z)$. Then, we can choose a neighbor, say x', of x in $G(B'_i)$ such that $x' \sim z'$, where $z' \in G(B'_z)$ is nonadjacent to both x and y. Similarly, we can choose a neighbor y' of y in $G(B'_j)$ such that $y' \sim z'$. Therefore, there exists a path of length 4 of the form $x - x' - z' - y' - y$, see Figure 5 (a). So, there are $(p-2)$ internally disjoint paths of length 4 between x and y through all common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}.
2. Let x and y do not have the same neighbors in $G(B'_z)$. By proof of Lemma 10, x and y are adjacent to all vertices in $G(B'_z)$ except a_z and b_z, respectively. Approaching the proof similarly as Part 1, we can choose a neighbor x' of x in $G(B'_i)$ and a neighbor y' of y in $G(B'_j)$, where $x' \sim a_z$ and $y' \sim b_z$. Since $a_z \sim y$ and $b_z \sim x$, two internally disjoint paths of length 3 exist. These paths are described in Figure 5 (b). Hence, there are $2(p-2)$ internally disjoint paths of length 3 between x and y through all common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. □

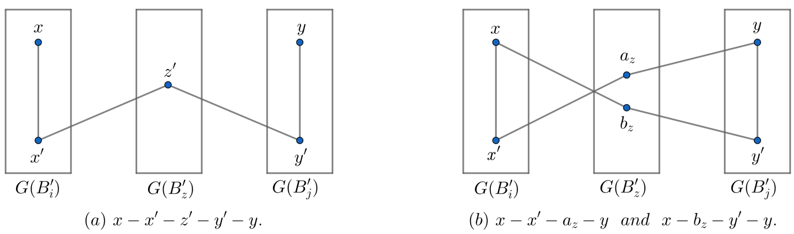

Figure 5. **(a)** $x - x' - z' - y' - y$. **(b)** $x - x' - a_z - y$ and $x - b_z - y' - y$.

The following results find the number of internally disjoint paths between nonadjacent vertices $x \in G(B'_i)$ and $y \in G(B'_j)$, where $G(B'_i)$ is adjacent to $G(B'_j)$ in G'_{pq}.

Lemma 15. *Let $x \in G(B'_i)$ and $y \in G(B'_j)$ be nonadjacent and $G(B'_i)$ be adjacent to $G(B'_j)$ in G'_{pq}. One of the following cases holds:*

1. *There are $(p-2)(q-1)$ internally disjoint paths of length 2 between x and y.*

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2. *There are $(p-2)(q-2)$ internally disjoint paths of length 2 between x and y.*

Proof. By Lemma 8, there are $(p-4)$ common neighbors between the adjacent vertices $G(B'_i)$ and $G(B'_j)$, where $i,j \neq 0$, in G'_{pq} and there are $(p-3)$ common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq}. Let $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Since x and y are nonadjacent, then x and y have the following probabilities:

1. Assume that x and y have the same neighbors in $G(B'_z)$. By Proposition 2 and Lemma 10, $x = x^*$ and $y = y^*$ and there are $(q-1)$ common neighbors between x and y through $G(B'_z)$. So, there are $(q-1)$ internally disjoint paths of length 2 between x and y through $G(B'_z)$. If $i,j \neq 0$, then there are $(p-4)(q-1)$ internally disjoint paths of length 2 between x and y through all common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Further, x (resp. y) is adjacent to all vertices in $G(B'_i)$ (resp. $G(B'_j)$). Also, x (resp. y) is adjacent to all vertices in $G(B'_j)$ (resp. $G(B'_i)$) except y (resp. x). So, the set of common neighbors between x and y in $G(B'_i)$ and $G(B'_j)$ is $G(B'_i) - \{x\}$ union $G(B'_j) - \{y\}$. Thus, there are $2(q-1)$ common neighbors between x and y in $G(B'_i)$ and $G(B'_j)$. Consequently, there are $2(q-1)$ internally disjoint paths of length 2 between x and y in $G(B'_i)$ and $G(B'_j)$. Therefore, the total number of internally disjoint paths of length 2 between x and y is

$$(p-4)(q-1) + 2(q-1) = (p-2)(q-1).$$

If $j = 0$, then there are $(p-3)(q-1)$ internally disjoint paths of length 2 between x and y through all common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq}. Further, x is adjacent to all vertices in $G(B'_i)$ and y is nonadjacent to any vertex in $G(B'_0)$. Also, x (resp. y) is adjacent to all vertices in $G(B'_0)$ (resp. $G(B'_i)$) except y (resp. x). So, the set of common neighbors between x and y in $G(B'_0)$ and $G(B'_i)$ are $G(B'_i) - \{x\}$. Thus, there are $(q-1)$ common neighbors between x and y in $G(B'_0)$ and $G(B'_i)$. Then, there are $(q-1)$ internally disjoint paths of length 2 between x and y in $G(B'_0)$ and $G(B'_i)$. Thus, the total number of internally disjoint paths of length 2 between x and y is

$$(p-3)(q-1) + (q-1) = (p-2)(q-1).$$

2. Assume that x and y do not have the same neighbors in $G(B'_z)$. By Proposition 2 and Lemma 10, $x \neq x^*$ and $y \neq y^*$ and there are $(q-2)$ common neighbors between x and y in $G(B'_z)$. Thus, there are $(q-2)$ internally disjoint paths of length 2 between x and y through $G(B'_z)$. If $i,j \neq 0$, then there are $(p-4)(q-2)$ internally disjoint paths of length 2 between x and y through all common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Furthermore, x (resp. y) is adjacent to all vertices except only one vertex x' (resp. y') in $G(B'_i)$ (resp. $G(B'_j)$). Also, x (resp. y) is adjacent to all vertices in $G(B'_j)$ (resp. $G(B'_i)$) except y (resp. x). Thus, the set of common neighbors between x and y in $G(B'_i)$ and $G(B'_j)$ is $G(B'_i) - \{x, x'\}$ union $G(B'_j) - \{y, y'\}$. So, there are $2(q-2)$ common neighbors between x and y in $G(B'_i)$ and $G(B'_j)$. As a result, there are $2(q-2)$ internally disjoint paths of length 2 between x and y in $G(B'_i)$ and $G(B'_j)$. Therefore, the total number of internally disjoint paths of length 2 between x and y is

$$(p-4)(q-2) + 2(q-2) = (p-2)(q-2).$$

If $j = 0$, then there are $(p-3)(q-2)$ internally disjoint paths of length 2 between x and y through all common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq}. Further, x is adjacent to all vertices except only one vertex x' in $G(B'_i)$, and y is nonadjacent to any vertex in $G(B'_0)$. Also, x (resp. y) is adjacent to all vertices in $G(B'_0)$ (resp. $G(B'_i)$) except y (resp. x). Thus, the set of common neighbors between x and y in $G(B'_0)$ and $G(B'_i)$ is $G(B'_i) - \{x, x'\}$. So, there are $(q-2)$ common neighbors between x and y in $G(B'_0)$ and $G(B'_i)$. Then, there are $(q-2)$ internally disjoint paths of

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length 2 between x and y in $G(B'_0)$ and $G(B'_i)$. Thus, the total number of internally disjoint paths of length 2 between x and y is

$$(p-3)(q-2)+(q-2)=(p-2)(q-2). \quad \square$$

Lemma 16. *Let $x \in G(B'_i)$ and $y \in G(B'_j)$ be nonadjacent, where $G(B'_i)$ is adjacent to $G(B'_j)$ in G'_{pq}. One of the following cases holds:*

1. *There are $(q-1)$ internally disjoint paths of length 3 between x and y.*
2. *There are $(p+q-5)$ internally disjoint paths of length 3 between x and y.*
3. *There are $(p+q-4)$ internally disjoint paths of length 3 between x and y.*

Proof. We need to examine whether any xy-path passes through $G(B'_{p-j})$ and $G(B'_{p-i})$ because we are sure that $G(B'_{p-j})$ (resp. $G(B'_{p-i})$) is adjacent to $G(B'_i)$ (resp. $G(B'_j)$) and nonadjacent to $G(B'_j)$ (resp. $G(B'_i)$) in G'_{pq}. Let $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Since x and y are nonadjacent, then x and y have the following cases:

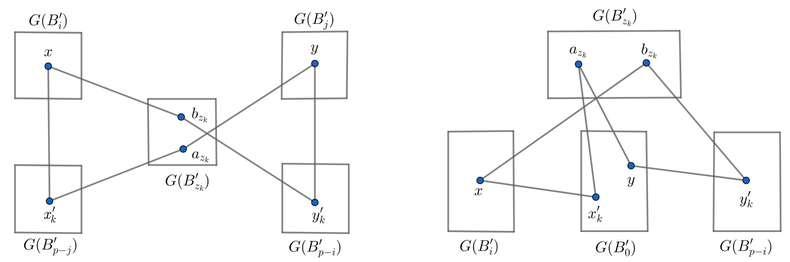
1. Assume that x and y have the same neighbors in $G(B'_z)$. Indeed, $x = x^*$ and $y = y^*$ by Proposition 2. If $i, j \neq 0$, there are $(q-1)$ neighbors of x in $G(B'_{p-j})$, denote these neighbors by x'_k such that $k = 1, 2, ..., q-1$, and each of them is adjacent to $(q-1)$ vertices of $G(B'_{p-i})$ by Part 2 of Lemma 6. Similarly, there are $(q-1)$ neighbors of y in $G(B'_{p-i})$, and each of them is adjacent to $(q-1)$ vertices of $G(B'_{p-j})$. To get the internally disjoint paths of length 3 between x and y, we choose one of the neighbors of x'_k, say y'_k, in $G(B'_{p-i})$ such that y'_k is a neighbor of y. Indeed, for each x'_k in $G(B'_{p-j})$ there is one internally disjoint path between x and y through $G(B'_{p-j})$ and $G(B'_{p-i})$. Therefore, the total number of internally disjoint paths of length 3 between x and y through $G(B'_{p-j})$ and $G(B'_{p-i})$ together is equal to the number of neighbors of x in $G(B'_{p-j})$, which is $(q-1)$. Now let $j = 0$. There are $(q-1)$ neighbors of x in $G(B'_0)$ and each of them is adjacent to $(q-1)$ vertices of $G(B'_{p-i})$. Since there are $(q-1)$ neighbors of y in $G(B'_{p-i})$, so there are more than $(q-1)$ paths of length 3 between x and y through $G(B'_0)$ and $G(B'_{p-i})$ together. By applying the same method in the case where $i, j \neq 0$, there are $(q-1)$ internally disjoint paths of length 3 between x and y.
2. Assume that x and y do not have the same neighbors in $G(B'_z)$. So, $x \neq x^*$ and $y \neq y^*$ by Proposition 2. Suppose that $G(B'_{z_k})$ is a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. By proof of Lemma 10, x and y are adjacent to all vertices in $G(B'_{z_k})$ except a_{z_k} and b_{z_k}, respectively. Let $i, j \neq 0$. Since x has $(q-1)$ neighbors in $G(B'_{p-j})$ and each of these neighbors is adjacent to $(q-1)$ vertices of $G(B'_{z_k})$, then we can choose a neighbor x'_k of x in $G(B'_{p-j})$ such that $x'_k \sim a_{z_K}$. Similarly, we can choose a neighbor y'_k of y in $G(B'_{p-i})$ such that $y'_k \sim b_{z_K}$. Since $a_{z_k} \sim y$ and $b_{z_k} \sim x$, there exist two internally disjoint paths of length 3 between x and y, as illustrated in Figure 6 (a). By Part 2 of Lemma 8, there are $(p-4)$ common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. Then, there are $2(p-4)$ internally disjoint paths of length 3 between x and y through all common neighbors between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. After removing all x'_k and y'_k from $G(B'_{p-j})$ and $G(B'_{p-i})$, respectively, then the number of remaining neighbors of x and y in $G(B'_{p-j})$ and $G(B'_{p-i})$, respectively, is $(q-1)-(p-4) = q-p+3$. So, there are $(q-p+3)$ internally disjoint paths length 3 between x and y that pass through the remaining of neighbors of x and y in $G(B'_{p-j})$ and $G(B'_{p-i})$, respectively, together. So, the total number of internally disjoint paths of length 3 between x and y is

$$2(p-4)+(q-p+3)=p+q-5.$$

Let $j = 0$. Since x has $(q-1)$ neighbors in $G(B'_0)$ and each of these neighbors is adjacent to $(q-1)$ vertices of $G(B'_{z_k})$, then we can choose a neighbor x'_k of x in $G(B'_0)$ such that $x'_k \sim a_{z_k}$. Similarly, we can choose a neighbor y'_k of y in $G(B'_{p-i})$ such that $y'_k \sim b_{z_K}$. Since $a_{z_k} \sim y$ and

$b_{z_k} \sim x$, there exist two internally disjoint path of length 3 between x and y, as illustrated in Figure 6 (b). By Part 3 of Lemma 8, there are $(p-3)$ common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq}. Consequently, there are $2(p-3)$ internally disjoint paths of length 3 between x and y through all common neighbors between $G(B'_0)$ and $G(B'_i)$ in G'_{pq}. After removing all x'_k and y'_k from $G(B'_0)$ and $G(B'_{p-i})$, respectively, then the number of remaining neighbors of x and y in $G(B'_0)$ and $G(B'_{p-i})$, respectively, is $(q-1)-(p-3) = q-p+2$. So, there are $(q-p+2)$ internally disjoint paths length 3 between x and y that pass through the rest of neighbors of x and y in $G(B'_0)$ and $G(B'_{p-i})$, respectively, together. So, the total number of internally disjoint paths of length 3 between x and y is

$$2(p-3) + (q-p+2) = p+q-4. \quad \square$$



(a) $x - x'_k - a_{z_k} - y$ and $x - b_{z_k} - y'_k - y$ when $i, j \neq 0$. (b) $x - x'_k - a_{z_k} - y$ and $x - b_{z_k} - y'_k - y$ when $j = 0$.

Figure 6. (a) $x - x'_k - a_{z_k} - y$ and $x - b_{z_k} - y'_k - y$ when $i, j \neq 0$. **(b)** $x - x'_k - a_{z_k} - y$ and $x - b_{z_k} - y'_k - y$ when $j = 0$.

5.3. Vertex Connectivity of $G(\mathbb{Z}_{p^r q^s})$

The following result is of crucial importance to our study in this section.

Theorem 6. *Let $2 \neq p < q$ be distinct primes. The vertex connectivity of $G(\mathbb{Z}_{pq})$ is*

$$\kappa(G(\mathbb{Z}_{pq})) = (p-2)q.$$

Proof. Let x and y be nonadjacent in $G(\mathbb{Z}_{pq})$. In this proof, we will calculate the maximum number of internally disjoint paths between any two nonadjacent vertices. There are several cases for x and y, as follows:

Case 1: Let $x, y \in G(B'_i)$. By Lemma 11, then there are $(p-1)(q-2)$ internally disjoint paths of length 2 between x and y. In addition, there are other internally disjoint paths depending on the following cases for i:

(a) Let $i \neq 0$. By lemma 12, there are $(p-2)$ internally disjoint paths of length 4 between x and y. So, the maximum number of internally disjoint paths between x and y is

$$(p-1)(q-2) + (p-2) = pq - p - q.$$

(b) Let $i = 0$. By Lemma 12, there are $(p-1)$ internally disjoint paths of length 4 between x and y. So, the maximum number of internally disjoint paths between x and y is

$$(p-1)(q-2) + (p-1) = (p-1)(q-1).$$

Case 2: Let $x \in G(B'_i)$ and $y \in G(B'_j)$, where $G(B'_i)$ is nonadjacent to $G(B'_j)$ in G'_{pq}. Let $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. The following cases arise for x and y:

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(a) If y has the same neighbors as x in $G(B'_z)$, then there are $(p-2)(q-1)$ internally disjoint paths of length 2 between x and y by Lemma 13. According to Lemma 14, there are $(p-2)$ internally disjoint paths of length 4 between x and y. Hence, the maximum number of internally disjoint paths between x and y is
$$(p-2)(q-1)+(p-2)=(p-2)q.$$

(b) If x and y do not have the same neighbors in $G(B'_z)$, there are $(p-2)(q-2)$ internally disjoint paths of length 2 between x and y by Lemma 13. According to Lemma 14, there are $2(p-2)$ internally disjoint paths of length 3 between x and y. Hence, the maximum number of internally disjoint paths between x and y is
$$(p-2)(q-2)+2(p-2)=(p-2)q.$$

Case 3: Let $x \in G(B'_i)$ and $y \in G(B'_j)$, where $G(B'_i)$ is adjacent to $G(B'_j)$ in G'_{pq}. Let $G(B'_z)$ be a common neighbor between $G(B'_i)$ and $G(B'_j)$ in G'_{pq}. There are the following cases for x and y:

(a) If y has the same neighbors as x in $G(B'_z)$, then there are $(p-2)(q-1)$ internally disjoint paths of length 2 between x and y by proof of Lemma 15. According to proof of Lemma 16, there are $(q-1)$ internally disjoint paths of length 3 between x and y. Hence, the maximum number of internally disjoint paths between x and y is
$$(p-2)(q-1)+(q-1)=(p-1)(q-1).$$

(b) If x and y do not have the same neighbors in $G(B'_z)$, then there are $(p-2)(q-2)$ internally disjoint paths of length 2 between x and y by proof of Lemma 15. In addition, there are other internally disjoint paths depending on the following cases for i and j:

   (1) Let $i,\ j \neq 0$. According to proof of Lemma 16, there are $(p+q-5)$ internally disjoint paths of length 3 between x and y. Therefore, the maximum number of internally disjoint paths between x and y is
   $$(p-2)(q-2)+p+q-5=pq-p-q-1.$$

   (2) Let $j=0$. According to proof of Lemma 16, there are $(p+q-4)$ internally disjoint paths of length 3 between x and y. So, the maximum number of internally disjoint paths between x and y is
   $$(p-2)(q-2)+(p+q-4)=pq-p-q.$$

From the above cases and by Menger's theorem, we have
$$\begin{aligned}\kappa(G(\mathbb{Z}_{pq})) &= \min\{pq-p-q,\ (p-1)(q-1),\ (p-2)q,\ pq-p-q-1\}\\ &= (p-2)q. \quad \square\end{aligned}$$

Now, let us explore the vertex connectivity of $G(\mathbb{Z}_n)$ if $n=p^rq^s$, where $2\neq p<q$ are primes and r,s are positive integers such that at least one of r,s must be greater than 1.

Theorem 7. *Let $n=p^rq^s$, where $2\neq p<q$ are distinct primes, r and s are positive integers. Then, the vertex connectivity of $G(\mathbb{Z}_n)$ is given by*
$$\kappa(G(\mathbb{Z}_n))=(p-2)p^{(r-1)}q^s.$$

Proof. By Lemma 4, the unit graph $G(\mathbb{Z}_n)$ is
$$G(\mathbb{Z}_n)=G_{p^rq^s}[G(B_0),G(B_1),\ldots,G(B_{pq-1})].$$

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According to Lemma 3, $G_{p^r q^s}$ is isomorphic to $G(\mathbb{Z}_{pq})$. Hence, by Theorem 6, we get

$$\kappa(G_{p^r q^s}) = (p-2)q.$$

Note that, for every vertex i of $G_{p^r q^s}$, we have $|G(B_i)|$ vertices in $G(\mathbb{Z}_n)$. Since $|G(B_i)| = p^{r-1}q^{s-1}$ for $0 \leq i \leq pq-1$, then the vertex connectivity of $G(\mathbb{Z}_n)$ is

$$\begin{aligned} \kappa(G(\mathbb{Z}_n)) &= p^{r-1}q^{s-1}\kappa(G_{p^r q^s}) \\ &= (p-2)p^{(r-1)}q^s. \quad \square \end{aligned}$$

Based on our results for the vertex connectivity of the unit graph, in Theorems 6 and 7, we state the following conjectures:

Conjecture I: Let $n = p_1 p_2 ... p_k$, where $2 \neq p_1 < p_2 < ... < p_k$ are distinct primes. The vertex connectivity of $G(\mathbb{Z}_n)$ is

$$\kappa(G(\mathbb{Z}_n)) = (p_1 - 2) \prod_{2 \leq i \leq k} p_i.$$

Conjecture II: Let $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where $2 \neq p_1 < p_2 < ... < p_k$ are distinct primes, r_i and k are positive integers, and $1 \leq i \leq k$. Then, the vertex connectivity of $G(\mathbb{Z}_n)$ is given by

$$\kappa(G(\mathbb{Z}_n)) = (p_1 - 2)p_1^{(r_1-1)} \prod_{2 \leq i \leq k} p_i^{r_i}.$$

6. Conclusions

In this paper, we have investigated the structure of $G(\mathbb{Z}_n)$. Based on this structure, the Laplacian spectrum and vertex connectivity of $G(\mathbb{Z}_n)$ have been determined for various n. First, we study the structure of $G(\mathbb{Z}_n)$ for $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where $p_1, p_2, ..., p_k$ are distinct primes and $k, r_1, r_2, ..., r_k$ are positive integers such that at least one of the r_i must be greater than 1, and we prove that the graph $G(\mathbb{Z}_n)$ is a generalized join of certain complete graphs and null graphs. Then, we determine the Laplacian spectrum of $G(\mathbb{Z}_n)$, we prove that $G(\mathbb{Z}_n)$ is Laplacian integral, and we deduce the algebraic connectivity and Laplacian spectral radius of $G(\mathbb{Z}_n)$. Furthermore, we examine the vertex connectivity of $G(\mathbb{Z}_{pq})$ and $G(\mathbb{Z}_{p^r q^s})$, where $2 \neq p < q$ are primes and r and s are positive integers by using their structure and Menger's theorem. Finally, we present conjectures about the vertex connectivity of $G(\mathbb{Z}_n)$ when $n = p_1 p_2 ... p_k$ and $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, where p_i are distinct primes, r_i are positive integers, and $1 \leq i \leq k$. Our results are precise and dependable, as verified by Python programming (see Appendix A).

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Appendix A

Pranjali et al. [19] provided the generation code of the unit graph of $\mathbb{Z}_n$. We utilize this code in Python programming to create the following algorithm that verifies the validity of our results.

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Algorithm: Unit Graph Generation and Analysis

Input:

- **n**: integer representing the modulus for the ring of integers modulo **n**.

Output:

- Displays the unit graph $G(R)$ for the ring $\mathbb{Z}_n$.
- Outputs the Laplacian matrix of the graph.
- Displays the eigenvalues of the Laplacian matrix with their multiplicities.
- Reports the vertex connectivity of the graph.

Steps:

1. **Initialization:**
   - Define R as the range of integers from 0 to $n-1$.
   - Initialize a graph G.
2. **Check Unit Function:**
   - Define a function $\textbf{is\_unit}(\mathbf{x}, \mathbf{n})$ that returns **True** if the greatest common divisor of x and n is 1, indicating that x is a unit in $\mathbb{Z}_n$.
3. **Graph Construction:**
   - Iterate through all pairs of elements i and j in R.
   - For each pair, if $i \neq j$ and $(i+j)\%n$ is a unit in $\mathbb{Z}_n$, add an edge between i and j in G.
4. **Laplacian Matrix Calculation:**
   - Compute the Laplacian matrix L of the graph G.
5. **Eigenvalue Calculation:**
   - Calculate the eigenvalues of L and round them to two decimal places.
6. **Eigenvalue Output:**
   - Display the Laplacian matrix.
   - Print a table listing each eigenvalue and its multiplicity.
7. **Vertex Connectivity Calculation:**
   - Calculate the vertex connectivity of the graph G.
   - Print the vertex connectivity.

End Algorithm

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