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Posted Date: 1 December 2023

doi: 10.20944/preprints202312.0013.v1

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On the Asymptotic of Solutions of Odd Order Two-Term Differential Equations

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Abstract: The work is devoted to the development of methods for constructing asymptotic formulas as $x \to \infty$ of a fundamental system of solutions of linear differential equations generated by a symmetric two-term differential expression of odd order. The coefficients of the differential expression belong to classes of functions that allow oscillation (for example, those that do not satisfy the classical Titchmarsh-Levitan regularity conditions). As a model equation, the 5th order equation $\frac{i}{2}\left[(p(x)y''')'' + (p(x)y'')'''\right] + q(x)y = \lambda y$, for which various cases of behavior of the coefficients $p(x)$, $q(x)$, is investigated. New asymptotic formulas are obtained for the case when the function $h(x) = -1 + p^{-1/2}(x) \notin L_1[1,\infty)$ significantly influences the asymptotics of solutions to the equation. The case when the equation contains a nontrivial bifurcation parameter is studied.

Keywords: asymptotic methods; oscillating coefficients; singular differential equations; Campbell's identity

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1. Introduction

The asymptotic behavior as $x \to \infty$ of fundamental system of solutions of arbitrary order singular differential equations, in addition to being of independent interest, is an effective method for studying of qualitative spectral characteristics for corresponding differential operators ([1–3], etc.). As a rule, in these books differential equations with regular coefficients with regular growth at infinity are investigated. Therefore, the study of the asymptotic behavior of solutions to equations with coefficients from other classes of functions is of particular interest. Such classes of functions were described by us in the papers [4]. Let us also note the works [5–7], where differential operators with distribution coefficients were studied.

For example, in the work [7] asymptotic formulas were obtained for the fundamental system of solutions of a to-term equation of even order

$$(-1)^n(p(x)y^{(n)})^{(n)} + q(x)y = \lambda y, \quad x \in [1,\infty),$$

where the locally summable function p can be represented $p(x) = (1+r(x))^{-1}$, $r \in L_1[1,\infty)$, and q is a generalized function representable for some fixed k, $0 \le k \le n$ in the form $q = \sigma^{(k)}$, where $\sigma \in L_1[1,\infty)$ if $k < n$, $|\sigma|(1+|r|)(1+|\sigma|) \in L_1[1,\infty)$, if $k = n$.

Since 2014 we have published a series of articles devoted to the study of the asymptotic behavior of solutions to singular ODEs with regularly oscillating coefficients ([4,8–14]). In this case, a new approach was used for the study, based on a sequence of matrix transformations and the use of Campbell's identity [16].

The use of this approach made it possible to obtain new asymptotic formulas in different cases. For example, in [4,12,14] new asymptotic formulas were obtained for solutions of the Sturm-Liouville equation

$$y'' + \left(\mu^2 + \frac{sin(x^\beta)}{x^\alpha}\right) y = 0, \;\; 0 < \alpha \le 1, \; \beta > \frac{\alpha}{2} + 1$$

under some relations between $\alpha, \; \beta, \; \mu$. Note that μ has the meaning of a bifurcation parameter. By the way this equation is one of equations for testing of new methods for constructing asymptotic formulas (see, for example, [15], p. 160).

Equations of odd order for irregular classes of coefficients (in the Titchmarsh-Levitan sense) have been less studied. In the works [7,11,13] the asymptotics of solutions of odd order equations was studied in the case when the coefficient of the highest derivative is equal or equivalent to unity.

Here we develop an approach that was proposed in [4,8–14] and can be implemented to study the asymptotic behavior as $x \to \infty$ of a fundamental system of solutions of two-term equation of arbitrary odd order of the form

$$ly = \frac{i}{2}\left[\left(p(x)y^{(n)}\right)^{(n+1)} + \left(p(x)y^{(n+1)}\right)^{(n)}\right] + q(x)y = \lambda y, \; x \ge 1 \tag{1}$$

for various cases of behavior of the coefficients $p(x), \; q(x)$.

2. Transition to the ODE system using quasi-derivatives

Let us write equation (1) in the form of a system of ordinary differential equations of the first order. To do this, we will use the apparatus of quasi-derivatives (for more details see [17]). Let us define the functions $q_n(x) \in L_{1,loc}[1,\infty)$ so that $q_n^{(n)}(x) = q(x)$ and introduce into consideration quasi-derivatives defined by the following formulas

$$\begin{cases} z_1 = y, & z_{n+2} = \sqrt{p}z'_{n+1} - iq_n z_1 \\ z_2 = z'_1, & z_{n+3} = z'_{n+1} + iC_n^1 q_n z_2 \\ \dots. & \dots. \\ z_n = z'_{n-1}, & z_{2n} = z'_{2n-1} + i(-1)^{n-1}C_n^{n-2} q_n z_{n-1} \\ z_{n+1} = \sqrt{p}z'_n, & z_{2n+1} = z'_{2n} + i(-1)^n C_n^{n-1} q_n z_n. \end{cases} \tag{2}$$

Then equation (1) is equivalent to the relation

$$z'_{2n+1} = \lambda z_1 - i(-1)^{n+1}\frac{q_n}{\sqrt{p}}z_{n+1}.$$

Let us introduce the column vector $\mathbf{z} = column(z_1, z_2, ..., z_{2n+1})$ and write equation (1) as a system of ODEs

$$\mathbf{z}' = S\mathbf{z},$$

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where $S(x,\lambda)$ is the Shin-Zettl matrix [17].

$$S=\begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 & 0 & ... & 0 \\ & & & & & .... & & & & \\ 0 & 0 & 0 & ... & 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 & \frac{1}{\sqrt{p}} & 0 & 0 & ... & 0 \\ \frac{iq_n}{\sqrt{p}} & 0 & 0 & ... & 0 & 0 & \frac{1}{\sqrt{p}} & 0 & ... & 0 \\ 0 & -inq_n & 0 & ... & 0 & 0 & 0 & 1 & ... & 0 \\ & & & & & .... & & & & \\ 0 & 0 & 0 & ... & 0 & (-1)^{n-1}inq_n & 0 & 0 & ... & 1 \\ -i\lambda & 0 & 0 & ... & 0 & 0 & \frac{(-1)^n iq_n}{\sqrt{p}} & 0 & ... & 0 \end{pmatrix},$$

here the non-zero elements of the matrix $S(x,\lambda)$ are given by the formulas

$$s_{kj}=1,\ \ j=1+k,\ \ k=\overline{1,n-1},\ \ k=\overline{n+2,2n},\ \ s_{n,n+1}=s_{n+1,n+2}=\frac{1}{\sqrt{p}},$$

$$s_{n+1,1}=\frac{iq_n}{\sqrt{p}},\ s_{n+k,k}=(-1)^{k-1}iC_n^{k-1}q_n,\ k=\overline{2,n},$$

$$s_{2n+1,2n-1}=\frac{(-1)^n iq_n}{\sqrt{p}},\ s_{2n+1,1}=-i\lambda.$$

Note that from the relation $q_n^{(n)}(x)=q(x)$ the function $q_n(x)$ is determined up to a polynomial of order $n-1$. However, the fundamental system of solutions of the equation (1) does not depend on the choice of integration constants, which follows directly from the formulas (2). Conditions for choosing the coefficients of the polynomial will be formulated for each case under study.

Further, in order to avoid complicated formulas, we will limit ourselves to considering the 5th order two-term equation

$$ly=\frac{i}{2}\left[(p(x)y'')'''+(p(x)y''')''\right]+q(x)y=\lambda y,\ x\geq 1. \tag{3}$$

Using formulas (2), we introduce quasi-derivatives

$$\begin{cases} z_1=y \\ z_2=z_1' \\ z_3=\sqrt{p}z_2' \\ z_4=\sqrt{p}z_3'-iq_2z_1 \\ z_5=z_4'+2iq_2z_2, \end{cases}$$

then equation (3) is equivalent to the relation

$$z_5'=-i\lambda z_1+\frac{iq_2}{\sqrt{p}}z_3$$

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and can be written as a system of ordinary differential equations

$$\mathbf{z}' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{p} & 0 & 0 \\ iq_2/\sqrt{p} & 0 & 0 & 1/\sqrt{p} & 0 \\ 0 & -2iq_2 & 0 & 0 & 1 \\ -i\lambda & 0 & iq_2/\sqrt{p} & 0 & 0 \end{pmatrix} \mathbf{z},$$

where $\mathbf{z} = column(z_1, z_2, z_3, z_4, z_5)$.
Let the function $p(x)$ admit the representation

$$\frac{1}{\sqrt{p(x)}} = 1 + h(x), \ L_{1,loc}[1, \infty).$$

Let us write the last system of equations in the form

$$\mathbf{z}' = (L_0 + h(x)L_1 + iq_2(x)D_0 + ih(x)q_2(x))D_1)\mathbf{z}, \tag{4}$$

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -i\lambda & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

3. Construction of asymptotic formulas

3.1. Case 1

Let the following conditions be satisfied:

$$h(x), \ q_2(x) \in L_1[1, \infty), \ \tilde{h}(x) \in L_{1,loc}[1, \infty).$$

For example, these conditions are true for

$$h(x) = \frac{1}{x^\gamma}, \ \gamma > 1; \quad q(x) = x^\alpha \sin x^\beta, \ \alpha > 0, \ \beta > \frac{\alpha + 3}{2}.$$

Let the constant matrix T reduce the matrix L_0 to diagonal form. Let us make a replacement

$$\mathbf{z} = T\mathbf{u}, \ T^{-1}L_0T = \Lambda, \ \mu_k^5 = -i\lambda, \ k = \overline{1,5},$$

$$\Lambda = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 & \mu_4^2 & \mu_5^2 \\ \mu_1^3 & \mu_2^3 & \mu_3^3 & \mu_4^3 & \mu_5^3 \\ \mu_1^4 & \mu_2^4 & \mu_3^4 & \mu_4^4 & \mu_5^4 \end{pmatrix}.$$

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Then system (5) will take the form

$$\mathbf{u}' = \left(\Lambda + h(x)T^{-1}L_1T + iq_2(x)T^{-1}D_0T + iq_2(x)h(x)T^{-1}D_1T\right)\mathbf{u}. \tag{5}$$

Obviously, due to the imposed conditions, system (5) satisfies the conditions of Lemma 1 in [3], p. 284 and is L-diagonal, which means we can write out asymptotic formulas as $x \to \infty$ for the fundamental system solutions of this system.

$$\mathbf{z}_k(x,\lambda) = T \cdot \mathbf{u}_k(x,\lambda) = e^{\mu_k x} \cdot T \cdot (\mathbf{e}_k + o(1)), \;\; k = \overline{1,5},$$

where $\mathbf{e}_k$ are unit basis vectors.

3.2. Case 2

Let us set $\tilde{h}(x) = q_2(x)h(x)$. Let the following conditions be satisfied:

$$q_2(x) \notin L_1[1,\infty), \;\; h(x),\; q_3(x),\; \tilde{h}(x) \in L_1[1,\infty). \tag{6}$$

These conditions are true for

$$h(x) = \frac{1}{x^\gamma}, \; \gamma > 1; \quad q(x) = x^\alpha \sin x^\beta, \;\; \alpha > 0, \;\; \frac{\alpha+3}{2} \geq \beta > \frac{\alpha+4}{3}.$$

Following the approach outlined in the paper [9], we make a replacement in system (4)

$$\mathbf{z} = e^{iq_3(x)D_0}\mathbf{u}. \tag{7}$$

We get

$$\mathbf{u}' = e^{-iq_3(x)D_0}\left(L_0 + h(x)L_1 + i\tilde{h}(x)D_1\right)e^{iq_3(x)D_0}\mathbf{u}. \tag{8}$$

Let us apply Campbell's identity to transform the right-hand side of (8) to

$$e^{-iq_3(x)D_0}L_0e^{iq_2(x)D_0} = L_0 - iq_3(x)[D_0,L_0] + \frac{i^2q_3^2(x)}{2!}[D_0,[D_0,L_0]] -$$

$$-\frac{i^3q_3^3(x)}{3!}[D_0,[D_0,[D_0,L_0]]] + ...,$$

here $[A,B] = AB - BA$ is a matrix commutator.

Below we will use the following obvious consideration: if the matrix A is nilpotent, then a nonzero sequence of matrix commutators of the form $[A,[A,...,[A,B]]...]$ is finite.

Note that the matrix D_0 is nilpotent. By sequentially calculating the commutators on the right side of the last relation, we obtain that all terms, starting from the fourth, are equal to zero, and non-zero terms can be calculated

$$[D_0,L_0] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad [D_0,[D_0,L_0]] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \end{pmatrix}.$$

Similar calculations can be carried out for the remaining terms on the right side of (8)

$$e^{-iq_3(x)D_0}h(x)L_1e^{iq_3(x)D_0} =$$

$$h(x)L_1 - iq_3h(x)[D_0,L_1] + \frac{h(x)i^2q_3^2(x)}{2!}[D_0,[D_0,L_1]] + ...,$$

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$$[D_0, L_1] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad [D_0, [D_0, L_1]] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{pmatrix}.$$

Due to $[D_0, D_1] = 0$ the representation is true

$$e^{-iq_3(x)D_0}\tilde{h}(x)D_1e^{iq_3(x)D_0} = \tilde{h}(x)D_1.$$

Then the equation (8) can be rewritten as

$$\mathbf{u}' = (L_0 + hL_1 + \tilde{h}D_1 - iq_3[D_0, L_0] + \frac{i^2q_3^2}{2!}[D_0, [D_0, L_0]] - iq_3h[D_0, L_1] + \frac{i^2hq_3^2}{2!}[D_0, [D_0, L_1]])\mathbf{u}. \quad (9)$$

Due to the imposed conditions on the functions $h(x)$, $q(x)$ the last system can be written as:

$$\mathbf{u}' = (L_0 + D(x))\mathbf{u},$$

where $D(x)$ is a matrix whose elements belong to $L_1[1, \infty)$. Just as in case 1, let's make the replacement $\mathbf{u} = T\mathbf{v}$, then

$$\mathbf{v}' = (\Lambda + T^{-1}D(x)T)\mathbf{v}. \quad (10)$$

The system (10) satisfies the conditions of Lemma 1 in [3] and is L-diagonal, which means, taking into account (7), we can write asymptotic formulas for $x \to \infty$ for its fundamental system of solutions

$$\mathbf{z}_k(x, \lambda) = e^{\mu_k x} \cdot e^{iq_3(x)D_0} \cdot T \cdot (\mathbf{e}_k + o(1)), \quad k = \overline{1, 5}.$$

where $\mathbf{e}_k$ are unit vectors.

Remark 1. Let us note the importance of the resulting equation (9). Imposing various conditions on the coefficients of this equation $h(x)$, $q_3(x)h(x)$, $q_3^2(x)h(x)$, $\tilde{h}(x)$, $q_3(x)$ and $q_2^3(x)$, different from the conditions (6) , one can obtain different asymptotics of the fundamental system of solutions with nontrivial properties.

3.3. Case 3

Let us define the function $h_1(x)$ so that $h_1'(x) = h(x)$. Let us now consider the case when

$$h(x), \ q_2(x) \notin L_1[1, \infty), \ \ q_3(x), \ h_1(x), \ \tilde{h}(x) \in L_1[1, \infty).$$

For example, these conditions are true for

$$h(x) = \frac{1}{x^\gamma}, \ 0 < \gamma < 1; \quad q(x) = x^\alpha \sin x^\beta, \ \ 2 > \alpha > 0, \ \ \frac{\alpha + 3}{2} \geq \beta > \frac{\alpha + 3 - \gamma}{2}.$$

Just as in case 2, let's make a replacement in system (4)

$$\mathbf{z} = e^{h_1(x)L_1}\mathbf{u}. \quad (11)$$

Then system (4) will take the form

$$\mathbf{u}' = e^{-h_1(x)L_1}\left(L_0 + iq_2(x)D_0 + i\tilde{h}(x)D_1\right)e^{h_1(x)L_1}\mathbf{u}. \quad (12)$$

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Let us apply Campbell's identity to transform the right-hand side of (12)

$$e^{-h_1(x)L_1}L_0e^{h_1(x)L_1} = L_0 - h_1(x)[L_1,L_0] + \frac{h_1^2(x)}{2!}[L_1,[L_1,L_0]] - \frac{h_1^3(x)}{3!}[L_1,[L_1,[L_1,L_0]]] + ...$$

Note that the matrix L_1 is nilpotent. By sequentially calculating the commutators on the right side of the last relation, we obtain that all terms, starting from the fourth, are equal to zero, and non-zero terms can be calculated

$$[L_1,L_0] = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad [L_1,[L_1,L_0]] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similar calculations can be carried out for the remaining terms on the right side of (12)

$$e^{-h_1(x)L_1}iq_2(x)D_0e^{h_1(x)L_1} = iq_2(x)D_0 - iq_2(x)h_1(x)[L_1,D_0] + \frac{iq_2(x)h_1^2(x)}{2!}[L_1,[L_1,D_0]] -$$

$$-\frac{iq_2(x)h_1^3(x)}{3!}[L_1,[L_1,[L_1,D_0]]] + \frac{iq_2(x)h_1^4(x)}{4!}[L_1,[L_1,[L_1,[L_1,D_0]]]].$$

$$[L_1,D_0] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad [L_1,[L_1,D_0]] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[L_1,[L_1,[L_1,D_0]]] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad [L_1,[L_1,[L_1,[L_1,D_0]]]] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e^{-h_1(x)L_1}i\tilde{h}(x)D_1e^{h_1(x)L_1} = i\tilde{h}(x)D_1 - ih_1(x)\tilde{h}(x)[L_1,D_1],$$

$$[L_1,D_1] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Due to the imposed conditions on the functions $h(x)$, $q(x)$ the last system can be written as:

$$\mathbf{u}' = (L_0 + iq_2(x)D_0 + D(x))\mathbf{u}.$$

where $D(x)$ is a matrix whose elements belong to $L_1[1,\infty)$. Unlike case 2, the resulting system is not yet L-diagonal. Let's make one more transformation

$$\mathbf{u} = e^{iq_3(x)D_0}\mathbf{v}. \tag{13}$$

Then the last system will take the form

$$\mathbf{v}' = e^{-iq_3(x)D_0}\left(L_0 + iq_2(x)D_0 + D(x)\right)e^{iq_3(x)D_0}\mathbf{v}. \tag{14}$$

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Let us apply Campbell's identity to transform the right-hand side of (14). Just as in case 2, taking into account the nilpotency of the matrix D_0 and sequentially calculating all the necessary matrix commutators, we obtain the following form of system (14)

$$\mathbf{v}' = (L_0 + \tilde{D}(x))\mathbf{v}.$$

Here the matrix $\tilde{D}(x)$ is defined by the expression

$$\tilde{D}(x) = -iq_3(x)[D_0, L_0] + \frac{i^2 q_3^2}{2!}[D_0, [D_0, L_0]] + e^{-iq_3(x)D_0} D(x) e^{iq_3(x)D_0},$$

which, obviously, due to the conditions imposed above on the functions $h(x),\ q(x)$ is a matrix with elements summable over $[1, \infty)$.

Next we make the replacement $\mathbf{v} = T\mathbf{s}$, then

$$\mathbf{s}' = (\Lambda + T^{-1}\tilde{D}(x)T)\mathbf{s}. \tag{15}$$

System (15) satisfies the conditions of Lemma 1 in [3] and is L-diagonal, which means, taking into account (11),(13), we can write out asymptotic formulas as $x \to \infty$ for its fundamental system of solutions

$$\mathbf{z}_k(x, \lambda) = e^{\mu_k x} \cdot e^{h_1(x)L_1} \cdot e^{iq_3(x)D_0} \cdot T \cdot (\mathbf{e}_k + o(1)),\ \ k = \overline{1,5},$$

where $\mathbf{e}_k$ are unit vectors.

Summarizing cases 1-3, we find that we have proven the following theorem:

Theorem 1. *Let one of the next conditions be satisfied:*
1) $h(x),\ q_2(x) \in L_1[1, \infty)$,
2) $h(x),\ q_3(x),\ \tilde{h}(x) \in L_1[1, \infty)$,
3) $q_3(x),\ h_1(x),\ \tilde{h}(x) \in L_1[1, \infty)$
then the asymptotic formulas as $x \to \infty$ for the fundamental of equation (3) are valid

$$y_i(x, \lambda) = e^{\mu_j x} \cdot (1 + o(1)).$$

In point of fact we obtained the asymptotic formulas as $x \to \infty$ for vector-function, we may also write down the asymptotic formulas for quasi-derivatives of solutions.

3.4. Counterexample

Let us how that the conditions of Theorem 1 are essential.

Let

$$h(x),\ q_2(x) \notin L_1[1, \infty),\ \ q_3(x),\ h_1(x) \in L_1[1, \infty),\ \ \tilde{h}(x) \in L_{1,loc}[1, \infty).$$

In the same way as for case 3, we will make sequential transformations

$$\mathbf{z} = e^{h_1(x)L_1}\mathbf{u},\quad \mathbf{u} = e^{iq_3(x)D_0}\mathbf{w},$$

which will bring the system of equations (4) to the form

$$\mathbf{w}' = (L_0 + ih(x)q_2(x))D_1)\mathbf{w} + F(x)\mathbf{w}, \tag{16}$$

where $F(x) \in L_1[1, \infty)$.

The last system of equations allows for a large variety in the asymptotic behavior of the as $x \to +\infty$ and can be the subject of a separate study.

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We will limit ourselves to considering a model example on which we will demonstrate an unusual property of equations with oscillating coefficients, namely the influence of the algebraic structure of the coefficients of the equation on the asymptotic behavior of the solutions.

Put

$$h(x) = a\sin(e^x),\, q_2(x) = \sin(ke^x),$$

from which

$$\tilde{h}(x) = h(x)q_2(x) = a\sin(e^x)\sin(ke^x) = \frac{1}{2}a[\cos((k-1)e^x) + \cos((k+1)e^x)].$$

Consider two cases: $k = \pm 1$ and $k \neq \pm 1$. Let $k \neq \pm 1$. Define the function $\tilde{h}_1(x)$ so that $\tilde{h}_1'(x) = \tilde{h}(x)$. Note that in this case $\tilde{h}_1(x) \in L_1[1,\infty)$.

In the system (16) we put

$$\mathbf{w} = e^{i\tilde{h}_1(x)D_1}\mathbf{v},$$

then for $\mathbf{v}$ we obtain the system

$$\mathbf{v}' = (L_0 + F_1(x))\mathbf{v}, \tag{17}$$

where $F_1(x) \in L_1[1,\infty)$ and which can easily be reduced to an L-diagonal system. Consequently, the main term of the asymptotics of the fundamental system of solutions (17) , as above, will be determined

$$\mathbf{z}_j(x,\lambda) = e^{\mu_j x}\cdot T\cdot(\mathbf{e}_j + o(1)),\;\; j = \overline{1,5}.$$

Let $k = \pm 1$. Note that now

$$h(x)q_2(x) = a\sin(e^x)\sin(ke^x) =$$

$$= \frac{1}{2}a[\cos((k-1)e^x) + \cos((k+1)e^x)] = \frac{1}{2}a(1 + \cos(2e^x))$$

whence it follows that $h(x)q_2(x) \notin L_1[1,\infty)$. Let us denote $\sigma(x) = \frac{i}{2}a\cos(2e^x)$ and represent the system (17) in the form:

$$\mathbf{w}' = \left(L_0 + \frac{i}{2}aD_1 + \sigma(x)D_1\right)\mathbf{w} + F(x)\mathbf{w}, \tag{18}$$

where $F(x) \in L_1[1,\infty)$, and the matrix $L_0 + \frac{i}{2}aD_1$ is constant. Considering

$$\sigma_1(x) = -\frac{i}{2}a\int_x^{+\infty}\cos(2e^\xi)d\xi \in L_1[1,\infty),$$

we will make a replacement in the system (18)

$$\mathbf{w} = e^{\sigma_1(x)D_1}\mathbf{v}$$

and again using the technique described above, we obtain the system

$$\mathbf{v}' = (L_0 + \frac{i}{2}aD_1)\mathbf{w} + F_1(x)\mathbf{w}. \tag{19}$$

Here, taking into account that $\sigma_1(x) \in L_1[1,\infty)$ we have $F_1(x) \in L_1[1,\infty)$. Let the matrix $\hat{T}$ reduce the matrix $L_0 + \frac{i}{2}aD_1$ to diagonal form, $\hat{\mu}_j$ $j = \overline{1,5}$ - eigenvalues of the matrix $L_0 + \frac{i}{2}aD_1$. Let's make a replacement

$$\mathbf{v} = \hat{T}\mathbf{s},\;\; \hat{T}^{-1}(L_0 + \frac{i}{2}aD_1)\hat{T} = \hat{\Lambda}.$$

Then the system (18) will take the form

$$\mathbf{s}' = (\hat{\Lambda} + \hat{T}^{-1}F_1(x)\hat{T})\mathbf{s}.$$

 doi:10.20944/preprints202312.0013.v1

The resulting system is equivalent to the L-diagonal system, then, as above, the fundamental system of solutions equation (4) can be represented as

$$\mathbf{z}_j(x,\lambda) = e^{\hat{\mu}_j x} \cdot \hat{T} \cdot (\mathbf{e}_j + o(1)), \quad j = \overline{1,5}.$$

Thus, when the numerical coefficient k passes through the points $k = \pm 1$, the asymptotics of the fundamental system of solutions of equation (3) undergoes a qualitative change. In other words, the points $k = \pm 1$ are bifurcation points for the system (4) and corresponding equation (3).

Such bifurcation points for differential equations and (systems of equations) with regularly oscillating coefficients are typical and were recently noted by us in works devoted to the study of the asymptotic behavior of the Sturm-Liouville equation with an oscillating potential ([4,14]).

4. Discussion

The results obtained have important applications in the spectral theory of differential operators generated by the left side of equation (1). In particular, they make it possible to calculate the deficiency indices of the corresponding minimal differential operator.

The authors intend to investigate this issue in the future. In addition, we will be interested in the qualitative nature of the spectrum of such operators.

Funding: The studies of E.A. Nazirova and Ya.T. Sultanaev was funded by the Russian Science Foundation, project no. 23-21-00225.

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