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Article

The Collatz Conjecture and the Spectral Calculus for Arithmetic Dynamics

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Abstract

We develop an operator–theoretic framework for the Collatz map based on its backward transfer operator acting on weighted Banach spaces of arithmetic functions. The associated Dirichlet transforms form a holomorphic family that captures the complex–analytic evolution of iterates and admits a decomposition into a zeta–type pole at $s = 1$ and a holomorphic remainder. Within a finer multiscale space adapted to the Collatz preimage tree, we establish a Lasota–Yorke inequality with an explicit contraction constant $\lambda < 1$, giving quasi–compactness and a spectral gap at the dominant eigenvalue. The resulting invariant density is strictly positive and exhibits a c/n decay profile. We formulate a general criterion showing that, under a verified quasi–compactness hypothesis with isolated eigenvalue 1, the forward dynamics admit no infinite trajectories. The framework provides a coherent spectral perspective on the Collatz operator and suggests a broader analytic approach to arithmetic dynamical systems.

Keywords: Collatz conjecture; transfer operators; Lasota–Yorke inequality; invariant densities; dirichlet transforms; nonlinear integer dynamics; quasi-compactness

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1. Introduction

The Collatz conjecture asserts that every positive integer n eventually reaches the 1–2 cycle under repeated application of

$$T(n) = \begin{cases} n/2, & n \text{ even}, \\ 3n+1, & n \text{ odd}. \end{cases} \tag{1}$$

Equivalently, every forward orbit $\mathcal{O}^+(n) = \{T^k(n) : k \geq 0\}$ is conjectured to terminate in $\{1, 2\}$. Despite its elementary definition, the iteration exhibits striking irregularity, with long sequences of expansions and contractions that have motivated extensive probabilistic, analytic, and computational study over many decades. Classical work of Terras [15,16] established early density results and stopping-time estimates, while the surveys of Lagarias [7,8] synthesized a wide range of heuristic and structural approaches. Subsequent analytic contributions, including those of Meinardus [11] and Applegate–Lagarias [1], have developed refined density bounds and asymptotic estimates for the distribution of orbits. Nevertheless, the global termination problem remains open, and the intricate behavior of Collatz trajectories continues to motivate the search for structural or spectral frameworks capturing the underlying arithmetic dynamics.

The purpose of this paper is to recast the Collatz problem in an analytic and operator–theoretic framework, and to show that the conjecture follows from a verifiable spectral–gap property of an associated *backward transfer operator*. Instead of studying T directly, we analyze its inverse dynamics through the operator

$$(Pf)(n) := \sum_{m:T(m)=n} \frac{f(m)}{m}, \tag{2}$$

acting on arithmetic functions $f : \mathbb{N} \to \mathbb{C}$. Transfer–operator methods of this type originate in statistical mechanics and dynamical systems [13,14], and have more recently been applied to $3x + 1$–type maps

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in various analytic and functional–analytic contexts [10,12]. For the Collatz map (1), each n has an even preimage $2n$ and an additional odd preimage $(n-1)/3$ whenever $n \equiv 4 \pmod 6$, giving

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4 \pmod 6\}} \frac{f((n-1)/3)}{(n-1)/3}. \tag{3}$$

The weights $1/m$ normalize the operator so that P acts as a mass–preserving average on non-negative ℓ^1 sequences, reflecting the logarithmic contraction inherent in the preimage structure of T.

Remark 1 (Invariant density and logarithmic mass balance)**.** *Although P preserves total mass only up to a logarithmic factor, it does not fix the constant function. Indeed,*

$$(P\mathbf{1})(n) = \frac{1}{2n} + \mathbf{1}_{\{n\equiv 4 \pmod 6\}} \frac{3}{n-1} \sim \frac{C}{n} \qquad (n \to \infty),$$

so $(P\mathbf{1}) \neq \mathbf{1}$. More generally,

$$\sum_{n\geq 1} (Pf)(n) = \sum_{m\geq 1} \frac{f(m)}{m}, \tag{4}$$

which shows that P is logarithmically mass–preserving*: the pushforward of mass is reweighted by the harmonic kernel $m \mapsto 1/m$.*

This logarithmic balance forces any P–invariant density h to satisfy $Ph = h$ with a decay of order $1/n$ as $n \to \infty$. In particular, the explicit block recursion developed in Section 5.2, together with the oscillation control provided by the Lasota–Yorke inequality [9], yields the precise asymptotic profile

$$h(n) \sim \frac{c}{n}, \qquad n \to \infty,$$

consistent with Tauberian heuristics of Delange type [3]. All spectral decompositions in the sequel are expressed relative to this nonconstant $1/n$–type invariant profile.

The operator P induces a rich spectral structure on weighted sequence spaces. On ℓ^1_σ, defined by $\|f\|_\sigma = \sum_{n\geq 1} |f(n)|\, n^{-\sigma}$, the Dirichlet transform

$$Df(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}, \tag{5}$$

intertwines P with analytic continuation in the half-plane $\Re(s) > \sigma$. Uniform ℓ^1_σ bounds on P^k translate into exponential envelopes for $D(P^k f)(s)$ and yield meromorphic continuations of the corresponding Collatz–Dirichlet series, whose pole at $s = 1$ reflects the average branching behavior [2,5]. The spectral radius of P on ℓ^1_σ captures the global weighted expansion rate of inverse branches and determines the analytic location of dominant singularities.

To resolve finer dynamical properties, we refine this setting to a multiscale Banach space $B_{\mathrm{tree},\sigma}$ built from dyadic–triadic block averages and oscillation seminorms that encode the hierarchical structure of the Collatz preimage tree. On this space, P satisfies a two-norm Lasota–Yorke inequality,

$$[Pf]_{\mathrm{tree},\sigma} \leq \lambda_{\mathrm{LY}} [f]_{\mathrm{tree},\sigma} + C\|f\|_\sigma, \qquad 0 < \lambda_{\mathrm{LY}} < 1,$$

placing the dynamics within the classical Ionescu–Tulcea–Marinescu and Hennion spectral frameworks for quasi–compact operators [4,6]. The precise Lasota–Yorke bounds, including the explicit contraction of the odd branch, are developed in Sections 4–6.

The main theorem of the paper establishes that when the odd-branch contraction constant $\lambda_{\mathrm{odd}}(\alpha, \vartheta)$ satisfies $\lambda_{\mathrm{odd}} < 1$ for specific parameters $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$, the backward Collatz operator P possesses a strict spectral gap on $B_{\mathrm{tree},\sigma}$. The spectral decomposition then implies that every invariant measure of P is supported on the 1–2 cycle, ruling out any positive-density family of diver-

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gent or periodic orbits. A strengthened criterion shows that a non-trivial invariant functional in $B^*_{\text{tree},\sigma}$ would contradict the spectral gap, hence all Collatz trajectories must terminate.

The remainder of the paper is organized as follows. Section 2 establishes notation and basic properties of the weighted ℓ^1_σ spaces together with the associated Dirichlet transforms. Section 3 introduces the backward transfer operator P and its analytic representation. Section 4 constructs the multiscale space $B_{\text{tree},\sigma}$ adapted to the Collatz preimage tree and proves the corresponding Lasota–Yorke inequalities. Section 6 verifies that the odd branch admits an explicit contraction constant $\lambda_{\text{odd}} < 1$ for the chosen parameters, yielding quasi–compactness and a spectral gap. Finally, Section 7 develops the resulting spectral consequences, formulating a general criterion that links quasi–compactness with the absence of infinite forward trajectories, and situating the Collatz operator within a broader analytical framework for arithmetic dynamical systems.

2. Preliminaries

The analysis begins with a careful description of the function spaces, Dirichlet transforms, and basic structural features of the Collatz map that underlie the spectral study of the backward operator P. Throughout we work with complex-valued arithmetic functions $f : \mathbb{N} \to \mathbb{C}$.

2.1. Weighted ℓ^1 Spaces and Dirichlet Transforms

For $\sigma > 0$ we define the weighted ℓ^1 space

$$\ell^1_\sigma := \left\{ f : \mathbb{N} \to \mathbb{C} : \|f\|_\sigma := \sum_{n \geq 1} \frac{|f(n)|}{n^\sigma} < \infty \right\}. \tag{6}$$

The weight exponent σ measures polynomial decay and is chosen so that Dirichlet series associated with f converge absolutely in a half-plane $\Re(s) > \sigma$.

Given $f \in \ell^1_\sigma$, we define its Dirichlet transform

$$Df(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}, \qquad \Re(s) > \sigma. \tag{7}$$

Lemma 1 (Dirichlet convergence). *Let $\sigma > 0$ and $f \in \ell^1_\sigma$. Then $Df(s)$ in (7) converges absolutely for $\Re(s) > \sigma$ and defines a bounded holomorphic function on every half-plane $\Re(s) \geq \sigma + \varepsilon$, $\varepsilon > 0$. Moreover,*

$$|Df(s)| \leq \|f\|_\sigma \sup_{n \geq 1} n^{\sigma - \Re(s)}. \tag{8}$$

Proof. For $\Re(s) > \sigma$,

$$\sum_{n \geq 1} \left| \frac{f(n)}{n^s} \right| = \sum_{n \geq 1} \frac{|f(n)|}{n^{\Re(s)}} = \sum_{n \geq 1} \frac{|f(n)|}{n^\sigma} n^{\sigma - \Re(s)} \leq \|f\|_\sigma \sup_{n \geq 1} n^{\sigma - \Re(s)} < \infty,$$

so the series converges absolutely and locally uniformly in $\Re(s) \geq \sigma + \varepsilon$, giving holomorphy and the bound (8). □

We write $\ell^1 = \ell^1_0$ for the unweighted space with norm $\|f\|_1 = \sum_{n \geq 1} |f(n)|$.

2.2. Coarse Forward Envelopes

Lemma 2 (Coarse k-step envelopes). *Let $T : \mathbb{N} \to \mathbb{N}$ denote the Collatz map, (1). For every $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$,*

$$\frac{n}{2^k} \leq T^k(n) \leq 3^k n + \frac{3^k - 1}{2}. \tag{9}$$

Proof. The proof proceeds as before: $T(x) \geq x/2$ and $T(x) \leq 3x + 1$, and induction gives the bounds (9). □

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These envelopes are intentionally crude, yet they ensure that forward iterates of typical arithmetic weights remain controlled on the scales relevant for our Dirichlet and transfer-operator analysis.

2.3. Backward Preimages and the Transfer Recursion

For each $n \geq 1$, define the even and odd preimage sets

$$E(n) := \{m \in \mathbb{N} : T(m) = n,\ m \text{ even}\}, \qquad O(n) := \{m \in \mathbb{N} : T(m) = n,\ m \text{ odd}\}.$$

Lemma 3 (Preimage structure). *For every $n \in \mathbb{N}$,*

$$E(n) = \{2n\}, \qquad O(n) = \begin{cases} \{(n-1)/3\}, & n \equiv 4 \pmod 6, \\ \varnothing, & \textit{otherwise}, \end{cases} \tag{10}$$

and in the first case $(n-1)/3$ is odd. In particular, each n has either one preimage (even) or two preimages (one even and one odd), and the odd preimage occurs with natural density $1/6$.

Proof. If m is even and $T(m) = n$, then $m/2 = n$, so $m = 2n$, establishing $E(n) = \{2n\}$.

If m is odd and $T(m) = n$, then $3m + 1 = n$, so $m = (n-1)/3$. This is an integer precisely when $n \equiv 1 \pmod 3$. For m to be odd, $n - 1$ must be divisible by 3 but not by 6, so $n \equiv 4 \pmod 6$. In that case $(n-1)/3$ is odd. The density statement follows since the congruence class $n \equiv 4 \pmod 6$ has natural density $1/6$. □

Hence each n admits exactly one even preimage and possibly one odd preimage when $n \equiv 4 \pmod 6$. The corresponding backward transfer operator is defined as

$$(Pf)(n) := \sum_{m:T(m)=n} \frac{f(m)}{m} = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}. \tag{11}$$

The normalization by $1/m$ reflects the logarithmic contraction of the forward map and ensures a natural mass-balance property.

Lemma 4 (Mass preservation on ℓ^1). *If $f \geq 0$ and $f \in \ell^1$, then*

$$\sum_{n\geq 1} (Pf)(n) = \sum_{m\geq 1} f(m). \tag{12}$$

Proof. Each m contributes exactly once to the double sum $\sum_{n\geq 1} \sum_{m:T(m)=n} \frac{f(m)}{m}$, so equality (12) follows directly from (11). □

2.4. Dirichlet Envelope for Iterates of the Backward Operator

The preimage structure allows a crude but useful bound on P acting on ℓ^1_σ.

Proposition 1 (Backward operator bound). *Let $\sigma > 0$ and let P be defined by (11). Then $P : \ell^1_\sigma \to \ell^1_\sigma$ is bounded and*

$$\|Pf\|_\sigma \leq C_\sigma \|f\|_\sigma, \qquad C_\sigma := 2^\sigma + 3^{-\sigma}, \tag{13}$$

for all $f \in \ell^1_\sigma$. Consequently, for every $k \geq 1$,

$$\|P^k f\|_\sigma \leq C_\sigma^k \|f\|_\sigma. \tag{14}$$

Proof. From (11),

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}.$$

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Hence

$$\|Pf\|_\sigma \le S_{\text{even}} + S_{\text{odd}},$$

with

$$S_{\text{even}} := \sum_{n\ge 1} \frac{|f(2n)|}{2n\, n^\sigma}, \qquad S_{\text{odd}} := \sum_{\substack{n\ge 1\\ n\equiv 4\,(6)}} \frac{\left|f\left(\frac{n-1}{3}\right)\right|}{\left(\frac{n-1}{3}\right) n^\sigma}.$$

For the even branch, set $m = 2n$, so $n = m/2$ and

$$S_{\text{even}} = \sum_{\substack{m\ge 1\\ m \text{ even}}} \frac{|f(m)|}{m\,(m/2)^\sigma} = \sum_{\substack{m\ge 1\\ m \text{ even}}} \frac{2^\sigma\,|f(m)|}{m^{\sigma+1}} \le 2^\sigma \sum_{m\ge 1} \frac{|f(m)|}{m^\sigma} = 2^\sigma \|f\|_\sigma.$$

For the odd branch, write $m = (n-1)/3$, so $n = 3m+1$ and m is odd. Then

$$S_{\text{odd}} = \sum_{\substack{m\ge 1\\ m \text{ odd}}} \frac{|f(m)|}{m\,(3m+1)^\sigma} \le \sum_{m\ge 1} \frac{|f(m)|}{m\,(3m)^\sigma} = 3^{-\sigma} \sum_{m\ge 1} \frac{|f(m)|}{m^{\sigma+1}} \le 3^{-\sigma}\|f\|_\sigma.$$

Combining the two estimates gives (13), and iterating yields (14). □

The constant $C_\sigma = 2^\sigma + 3^{-\sigma}$ is an explicit growth factor for P on ℓ^1_σ. It is not < 1 in this normalization, so no contraction is claimed at this level. The genuine contraction mechanism is obtained later on the multiscale Banach space B_{tree}, where a strong seminorm captures oscillatory decay along the Collatz tree while the ℓ^1 component provides compactness.

3. Transfer Operator Formulation

We now reformulate the Collatz dynamics in terms of the *backward transfer operator* associated with the map (1). This operator-theoretic viewpoint provides an analytic bridge between the discrete recurrence and the functional framework developed in later sections. The transfer operator encodes the inverse–branching structure of the map and propagates densities backward along the Collatz tree, in a form compatible with logarithmic weighting and Dirichlet series.

Recall that the Collatz map, (1), by Lemma 3, each $n \ge 1$ has the even preimage $2n$, together with an additional odd preimage $(n-1)/3$ precisely when $n \equiv 4 \pmod 6$.

3.1. Backward Transfer Operator

Definition 1 (Backward transfer operator)**.** *For an arithmetic function $f : \mathbb{N} \to \mathbb{C}$, define*

$$(Pf)(n) := \sum_{m:\,T(m)=n} \frac{f(m)}{m} = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}, \qquad n \in \mathbb{N}, \tag{15}$$

where $\mathbf{1}_A$ denotes the indicator of the condition A.

The multiplicative factor $1/m$ assigns to each inverse branch a logarithmic weight, so that P acts as a normalized backward average along preimages. This normalization aligns the discrete dynamics with Dirichlet weights and will be crucial for analytic continuation and spectral estimates below.
Positivity. If $f(n) \ge 0$ for all n, then $(Pf)(n) \ge 0$ for all n, since P is a positive linear combination of values of f.
Weighted mass preservation. A direct change of variables shows that for every nonnegative f satisfying $\sum_{m\ge 1} |f(m)|/m < \infty$,

$$\sum_{n\ge 1} (Pf)(n) = \sum_{m\ge 1} \frac{f(m)}{m}. \tag{16}$$

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Thus P preserves the logarithmically weighted mass $\sum f(m)/m$; plain ℓ^1 mass is not preserved under this normalization.

Boundedness on weighted spaces. Let

$$\ell^1_\sigma := \Big\{ f : \mathbb{N} \to \mathbb{C} \;:\; \|f\|_{\ell^1_\sigma} := \sum_{n\ge 1} \frac{|f(n)|}{n^\sigma} < \infty \Big\}, \qquad \sigma > 0.$$

A direct change of variables in (15) yields, for all $f \in \ell^1_\sigma$,

$$\begin{aligned} \|Pf\|_{\ell^1_\sigma} &= \sum_{n\ge 1} \frac{|(Pf)(n)|}{n^\sigma} \le \sum_{n\ge 1} \left(\frac{|f(2n)|}{2n^{1+\sigma}} + \mathbf{1}_{\{n\equiv 4(6)\}} \frac{|f((n-1)/3)|}{((n-1)/3)^{1+\sigma}} \right) \\ &= \frac{1}{2} \sum_{n\ge 1} \frac{|f(2n)|}{n^{1+\sigma}} + 3^{1+\sigma} \sum_{\substack{n\ge 1 \\ n\equiv 4(6)}} \frac{|f((n-1)/3)|}{(n-1)^{1+\sigma}}. \end{aligned} \tag{17}$$

Changing variables $m = 2n$ in the first sum and $m = (n-1)/3$ in the second gives

$$\sum_{n\ge 1} \frac{|f(2n)|}{2n^{1+\sigma}} = 2^\sigma \sum_{\substack{m\ge 1 \\ m \text{ even}}} \frac{|f(m)|}{m^{1+\sigma}} \le 2^\sigma \|f\|_{\ell^1_\sigma},$$

$$3^{1+\sigma} \sum_{\substack{n\ge 1 \\ n\equiv 4(6)}} \frac{|f((n-1)/3)|}{(n-1)^{1+\sigma}} = 3^{-\sigma} \sum_{\substack{m\ge 1 \\ 3m+1\equiv 4(6)}} \frac{|f(m)|}{m^\sigma} \le 3^{-\sigma} \|f\|_{\ell^1_\sigma}.$$

Hence

$$\|Pf\|_{\ell^1_\sigma} \le (2^\sigma + 3^{-\sigma}) \|f\|_{\ell^1_\sigma}, \tag{18}$$

and therefore

$$\|P^k f\|_{\ell^1_\sigma} \le (2^\sigma + 3^{-\sigma})^k \|f\|_{\ell^1_\sigma}, \qquad k \ge 0. \tag{19}$$

Action on the weighted sup space. For the Banach space

$$B_\sigma := \Big\{ f : \mathbb{N} \to \mathbb{C} \;:\; \|f\|_{B_\sigma} := \sup_{n\ge 1} n^\sigma |f(n)| < \infty \Big\},$$

the normalization factor $1/m$ in (15) improves decay at each branch but does not make P a contraction. Setting $g(n) := nf(n)$, one obtains

$$n(Pf)(n) = g(2n) + \mathbf{1}_{\{n\equiv 4(6)\}}\, g\!\left(\tfrac{n-1}{3}\right), \qquad (Pf)(n) = \frac{(Qg)(n)}{n}, \quad (Qg)(n) := g(2n) + \mathbf{1}_{\{n\equiv 4(6)\}} g\!\left(\tfrac{n-1}{3}\right).$$

Using $\|f\|_{B_\sigma} = \|g\|_{B_{\sigma-1}}$, one obtains the bound

$$\begin{aligned} \|Pf\|_{B_\sigma} &= \sup_{n\ge 1} n^{\sigma-1} |(Qg)(n)| \le \sup_{n\ge 1} \Big(n^{\sigma-1} |g(2n)| + n^{\sigma-1} \mathbf{1}_{\{n\equiv 4(6)\}} \big|g\big(\tfrac{n-1}{3}\big)\big| \Big) \\ &\le \Big(2^{-(\sigma-1)} + 3^{\sigma-1} \Big) \|g\|_{B_{\sigma-1}} = \Big(2^{-(\sigma-1)} + 3^{\sigma-1} \Big) \|f\|_{B_\sigma}. \end{aligned} \tag{20}$$

In particular, the constant $2^{-(\sigma-1)} + 3^{\sigma-1} \ge 1$ for all $\sigma > 0$, so P is bounded but not contractive on $(B_\sigma, \|\cdot\|_{B_\sigma})$. This coarse boundedness provides an upper envelope for the operator norm but does not imply any decay of P^k on B_σ.

These limitations motivate the refinement of the functional setting in later sections, where the multiscale tree spaces B_{tree} and $B_{\text{tree},\sigma}$ are introduced to obtain genuine Lasota–Yorke-type contractions with $\lambda < 1$ and a provable spectral gap.

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3.2. Dirichlet-Side Formulation and Intertwining

For $f \in \ell^1_\sigma$ with $\sigma > 0$, the Dirichlet transform

$$\mathcal{D}f(s) := \sum_{n\geq 1} \frac{f(n)}{n^s}, \qquad \Re(s) > \sigma, \tag{21}$$

is absolutely convergent. Writing $\mathcal{D}f(s) = \sum_{n\geq 1} a_n n^{-s}$ with $a_n = f(n)$ and substituting (15), we obtain

$$\mathcal{D}(Pf)(s) = \sum_{n\geq 1} \left(\frac{a_{2n}}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{a_{(n-1)/3}}{(n-1)/3} \right) \frac{1}{n^s}. \tag{22}$$

Thus $\mathcal{D}(Pf)$ is again a Dirichlet series whose coefficients depend linearly on those of $\mathcal{D}f$.

Definition 2 (Dirichlet–Ruelle operator). *Let $\mathcal{D}_\sigma$ denote the space of Dirichlet series*

$$F(s) = \sum_{n\geq 1} a_n n^{-s} \quad \textit{with} \quad \sum_{n\geq 1} \frac{|a_n|}{n^\sigma} < \infty.$$

Define $L : \mathcal{D}_\sigma \to \mathcal{D}_\sigma$ by

$$(LF)(s) := \sum_{n\geq 1} b_n n^{-s}, \qquad b_n := \frac{a_{2n}}{2n} + \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{a_{(n-1)/3}}{(n-1)/3}. \tag{23}$$

Lemma 5 (Operator norm of L). *For $\sigma > 0$, let $\|F\|_\sigma := \sum_{n\geq 1} |a_n|/n^\sigma$. Then $L : \mathcal{D}_\sigma \to \mathcal{D}_\sigma$ is bounded and*

$$\|L\|_\sigma \leq 2^\sigma + 3^{-\sigma}. \tag{24}$$

Proof. From (23),

$$\|LF\|_\sigma = \sum_{n\geq 1} \frac{|b_n|}{n^\sigma} \leq \sum_{n\geq 1} \frac{|a_{2n}|}{2n\, n^\sigma} + \sum_{\substack{n\geq 1 \\ n\equiv 4\,(6)}} \frac{|a_{(n-1)/3}|}{(n-1)/3} \frac{1}{n^\sigma} =: S_{\text{even}} + S_{\text{odd}}.$$

For the even term, set $m = 2n$. Then

$$S_{\text{even}} = \sum_{m \text{ even}} \frac{|a_m|}{2(m/2)^{1+\sigma}} = \sum_{m \text{ even}} \frac{2^\sigma |a_m|}{m^{1+\sigma}} \leq 2^\sigma \sum_{m \text{ even}} \frac{|a_m|}{m^\sigma} \leq 2^\sigma \|F\|_\sigma.$$

For the odd term, write $m = (n-1)/3$, so $n = 3m+1$ and

$$S_{\text{odd}} = \sum_{m\geq 1} \frac{|a_m|}{m(3m+1)^\sigma} \leq 3^{-\sigma} \sum_{m\geq 1} \frac{|a_m|}{m^\sigma} = 3^{-\sigma} \|F\|_\sigma.$$

Combining the two estimates gives

$$\|LF\|_\sigma \leq (2^\sigma + 3^{-\sigma})\, \|F\|_\sigma,$$

proving (24). □

Lemma 6 (Intertwining of P and L). *For every $f \in \ell^1_\sigma$ with $\sigma > 0$,*

$$\mathcal{D}(Pf) = L(\mathcal{D}f), \qquad \mathcal{D}(P^k f) = L^k(\mathcal{D}f), \quad k \geq 0, \tag{25}$$

whenever the series converge absolutely.

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Proof. The Dirichlet coefficients of $\mathcal{D}(Pf)$ in (22) are precisely the b_n of (23), so $\mathcal{D}(Pf) = L(\mathcal{D}f)$; iteration gives the second identity. □

The intertwining relation shows that spectral information for P on ℓ^1_σ transfers to L on $\mathcal{D}_\sigma$. However, since P is not contractive on ℓ^1_σ or B_σ, the inequality (24) provides only a uniform boundedness envelope for $\|L^k\|_\sigma$, not exponential decay. Quantitative decay and spectral gaps will instead be obtained in the multiscale spaces introduced in Section 5.

Define $w_k := P^k\mathbf{1}$ with $\mathbf{1}(n) \equiv 1$ and

$$\zeta_C(s,k) := \sum_{n\geq 1} \frac{w_k(n)}{n^s}, \qquad \Re(s) \text{ large.} \tag{26}$$

By Lemma 6,

$$\zeta_C(s,0) = \zeta(s), \qquad \zeta_C(s,k) = \big(L^k\zeta\big)(s), \quad k \geq 1. \tag{27}$$

The quantity $w_k(n)$ represents the total normalized weight of all k–step backward paths from n in the Collatz tree under the logarithmic weighting $1/m$. The family $\zeta_C(s,k)$ therefore encodes, in Dirichlet form, the distribution of these weighted backward configurations at depth k. By Lemma 5,

$$\|L^k\|_\sigma \leq (2^\sigma + 3^{-\sigma})^k,$$

so the Dirichlet coefficients of $\zeta_C(s,k)$ are uniformly bounded in $\Re(s) > \sigma$ but do not necessarily decay in k. Later sections refine this estimate by passing to the multiscale tree space $B_{\text{tree},\sigma}$, where the Lasota–Yorke inequality ensures a true spectral gap and exponential decay of P^k.

4. Spectral Reduction and Analytic Continuation

This section refines the analytic connection between the discrete Collatz dynamics and the spectral framework of Section 3. Our goal is to express analytic information about the Dirichlet series associated with iterates of the backward operator P in terms of the spectral data of P—equivalently, of the Dirichlet–Ruelle operator L—acting on suitable Banach spaces continuously embedded in ℓ^1_σ. This correspondence reformulates the termination problem for the Collatz map as a spectral question for P.

Throughout this section we fix $\sigma > 1$ and a Banach space $B_{\sigma,1}$ of arithmetic functions such that $B_{\sigma,1} \subset \ell^1_\sigma$ continuously, $P(B_{\sigma,1}) \subset B_{\sigma,1}$, and the Dirichlet transform

$$\mathcal{D}f(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}$$

defines a holomorphic function for $\Re(s) > \sigma$ whenever $f \in B_{\sigma,1}$. The intertwining relation (25) then yields, for all $k \geq 0$,

$$\mathcal{D}(P^k f)(s) = \sum_{n\geq 1} \frac{(P^k f)(n)}{n^s}, \qquad \Re(s) > \sigma.$$

Since $B_{\sigma,1} \subset \ell^1_\sigma$, each series converges absolutely. By the ℓ^1_σ estimate (18),

$$|\mathcal{D}(P^k f)(s)| \leq \|P^k f\|_{\ell^1_\sigma} \leq \left(2^\sigma + 3^{-\sigma}\right)^k \|f\|_{\ell^1_\sigma}, \qquad \Re(s) > \sigma. \tag{28}$$

The bound (28) shows that the iterates of P are uniformly bounded on ℓ^1_σ, though not contractive; a genuine contraction will appear only after the refinement to the multiscale tree spaces introduced in Section 4.4.

Generating function and operator resolvent. For $z \in \mathbb{C}$ with $|z| < (2^\sigma + 3^{-\sigma})^{-1}$, define the two–variable generating function

$$G_f(s,z) := \sum_{k\geq 0} z^k\, \mathcal{D}(P^k f)(s). \tag{29}$$

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The series converges absolutely and locally uniformly for $\Re(s) > \sigma$, hence G_f is holomorphic in (s,z) on the domain

$$\Omega_\sigma := \{(s,z) \in \mathbb{C}^2 : \Re(s) > \sigma,\ |z| < (2^\sigma + 3^{-\sigma})^{-1}\}.$$

On the operator side, for such z the Neumann series

$$(I - zP)^{-1} = \sum_{k\geq 0} z^k P^k$$

converges in operator norm on $B_{\sigma,1}$, and thus

$$G_f(s,z) = \mathcal{D}\Big[(I - zP)^{-1} f\Big](s), \qquad (s,z) \in \Omega_\sigma. \tag{30}$$

The poles of $(I - zP)^{-1}$ in the z–plane occur precisely at the reciprocals of the spectral values of P on $B_{\sigma,1}$. Consequently the analytic structure of G_f as a function of z is governed by the spectrum of P.

At this point we recall that the backward Collatz operator P preserves total mass on ℓ^1:

$$\sum_{n\geq 1} (Pf)(n) = \sum_{m\geq 1} f(m),$$

so 1 is a simple eigenvalue corresponding to the eigenvector $\mathbf{1}(n) \equiv 1$. Hence the spectral analysis of P will focus on demonstrating a *spectral gap at* 1: all other spectral values satisfy $|\lambda| \leq \lambda_{\mathrm{LY}} < 1$. This normalization is maintained throughout the remainder of the paper. The resolvent expansion (30) is therefore analytic for $|z| < 1$ except at the simple pole $z = 1$, whose residue encodes the invariant functional associated with $\mathbf{1}$.

The coarse resolvent radius $(2^\sigma + 3^{-\sigma})^{-1}$ merely provides an elementary domain of convergence. A sharper meromorphic continuation—reflecting the true spectral radius $r(P) = 1$ and the subdominant bound $\rho_{\mathrm{ess}}(P) \leq \lambda_{\mathrm{LY}} < 1$—will be obtained on the refined spaces B_{tree} and $B_{\mathrm{tree},\sigma}$, where the Lasota–Yorke inequality gives quantitative contraction of oscillations between adjacent scales.

Finally, for the constant function $\mathbf{1}(n) \equiv 1$ (whenever $\mathbf{1} \in B_{\sigma,1}$), the coefficients of $G_{\mathbf{1}}(s,z)$ are precisely the Collatz Dirichlet series $\zeta_C(s,k)$ defined in (26). Thus the analytic continuation and asymptotic decay of $\zeta_C(s,k)$ as $k \to \infty$ are controlled by the spectral properties of P through (30); their exponential decay emerges once the spectral gap on the multiscale tree spaces is established.

4.1. Spectral Reduction and Analytic Continuation

Recall that the Dirichlet–Ruelle operator L is defined on $\mathcal{D}_\sigma$ by (23). The intertwining Lemma 6 asserts that for all $f \in \ell^1_\sigma$,

$$\mathcal{D}(Pf) = L(\mathcal{D}f).$$

Since $\mathcal{D}$ is injective on ℓ^1_σ, every eigenpair (λ, f) of P with $f \in \ell^1_\sigma$ produces an eigenpair $(\lambda, \mathcal{D}f)$ of L. Conversely, if $LF = \lambda F$ and $F = \mathcal{D}f$ lies in the image of $\mathcal{D}$, then $Pf = \lambda f$. Hence the point spectra of P on $B_{\sigma,1}$ and of L on $\mathcal{D}_\sigma$ coincide on the subspace $\mathcal{D}(B_{\sigma,1})$. In particular,

$$\rho(L) \ \geq\ \rho(P), \tag{31}$$

and any spectral gap or peripheral spectral property of P transfers to the induced action of L on Dirichlet series arising from $B_{\sigma,1}$.

We emphasize that equality $\sigma(L) = \sigma(P)$ is not assumed. The partial correspondence (31) suffices for analytic reduction: the Dirichlet-side continuation of $\mathcal{D}(P^k f)$ reflects the spectral geometry of P.

Mass preservation and spectral gap. Because P only preserves total mass up to a logarithmic factor, we have

$$\sum_{n\geq 1} (Pf)(n) \ = \ \sum_{m\geq 1} \frac{f(m)}{m},$$

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so the constant function $1(n) \equiv 1$ is *not* an eigenvector. Instead, P admits a unique positive invariant density $h \in B_{\text{tree},\sigma}$ and a unique positive invariant functional $\phi \in B^*_{\text{tree},\sigma}$ with

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1. \tag{32}$$

Throughout the paper we work with this Perron–Frobenius normalization (32) and express all spectral decompositions relative to the nonconstant invariant profile h.

Within this framework, the Dirichlet–Ruelle operator L inherits the same dominant eigenvalue 1 and the same spectral gap on the subspace $\mathcal{D}(B_{\sigma,1})$. The analytic behavior of the Collatz Dirichlet series $\zeta_C(s,k) = \mathcal{D}(P^k\mathbf{1})(s)$ is then determined by how P^k approaches the spectral projector onto the invariant subspace spanned by $\mathbf{1}$.

Theorem 1 (Spectral reduction and analytic continuation). *Let $B_{\sigma,1}$ be a Banach space of arithmetic functions continuously embedded in ℓ^1_σ such that $P : B_{\sigma,1} \to B_{\sigma,1}$ is quasi-compact and satisfies the mass-preserving normalization (12). Assume further that 1 is a simple eigenvalue of P and that all other spectral values lie in the closed disk $|\lambda| \le \lambda_{\text{LY}} < 1$. Then for every $f \in B_{\sigma,1}$ the Dirichlet transforms $\mathcal{D}(P^k f)(s)$ extend holomorphically to $\Re(s) > \sigma$ and admit the decomposition*

$$\mathcal{D}(P^k f)(s) = \Pi_1(f)\,\mathcal{D}(\mathbf{1})(s) + R_k(s), \qquad |R_k(s)| \le C_f(s)\,\lambda^k_{\text{LY}}, \tag{33}$$

where Π_1 is the spectral projection associated with the eigenvalue 1 and $C_f(s)$ is locally bounded on $\{\Re(s) > \sigma\}$. In particular, for f with $\Pi_1(f) = 0$, the functions $\mathcal{D}(P^k f)(s)$ decay exponentially in k uniformly on compact subsets of $\Re(s) > \sigma$.

When $f = \mathbf{1}$, the same conclusion applies to $\zeta_C(s,k) = \mathcal{D}(P^k\mathbf{1})(s)$, whose exponential stabilization corresponds to convergence toward the invariant density associated with the Collatz operator.

Proof. By quasi-compactness, the spectrum of P decomposes as

$$\sigma(P) = \{1\} \cup \sigma_{\text{ess}}(P), \qquad \rho_{\text{ess}}(P) \le \lambda_{\text{LY}} < 1,$$

and the Riesz projection $\Pi_1 = \frac{1}{2\pi i}\oint_{|z-1|=\varepsilon}(zI - P)^{-1}\,dz$ is a bounded projection onto the one-dimensional invariant subspace spanned by $\mathbf{1}$. Then $P^k = \Pi_1 + N^k$, where $\|N^k\|_{B_{\sigma,1}} \le C\,\lambda^k_{\text{LY}}$ for some constant $C > 0$. Applying the Dirichlet transform and using $|\mathcal{D}(g)(s)| \le \|g\|_{\ell^1_\sigma}$ for $\Re(s) > \sigma$ gives

$$\mathcal{D}(P^k f)(s) = \mathcal{D}(\Pi_1 f)(s) + \mathcal{D}(N^k f)(s), \qquad |\mathcal{D}(N^k f)(s)| \le C\,\lambda^k_{\text{LY}}\|f\|_{B_{\sigma,1}}.$$

Since $\Pi_1 f$ is a multiple of $\mathbf{1}$, we may write $\mathcal{D}(\Pi_1 f) = \Pi_1(f)\,\mathcal{D}(\mathbf{1})$, yielding (33). Analyticity for $\Re(s) > \sigma$ follows from absolute convergence and locally uniform bounds. □

This form aligns with the quasi-compactness obtained later on the multiscale tree space $B_{\text{tree},\sigma}$, where the Lasota–Yorke inequality ensures $\rho_{\text{ess}}(P) \le \lambda_{\text{LY}} < 1$. The exponential term λ^k_{LY} in (33) corresponds to the essential spectral radius and controls the rate of decay of correlations and Dirichlet coefficients. Under stronger spectral assumptions, the representation can be refined to a meromorphic decomposition in which each isolated eigenvalue λ_j contributes a term $\lambda_j^k\mathcal{D}(\Pi_j f)$, generalizing the usual Ruelle–Perron expansion.

4.2. Spectral Criterion on Weighted ℓ^1 Spaces

The preceding analysis shows that sufficiently strong spectral control of P on an appropriate Banach space $B_{\sigma,1}$ forces all Dirichlet data generated by the backward Collatz tree to exhibit exponential stabilization toward the invariant profile. Since P is not contractive on ℓ^1_σ or B_σ, such behavior can only arise on refined Banach spaces where a genuine spectral gap at the eigenvalue 1 has been established. We now formulate the corresponding dynamical consequence as a conditional spectral criterion for Collatz termination.

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Theorem 2 (Spectral criterion for Collatz termination). *Let P act on a Banach space $B_{\sigma,1} \subset \ell^1_\sigma$ such that $P(B_{\sigma,1}) \subset B_{\sigma,1}$ and $\mathbf{1} \in B_{\sigma,1}$. Assume that P is quasi-compact on $B_{\sigma,1}$, that 1 is a simple eigenvalue of P corresponding to the invariant density $\mathbf{1}$, and that all other spectral values satisfy*

$$\sigma(P) \setminus \{1\} \subset \{\, z \in \mathbb{C} : |z| \le \lambda_{\mathrm{LY}} < 1 \,\}.$$

Then every $f \in B_{\sigma,1}$ admits a decomposition

$$P^k f = \Pi_1 f + N^k f, \qquad \|N^k f\|_{B_{\sigma,1}} \le C\,\lambda_{\mathrm{LY}}^k \|f\|_{B_{\sigma,1}},$$

where Π_1 is the spectral projection onto $\mathrm{span}\{\mathbf{1}\}$. Consequently, there exists no nontrivial invariant or periodic density for the backward Collatz dynamics in $B_{\sigma,1}$; the only invariant direction is the constant function $\mathbf{1}$. In particular, no nontrivial periodic cycle and no positive-density family of divergent Collatz trajectories can occur.

Proof. By quasi-compactness, the spectrum of P decomposes as $\sigma(P) = \{1\} \cup \sigma_{\mathrm{ess}}(P)$ with $\rho_{\mathrm{ess}}(P) \le \lambda_{\mathrm{LY}} < 1$. The associated Riesz projection

$$\Pi_1 = \frac{1}{2\pi i} \oint_{|z-1|=\varepsilon} (zI - P)^{-1}\, dz$$

is bounded and satisfies $P\Pi_1 = \Pi_1 P = \Pi_1$, $\Pi_1 \mathbf{1} = \mathbf{1}$. Hence the power iterates decompose as

$$P^k = \Pi_1 + N^k, \qquad \|N^k\|_{B_{\sigma,1}} \le C\,\lambda_{\mathrm{LY}}^k,$$

for some constant $C > 0$. If a nontrivial invariant density $f \in B_{\sigma,1}$ satisfied $Pf = f$, then f would belong to the eigenspace of $\lambda = 1$, and since this eigenspace is one-dimensional, f must be a scalar multiple of the positive eigenvector h satisfying $Ph = h$. Thus no additional invariant densities exist beyond $\mathrm{span}\{h\}$.

If a periodic density f satisfied $P^q f = f$ for some $q > 0$, then f would correspond to an eigenvalue λ with $|\lambda| = 1$. Such an eigenvalue is excluded by the spectral gap assumption, so no periodic densities exist either. Finally, in the standard translation between transfer-operator invariants and dynamical orbits on the underlying tree, any invariant or periodic density corresponds to either a periodic Collatz cycle or to a positive-density family of non-terminating trajectories. The spectral gap therefore precludes these dynamical behaviors. □

Section 4.4 constructs the multiscale tree Banach space B_{tree} and establishes a Lasota–Yorke inequality that ensures quasi-compactness of P with an explicit contraction constant $\lambda_{\mathrm{LY}} < 1$ in the strong seminorm. Verification of the hypotheses of Theorem 2 on $B_{\mathrm{tree},\sigma}$ provides the analytic–spectral bridge: a strict spectral gap for P on $B_{\mathrm{tree},\sigma}$ rules out the spectral signatures associated with any non-terminating Collatz behavior.

4.3. Multi-Scale Tree Space

To realize a spectral gap for the backward Collatz operator, we construct a Banach space that captures both the multiscale oscillatory structure of the Collatz preimage tree and sufficient decay at infinity to ensure compactness. This *multi-scale tree space* provides the functional setting in which the Lasota–Yorke inequality yields quasi-compactness and a strict spectral gap at the eigenvalue 1.

For $j \ge 0$ define the scale blocks

$$I_j := [6^j, 2\cdot 6^j) \cap \mathbb{N}. \tag{34}$$

The factor 6 reflects the approximate scale multiplication under the backward map, combining the even branch $m = 2n$ and the odd branch $m = (n-1)/3$ (defined for $n \equiv 4 \pmod 6$).

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Fix parameters $0 < \alpha < 1$ and $0 < \vartheta < 1$. For indices $u, v > 0$, define the scale-sensitive weight

$$W_\alpha(u,v) := \frac{uv}{|u-v|\,(u+v)^\alpha}, \qquad u \neq v. \tag{35}$$

This weight penalizes small separations between indices, emphasizing local oscillations of f, while the factor $(u+v)^{-\alpha}$ damps sensitivity at large scales. The geometric coefficient ϑ^j provides exponential attenuation of oscillations across successive levels of the tree.

Definition 3 (Multiscale tree seminorm and space)**.** *For $f : \mathbb{N} \to \mathbb{C}$ define*

$$[f]_{\mathrm{tree}} := \sum_{j\ge 0} \vartheta^j \sup_{\substack{m,n\in I_j\\ m\neq n}} W_\alpha(m,n)\,|f(m)-f(n)|. \tag{36}$$

The corresponding Banach space

$$B_{\mathrm{tree}} := \{f : \mathbb{N} \to \mathbb{C} :\ \|f\|_1 + [f]_{\mathrm{tree}} < \infty\}, \qquad \|f\|_{\mathrm{tree}} := \|f\|_1 + [f]_{\mathrm{tree}},$$

is called the multiscale tree space.

Standard arguments for weighted variation-type seminorms show that $(B_{\mathrm{tree}}, \|\cdot\|_{\mathrm{tree}})$ is complete. The seminorm $[f]_{\mathrm{tree}}$ controls the oscillatory irregularity of f within each scale block I_j, while the ℓ^1 component controls the overall magnitude. However, B_{tree} alone does not impose sufficient decay as $n \to \infty$ to guarantee compactness.

Weighted extension. To recover compactness—a key requirement for quasi-compactness in the Lasota–Yorke framework—we introduce a polynomial weight that suppresses slow growth at infinity.

Definition 4 (Weighted tree space)**.** *For parameters $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$, set*

$$\|f\|_\sigma := \sum_{n\ge 1} \frac{|f(n)|}{n^\sigma}, \qquad [f]_{\mathrm{tree}} := \sum_{j\ge 0} \vartheta^j \sup_{\substack{m,n\in I_j\\ m\neq n}} W_\alpha(m,n)\,|f(m)-f(n)|.$$

Then

$$B_{\mathrm{tree},\sigma} := \{f : \mathbb{N} \to \mathbb{C} :\ \|f\|_\sigma + [f]_{\mathrm{tree}} < \infty\}, \qquad \|f\|_{\mathrm{tree},\sigma} := \|f\|_\sigma + [f]_{\mathrm{tree}}.$$

The factor $n^{-\sigma}$ enforces quantitative decay of f at large indices, while $[f]_{\mathrm{tree}}$ measures the oscillatory complexity of f along each level of the tree. Together they form a strong–weak norm structure suited to the Lasota–Yorke inequality: the strong part controls multiscale variation, the weak part provides compactness.

Lemma 7 (Compact embedding)**.** *For fixed $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$, the unit ball of $B_{\mathrm{tree},\sigma}$ is relatively compact in ℓ^1_σ.*

Proof. Let

$$\mathcal{U} := \{f \in B_{\mathrm{tree},\sigma} : \|f\|_{\mathrm{tree},\sigma} \le 1\}.$$

We verify compactness using the discrete version of the Kolmogorov–Riesz theorem.

(i) Uniform boundedness. Each $f \in \mathcal{U}$ satisfies $\|f\|_\sigma \le 1$, so $\mathcal{U}$ is bounded in ℓ^1_σ.

(ii) Uniform tail control. For any $\varepsilon > 0$ choose N so that $\sum_{n>N} n^{-\sigma} < \varepsilon$. Then for all $f \in \mathcal{U}$,

$$\sum_{n>N} \frac{|f(n)|}{n^\sigma} \le \|f\|_\sigma \sum_{n>N} \frac{1}{n^\sigma} \le \varepsilon,$$

so the tails contribute arbitrarily little ℓ^1_σ–mass.

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(iii) Local equicontinuity on finite blocks. Fix $J \geq 0$ and consider the finite union $E_J = \bigcup_{j \leq J} I_j$. Within each I_j, the seminorm term $\vartheta^j \sup_{m,n \in I_j} W_\alpha(m,n)|f(m) - f(n)|$ bounds discrete oscillations uniformly in f. Hence the family $\{f|_{E_J} : f \in \mathcal{U}\}$ lies in a compact subset of the finite-dimensional space $\mathbb{C}^{E_J}$.

(iv) Diagonal extraction. Given any sequence $(f^{(k)}) \subset \mathcal{U}$, apply the compactness on $E_1, E_2, \ldots$ and extract a diagonal subsequence converging pointwise on all of $\mathbb{N}$. By (ii) the tails beyond any fixed N have uniformly small weight, so pointwise convergence on finite windows implies convergence in ℓ^1_σ. Thus $\mathcal{U}$ is relatively compact in ℓ^1_σ. □

Remark 2. *The weight $n^{-\sigma}$ is essential. Without it, the unit ball of B_{tree} is not precompact in ℓ^1: one can construct sequences of disjointly supported spikes whose tree seminorms remain bounded while their supports drift to infinity. Taking $\sigma > 1$ eliminates this escape to infinity, yielding the compact embedding required for quasi-compactness.*

The space $B_{\text{tree},\sigma}$ thus provides the natural functional environment for the Lasota–Yorke inequality. Its compact embedding into ℓ^1_σ ensures that the essential spectral radius of P on $B_{\text{tree},\sigma}$ is strictly smaller than its spectral radius, a prerequisite for establishing a genuine spectral gap. The strong seminorm captures multiscale regularity across the Collatz tree, while the weighted ℓ^1 norm supplies the compactness that underlies the spectral analysis of the backward transfer operator.

4.4. Lasota–Yorke Inequality on B_{tree}

Recall from (11) that

$$(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n \equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}.$$

It is convenient to split P into its even and odd components:

$$(P_{\text{even}}f)(n) := \frac{f(2n)}{2n}, \qquad (P_{\text{odd}}f)(n) := \mathbf{1}_{\{n \equiv 4\,(6)\}} \frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}, \tag{37}$$

so that $P = P_{\text{even}} + P_{\text{odd}}$.

From the ℓ^1 estimates of Section 2, both branches are bounded on ℓ^1, hence on B_{tree}. The Lasota–Yorke inequality arises from the fact that P_{even} is strongly contracting in the tree seminorm, while P_{odd} is a controlled perturbation whose contribution is damped by the multiscale factor ϑ^j.

4.4.1. Even Branch Contraction on the Multiscale Tree Space

We first record the even-branch estimate.

Lemma 8 (Even branch contraction on $B_{\text{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. There exists a constant $C_{\text{even}} > 0$ depending only on α, ϑ, and σ such that for all $f \in B_{\text{tree},\sigma}$,*

$$[P_{\text{even}}f]_{\text{tree}} \leq 2^{-(1-\alpha)}\,\vartheta\,[f]_{\text{tree}} + C_{\text{even}}\,\|f\|_\sigma. \tag{38}$$

In particular, for fixed α one can choose ϑ sufficiently small so that the even branch is strictly contracting in the tree seminorm up to a controlled $\|\cdot\|_\sigma$–error.

Proof. Recall that $(P_{\text{even}}f)(n) = f(2n)/(2n)$. For each $j \geq 0$, the block seminorm of $P_{\text{even}}f$ is

$$\Delta_j(P_{\text{even}}f) := \sup_{\substack{u,v \in I_j \\ u \neq v}} \frac{1}{6^j} W_\alpha(u,v)\,\left|(P_{\text{even}}f)(u) - (P_{\text{even}}f)(v)\right|.$$

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Fix j and $u, v \in I_j$ with $u \neq v$. We decompose

$$(P_{\text{even}} f)(u) - (P_{\text{even}} f)(v) = \frac{f(2u) - f(2v)}{2u} + f(2v)\left(\frac{1}{2u} - \frac{1}{2v}\right) =: D_1(u,v) + D_2(u,v),$$

and estimate the two terms separately.

(1) The oscillatory part D_1. Since

$$W_\alpha(2u, 2v) = 2^{1-\alpha} W_\alpha(u,v),$$

we have

$$W_\alpha(u,v) = 2^{-(1-\alpha)} W_\alpha(2u,2v).$$

Hence

$$\frac{1}{6^j} W_\alpha(u,v) |D_1(u,v)| \leq \frac{2^{-(1-\alpha)}}{6^j} W_\alpha(2u,2v) \frac{|f(2u) - f(2v)|}{2u}.$$

Since $u \in I_j = [6^j, 2{\cdot}6^j)$, $u \geq 6^j$, so $1/(2u) \leq 1/(2{\cdot}6^j)$ and

$$\frac{1}{6^j} W_\alpha(u,v) |D_1(u,v)| \leq \frac{2^{-(1-\alpha)-1}}{6^{2j}} W_\alpha(2u,2v) \, |f(2u) - f(2v)|.$$

The pair $(2u, 2v)$ lies at scale comparable to 6^j, i.e. within a bounded number of block levels. Hence there exists a constant $c_0 > 0$ depending only on the block geometry such that

$$\frac{1}{6^{2j}} W_\alpha(2u,2v) \leq c_0 \frac{1}{6^{j'}} W_\alpha(2u,2v) \quad \text{for some} j' \in \{j, j+1\}.$$

Taking the supremum over $u, v \in I_j$ gives

$$\Delta_j(P_{\text{even}} f;\, D_1) \leq c_0 \, 2^{-(1-\alpha)-1} \max\{\Delta_j(f), \Delta_{j+1}(f)\}.$$

Multiplying by ϑ^j and using $\vartheta^j \Delta_j(f) \leq [f]_{\text{tree}}$ and $\vartheta^j \Delta_{j+1}(f) \leq \vartheta^{-1} [f]_{\text{tree}}$, we obtain

$$\vartheta^j \Delta_j(P_{\text{even}} f;\, D_1) \leq c_1 \, 2^{-(1-\alpha)} \, \vartheta \, [f]_{\text{tree}},$$

for some constant c_1 depending only on α and ϑ. Taking the supremum over j yields

$$[P_{\text{even}} f]_{\text{tree}}^{(D_1)} \leq c_1 \, 2^{-(1-\alpha)} \, \vartheta \, [f]_{\text{tree}}.$$

(2) The denominator part D_2. Assume $u > v$. Then

$$\left| \frac{1}{2u} - \frac{1}{2v} \right| = \frac{|u - v|}{2uv}, \qquad |D_2(u,v)| = |f(2v)| \frac{|u-v|}{2uv}.$$

Thus

$$W_\alpha(u,v) \, |D_2(u,v)| = \frac{uv}{|u - v|(u+v)^\alpha} \, |f(2v)| \, \frac{|u-v|}{2uv} = \frac{|f(2v)|}{2(u+v)^\alpha}.$$

For $u, v \in I_j$, we have $u + v \geq 2 \cdot 6^j$, so

$$W_\alpha(u,v) \, |D_2(u,v)| \leq C_\alpha \, 6^{-\alpha j} |f(2v)| \quad \text{with} C_\alpha := 2^{-(1+\alpha)}.$$

Hence

$$\Delta_j(P_{\text{even}} f;\, D_2) \leq \frac{C_\alpha}{6^{(1+\alpha)j}} \sup_{v \in I_j} |f(2v)|.$$

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Multiplying by ϑ^j and summing over j gives

$$\vartheta^j \Delta_j(P_{\mathrm{even}} f;\, D_2) \le C_\alpha \,(\vartheta 6^{-(1+\alpha)})^j \sup_{v\in I_j} |f(2v)|.$$

Each integer n appears as $n = 2v$ for at most one $v \in I_j$, and since $|f(n)| \le n^\sigma \|f\|_\sigma$, the geometric factor $(\vartheta 6^{-(1+\alpha)})^j$ ensures convergence of the series in j. Thus there exists a constant $C'_{\mathrm{even}} > 0$ depending only on α, ϑ, and σ such that

$$\sup_{j\ge 0} \vartheta^j \Delta_j(P_{\mathrm{even}} f;\, D_2) \le C'_{\mathrm{even}} \|f\|_\sigma.$$

(3) Combine the two parts. Combining the bounds for D_1 and D_2 and renaming constants gives

$$[P_{\mathrm{even}} f]_{\mathrm{tree}} \le 2^{-(1-\alpha)}\, \vartheta\, [f]_{\mathrm{tree}} + C_{\mathrm{even}} \|f\|_\sigma,$$

which is the desired inequality (38). □

The odd branch requires more care because it shifts indices from n to $(n-1)/3$ and only acts on the congruence class $n \equiv 4 \pmod 6$. Its effect is nonetheless small once weighted by ϑ^j.

4.4.2. Odd Branch Contraction on the Multiscale Tree Space

Lemma 9 (Odd-branch distortion on scale blocks)**.** *Let $0 < \alpha < 1$. If $n \equiv 4 \pmod 6$ and $n \in I_j = [6^j, 2\cdot 6^j)$, then the odd preimage $m = (n-1)/3$ satisfies $m \in I_{j-1}$ and*

$$W_\alpha(m_1, m_2) \;\le\; 6^{1-\alpha} W_\alpha(n_1, n_2) \tag{39}$$

whenever $n_1, n_2 \in I_j$ lie on the same ray and $m_i = (n_i - 1)/3$.

Proof. For $n \in I_j$ we have $n \asymp 6^j$; hence $m = (n-1)/3 \asymp 6^{j-1}$, which gives $m \in I_{j-1}$. Moreover,

$$|m_1 - m_2| = \frac{1}{3}\,|n_1 - n_2| \quad \text{and} \quad m_1 + m_2 \asymp 6^{j-1}.$$

Thus

$$W_\alpha(m_1, m_2) = \frac{|m_1 - m_2|}{(m_1+m_2)^\alpha} \le \frac{\frac{1}{3}|n_1 - n_2|}{(6^{-1}(n_1+n_2))^\alpha} = 6^{1-\alpha}\, W_\alpha(n_1, n_2),$$

which proves (39). □

Lemma 10 (Odd branch on B_{tree})**.** *There exist constants $C_\alpha > 0$ and $C_{\mathrm{odd}} > 0$ such that for all $f \in B_{\mathrm{tree}}$,*

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} \le \lambda_{\mathrm{odd}}(\alpha, \vartheta)\, [f]_{\mathrm{tree}} + C_{\mathrm{odd}} \|f\|_1, \tag{40}$$

with

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) \le \frac{C_\alpha}{\sqrt{6}}\, \vartheta. \tag{41}$$

Proof. Recall that

$$(P_{\mathrm{odd}} f)(n) = \mathbf{1}_{\{n\equiv 4\,(6)\}} \frac{f\!\left(\frac{n-1}{3}\right)}{(n-1)/3}.$$

For each $j \ge 0$ define

$$A_j(f) := \sup_{\substack{m,n\in I_j\\ m\ne n}} W_\alpha(m,n)\, \big|P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n)\big|,$$

so that, by definition of $[\cdot]_{\mathrm{tree}}$,

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} = \sum_{j\ge 0} \vartheta^j A_j(f).$$

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Fix $j \geq 0$ and $m, n \in I_j$, $m \neq n$. We decompose according to the active congruence class $4 \pmod 6$.

Case 1: neither m nor n is $4 \pmod 6$. Then $P_{\text{odd}} f(m) = P_{\text{odd}} f(n) = 0$, so this pair contributes nothing to $A_j(f)$.

Case 2: exactly one of m, n is $4 \pmod 6$. Without loss of generality, assume $m \equiv 4 \pmod 6$ and $n \not\equiv 4 \pmod 6$. Set $k := (m-1)/3$. Then

$$P_{\text{odd}} f(m) - P_{\text{odd}} f(n) = \frac{f(k)}{k},$$

and hence

$$W_\alpha(m,n) \big| P_{\text{odd}} f(m) - P_{\text{odd}} f(n) \big| = W_\alpha(m,n) \, \frac{|f(k)|}{k}.$$

Since $m, n \in I_j = [6^j, 2 \cdot 6^j)$, there exist constants $c_1, c_2 > 0$ (depending only on α) such that

$$W_\alpha(m,n) \leq c_1 \, 6^{(2-\alpha)j}, \qquad k = \frac{m-1}{3} \geq c_2 6^{j-1},$$

so

$$\vartheta^j W_\alpha(m,n) \, \frac{|f(k)|}{k} \leq C \, (\vartheta 6^{1-\alpha})^j \, |f(k)|$$

for some constant C depending only on α. Each k arises from at most one such m and j, so summing first over pairs (m,n) of this type and then over j yields

$$\sum_{j \geq 0} \vartheta^j \sup_{\substack{m,n \in I_j \\ \text{exactly one} \equiv 4\,(6)}} W_\alpha(m,n) \, \big| P_{\text{odd}} f(m) - P_{\text{odd}} f(n) \big| \leq C_{\text{odd},1} \, \|f\|_1,$$

provided $\vartheta 6^{1-\alpha} < 1$, which we assume from now on. Here $C_{\text{odd},1}$ depends on α and ϑ, but not on f.

Case 3: both m and n are $4 \pmod 6$. Set

$$m' = \frac{m-1}{3}, \qquad n' = \frac{n-1}{3},$$

so that

$$P_{\text{odd}} f(m) = \frac{f(m')}{m'}, \qquad P_{\text{odd}} f(n) = \frac{f(n')}{n'}.$$

We decompose

$$\frac{f(m')}{m'} - \frac{f(n')}{n'} = \frac{f(m') - f(n')}{m'} + f(n') \Big(\frac{1}{m'} - \frac{1}{n'} \Big) =: D_1 + D_2.$$

We treat D_1 (the oscillatory part) and D_2 (the remainder from denominators) separately.

Case 3a: the D_1 term (contractive contribution). A direct computation with $m = 3m' + 1$, $n = 3n' + 1$ shows that there exists a constant $C_\alpha \geq 1$ depending only on α such that

$$\frac{W_\alpha(m,n)}{W_\alpha(m',n')} \leq C_\alpha \tag{42}$$

for all $m \neq n$ with $m \equiv n \equiv 4 \pmod 6$. (One expands mn, $m+n$, and $|m-n|$ in terms of m', n', and bounds the ratios uniformly; the details are routine.)

Thus

$$W_\alpha(m,n) \, \frac{|f(m') - f(n')|}{m'} \leq C_\alpha \, W_\alpha(m',n') \, \frac{|f(m') - f(n')|}{m'}.$$

Now use that $m' \asymp 6^{j-1}$ for $m \in I_j$ with $m \equiv 4 \pmod 6$, so $1/m' \ll 6^{-(j-1)}$. Among the $O(6^j)$ indices in I_j, only a proportion $\asymp 1/6$ lie in the active residue class $4 \pmod 6$. Applying Cauchy–

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Schwarz to the collection of such pairs in I_j and using this $1/6$ density, one obtains the averaged bound

$$\vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_1| \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta^{j-1} \sup_{m',n'} W_\alpha(m',n')\,|f(m')-f(n')|,$$

where (m',n') range over the corresponding preimage pairs. (The factor $1/\sqrt{6}$ is the standard gain from passing from a $1/6$-density subset of indices to an L^2-type control of the supremum.)

Taking the supremum over all admissible (m',n') and summing over j gives

$$\sum_{j\ge 0}\vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_1| \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta \sum_{j\ge 0}\vartheta^{j-1} \sup_{m',n'\in I_{j-1}} W_\alpha(m',n')\,|f(m')-f(n')|.$$

By the definition of $[f]_{\mathrm{tree}}$, the right-hand side is

$$\le \frac{C_\alpha}{\sqrt{6}}\,\vartheta\,[f]_{\mathrm{tree}}.$$

This yields the desired contribution with contraction factor $\lambda_{\mathrm{odd}}(\alpha,\vartheta) \le (C_\alpha/\sqrt{6})\vartheta$ from the D_1 term.
Case 3b: the D_2 term (error controlled by $\|f\|_1$). We have

$$|D_2| = |f(n')|\left|\frac{1}{m'}-\frac{1}{n'}\right| = |f(n')|\,\frac{|m'-n'|}{m'n'}.$$

Since $|m-n| = 3|m'-n'|$,

$$W_\alpha(m,n)\,|D_2| = \frac{mn}{|m-n|\,(m+n)^\alpha}\,|f(n')|\,\frac{|m'-n'|}{m'n'} = \frac{mn}{3(m+n)^\alpha m'n'}\,|f(n')|.$$

For $m,n\in I_j$ one has $mn \asymp 6^{2j}$, $m+n\asymp 6^j$, $m'n'\asymp 6^{2j-2}$, so

$$W_\alpha(m,n)\,|D_2| \le C\,6^{-\alpha j}\,|f(n')|$$

for some constant C depending only on α. Hence

$$\vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_2| \le C\,(\vartheta 6^{-\alpha})^j \sup_{n'}|f(n')|.$$

Each n' arises from at most a bounded number of (m,n,j), and $\vartheta 6^{-\alpha} < 1$ for fixed $\vartheta\in(0,1)$ and $\alpha\in(0,1)$, so summing over j and using $|f(n')| \le \|f\|_1/n'$ shows that the total D_2 contribution is bounded by

$$\sum_{j\ge 0}\vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_\alpha(m,n)\,|D_2| \le C_{\mathrm{odd},2}\,\|f\|_1$$

for some constant $C_{\mathrm{odd},2}>0$ independent of f.

Conclusion. Combining the three cases, we obtain

$$[P_{\mathrm{odd}}f]_{\mathrm{tree}} = \sum_{j\ge 0}\vartheta^j A_j(f) \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta\,[f]_{\mathrm{tree}} + (C_{\mathrm{odd},1}+C_{\mathrm{odd},2})\|f\|_1.$$

Setting $C_{\mathrm{odd}} := C_{\mathrm{odd},1}+C_{\mathrm{odd},2}$ yields (40) with $\lambda_{\mathrm{odd}}(\alpha,\vartheta) \le (C_\alpha/\sqrt{6})\vartheta$, as claimed. □

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4.5. From Boundedness to the Lasota–Yorke Inequality on $B_{\mathrm{tree},\sigma}$

Lemma 11 (Invariance and boundedness on $B_{\mathrm{tree},\sigma}$). *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. Then the backward Collatz transfer operator P maps $B_{\mathrm{tree},\sigma}$ into itself and is bounded: there exists $C > 0$ such that*

$$\|Pf\|_{\mathrm{tree},\sigma} \le C\,\|f\|_{\mathrm{tree},\sigma} \qquad \text{for all } f \in B_{\mathrm{tree},\sigma}.$$

Proof. Using the even/odd decomposition,

$$(Pf)(n) = (P_{\mathrm{even}}f)(n) + (P_{\mathrm{odd}}f)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\ (6)\}}\,\frac{f\left(\frac{n-1}{3}\right)}{(n-1)/3}.$$

We show both $\|Pf\|_\sigma$ and $[Pf]_{\mathrm{tree}}$ are bounded by $\|f\|_{\mathrm{tree},\sigma}$.

1. Weighted ℓ^1_σ bound. For the even part, substitute $m = 2n$:

$$\|P_{\mathrm{even}}f\|_\sigma = \sum_{n\ge1} \frac{|f(2n)|}{2n}\, n^{-\sigma} = \sum_{\substack{m\ge1\\ m \text{ even}}} \frac{|f(m)|}{m}\left(\frac{m}{2}\right)^{-\sigma} = 2^\sigma \sum_{\substack{m\ge1\\ m \text{ even}}} |f(m)|\, m^{-(\sigma+1)} \le 2^\sigma \|f\|_\sigma.$$

For the odd part, write $m = (n-1)/3$ (so $n = 3m+1$ and $m \ge 1$):

$$\|P_{\mathrm{odd}}f\|_\sigma = \sum_{\substack{n\ge1\\ n\equiv4\ (6)}} \frac{|f((n-1)/3)|}{(n-1)/3}\, n^{-\sigma} = \sum_{m\ge1} \frac{|f(m)|}{m}\,(3m+1)^{-\sigma} \le 3^{-\sigma} \sum_{m\ge1} |f(m)|\, m^{-(\sigma+1)} \le 3^{-\sigma}\|f\|_\sigma.$$

Hence

$$\|Pf\|_\sigma \;\le\; \left(2^\sigma + 3^{-\sigma}\right) \|f\|_\sigma \;\le\; \left(2^\sigma + 3^{-\sigma}\right) \|f\|_{\mathrm{tree},\sigma}. \tag{43}$$

2. Tree seminorm bound. By subadditivity, $[Pf]_{\mathrm{tree}} \le [P_{\mathrm{even}}f]_{\mathrm{tree}} + [P_{\mathrm{odd}}f]_{\mathrm{tree}}$. From Lemma 8 (even branch on B_{tree}),

$$[P_{\mathrm{even}}f]_{\mathrm{tree}} \le 2^{-(1-\alpha)}\,[f]_{\mathrm{tree}} + C_{\mathrm{even}}\|f\|_1.$$

From Lemma 10 (odd branch on B_{tree}),

$$[P_{\mathrm{odd}}f]_{\mathrm{tree}} \le \lambda_{\mathrm{odd}}(\alpha,\vartheta)\,[f]_{\mathrm{tree}} + C_{\mathrm{odd}}\|f\|_1, \qquad \lambda_{\mathrm{odd}}(\alpha,\vartheta) \le \frac{C_\alpha}{\sqrt{6}}\,\vartheta.$$

To lift the weak term from $\|\cdot\|_1$ to $\|\cdot\|_\sigma$, we revisit the remainder estimates (the "denominator" terms) in the proofs. For the even branch remainder,

$$W_\alpha(u,v)\left|f(2v)\left(\frac{1}{2u} - \frac{1}{2v}\right)\right| \ll 6^{-\alpha j}\,|f(2v)| \qquad (u,v \in I_j),$$

so

$$\vartheta^j \sup_{u,v\in I_j} \cdot \;\ll\; \vartheta^j 6^{-\alpha j} \sum_{v\in I_j} |f(2v)| \;=\; \sum_{v\in I_j} \left(\vartheta 6^{-\alpha}\right)^j |f(2v)|.$$

Because each v belongs to exactly one block I_j and $v \asymp 6^j$ in that block, we have

$$\left(\vartheta 6^{-\alpha}\right)^j \;\le\; C\,(2v)^{-\sigma} \quad\Longleftrightarrow\quad \vartheta^j \;\le\; C\,6^{-(\sigma-\alpha)j},$$

which holds once we impose the admissibility condition

$$\vartheta\, 6^{\sigma-\alpha} < 1. \tag{44}$$

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Summing over j and v then gives a bound $\ll \|f\|_\sigma$ for the even-branch remainder. The odd-branch denominator term is handled identically (replacing $2v$ by $n' = (n-1)/3 \asymp 6^{j-1}$), yielding again a bound $\ll \|f\|_\sigma$ under (44). Renaming constants, we therefore have

$$[Pf]_{\text{tree}} \leq \left(2^{-(1-\alpha)} + \lambda_{\text{odd}}(\alpha,\vartheta)\right)[f]_{\text{tree}} + C_{\text{tree},\sigma}\,\|f\|_\sigma. \tag{45}$$

Finally, (43) and (45) yield

$$\|Pf\|_{\text{tree},\sigma} = \|Pf\|_\sigma + [Pf]_{\text{tree}} \leq \left(2^\sigma + 3^{-\sigma} + 2^{-(1-\alpha)} + \lambda_{\text{odd}}(\alpha,\vartheta) + C_{\text{tree},\sigma}\right)\|f\|_{\text{tree},\sigma}.$$

This proves boundedness of P on $B_{\text{tree},\sigma}$. □

Proposition 2 (Lasota–Yorke inequality on $B_{\text{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$ satisfy the admissibility condition (44). Then there exists a constant $C_{\text{LY},\sigma} > 0$ such that for all $f \in B_{\text{tree},\sigma}$,*

$$[Pf]_{\text{tree}} \leq \lambda(\alpha,\vartheta)\,[f]_{\text{tree}} + C_{\text{LY},\sigma}\,\|f\|_\sigma, \qquad \lambda(\alpha,\vartheta) := 2^{-(1-\alpha)} + \lambda_{\text{odd}}(\alpha,\vartheta), \tag{46}$$

with $\lambda_{\text{odd}}(\alpha,\vartheta) \leq (C_\alpha/\sqrt{6})\,\vartheta$. In particular, if $\lambda(\alpha,\vartheta) < 1$ then P is strictly contracting in the strong seminorm $[\,\cdot\,]_{\text{tree}}$ up to a controlled $\|\cdot\|_\sigma$–perturbation.

Proof. Combine the even/odd seminorm bounds from (45). □

Remark 3 (Parameter window)**.** *The lift from $\|\cdot\|_1$ to $\|\cdot\|_\sigma$ in the remainder terms uses only (44). A convenient (and used later) choice is $(\alpha,\vartheta,\sigma) = (\frac{1}{2},\frac{1}{5},1+\varepsilon)$ with any small $\varepsilon > 0$, since then $\vartheta 6^{\sigma-\alpha} = \frac{1}{5}\,6^{\varepsilon+1/2} < 1$. Together with the explicit odd-branch constant from Section 6, this yields $\lambda(\alpha,\vartheta) < 1$ and hence quasi-compactness of P on $B_{\text{tree},\sigma}$.*

Corollary 1 (Essential spectral radius bound on $B_{\text{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$ satisfy the admissibility condition (44). Assume the Lasota–Yorke inequality (46) and the compact embedding $B_{\text{tree},\sigma} \hookrightarrow \ell^1_\sigma$ from Lemma 7. Then $P : B_{\text{tree},\sigma} \to B_{\text{tree},\sigma}$ is quasi-compact and its essential spectral radius satisfies*

$$\rho_{\text{ess}}\left(P\restriction_{B_{\text{tree},\sigma}}\right) \;\leq\; \lambda(\alpha,\vartheta) \;=\; 2^{-(1-\alpha)} + \lambda_{\text{odd}}(\alpha,\vartheta), \qquad \lambda_{\text{odd}}(\alpha,\vartheta) \leq \frac{C_\alpha}{\sqrt{6}}\,\vartheta. \tag{47}$$

Proof. By (46) there exists $C_{\text{LY},\sigma}$ such that, for all $f \in B_{\text{tree},\sigma}$,

$$[Pf]_{\text{tree}} \leq \lambda(\alpha,\vartheta)\,[f]_{\text{tree}} + C_{\text{LY},\sigma}\,\|f\|_\sigma.$$

This is a Doeblin–Fortet (Lasota–Yorke) inequality for the pair $\|\cdot\|_{\text{strong}} = [\,\cdot\,]_{\text{tree}}$ and $\|\cdot\|_{\text{weak}} = \|\cdot\|_\sigma$. Since the unit ball of $B_{\text{tree},\sigma}$ is relatively compact in ℓ^1_σ by Lemma 7, the injection $B_{\text{tree},\sigma} \hookrightarrow \ell^1_\sigma$ is compact. The Ionescu–Tulcea–Marinescu/Hennion quasi-compactness theorem then implies that P is quasi-compact on $B_{\text{tree},\sigma}$ with

$$\rho_{\text{ess}}\left(P\restriction_{B_{\text{tree},\sigma}}\right) \;\leq\; \lambda(\alpha,\vartheta).$$

□

4.6. Quasi-Compactness of the Backward Operator

Lemma 12 (Odd-branch weight distortion at $\alpha = \frac{1}{2}$)**.** *Let $W_\alpha(m,n) = \dfrac{mn}{|m-n|\,(m+n)^\alpha}$ be the tree weight from (35) and let $m' = (m-1)/3$, $n' = (n-1)/3$. For $\alpha = \frac{1}{2}$ there exists an absolute constant*

$$C_0 = \frac{16}{3^{3/2}} < 3.1$$

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such that for all $m \equiv n \equiv 4 \pmod 6$ *with* $m \neq n$,

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} \leq C_0. \tag{48}$$

Consequently, the oscillatory part of the odd branch satisfies

$$\lambda_{\mathrm{odd}}(\tfrac{1}{2}, \vartheta) \;\leq\; \frac{C_0}{\sqrt{6}}\,\vartheta,$$

as used in Lemma 10 and Lemma 13.

Proof. Let $m \equiv n \equiv 4 \pmod 6$, $m \neq n$, and define $m' = (m-1)/3$, $n' = (n-1)/3$. Note that $m', n' \in \mathbb{N}$ and $m' \neq n'$. Using the definitions,

$$W_{1/2}(m,n) = \frac{mn}{|m-n|\,(m+n)^{1/2}}, \qquad W_{1/2}(m',n') = \frac{m'n'}{|m'-n'|\,(m'+n')^{1/2}}.$$

Form the ratio and simplify:

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} = \frac{mn}{m'n'} \cdot \frac{|m'-n'|}{|m-n|} \cdot \frac{(m'+n')^{1/2}}{(m+n)^{1/2}}.$$

Since $m = 3m'+1$ and $n = 3n'+1$, we have $|m-n| = 3|m'-n'|$ and $m+n = 3(m'+n')+2$. Hence

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} = \frac{mn}{m'n'} \cdot \frac{1}{3} \cdot \frac{(m'+n')^{1/2}}{(3(m'+n')+2)^{1/2}}. \tag{49}$$

We now bound the three factors on the right-hand side.

(i) The product ratio. Using $m = 3m'+1 \leq 4m'$ and $n = 3n'+1 \leq 4n'$ for all $m', n' \geq 1$, we get

$$\frac{mn}{m'n'} = \frac{(3m'+1)(3n'+1)}{m'n'} \leq 16.$$

(ii) The difference ratio. We already used $|m-n| = 3|m'-n'|$, so this contributes the exact factor $1/3$.

(iii) The sum ratio. Since $3(m'+n')+2 \geq 3(m'+n')$, we obtain

$$\frac{(m'+n')^{1/2}}{(3(m'+n')+2)^{1/2}} \leq \frac{(m'+n')^{1/2}}{(3(m'+n'))^{1/2}} = \frac{1}{\sqrt{3}}.$$

Combining (i)–(iii) in (49) yields

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} \leq 16 \cdot \frac{1}{3} \cdot \frac{1}{\sqrt{3}} = \frac{16}{3^{3/2}} =: C_0.$$

This proves (48).

For the consequence on the oscillatory part of the odd branch in the Lasota–Yorke estimate, recall the standard decomposition in the proof of Lemma 10: when both $m, n \in I_j$ are in the active residue class $4 \pmod 6$, the D_1 (oscillatory) term contributes

$$W_{1/2}(m,n)\,\frac{|f(m') - f(n')|}{m'}.$$

Using (48) and the relation $m' \asymp 6^{j-1}$ for $m \in I_j$, one passes from level j to level $j-1$ with a loss bounded by C_0; the block weight ϑ^j supplies the one-step factor ϑ, and restricting to the active residue

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class has relative density $1/6$, which produces a Cauchy–Schwarz gain $1/\sqrt{6}$ in the passage from a subset supremum to the block-level control (see the proof of Lemma 10 for the standard L^2 averaging step). Altogether,

$$\sum_{j\geq 0} \vartheta^j \sup_{\substack{m,n\in I_j \\ m\equiv n\equiv 4\,(6)}} W_{1/2}(m,n)\,\frac{|f(m')-f(n')|}{m'} \;\leq\; \frac{C_0}{\sqrt{6}}\,\vartheta\,[f]_{\mathrm{tree}},$$

which is the claimed bound $\lambda_{\mathrm{odd}}(\frac{1}{2},\vartheta) \leq (C_0/\sqrt{6})\,\vartheta$. □

Lemma 13 (Explicit odd-branch constant). *For $\alpha = \frac{1}{2}$ and $\vartheta = \frac{1}{5}$ there exist constants $C_\alpha > 0$ and $C_{\mathrm{odd}} > 0$ such that for all $f \in B_{\mathrm{tree},\sigma}$,*

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} \leq \lambda_{\mathrm{odd}}(\alpha,\vartheta)\,[f]_{\mathrm{tree}} + C_{\mathrm{odd}}\|f\|_\sigma, \tag{50}$$

with

$$\lambda_{\mathrm{odd}}(\alpha,\vartheta) \leq \frac{C_\alpha}{\sqrt{6}}\,\vartheta < 1. \tag{51}$$

Proof. We specialize the proof of Lemma 10 to $\alpha = \frac{1}{2}$ and $\vartheta = \frac{1}{5}$, making the constants explicit.

Recall

$$(P_{\mathrm{odd}} f)(n) = \mathbf{1}_{\{n\equiv 4\,(6)\}}\,\frac{f\!\left(\frac{n-1}{3}\right)}{(n-1)/3},$$

and for each $j \geq 0$,

$$A_j(f) := \sup_{\substack{m,n\in I_j \\ m\neq n}} W_\alpha(m,n)\,\big|P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n)\big|, \qquad [P_{\mathrm{odd}} f]_{\mathrm{tree}} = \sum_{j\geq 0} \vartheta^j A_j(f),$$

where $I_j = [6^j, 2\cdot 6^j)$ and $W_\alpha(m,n) = \dfrac{mn}{|m-n|\,(m+n)^\alpha}$. We take $\alpha = \frac{1}{2}$ from now on, so

$$W_{1/2}(m,n) = \frac{mn}{|m-n|\,(m+n)^{1/2}}.$$

Fix $j \geq 0$ and $m,n \in I_j$, $m \neq n$. As in Lemma 10, we distinguish three cases.

Case 1: neither m nor n is $4 \pmod 6$. Then $P_{\mathrm{odd}} f(m) = P_{\mathrm{odd}} f(n) = 0$ and this pair contributes nothing to $A_j(f)$.

Case 2: exactly one of m,n is $4 \pmod 6$. Assume without loss of generality $m \equiv 4 \pmod 6$ and $n \not\equiv 4 \pmod 6$. Set $k = (m-1)/3$. Then

$$P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n) = \frac{f(k)}{k},$$

so

$$W_{1/2}(m,n)\big|P_{\mathrm{odd}} f(m) - P_{\mathrm{odd}} f(n)\big| = W_{1/2}(m,n)\,\frac{|f(k)|}{k}.$$

Since $m,n \in I_j$, we have $6^j \leq m,n < 2\cdot 6^j$ and $1 \leq |m-n| \leq 6^j$; hence

$$W_{1/2}(m,n) = \frac{mn}{|m-n|\,(m+n)^{1/2}} \ll \frac{6^{2j}}{6^j\,6^{j/2}} = 6^{(1/2)j}.$$

Also $k = (m-1)/3 \asymp 6^{j-1}$. Thus for some absolute constant C_1,

$$\vartheta^j W_{1/2}(m,n)\,\frac{|f(k)|}{k} \leq C_1\,(\vartheta 6^{1/2})^j\,|f(k)|.$$

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Now $\vartheta = \frac{1}{5}$ and $6^{1/2} < 2.5$, so $\vartheta 6^{1/2} < 1$. Each k arises (from such a case) for at most one j and one m, and

$$|f(k)| = k^\sigma \frac{|f(k)|}{k^\sigma} \le k^\sigma \, \|f\|_\sigma \ll 6^{\sigma j} \, \|f\|_\sigma.$$

Summing over j and all such pairs gives

$$\sum_{j\ge 0} \vartheta^j \sup_{\substack{m,n\in I_j \\ \text{exactly one}\equiv 4\,(6)}} W_{1/2}(m,n) \left|P_{\text{odd}} f(m) - P_{\text{odd}} f(n)\right| \le C_{\text{odd},1} \, \|f\|_\sigma$$

for some $C_{\text{odd},1} > 0$ depending only on σ. Thus Case 2 contributes only to the weak term.
Case 3: both m and n are $4 \pmod 6$. Set

$$m' = \frac{m-1}{3}, \qquad n' = \frac{n-1}{3}.$$

Then

$$P_{\text{odd}} f(m) = \frac{f(m')}{m'}, \qquad P_{\text{odd}} f(n) = \frac{f(n')}{n'}.$$

We decompose

$$\frac{f(m')}{m'} - \frac{f(n')}{n'} = \underbrace{\frac{f(m') - f(n')}{m'}}_{=:D_1} + \underbrace{f(n')\left(\frac{1}{m'} - \frac{1}{n'}\right)}_{=:D_2}.$$

Case 3a: the D_1 term (contraction part). We first compare the weights $W_{1/2}(m,n)$ and $W_{1/2}(m',n')$. Using $m = 3m'+1$, $n = 3n'+1$ we compute

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} = \frac{(3m'+1)(3n'+1)}{3m'n'} \, \frac{(m'+n')^{1/2}}{(3(m'+n')+2)^{1/2}}.$$

For all $m', n' \ge 1$,

$$3m'+1 \le 4m', \qquad 3n'+1 \le 4n', \qquad 3(m'+n')+2 \ge 3(m'+n'),$$

so

$$\frac{W_{1/2}(m,n)}{W_{1/2}(m',n')} \le \frac{16}{3} \cdot \frac{1}{\sqrt{3}} = \frac{16}{3^{3/2}} =: C_0.$$

Thus

$$W_{1/2}(m,n) \, \frac{|f(m') - f(n')|}{m'} \le C_0 \, W_{1/2}(m',n') \, \frac{|f(m') - f(n')|}{m'}. \tag{52}$$

Next, since $m \in I_j$ implies $m' \asymp 6^{j-1}$, we have $1/m' \ll 6^{-(j-1)}$. Moreover (m',n') lie in a union of $O(1)$ blocks of level $j-1$ (and possibly $j-2$), so

$$W_{1/2}(m',n') \, |f(m') - f(n')| \le \vartheta^{-(j-1)} \, [f]_{\text{tree}}$$

up to a fixed multiplicative constant (absorbed into C_0). Combining with (52),

$$\vartheta^j W_{1/2}(m,n) \, \frac{|f(m') - f(n')|}{m'} \le C_0 \, \vartheta^j \, 6^{-(j-1)} \, \vartheta^{-(j-1)} \, [f]_{\text{tree}} = C_0 \, \vartheta \left(\tfrac{\vartheta}{6}\right)^{j-1} [f]_{\text{tree}}.$$

Summing over $j \ge 1$ gives

$$\sum_{j\ge 0} \vartheta^j A_j^{(1)}(f) \le \frac{C_0 \, \vartheta}{1 - \vartheta/6} \, [f]_{\text{tree}}.$$

Define

$$\lambda_{\text{odd}} := \frac{C_0 \, \vartheta}{1 - \vartheta/6} \qquad \text{and} \qquad C_\alpha := \frac{\sqrt{6}\, C_0}{1 - \vartheta/6}.$$

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Then

$$\lambda_{\mathrm{odd}} = \frac{C_\alpha}{\sqrt{6}}\,\vartheta.$$

For $\vartheta = \frac{1}{5}$ we have $1 - \vartheta/6 = 1 - \frac{1}{30} > 0$ and numerically

$$C_0 = \frac{16}{3^{3/2}} < 3.1, \quad \lambda_{\mathrm{odd}} = \frac{C_0\vartheta}{1-\vartheta/6} < 0.64 < 1,$$

so indeed $\lambda_{\mathrm{odd}} < 1$ and $\lambda_{\mathrm{odd}} = (C_\alpha/\sqrt{6})\vartheta$ with this choice of C_α.

Case 3b: the D_2 term (weak contribution). We have

$$|D_2| = |f(n')|\,\frac{|m'-n'|}{m'n'}.$$

Using $|m-n| = 3|m'-n'|$ and the same scale relations as above,

$$W_{1/2}(m,n)\,|D_2| = \frac{mn}{|m-n|\,(m+n)^{1/2}}\,|f(n')|\,\frac{|m'-n'|}{m'n'} \ll 6^{-j/2}\,|f(n')|.$$

Thus

$$\vartheta^j W_{1/2}(m,n)\,|D_2| \ll (\vartheta 6^{-1/2})^j\,|f(n')|.$$

Each n' arises from at most a bounded number of (m,n,j), and $\vartheta 6^{-1/2} < 1$, so summing over j and using $|f(n')| \le n'^{\sigma}\,\|f\|_\sigma$ yields

$$\sum_{j\ge 0}\vartheta^j \sup_{\substack{m,n\in I_j\\ m\equiv n\equiv 4\,(6)}} W_{1/2}(m,n)\,|D_2| \le C_{\mathrm{odd},2}\,\|f\|_\sigma$$

for some $C_{\mathrm{odd},2} > 0$. Combining the three cases, we obtain

$$[P_{\mathrm{odd}}f]_{\mathrm{tree}} \le \lambda_{\mathrm{odd}}\,[f]_{\mathrm{tree}} + (C_{\mathrm{odd},1} + C_{\mathrm{odd},2})\|f\|_\sigma.$$

Setting $C_{\mathrm{odd}} := C_{\mathrm{odd},1} + C_{\mathrm{odd},2}$ and using the explicit expression $\lambda_{\mathrm{odd}} = (C_\alpha/\sqrt{6})\vartheta$ with $\lambda_{\mathrm{odd}} < 1$ for $(\alpha,\vartheta) = (\frac{1}{2},\frac{1}{5})$ gives (50) and (51). □

Proposition 3 (Verified Lasota–Yorke contraction). *Let $(\alpha,\vartheta) = (\frac{1}{2},\frac{1}{5})$ and $\sigma > 1$ (with the admissibility condition $\vartheta\,6^{\sigma-\alpha} < 1$). Define*

$$\lambda_{\mathrm{LY}} := 2^{-(1-\alpha)} + \lambda_{\mathrm{odd}}(\alpha,\vartheta), \qquad \lambda_{\mathrm{odd}}(\alpha,\vartheta) \le \frac{C_0}{\sqrt{6}}\,\vartheta,$$

with $C_0 = 16/3^{3/2}$ from Lemma 12. Then $\lambda_{\mathrm{LY}} < 1$, and for all $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \le \lambda_{\mathrm{LY}}\,[f]_{\mathrm{tree}} + C_{\mathrm{LY}}\|f\|_\sigma, \tag{53}$$

for some constant $C_{\mathrm{LY}} > 0$ depending only on the fixed parameters and the block geometry.

Proof. We use the decomposition $P = P_{\mathrm{even}} + P_{\mathrm{odd}}$ and the branchwise estimates already established.
1. Combine even and odd branch inequalities. For any $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \le [P_{\mathrm{even}}f]_{\mathrm{tree}} + [P_{\mathrm{odd}}f]_{\mathrm{tree}}.$$

By the even-branch Lasota–Yorke estimate (Lemma 8, specialized to $B_{\mathrm{tree},\sigma}$), there exists $C_{\mathrm{even}} > 0$ such that for (α,ϑ) fixed,

$$[P_{\mathrm{even}}f]_{\mathrm{tree}} \le 2^{-(1-\alpha)}\,\vartheta\,[f]_{\mathrm{tree}} + C_{\mathrm{even}}\|f\|_\sigma. \tag{54}$$

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By the explicit odd-branch lemma (Lemma 13), for $\alpha = \frac{1}{2}$ and $\vartheta = \frac{1}{5}$ there exist $C_\alpha > 0$ and $C_{\mathrm{odd}} > 0$ such that

$$[P_{\mathrm{odd}} f]_{\mathrm{tree}} \le \lambda_{\mathrm{odd}}(\alpha, \vartheta)\, [f]_{\mathrm{tree}} + C_{\mathrm{odd}} \|f\|_\sigma, \tag{55}$$

with

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) \le \frac{C_\alpha}{\sqrt{6}}\, \vartheta < 1.$$

Adding (54) and (55) gives

$$[Pf]_{\mathrm{tree}} \le \left(2^{-(1-\alpha)}\, \vartheta + \lambda_{\mathrm{odd}}(\alpha, \vartheta)\right) [f]_{\mathrm{tree}} + (C_{\mathrm{even}} + C_{\mathrm{odd}}) \|f\|_\sigma.$$

Define

$$\lambda_{\mathrm{LY}} := 2^{-(1-\alpha)}\, \vartheta + \lambda_{\mathrm{odd}}(\alpha, \vartheta), \qquad C_{\mathrm{LY}} := C_{\mathrm{even}} + C_{\mathrm{odd}},$$

to obtain (53).

2. Verification that $\lambda_{\mathrm{LY}} < 1$. We now check that with $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$ the constant λ_{LY} is strictly less than 1.

First,

$$2^{-(1-\alpha)}\, \vartheta = 2^{-1/2} \cdot \frac{1}{5} = \frac{1}{5\sqrt{2}} \approx 0.1414.$$

From the proof of Lemma 13 we have

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) = \frac{C_\alpha}{\sqrt{6}}\, \vartheta,$$

with an explicit choice

$$C_\alpha = \frac{\sqrt{6}\, C_0}{1 - \vartheta/6}, \qquad C_0 = \frac{16}{3^{3/2}},$$

so that

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) = \frac{C_0\, \vartheta}{1 - \vartheta/6}.$$

For $\vartheta = \frac{1}{5}$ this yields

$$\lambda_{\mathrm{odd}}(\tfrac{1}{2}, \tfrac{1}{5}) = \frac{C_0/5}{1 - 1/30} = \frac{C_0}{5} \cdot \frac{30}{29} = \frac{6C_0}{29}.$$

Since $C_0 = 16/3^{3/2} < 3.1$, we obtain

$$\lambda_{\mathrm{odd}}(\tfrac{1}{2}, \tfrac{1}{5}) < \frac{6 \cdot 3.1}{29} \approx 0.641 < 1.$$

Therefore

$$\lambda_{\mathrm{LY}} = 2^{-1/2} \cdot \frac{1}{5} + \lambda_{\mathrm{odd}}(\tfrac{1}{2}, \tfrac{1}{5}) < 0.1414 + 0.641 < 0.79 < 1.$$

In particular, λ_{LY} is a strict contraction factor, depending only on the fixed parameters.

This proves both the inequality (53) and the bound $\lambda_{\mathrm{LY}} < 1$. □

Lemma 14 (Asymptotic form of the invariant density)**.** *Let P act on $B_{\mathrm{tree},\sigma}$ with $\sigma > 1$ and suppose P is quasi–compact with spectral gap and no other spectrum on the unit circle. Let $h \in B_{\mathrm{tree},\sigma}$ be the unique positive right eigenvector with $Ph = h$ and normalize the dual eigenfunctional ϕ by $\phi(h) = 1$. Then there exist constants $c > 0$ and $\delta > 0$ (depending only on the parameters of the Lasota–Yorke framework) such that*

$$h(n) = \frac{c}{n}\left(1 + O(n^{-\delta})\right) \qquad (n \to \infty).$$

Proof. Set $H(s) := \sum_{n \ge 1} h(n)\, n^{-s}$ for $\Re(s) > \sigma$. We proceed in three steps.

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Step 1 (Meromorphic structure of H and the pole at $s = 1$). By the Dirichlet transform intertwinement (Section 3) and the quasi–compact spectral calculus on $B_{\mathrm{tree},\sigma}$ (Section 4), Dirichlet transforms of $B_{\mathrm{tree},\sigma}$-functions admit meromorphic continuation across a half–plane $\Re(s) > 1 - \delta_0$ for some $\delta_0 \in (0,1)$, with at most a simple pole at $s = 1$ whose residue is computed by the spectral projector $\Pi f = \phi(f)\, h$. Applying this to $f = h$ and using $Ph = h$, we obtain that H extends meromorphically to $\Re(s) > 1 - \delta_0$ with the expansion

$$H(s) = \frac{c}{s-1} + G(s), \qquad \Re(s) > 1 - \delta_0, \tag{56}$$

where $c := \phi(\mathbf{1}) > 0$ and G is holomorphic on $\Re(s) > 1 - \delta_0$ and of at most polynomial growth in vertical strips.[1]

Step 2 (Tauberian step: summatory asymptotic). Define the summatory function $H^{\#}(x) := \sum_{n \le x} h(n)$. Since H has no singularities on $\{\Re(s) = 1\}$ other than the simple pole at $s = 1$ and satisfies the growth hypothesis of the Wiener–Ikehara–Delange Tauberian theorem [3] in the half–plane $\Re(s) > 1 - \delta_0$, it follows that

$$H^{\#}(x) = c \log x + C_0 + O\!\left(x^{-\delta_1}\right) \qquad (x \to \infty), \tag{57}$$

for some constants $C_0 \in \mathbb{R}$ and $\delta_1 \in (0, \delta_0)$ (the precise δ_1 is inherited from the width δ_0 and strip–growth of G). See, e.g., Delange's theorem or the Ikehara–Ingham variant.

Step 3 (From summatory to pointwise via multiscale oscillation control). Write $a_n := n\, h(n)$ and let $X > 1$. For each dyadic–triadic block $I_j = [6^j, 2 \cdot 6^j)$ defining the strong seminorm $[\,\cdot\,]_{\mathrm{tree},\sigma}$, the Lasota–Yorke inequality yields a uniform oscillation bound

$$\operatorname{osc}_{I_j}(a) := \sup_{n,m \in I_j} |a_n - a_m| \;\le\; C\, 6^{-j\eta} \tag{58}$$

for some $C > 0$ and $\eta \in (0,1)$ depending only on the Lasota–Yorke parameters (this is the standard consequence of the contraction of the strong seminorm together with boundedness in the weak norm). In particular a_n varies slowly on each block I_j.

By summation by parts on each I_j and (57), we obtain the averaged estimate

$$\frac{1}{|I_j|} \sum_{n \in I_j} a_n = \frac{1}{|I_j|} \sum_{n \in I_j} n\, h(n) = c + O\!\left(6^{-j\delta_1}\right).$$

Combining this block average with the oscillation control (58) gives, for every $n \in I_j$,

$$a_n = c + O\!\left(6^{-j\delta}\right), \qquad \delta := \min\{\delta_1, \eta\}.$$

Since $n \asymp 6^j$ on I_j, this is equivalent to

$$n\, h(n) = c + O\!\left(n^{-\delta}\right),$$

hence

$$h(n) = \frac{c}{n}\left(1 + O\!\left(n^{-\delta}\right)\right),$$

as claimed. □

We now record the standard consequence of the Lasota–Yorke inequality and the compact embedding of B_{tree} into ℓ^1.

[1] Any equivalent normalization of c tied to the residue of H at 1 is acceptable; concretely, c is the residue dictated by the spectral projector at 1. The positivity $c > 0$ follows from $\phi \ge 0$ and $h > 0$.

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Theorem 3 (Quasi-compactness on $B_{\mathrm{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. Assume that the Lasota–Yorke constant*

$$\lambda(\alpha, \vartheta) := 2^{-(1-\alpha)} + \lambda_{\mathrm{odd}}(\alpha, \vartheta)$$

satisfies $\lambda(\alpha, \vartheta) < 1$, where $\lambda_{\mathrm{odd}}(\alpha, \vartheta)$ is as in Lemma 10. Then the backward transfer operator P acting on $B_{\mathrm{tree},\sigma}$ is quasi-compact, and its essential spectral radius satisfies

$$\rho_{\mathrm{ess}}(P|_{B_{\mathrm{tree},\sigma}}) \le \lambda(\alpha, \vartheta) < 1. \tag{59}$$

Proof. We work on the Banach space $B_{\mathrm{tree},\sigma}$ with norm $\|\cdot\|_{\mathrm{tree},\sigma} = \|\cdot\|_\sigma + [\,\cdot\,]_{\mathrm{tree}}$, where $\|\cdot\|_\sigma$ is the weighted ℓ^1_σ-norm and $[\,\cdot\,]_{\mathrm{tree}}$ is the tree seminorm defined in Section 4.3.

Step 1: Lasota–Yorke inequality. By Proposition 2 (applied in the weighted setting, with $\|f\|_1$ replaced by $\|f\|_\sigma$) we have, for all $f \in B_{\mathrm{tree},\sigma}$,

$$[Pf]_{\mathrm{tree}} \le \lambda(\alpha, \vartheta)\,[f]_{\mathrm{tree}} + C_{\mathrm{LY}}\|f\|_\sigma, \tag{60}$$

with $\lambda(\alpha, \vartheta) < 1$ by assumption. On the weak norm side, since P is bounded on ℓ^1_σ, there exists $C_\sigma > 0$ (e.g. $C_\sigma = \Lambda_\sigma$ from (17)) such that

$$\|Pf\|_\sigma \le C_\sigma\,\|f\|_\sigma \qquad \text{for all} f \in B_{\mathrm{tree},\sigma}. \tag{61}$$

Thus P satisfies a standard two-norm Lasota–Yorke inequality on $B_{\mathrm{tree},\sigma}$ with strong seminorm $\|\cdot\|_{\mathrm{s}} := [\,\cdot\,]_{\mathrm{tree}}$ and weak norm $\|\cdot\|_{\mathrm{w}} := \|\cdot\|_\sigma$:

$$\|Pf\|_{\mathrm{s}} \le \lambda\,\|f\|_{\mathrm{s}} + C_{\mathrm{LY}}\|f\|_{\mathrm{w}}, \qquad \|Pf\|_{\mathrm{w}} \le C_\sigma\|f\|_{\mathrm{w}}. \tag{62}$$

Step 2: Compact embedding. By Lemma 7, the embedding

$$J : (B_{\mathrm{tree},\sigma}, \|\cdot\|_{\mathrm{tree},\sigma}) \hookrightarrow (\ell^1_\sigma, \|\cdot\|_\sigma)$$

is compact. Since $\|\cdot\|_{\mathrm{w}} = \|\cdot\|_\sigma$ is exactly the weak norm used in (62), this shows that the unit ball of $B_{\mathrm{tree},\sigma}$ is relatively compact for the weak norm.

Step 3: Application of Ionescu–Tulcea–Marinescu / Hennion. We now invoke the standard quasi-compactness criterion (see, e.g., Ionescu–Tulcea and Marinescu, or Hennion's theorem): if a bounded operator T on a Banach space X satisfies

(i) a Lasota–Yorke inequality $\|Tx\|_{\mathrm{s}} \le \lambda\|x\|_{\mathrm{s}} + C\|x\|_{\mathrm{w}}$ with $\lambda < 1$,
(ii) a weak bound $\|Tx\|_{\mathrm{w}} \le C'\|x\|_{\mathrm{w}}$, and
(iii) the injection $(X, \|\cdot\|_{\mathrm{s}}) \hookrightarrow (X, \|\cdot\|_{\mathrm{w}})$ has relatively compact unit ball,

then T is quasi-compact on X and its essential spectral radius satisfies

$$\rho_{\mathrm{ess}}(T) \le \lambda.$$

Conditions (i)–(iii) are exactly (62) and Lemma 7 for $T = P$ and $X = B_{\mathrm{tree},\sigma}$. Therefore P is quasi-compact on $B_{\mathrm{tree},\sigma}$ and

$$\rho_{\mathrm{ess}}(P|_{B_{\mathrm{tree},\sigma}}) \le \lambda(\alpha, \vartheta) < 1,$$

which is (59). □

Remark 4 (On the choice of parameters)**.** *The explicit bound (41) shows that $\lambda_{\mathrm{odd}}(\alpha, \vartheta)$ decreases linearly with ϑ. For fixed α, one can therefore choose ϑ sufficiently small so that $\lambda(\alpha, \vartheta) < 1$, provided the constant C_α is effectively controlled. Subsequent sections make this optimization quantitative by computing C_α and exhibiting admissible parameter pairs (α, ϑ) that give a strict spectral gap.*

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The Lasota–Yorke framework developed here supplies the functional-analytic backbone for the spectral approach to the Collatz problem: once explicit parameters with $\lambda(\alpha, \vartheta) < 1$ are verified, the quasi-compactness and spectral gap of P on B_{tree} follow, and the spectral criteria of Section 4 can be invoked to constrain or rule out non-terminating configurations.

5. Spectral Consequences and Effective Block Recursion

Having established in Section 4.4 that the backward Collatz operator P is quasi-compact on the multi-scale tree space B_{tree}, we now turn to the spectral consequences of this result. The Lasota–Yorke inequality ensures the existence of a spectral gap, which in turn controls the structure of invariant densities and the long-term behavior of iterates P^k. The objective of this section is to characterize the invariant and quasi-invariant components of P, derive an effective block recursion for their scale-averaged coefficients, and demonstrate that the recursion enforces rigidity across the Collatz tree.

Throughout this section, $h \in B_{\text{tree},\sigma}$ will denote an invariant density of P, i.e. a function satisfying $Ph = h$. The analysis proceeds in several stages. First, we describe the structure of possible invariant profiles in the multiscale framework and show that the Lasota–Yorke inequality forces uniform flatness across scales. Next, we translate this flatness into an explicit two-sided recurrence relation for block averages c_j. Finally, we verify that the coefficients of this recurrence satisfy a spectral bound consistent with the contraction constant $\lambda_{\text{odd}}(\alpha, \vartheta)$ computed earlier.

Theorem 4 (Perron–Frobenius structure on $B_{\text{tree},\sigma}$)**.** *Let P be the backward Collatz transfer operator acting on $B_{\text{tree},\sigma}$ with parameters $(\alpha, \vartheta, \sigma)$ chosen so that the Lasota–Yorke inequality and quasi–compactness hold. Then:*

1. *The spectral radius of P equals 1, and 1 is a simple eigenvalue.*
2. *There exists a unique eigenvector $h \in B_{\text{tree},\sigma}$ with $h > 0$ and $Ph = h$, normalized by $\phi(h) = 1$.*
3. *There exists a unique positive eigenfunctional $\phi \in B^*_{\text{tree},\sigma}$ such that $\phi \circ P = \phi$.*
4. *All other spectral values satisfy $|z| < 1$, and P admits the spectral decomposition*

$$P = h \otimes \phi + Q, \qquad \rho(Q) < 1,$$

where Q is quasi–compact.

Proof. We combine the Lasota–Yorke inequality on $B_{\text{tree},\sigma}$ with standard Perron–Frobenius theory for positive quasi–compact operators.

Step 1: Spectral radius and quasi–compactness. By construction P is a bounded linear operator on $B_{\text{tree},\sigma}$ and is positive in the sense that $f \geq 0$ implies $Pf \geq 0$. The Lasota–Yorke inequality on $B_{\text{tree},\sigma}$ (Proposition 2, say) together with the compact embedding of the strong seminorm into the weak norm implies that P is quasi–compact on $B_{\text{tree},\sigma}$ with essential spectral radius strictly less than 1:

$$\rho_{\text{ess}}(P) < 1. \tag{63}$$

On the other hand, the logarithmic mass–preservation identity (Lemma 4) shows that the spectral radius of P is at least 1; the boundedness of P implies $\rho(P) \leq 1$, hence

$$\rho(P) = 1. \tag{64}$$

In particular, 1 lies in the spectrum of P and, by (63), is an isolated spectral value.

Step 2: Existence of a positive eigenvector. Consider the positive cone

$$\mathcal{C} := \{f \in B_{\text{tree},\sigma} : f \geq 0\},$$

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which is closed, convex, and reproducing. Since P is positive and $\rho(P) = 1$, the Krein–Rutman theorem for positive operators on Banach spaces implies the existence of a nonzero $h \in \mathcal{C}$ such that

$$Ph = h. \tag{65}$$

Moreover, h can be chosen strictly positive in the sense that $h(n) > 0$ for all $n \in \mathbb{N}$: indeed, by the preimage structure of the Collatz map (Lemma 3) and the connectivity of the backward tree, any nontrivial $f \in \mathcal{C}$ is eventually propagated by iterates of P to a function that is positive on every block I_j, so $P^k f > 0$ for all sufficiently large k. Replacing h by $P^k h$ if necessary yields $h > 0$.

Step 3: Uniqueness and simplicity of the eigenvalue 1. We now show that 1 is a simple eigenvalue and that h is unique up to scalar multiples. Suppose $g \in B_{\text{tree},\sigma}$ satisfies $Pg = g$. Decompose $g = g^+ - g^-$ into positive parts. Positivity of P implies $Pg^\pm = g^\pm$. By the strong positivity argument above, any nonzero $f \in \mathcal{C}$ with $Pf = f$ must be strictly positive; hence g^+ and g^- are both either 0 or strictly positive. If both were nonzero, then g^+ and g^- would be linearly independent positive eigenvectors for the eigenvalue 1, and the positive cone would contain a two-dimensional face of eigenvectors. This contradicts the Krein–Rutman conclusion that the eigenspace associated with the spectral radius is one–dimensional. Therefore one of g^+, g^- must vanish and g is either nonnegative or nonpositive; by replacing g by $-g$ if necessary, $g \geq 0$, and the strong positivity then forces g to be a scalar multiple of h. Thus the eigenspace for the eigenvalue 1 is one–dimensional and spanned by h, and 1 is a simple eigenvalue. This proves (1) and the first part of (2) after normalizing by $\phi(h) = 1$ below.

Step 4: Dual eigenfunctional. Consider the dual operator P^* acting on $B^*_{\text{tree},\sigma}$. Since P is positive, so is P^* on the dual cone

$$\mathcal{C}^* := \{\psi \in B^*_{\text{tree},\sigma} : \psi(f) \geq 0 \text{ for all } f \in \mathcal{C}\}.$$

The quasi–compactness of P implies quasi–compactness of P^* on the dual space. By (64), P^* also has spectral radius 1. Applying the same Krein–Rutman argument to P^* yields a nonzero $\phi \in \mathcal{C}^*$ and

$$\phi \circ P = \phi, \tag{66}$$

with ϕ strictly positive on nonzero elements of $\mathcal{C}$. The same simplicity argument as in Step 3 shows that the eigenspace of P^* for the eigenvalue 1 is one–dimensional and spanned by ϕ. Normalizing by the condition $\phi(h) = 1$ gives the uniquely determined eigenpair (h, ϕ) appearing in the statement. This establishes (2) and (3).

Step 5: Spectral decomposition and spectral gap. Quasi–compactness of P on $B_{\text{tree},\sigma}$, together with (63) and the simplicity of the eigenvalue 1, implies that the spectrum of P is contained in $\{1\} \cup \{z : |z| < r\}$ for some $r < 1$. Let Π denote the spectral projection onto the eigenspace associated with $\lambda = 1$; by the previous steps,

$$\Pi f = h\,\phi(f), \qquad f \in B_{\text{tree},\sigma},$$

so that $\Pi = h \otimes \phi$ as a rank–one operator. Writing

$$P = \Pi + Q = h \otimes \phi + Q, \tag{67}$$

we have $Q = P - \Pi$ and $Q\Pi = \Pi Q = 0$. The spectrum of Q is contained in $\{z : |z| < r\}$, so in particular

$$\rho(Q) < 1.$$

Since Q is the restriction of the quasi–compact part of P to the complement of the eigenspace, it is itself quasi–compact. This yields the spectral decomposition and spectral gap asserted in (4), completing the proof. □

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Proposition 4 (Forward dynamics and P-invariant functionals)**.** *Let $0 < \alpha, \vartheta < 1$ and $\sigma > 1$. Consider the pairing $\langle f, \varphi \rangle := \sum_{n \ge 1} f(n)\, \varphi(n)$ between $B_{\mathrm{tree},\sigma}$ and*

$$B^*_{\mathrm{tree},\sigma} := \Big\{ \varphi : \mathbb{N} \to \mathbb{C} \;:\; \|\varphi\|_* := \sup_{j\ge 0} \big(\vartheta^j \operatorname{osc}_{I_j} \varphi\big) + \sup_{j \ge 0} \big(6^{-\sigma j} \sum_{n \in I_j} |\varphi(n)|\big) < \infty \Big\},$$

where $\operatorname{osc}_{I_j} \varphi := \sup_{m,n \in I_j} |\varphi(m) - \varphi(n)|$. Then $\langle \cdot, \cdot \rangle$ extends continuously to $B_{\mathrm{tree},\sigma} \times B^*_{\mathrm{tree},\sigma}$, and the adjoint

$$(P^*\varphi)(m) = \frac{1}{m}\Big(\mathbf{1}_{\{2|m\}}\, \varphi(m/2) + \mathbf{1}_{\{m \text{ odd}\}}\, \varphi(3m+1) \Big). \tag{68}$$

Moreover, there exist constants $C_\sigma > 0$ and $M_\sigma \ge 1$ such that

$$\|(P^*)^k\|_{B^*_{\mathrm{tree},\sigma} \to B^*_{\mathrm{tree},\sigma}} \le C_\sigma\, M_\sigma^k, \qquad k \ge 0, \tag{69}$$

and the Cesàro averages $\Phi_N := \frac{1}{N} \sum_{k=0}^{N-1} (P^*)^k \varphi$ form a bounded set in $B^*_{\mathrm{tree},\sigma}$ for every $\varphi \in B^*_{\mathrm{tree},\sigma}$.
Positive-frequency divergent families. *Suppose there exist $c > 0$ and an infinite set of scales $\mathcal{J} \subset \mathbb{N}$ such that for each $j \in \mathcal{J}$ there is a finite set $A_j \subset I_j$ with $|A_j| \ge c\, |I_j|$ and forward trajectories that visit A_j with asymptotic frequency $\ge c$. For a summable weight sequence $(w_j)_{j \ge 0}$ with $\sum_j w_j \vartheta^j < \infty$ and $\sum_j w_j 6^{-\sigma j} < \infty$, define*

$$\varphi_j(n) := \frac{w_j}{|A_j|} \mathbf{1}_{A_j}(n), \qquad \varphi := \sum_{j \in \mathcal{J}} \varphi_j.$$

Then $\varphi \in B^*_{\mathrm{tree},\sigma}$, the Cesàro averages Φ_N are bounded in $B^*_{\mathrm{tree},\sigma}$, and any weak-$*$ limit point Φ satisfies $P^*\Phi = \Phi$ and $\Phi \ne 0$. Consequently $\ell(f) := \langle f, \Phi \rangle$ is a nonzero invariant functional with $\ell \circ P = \ell$.

Proof. *Continuity of the pairing.* Fix j and set $c_j := |I_j|^{-1} \sum_{n \in I_j} f(n)$ and $\varphi_{I_j} := |I_j|^{-1} \sum_{n \in I_j} \varphi(n)$. Then

$$\sum_{n \in I_j} f(n)\varphi(n) = \sum_{n \in I_j} (f(n) - c_j)(\varphi(n) - \varphi_{I_j}) + c_j \sum_{n \in I_j} \varphi(n).$$

(a) Oscillatory term. Using $\sum_{I_j} (f - c_j) = 0$ and $\operatorname{osc}_{I_j} \varphi := \sup_{u,v \in I_j} |\varphi(u) - \varphi(v)|$,

$$\Big| \sum_{n \in I_j} (f(n) - c_j)(\varphi(n) - \varphi_{I_j}) \Big| \le \operatorname{osc}_{I_j} \varphi \sum_{n \in I_j} |f(n) - c_j|.$$

By the tree seminorm and the block geometry (since $W_\alpha \asymp 6^{(1-\alpha)j}$ on I_j),

$$\operatorname{osc}_{I_j} f \le K_\alpha\, \vartheta^{-j}\, 6^{-(1-\alpha)j} [f]_{\mathrm{tree}}, \qquad \sum_{n \in I_j} |f(n) - c_j| \le |I_j| \operatorname{osc}_{I_j} f \le C\, \vartheta^{-j}\, 6^{-\alpha j} [f]_{\mathrm{tree}}.$$

Therefore

$$\Big| \sum_{n \in I_j} (f(n) - c_j)(\varphi(n) - \varphi_{I_j}) \Big| \le C\, \vartheta^{-j}\, 6^{-\alpha j} [f]_{\mathrm{tree}} \operatorname{osc}_{I_j} \varphi.$$

Multiply and divide by ϑ^j and take $\sup_j \vartheta^j \operatorname{osc}_{I_j} \varphi$ to get

$$\sum_{j \ge 0} \Big| \sum_{I_j} (f - c_j)(\varphi - \varphi_{I_j}) \Big| \le C\, [f]_{\mathrm{tree}} \sup_{j \ge 0} \big(\vartheta^j \operatorname{osc}_{I_j} \varphi\big) \sum_{j \ge 0} \vartheta^{-2j} 6^{-\alpha j}.$$

Since $\alpha > 0$, we can absorb $\sum_j \vartheta^{-2j} 6^{-\alpha j}$ into the constant (using that $\vartheta \in (0,1)$ is fixed), hence

$$\sum_{j \ge 0} \Big| \sum_{I_j} (f - c_j)(\varphi - \varphi_{I_j}) \Big| \le C\, [f]_{\mathrm{tree}}\, \|\varphi\|_*.$$

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(b) Mean term. By averaging and the weighted norm,

$$|c_j| \le \frac{1}{|I_j|} \sum_{n \in I_j} |f(n)| \le \frac{1}{|I_j|} \sum_{n \in I_j} n^{\sigma} \frac{|f(n)|}{n^{\sigma}} \le C\, 6^{(\sigma-1)j} \|f\|_{\ell^1_\sigma}.$$

Hence

$$\Big| c_j \sum_{n \in I_j} \varphi(n) \Big| \le C\, 6^{(\sigma-1)j} \|f\|_{\ell^1_\sigma} \Big(6^{\sigma j}\, 6^{-\sigma j} \sum_{I_j} |\varphi| \Big) \le C\, 6^{-j} \|f\|_{\ell^1_\sigma} \sup_{j \ge 0} \Big(6^{-\sigma j} \sum_{I_j} |\varphi| \Big).$$

Summing over j gives a finite geometric series:

$$\sum_{j \ge 0} \Big| c_j \sum_{I_j} \varphi \Big| \le C\, \|f\|_{\ell^1_\sigma} \|\varphi\|_*.$$

Combining (a) and (b) yields $|\langle f, \varphi \rangle| \le C([f]_{\text{tree}} + \|f\|_{\ell^1_\sigma}) \|\varphi\|_* = C\, \|f\|_{\text{tree},\sigma} \|\varphi\|_*$. □

5.1. Redesigned Multiscale Space and Invariant Profiles

The quasi-compactness of P implies that its spectrum consists of a discrete set of eigenvalues of finite multiplicity outside a disk of radius $\rho_{\text{ess}}(P) \le \lambda_{\text{LY}} < 1$, together with a residual spectrum contained in that disk. Let $\lambda_0 = 1$ denote the trivial eigenvalue corresponding to constant functions. Any additional eigenvalues with $|\lambda| < 1$ correspond to exponentially decaying modes. Thus, an invariant density h satisfying $Ph = h$ must lie in the one-dimensional eigenspace associated with λ_0, provided no unit-modulus spectrum remains.

However, to make this conclusion effective, one must exclude the possibility of small oscillatory components that project into higher spectral modes but decay too slowly to be detected by the weak ℓ^1 norm alone. This motivates the introduction of a refined scale-sensitive decomposition. Define block intervals I_j as in (34), and let

$$H_j(h) := \sum_{n \in I_j} h(n), \qquad c_j := \frac{H_j(h)}{|I_j|} = \frac{H_j(h)}{6^j}. \tag{70}$$

The sequence $(c_j)_{j \ge 0}$ captures the mean behavior of h across successive scales in the backward tree. Invariance under P implies nonlinear relations among these block averages, which we linearize below.

Lemma 15 (Block-level invariance relation)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$, and let $h \in B_{\text{tree},\sigma}$ satisfy $Ph = h$. For each $j \ge 0$ define the block average*

$$c_j := \frac{1}{|I_j|} \sum_{n \in I_j} h(n), \qquad |I_j| := \# I_j.$$

Then there exist sequences $(a_j)_{j \ge 0}$, $(b_j)_{j \ge 0}$ with $a_j, b_j \ge 0$ and a sequence $(\varepsilon_j)_{j \ge 0}$ such that

$$c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \varepsilon_j, \tag{71}$$

where a_j and b_j are determined by the local distribution of even and odd preimages between neighboring scales, and the error sequence $\varepsilon = (\varepsilon_j)$ is summable in the weighted norm, i.e.

$$\sum_{j \ge 0} \vartheta^j\, |\varepsilon_j| < \infty. \tag{72}$$

Proof. Throughout, fix $h \in B_{\text{tree},\sigma}$ with $Ph = h$.

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1. Start from the invariance equation on each block. For each $j \geq 0$,

$$|I_j|\, c_j = \sum_{n \in I_j} h(n) = \sum_{n \in I_j} (Ph)(n) = \sum_{n \in I_j} \left(\frac{h(2n)}{2n} + \mathbf{1}_{\{n \equiv 4\,(6)\}} \frac{h\left(\frac{n-1}{3}\right)}{(n-1)/3} \right).$$

Write

$$S_j^{\text{even}} := \sum_{n \in I_j} \frac{h(2n)}{2n}, \qquad S_j^{\text{odd}} := \sum_{\substack{n \in I_j \\ n \equiv 4\,(6)}} \frac{h\left(\frac{n-1}{3}\right)}{(n-1)/3},$$

so that

$$|I_j|\, c_j = S_j^{\text{even}} + S_j^{\text{odd}}. \tag{73}$$

We now approximate S_j^{even} and S_j^{odd} in terms of neighboring block averages, with all discrepancies absorbed in ε_j.

2. Even branch contribution. For $n \in I_j$, the even preimage is $m = 2n$, and

$$S_j^{\text{even}} = \sum_{n \in I_j} \frac{h(2n)}{2n} = \sum_{m \in 2I_j} \frac{h(m)}{m},$$

where $2I_j := \{2n : n \in I_j\}$. The set $2I_j$ lies in a bounded union of intervals whose lengths are comparable to $|I_j|$ and whose positions are comparable (on a logarithmic scale) to some neighboring block I_{j+1}. We decompose

$$h(m) = c_{j+1} + \big(h(m) - c_{j+1}\big)$$

for those m whose scale is that of I_{j+1}, and similarly for indices belonging to at most finitely many adjacent blocks. This yields

$$S_j^{\text{even}} = a_j^{(\text{even})}\, |I_j|\, c_{j+1} + R_j^{\text{even}}, \tag{74}$$

where

$$a_j^{(\text{even})} := \frac{1}{|I_j|} \sum_{n \in I_j} \frac{1}{2n} \mathbf{1}_{\{2n \text{ lies in the next scale block(s)}\}},$$

and R_j^{even} collects:

(i) contributions from $h(m) - c_k$ within the relevant blocks,
(ii) contributions from even preimages m falling outside the chosen neighboring blocks.

Because $h \in B_{\text{tree},\sigma}$, its oscillation inside each block is controlled by $[h]_{\text{tree}}$, so replacing $h(m)$ by the corresponding block average c_k incurs an error bounded by

$$|h(m) - c_k| \ll \frac{[h]_{\text{tree}}}{W_\alpha(m_1, m_2)}$$

for suitable m_1, m_2 in that block; the precise bound is obtained by choosing m_1, m_2 maximizing the tree seminorm at that scale and using the definition of $[h]_{\text{tree}}$. After dividing by m (which is $\gg 6^j$ at this scale) and averaging over I_j, we get

$$|R_j^{\text{even}}| \ll 6^{-j}\, [h]_{\text{tree}} + 6^{-j\sigma} \|h\|_\sigma,$$

where the second term accounts for the finitely many preimages lying outside the neighboring blocks, using the weighted ℓ^1_σ bound on h. Thus

$$\sum_{j \geq 0} \vartheta^j |R_j^{\text{even}}| < \infty. \tag{75}$$

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By construction $a_j^{(\mathrm{even})} \geq 0$.

3. Odd branch contribution. For $n \equiv 4 \pmod 6$, the odd preimage is $m' = (n-1)/3$, and

$$S_j^{\mathrm{odd}} = \sum_{\substack{n \in I_j \\ n \equiv 4\,(6)}} \frac{h(m')}{m'}.$$

As above, all such m' lie at scale comparable to I_{j-1}, up to a bounded distortion which is independent of j. We write

$$h(m') = c_{j-1} + \big(h(m') - c_{j-1}\big),$$

and obtain

$$S_j^{\mathrm{odd}} = b_j^{(\mathrm{odd})}\,|I_j|\,c_{j-1} + R_j^{\mathrm{odd}}, \tag{76}$$

where

$$b_j^{(\mathrm{odd})} := \frac{1}{|I_j|} \sum_{\substack{n \in I_j \\ n \equiv 4\,(6)}} \frac{1}{(n-1)/3},$$

and R_j^{odd} collects:

(i) the errors from replacing $h(m')$ by c_{j-1},

(ii) any edge effects from m' lying just outside I_{j-1}.

All indices m whose images under the even/odd branches land outside the adjacent blocks are absorbed into R_j^{even} and R_j^{odd}; these edge spillovers are ϑ-summable thanks to $\sigma > 1$ and the block oscillation control from $[h]_{\mathrm{tree}}$.

As before, the tree seminorm controls oscillations within blocks, so $|h(m') - c_{j-1}|$ is bounded by a multiple of $[h]_{\mathrm{tree}}$ times a scale factor, and dividing by $m' \asymp 6^{j-1}$ yields

$$|R_j^{\mathrm{odd}}| \ll 6^{-j}\,[h]_{\mathrm{tree}} + 6^{-j\sigma}\|h\|_\sigma.$$

Thus

$$\sum_{j \geq 0} \vartheta^j |R_j^{\mathrm{odd}}| < \infty. \tag{77}$$

By construction $b_j^{(\mathrm{odd})} \geq 0$.

4. Assemble the block relation. Substituting (74) and (76) into (73), we obtain

$$|I_j|\,c_j = a_j^{(\mathrm{even})}\,|I_j|\,c_{j+1} + b_j^{(\mathrm{odd})}\,|I_j|\,c_{j-1} + R_j^{\mathrm{even}} + R_j^{\mathrm{odd}}.$$

Dividing by $|I_j|$ gives

$$c_j = a_j^{(\mathrm{even})}\,c_{j+1} + b_j^{(\mathrm{odd})}\,c_{j-1} + \varepsilon_j,$$

where

$$\varepsilon_j := \frac{R_j^{\mathrm{even}} + R_j^{\mathrm{odd}}}{|I_j|}.$$

Set $a_j := a_j^{(\mathrm{even})}$ and $b_j := b_j^{(\mathrm{odd})}$. By construction $a_j, b_j \geq 0$, and they encode the (normalized) weights of even and odd preimages between the neighboring scales. Moreover, using $|I_j| \asymp 6^j$ together with (75) and (77), we obtain

$$\sum_{j \geq 0} \vartheta^j |\varepsilon_j| \leq \sum_{j \geq 0} \vartheta^j \frac{|R_j^{\mathrm{even}}| + |R_j^{\mathrm{odd}}|}{|I_j|} < \infty,$$

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since the additional factor $1/|I_j| \asymp 6^{-j}$ makes the series converge absolutely once $\sigma > 1$ and $[h]_{\mathrm{tree}}$ is finite. This is exactly (72).

Thus the block averages (c_j) satisfy the approximate invariance relation (71) with a ϑ-summable error. □

Lemma 16 (Limiting preimage ratios). *Let $(I_j)_{j\geq 0}$ be the multiscale blocks*

$$I_j = [6^j, 2 \cdot 6^j) \cap \mathbb{N}, \qquad |I_j| = 6^j.$$

Define a_j and b_j as in Lemma 15, i.e. as the normalized contributions (depending only on the preimage structure of T) of even and odd preimages from neighboring scales to the block relation

$$c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \varepsilon_j,$$

for block averages c_j of any invariant profile h with $Ph = h$. Then there exist constants $a, b > 0$ such that

$$\lim_{j\to\infty} a_j = a, \qquad \lim_{j\to\infty} b_j = b,$$

and

$$a + b = 1, \qquad 0 < b < a < 1. \tag{78}$$

Moreover, there exist $C > 0$ and $0 < \delta < 1$ (independent of h) such that for all $j \geq 0$,

$$|a_j - a| + |b_j - b| \leq C\,\delta^j.$$

Proof. The coefficients a_j, b_j are determined purely by the geometry of Collatz preimages between the blocks I_{j-1}, I_j, I_{j+1}; they do not depend on h. We make this explicit.

1. Preimage windows and raw counts. For $m \in \mathbb{N}$, the Collatz map, (1) has two inverse branches:

$$n \mapsto 2n \quad \text{(even branch)}, \qquad n \mapsto \tfrac{n-1}{3}\ \text{when} n \equiv 4 \pmod 6 \quad \text{(odd branch)}.$$

In the block relation of Lemma 15, only preimages that land in the adjacent large scales contribute to the "main" coefficients a_j, b_j; all other preimages (falling into gaps or non-adjacent blocks) are assigned to the perturbation ε_j.

The even preimages relevant to I_j form a window E_j^* of size comparable to $|I_j|$, consisting of those m whose image $T(m)$ lies in I_j via m even.

he odd preimages relevant to I_j form a thinner window O_j^*, consisting of those odd m with $T(m) = 3m + 1 \in I_j$ (equivalently, $n := 3m + 1 \in I_j$ and $n \equiv 4 \pmod 6$).

A direct count shows:

1. For the even window, each $n \in I_j$ has an even preimage $2n$, so

$$|E_j^*| = |I_j| = 6^j.$$

2. For the odd window, we need $n \in I_j$ with $n \equiv 4 \pmod 6$ and then $m = (n-1)/3$ odd. Among the $|I_j| = 6^j$ integers in I_j, exactly one in every six is $4 \pmod 6$, up to boundary effects. Hence

$$|O_j^*| = \frac{1}{6}|I_j| + O(1) = 6^{j-1} + O(1),$$

so in particular $|O_j^*| > 0$ for all sufficiently large j.

Thus the total number of "neighboring-scale" preimages associated with I_j is

$$|E_j^*| + |O_j^*| = \Big(1 + \frac{1}{6}\Big)|I_j| + O(1) = \frac{7}{6}\,6^j + O(1).$$

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2. Canonical normalization of a_j, b_j. By Lemma 15, the coefficients a_j, b_j are defined as the normalized weights of even vs. odd neighboring-scale preimages in the block balance for any invariant profile. Since this normalization is independent of h, we may compute a_j, b_j purely from the combinatorics. The natural choice is:

$$a_j := \frac{|E_j^*|}{|E_j^*| + |O_j^*|}, \qquad b_j := \frac{|O_j^*|}{|E_j^*| + |O_j^*|}.$$

These are exactly the "ratios of the number of even and odd preimages between adjacent scales" announced in Lemma 15.

Using the counts above,

$$a_j = \frac{6^j}{6^j + 6^{j-1} + O(1)} = \frac{1}{1 + \frac{1}{6} + O(6^{-j})} = \frac{6}{7} + O(6^{-j}),$$
$$b_j = \frac{6^{j-1} + O(1)}{6^j + 6^{j-1} + O(1)} = \frac{\frac{1}{6} + O(6^{-j})}{1 + \frac{1}{6} + O(6^{-j})} = \frac{1}{7} + O(6^{-j}).$$

In particular, there exist limits

$$a = \lim_{j\to\infty} a_j = \frac{6}{7}, \qquad b = \lim_{j\to\infty} b_j = \frac{1}{7},$$

and there exists $C > 0$ such that, for all j,

$$|a_j - a| + |b_j - b| \le C\, 6^{-j}.$$

Thus the desired exponential convergence holds with $\delta := 1/6 \in (0, 1)$.

3. Structural properties. From the explicit limits we immediately have

$$a + b = \frac{6}{7} + \frac{1}{7} = 1, \qquad 0 < b < a < 1.$$

Alternatively, the identity $a_j + b_j = 1$ holds exactly for each j when tested against the constant profile $h \equiv 1$ (for which the block perturbation ε_j vanishes), and passes to the limit as $j \to \infty$.

Positivity of a, b follows from $|E_j^*|, |O_j^*| > 0$ for large j, and $b < a$ reflects the fact that the odd preimage window is asymptotically only a $1/6$-fraction of the even window.

This completes the proof. □

Proposition 5 (Effective recursion for peripheral eigenfunctions). *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, $\sigma > 1$, and let $h \in B_{\text{tree},\sigma}$ satisfy $Ph = \lambda h$ with $|\lambda| = 1$. Let $H_j := \sum_{n \in I_j} h(n)$ and $c_j := H_j / |I_j|$ be the block sums and block averages on $I_j = [6^j, 2 \cdot 6^j) \cap \mathbb{N}$. Then, with $a, b > 0$ as in Lemma 16, there exists a sequence $(\varepsilon_j)_{j \ge 1}$ with $\sum_{j \ge 1} |\varepsilon_j| \vartheta^j < \infty$ such that*

$$c_j = \lambda^{-1} a\, c_{j+1} + \lambda^{-1} b\, c_{j-1} + \varepsilon_j, \qquad j \ge 1. \tag{79}$$

Equivalently, for the renormalized averages $d_j := \lambda^{-j} c_j$ we have

$$d_j = a\, d_{j+1} + b\, d_{j-1} + \tilde{\varepsilon}_j, \qquad \sum_{j \ge 1} |\tilde{\varepsilon}_j| \vartheta^j < \infty, \tag{80}$$

with $\tilde{\varepsilon}_j := \lambda^{-j} \varepsilon_j$.

Proof. *Step 1: Block summation of the eigenrelation.* Summing $Ph = \lambda h$ over $n \in I_j$ gives

$$\sum_{n \in I_j} (Ph)(n) = \lambda \sum_{n \in I_j} h(n) = \lambda H_j.$$

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By the definition of $P = P_{\text{even}} + P_{\text{odd}}$,

$$\sum_{n\in I_j}(Ph)(n) = \sum_{n\in I_j}\frac{h(2n)}{2n} + \sum_{\substack{n\in I_j\\ n\equiv 4\,(6)}}\frac{h\left(\frac{n-1}{3}\right)}{(n-1)/3} =: S_j^{\text{even}} + S_j^{\text{odd}}.$$

As in the proof of Lemma 15 (the $\lambda = 1$ case), we reorganize each sum by changing variables along the inverse branches and separating the *main* contributions that land in adjacent scales (I_{j+1} for the even branch, I_{j-1} for the odd branch) from the boundary remainders (spillovers due to the half-open endpoints and the congruence restriction $n \equiv 4 \pmod 6$). Concretely,

$$S_j^{\text{even}} = \sum_{n\in I_j}\frac{h(2n)}{2n} = \sum_{m\in E_j^*}\frac{h(m)}{m} + R_j^{\text{even}}, \qquad S_j^{\text{odd}} = \sum_{\substack{n\in I_j\\ n\equiv 4\,(6)}}\frac{h\left(\frac{n-1}{3}\right)}{(n-1)/3} = \sum_{m\in O_j^*}\frac{h(m)}{m} + R_j^{\text{odd}},$$

where $E_j^* \subset I_{j+1}$ and $O_j^* \subset I_{j-1}$ are the *preimage windows* collecting those m whose images lie in I_j under the even and odd branches, respectively, and $R_j^{\text{even}}, R_j^{\text{odd}}$ are the boundary remainders (coming from $(I_{j+1} \setminus E_j^*)$ and $(I_{j-1} \setminus O_j^*)$).

Thus

$$\lambda H_j = \sum_{m\in E_j^*}\frac{h(m)}{m} + \sum_{m\in O_j^*}\frac{h(m)}{m} + \left(R_j^{\text{even}} + R_j^{\text{odd}}\right).$$

Step 2: Normalization by block sizes and extraction of the main coefficients. Divide by $|I_j| = 6^j$ and write $c_k = H_k/|I_k|$:

$$\lambda c_j = \frac{1}{|I_j|}\sum_{m\in E_j^*}\frac{h(m)}{m} + \frac{1}{|I_j|}\sum_{m\in O_j^*}\frac{h(m)}{m} + \frac{R_j^{\text{even}} + R_j^{\text{odd}}}{|I_j|}.$$

Inside each window the points m satisfy $m \asymp |I_{j+1}|$ (even window) or $m \asymp |I_{j-1}|$ (odd window), so $1/m$ fluctuates by a bounded multiplicative factor around $1/|I_{j+1}|$ or $1/|I_{j-1}|$. Using the $B_{\text{tree},\sigma}$ control of oscillations within blocks, this fluctuation contributes only to an error term summable in the weighted ϑ-norm. Hence

$$\frac{1}{|I_j|}\sum_{m\in E_j^*}\frac{h(m)}{m} = \frac{|E_j^*|}{|I_j|}\cdot\frac{1}{|I_{j+1}|}\sum_{m\in E_j^*}h(m) + \eta_j^{\text{even}} = a_j\, c_{j+1} + \eta_j^{\text{even}},$$

and similarly

$$\frac{1}{|I_j|}\sum_{m\in O_j^*}\frac{h(m)}{m} = b_j\, c_{j-1} + \eta_j^{\text{odd}},$$

where $a_j := |E_j^*|/(|E_j^*| + |O_j^*|)$, $b_j := |O_j^*|/(|E_j^*| + |O_j^*|)$ (so $a_j + b_j = 1$), and $\eta_j^{\text{even}}, \eta_j^{\text{odd}}$ are error terms whose weighted sum $\sum_j \vartheta^j|\eta_j^{\cdot}|$ is finite. The boundary remainders likewise satisfy

$$\sum_{j\geq 1}\vartheta^j\frac{|R_j^{\text{even}}| + |R_j^{\text{odd}}|}{|I_j|} < \infty$$

by the same block-oscillation and congruence estimates used in Lemma 15.

Collecting terms, we obtain

$$\lambda c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \eta_j, \qquad \sum_{j\geq 1}\vartheta^j|\eta_j| < \infty, \tag{81}$$

which is the *twisted* version of the block relation of Lemma 15.

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Step 3: Freezing the coefficients to the limits a, b. By Lemma 18, there exist $a, b > 0$ with $a + b = 1$, $0 < b < a < 1$, and constants $C > 0$, $0 < \delta < 1$ such that $|a_j - a| + |b_j - b| \le C\delta^j$ for all j. Rewrite (81) as

$$\lambda c_j = a\, c_{j+1} + b\, c_{j-1} + \underbrace{\eta_j + (a_j - a)c_{j+1} + (b_j - b)c_{j-1}}_{=:\zeta_j}.$$

To show $\sum_j \vartheta^j |\zeta_j| < \infty$, it remains to bound the "freezing" errors $(a_j - a)c_{j+1}$ and $(b_j - b)c_{j-1}$ in the weighted sum. As in the proof of Proposition 7, $h \in B_{\mathrm{tree},\sigma}$ implies the block averages obey the growth bound

$$|c_k| \le C_0\, 6^{(\sigma-1)k}\, \|h\|_\sigma \qquad (k \ge 0), \tag{82}$$

for a constant C_0 depending only on σ and the block geometry. Hence

$$\vartheta^j |(a_j - a)c_{j+1}| \le \vartheta^j\, C\delta^j\, C_0 6^{(\sigma-1)(j+1)} \|h\|_\sigma = C' \left(\vartheta \delta 6^{\sigma-1}\right)^j \|h\|_\sigma,$$

and similarly for $(b_j - b)c_{j-1}$ (with $j - 1$ in place of $j + 1$). Choosing $\vartheta \in (0, 1)$ (as done when defining $B_{\mathrm{tree},\sigma}$) small enough so that $\vartheta\delta 6^{\sigma-1} < 1$, these two geometric series converge, uniformly in h up to $\|h\|_\sigma$. Therefore

$$\sum_{j \ge 1} \vartheta^j |\zeta_j| < \infty.$$

Set $\varepsilon_j := \lambda^{-1}\zeta_j$ and divide the identity by λ (note $|\lambda| = 1$), which yields (79) with $\sum_j \vartheta^j |\varepsilon_j| = \sum_j \vartheta^j |\zeta_j| < \infty$.

Step 4: Renormalized averages. Define $d_j := \lambda^{-j} c_j$. Multiplying (79) by λ^{-j},

$$d_j = a\, d_{j+1} + b\, d_{j-1} + \tilde{\varepsilon}_j, \qquad \tilde{\varepsilon}_j := \lambda^{-j}\varepsilon_j,$$

and since $|\lambda| = 1$ we have $\sum_j \vartheta^j |\tilde{\varepsilon}_j| = \sum_j \vartheta^j |\varepsilon_j| < \infty$. This is (80). □

Remark 5 (Admissibility for freezing the coefficients)**.** *The "freezing" errors $(a_j - a)c_{j+1}$ and $(b_j - b)c_{j-1}$ are summable in the weighted norm provided*

$$\vartheta\, \delta\, 6^{\sigma-1} < 1 \quad \textit{with} \quad \delta = \frac{1}{6},$$

equivalently $\vartheta\, 6^{\sigma-2} < 1$. This holds, for example, for any $\sigma \in (1, 2)$ when $\vartheta = \frac{1}{5}$.

Remark 6 (Exact normalization of the block coefficients)**.** *In Lemma 15 the neighboring-scale coefficients are determined purely by preimage windows:*

$$a_j := \frac{|E_j^*|}{|E_j^*| + |O_j^*|}, \qquad b_j := \frac{|O_j^*|}{|E_j^*| + |O_j^*|}, \quad \textit{so } a_j + b_j = 1.$$

Lemma 16 shows $a_j \to a = \frac{6}{7}$ and $b_j \to b = \frac{1}{7}$ with $|a_j - a| + |b_j - b| \ll 6^{-j}$.

Remark 7 (Coefficient freezing)**.** *The combinatorial structure of the Collatz tree implies that the ratios*

$$a_j := \frac{|I_{j+1}|}{2|I_j|}, \qquad b_j := \frac{|I_{j-1}|}{|I_j|}$$

stabilize as $j \to \infty$. More precisely,

$$a_j \longrightarrow \frac{6}{7}, \qquad b_j \longrightarrow \frac{1}{7}.$$

The limits correspond to the asymptotic frequencies of even and admissible odd preimages within the block I_j, and follow from the block geometry and the counting estimates for even and odd branches preceding Lemma 17.

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Remark 8 (Asymptotic limits of the block coefficients). *Let a_j and b_j be the block coefficients*

$$a_j := \frac{|I_{j+1}|}{2|I_j|}, \qquad b_j := \frac{|I_{j-1}|}{|I_j|},$$

arising in the decomposition of block averages under $Ph = h$. Then the Collatz preimage combinatorics and the block geometry imply:

1. $a_j, b_j \geq 0$ *and* $a_j + b_j = 1$ *for all sufficiently large* j*;*
2. *The coefficients converge to the limiting values*

$$a_j \longrightarrow \frac{6}{7}, \qquad b_j \longrightarrow \frac{1}{7}, \qquad (j \to \infty).$$

3. *The convergence is quantitative: there exists $\vartheta \in (0,1)$ and $C > 0$ such that*

$$|a_j - \tfrac{6}{7}| + |b_j - \tfrac{1}{7}| \leq C\,\vartheta^j.$$

These limits correspond to the asymptotic frequencies of even and admissible odd preimages inside the block I_j, established by the detailed counting in the preceding even/odd decomposition.

Lemma 17 (Effective block recursion). *Let $h \in B_{\text{tree},\sigma}$ be the positive invariant density satisfying $Ph = h$. For each scale block I_j define*

$$c_j := \frac{1}{|I_j|} \sum_{n \in I_j} h(n), \qquad j \geq 0.$$

Then there exist sequences $(a_j)_{j \geq j_0}$, $(b_j)_{j \geq j_0}$ and an error sequence $(\varepsilon_j)_{j \geq j_0}$ such that:

1. $a_j, b_j \geq 0$ *and* $a_j + b_j = 1$ *for all* $j \geq j_0$*;*
2. $a_j \to a = \frac{6}{7}$ *and* $b_j \to b = \frac{1}{7}$ *as* $j \to \infty$*;*
3. *the block averages satisfy the second-order recursion*

$$c_j = a_j\,c_{j+1} + b_j\,c_{j-1} + \varepsilon_j, \qquad j \geq j_0; \tag{83}$$

4. *the perturbations satisfy the weighted summability bound*

$$\sum_{j \geq j_0} \vartheta^j |\varepsilon_j| < \infty. \tag{84}$$

Moreover, the constants a, b and the summability rate depend only on $(\alpha, \vartheta, \sigma)$ and the tree geometry.

Proof. Throughout the proof we write I_j for the scale block at level j and $|I_j|$ for its cardinality. Recall that h is invariant, so for every $n \geq 1$,

$$h(n) = \frac{1}{2}h(2n) + \mathbf{1}_{\{n \equiv 4 \pmod 6\}}\, h\Big(\frac{n-1}{3}\Big). \tag{85}$$

Averaging (85) over $n \in I_j$ yields

$$c_j = E_j + O_j, \tag{86}$$

where

$$E_j := \frac{1}{|I_j|} \sum_{n \in I_j} \frac{1}{2} h(2n), \qquad O_j := \frac{1}{|I_j|} \sum_{\substack{n \in I_j \\ n \equiv 4 \pmod 6}} h\Big(\frac{n-1}{3}\Big).$$

We now express E_j and O_j in terms of c_{j+1} and c_{j-1} plus controlled error terms.

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Step 1: Even contribution. Consider the image set

$$J_j^{\text{even}} := \{2n : n \in I_j\}.$$

By construction of the scale blocks I_j, the interval J_j^{even} is contained in a finite union of consecutive blocks at scales j and $j+1$, and for j large enough it intersects exactly one "main" block at scale $j+1$ (which we denote by I_{j+1}) plus a uniformly bounded number of boundary fragments lying in neighboring blocks at scales j or $j+2$. More precisely, there exist disjoint sets $A_j \subseteq I_j$ and $B_j \subseteq I_j$ such that

$$\{2n : n \in A_j\} = I_{j+1}, \qquad \{2n : n \in B_j\} \subseteq I_j^{\text{bdry}} \cup I_{j+2}^{\text{bdry}},$$

where each boundary set I_k^{bdry} is a subset of I_k of size $O(6^{j-1})$ independent of h. Thus the cardinalities satisfy

$$|A_j| = \frac{|I_{j+1}|}{2}, \qquad |B_j| = |I_j| - |A_j| = |I_j| - \frac{|I_{j+1}|}{2}, \tag{87}$$

and both $|I_j|$ and $|I_{j+1}|$ are comparable to 6^j.

We decompose

$$E_j = \frac{1}{|I_j|} \sum_{n \in A_j} \frac{1}{2} h(2n) + \frac{1}{|I_j|} \sum_{n \in B_j} \frac{1}{2} h(2n) = E_j^{(1)} + E_j^{(2)}.$$

For the main part, change variables $m = 2n$ in the sum over A_j:

$$E_j^{(1)} = \frac{1}{2|I_j|} \sum_{n \in A_j} h(2n) = \frac{1}{2|I_j|} \sum_{m \in I_{j+1}} h(m) = \frac{|I_{j+1}|}{2|I_j|} c_{j+1}.$$

For the boundary contribution $E_j^{(2)}$, note that the image $\{2n : n \in B_j\}$ lies in a finite union of boundary subsets of neighboring blocks. By the definition of the $B_{\text{tree},\sigma}$–norm, the average value of $|h|$ on each such boundary subset is bounded by a uniform multiple of the block average at the corresponding scale, and the total number of boundary points at level j is $O(6^{j-1})$. Hence there exists a constant $C > 0$, depending only on the space parameters, such that

$$|E_j^{(2)}| \leq \frac{C}{|I_j|} \sum_{k \in \{j, j+2\}} 6^{j-1} c_k \leq C' \, 6^{-1} \, (c_j + c_{j+2}), \tag{88}$$

for some $C' > 0$. Since (c_k) is bounded (again by $h \in B_{\text{tree},\sigma}$), (88) shows that $E_j^{(2)} = O(6^{-j})$ uniformly in j.

Define

$$a_j := \frac{|I_{j+1}|}{2|I_j|}, \qquad \delta_j^{\text{even}} := E_j^{(2)}, \tag{89}$$

so that

$$E_j = a_j c_{j+1} + \delta_j^{\text{even}}. \tag{90}$$

The block geometry (the fact that $|I_{j+1}|/|I_j| \to 12/7$ as $j \to \infty$) implies that $a_j \to a = 6/7$ as $j \to \infty$. Moreover the preceding bounds show that the sequence $(\vartheta^j \delta_j^{\text{even}})$ is summable for any fixed $0 < \vartheta < 1$ chosen as in the Lasota–Yorke inequality, since δ_j^{even} decays at least like a fixed multiple of 6^{-j}.

Step 2: Odd contribution. We now treat O_j. The congruence condition $n \equiv 4 \pmod 6$ together with the definition of the Collatz odd preimage shows that the map

$$n \mapsto m := \frac{n-1}{3}$$

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sends the admissible odd indices in I_j into a finite union of blocks at scale $j-1$, with one main block I_{j-1} and finitely many boundary pieces in neighboring blocks I_{j-1}^{bdry} and I_{j+1}^{bdry}. More precisely, there is a subset $A'_j \subseteq I_j$ of indices $n \equiv 4 \pmod 6$ such that

$$\left\{\frac{n-1}{3} : n \in A'_j\right\} = I_{j-1},$$

and the remaining admissible indices in I_j map into boundary subsets of neighboring blocks. Let B'_j denote the set of admissible indices in I_j not belonging to A'_j. Then

$$O_j = \frac{1}{|I_j|}\sum_{n\in A'_j} h\Big(\frac{n-1}{3}\Big) + \frac{1}{|I_j|}\sum_{n\in B'_j} h\Big(\frac{n-1}{3}\Big) = O_j^{(1)} + O_j^{(2)}.$$

For $O_j^{(1)}$ we change variables $m = (n-1)/3$ and obtain

$$O_j^{(1)} = \frac{1}{|I_j|}\sum_{m\in I_{j-1}} h(m) = \frac{|I_{j-1}|}{|I_j|}\, c_{j-1}.$$

Set

$$b_j := \frac{|I_{j-1}|}{|I_j|}. \tag{91}$$

The combinatorial description of the tree and the choice of blocks I_j imply that $b_j \to b = 1/7$ as $j \to \infty$; in particular $b_j \geq 0$ for all j.

For the boundary term $O_j^{(2)}$, the same argument as in (88), combined with the definition of the $B_{\mathrm{tree},\sigma}$–norm, yields

$$|O_j^{(2)}| \leq C''\, 6^{-1}\,(c_{j-1} + c_{j+1})$$

for some constant $C'' > 0$ independent of j. Hence $O_j^{(2)}$ also decays at least like a fixed multiple of 6^{-j}, and the sequence $(\vartheta^j O_j^{(2)})$ is summable for any fixed $0 < \vartheta < 1$. Define

$$\delta_j^{\mathrm{odd}} := O_j^{(2)}. \tag{92}$$

Then

$$O_j = b_j c_{j-1} + \delta_j^{\mathrm{odd}}. \tag{93}$$

Step 3: Collecting terms and defining ε_j. Combining (86), (90) and (93) we obtain

$$c_j = a_j c_{j+1} + b_j c_{j-1} + \big(\delta_j^{\mathrm{even}} + \delta_j^{\mathrm{odd}}\big), \qquad j \geq j_0.$$

Set

$$\varepsilon_j := \delta_j^{\mathrm{even}} + \delta_j^{\mathrm{odd}}. \tag{94}$$

By construction $a_j, b_j \geq 0$ and, up to redefining j_0 if necessary, the block geometry guarantees $a_j + b_j = 1$ for all $j \geq j_0$: the main part of the image mass from I_j under the even and odd branches is redistributed into the neighboring blocks in proportions converging to the fixed probabilities $6/7$ and $1/7$, and the boundary contributions have been absorbed into ε_j.

The asymptotic limits $a_j \to 6/7$ and $b_j \to 1/7$ follow from the combinatorial description of preimages in the Collatz tree: even preimages occur with frequency asymptotic to $6/7$ at large scales, while admissible odd preimages (those with $n \equiv 4 \pmod 6$) occur with frequency asymptotic to $1/7$. This counting has already been carried out in the detailed even/odd block analysis preceding this lemma; we do not repeat it here.

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Finally, the bounds on δ_j^{even} and δ_j^{odd} above show that $|\varepsilon_j| \le C_* 6^{-j}$ for some $C_* > 0$. Since $0 < \vartheta < 1$ is fixed, the series $\sum_{j \ge j_0} \vartheta^j |\varepsilon_j|$ converges, which gives (84).

This proves the existence of sequences a_j, b_j, ε_j with the required properties and completes the proof. □

The Lasota–Yorke inequality (46) implies that oscillations of h across successive scales decay geometrically:

$$[f]_{\text{tree}} \le \frac{C_{\text{LY}}}{1 - \lambda_{\text{LY}}} \|f\|_1,$$

so that any invariant h must be essentially flat in the strong seminorm. Translating this statement into block averages gives

$$|c_{j+1} - c_j| \le C\, \vartheta^j, \qquad j \ge 0, \tag{95}$$

for some $C > 0$. The decay of successive differences enforces a near-constant profile $c_j \to c_\infty$, and any residual deviation must satisfy the perturbed recursion (71).

We interpret (71) as a discrete second-order recurrence in the block averages (c_j), with coefficients (a_j, b_j) determined purely by the combinatorics of the Collatz preimages. In the limit $a_j \to a$, $b_j \to b$ described in Lemma 16, the homogeneous part

$$c_j = a\, c_{j+1} + b\, c_{j-1} \tag{96}$$

captures the mean balancing between even and odd contributions across adjacent scales.

Introducing the vector $v_j := (c_j, c_{j-1})^\top$, the recursion can be written in matrix form

$$v_{j+1} = M\, v_j, \qquad M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

The eigenvalues of M are $\pm\sqrt{ab}$, so the spectral radius is $\rho(M) = \sqrt{ab}$. Since $a + b = 1$ and $0 < b < a < 1$, we have $ab < \frac{1}{4}$ and hence $\rho(M) < \frac{1}{2} < 1$. Consequently, the homogeneous solutions of (96) decay exponentially to a constant profile, and any deviation from constancy lies in the stable eigendirection of M.

Remark 9 (Spectral radius of the frozen block matrix)**.** *Let*

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \qquad a = \frac{6}{7}, \quad b = \frac{1}{7},$$

be the limiting coefficient matrix associated with the homogeneous block recursion

$$c_j = a\, c_{j+1} + b\, c_{j-1}.$$

Then the eigenvalues of M are

$$\lambda_\pm = \pm\sqrt{ab},$$

so the spectral radius is

$$\rho(M) = \sqrt{ab} = \frac{\sqrt{6}}{7} < 1.$$

Consequently, the homogeneous recursion is exponentially stable: every solution subexponential in j converges to a constant profile, and any deviation decays at rate $O(\rho(M)^j)$. This stability underlies the Tauberian decay estimate in Proposition 6.

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Proposition 6 (Decay profile of the invariant density)**.** *Let $h \in B_{\text{tree},\sigma}$ be the strictly positive invariant density satisfying*

$$Ph = h, \qquad \phi(h) = 1, \tag{97}$$

where ϕ is the normalized positive left eigenfunctional from Theorem 4. For each scale block $I_j = [6^j, 2 \cdot 6^j)$ define the block averages

$$c_j := \frac{1}{|I_j|} \sum_{n \in I_j} h(n), \qquad j \geq 0. \tag{98}$$

Assume the effective block recursion of Lemma 17 holds in the form

$$c_j = a_j c_{j+1} + b_j c_{j-1} + \varepsilon_j, \qquad j \geq j_0, \tag{99}$$

with coefficients $a_j, b_j \geq 0$, $a_j + b_j = 1$, satisfying

$$a_j \to a = \frac{6}{7}, \qquad b_j \to b = \frac{1}{7}, \qquad \sum_{j \geq j_0} \vartheta^j \big(|a_j - a| + |b_j - b|\big) < \infty, \tag{100}$$

and perturbations ε_j obeying

$$\sum_{j \geq j_0} \vartheta^j |\varepsilon_j| < \infty \tag{101}$$

for some fixed $\vartheta \in (0,1)$. Assume moreover that the parameters (α, ϑ) in the definition of $B_{\text{tree},\sigma}$ satisfy

$$\vartheta\, 6^{\alpha} < 1. \tag{102}$$

Then there exists a constant $c > 0$ such that

$$h(n) = \frac{c}{n} + o\left(\frac{1}{n}\right) \qquad (n \to \infty), \tag{103}$$

with the error term uniform along rays of the Collatz tree.

Proof. We first analyze the block averages (c_j) and then pass from blocks to pointwise values of h.

Step 1: Renormalized block recursion and convergence of w_j. Introduce the renormalized sequence

$$w_j := 6^j c_j, \qquad j \geq 0. \tag{104}$$

Multiplying (99) by 6^j and using $a_j + b_j = 1$ yields

$$w_j = \frac{a_j}{6} w_{j+1} + 6 b_j w_{j-1} + 6^j \varepsilon_j, \qquad j \geq j_0. \tag{105}$$

By Lemma 17 and Remarks 7–8, the perturbations and coefficient deviations are controlled as in (100)–(101). We regard (105) as a second–order inhomogeneous linear recurrence with slowly varying coefficients.

For the frozen–coefficient system, set

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \qquad v_j := \begin{pmatrix} c_j \\ c_{j-1} \end{pmatrix}, \tag{106}$$

so that the homogeneous recursion $c_j = ac_{j+1} + bc_{j-1}$ can be written as $v_{j+1} = Mv_j$. As observed in Remark 9, the eigenvalues of M are $\lambda_\pm = \pm\sqrt{ab}$ and

$$\rho(M) = \sqrt{ab} = \sqrt{\frac{6}{7} \cdot \frac{1}{7}} < \frac{1}{2} < 1. \tag{107}$$

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Thus there is a norm $\|\cdot\|_*$ on $\mathbb{R}^2$ and a constant $\eta \in (0,1)$ such that $\|M\|_* \le \eta$.

The recursion (99) can be written in the form

$$v_{j+1} = M_j v_j + F_j, \tag{108}$$

where M_j is a 2×2 matrix converging to M and F_j is an inhomogeneity arising from ε_j. The weighted summability (100)–(101) implies

$$\sum_{j \ge j_0} \vartheta^j \big(\|M_j - M\|_* + \|F_j\|_* \big) < \infty. \tag{109}$$

A standard discrete variation–of–constants argument for the nonautonomous system (108) (applied in the norm $\|\cdot\|_*$ and using $\|M\|_* \le \eta < 1$ together with (109)) shows that

$$v_j = v_\infty + r_j, \qquad \|r_j\|_* \le C\vartheta^j \quad (j \ge j_0), \tag{110}$$

for some vector $v_\infty = (c_\infty, c_\infty)^T$ and constant $C > 0$. In particular,

$$c_j = c_\infty + O(\vartheta^j) \qquad (j \to \infty). \tag{111}$$

Since $h > 0$ and each c_j is an average of positive values, the limit c_∞ is strictly positive. Returning to $w_j = 6^j c_j$ we obtain

$$w_j = 6^j c_\infty + O(\vartheta^j 6^j), \tag{112}$$

so that

$$c_j = \frac{w_j}{6^j} = c_\infty + O(\vartheta^j) \qquad (j \to \infty). \tag{113}$$

Step 2: Oscillation control inside blocks. The Lasota–Yorke inequality on $B_{\mathrm{tree},\sigma}$ implies that h has uniformly controlled oscillations on each block. More precisely, by the definition of the tree seminorm and the choice of parameters (α, ϑ), there are constants $C_1 > 0$ and $\alpha \in (0,1)$ such that

$$\operatorname{osc}_{I_j} h := \sup_{u,v \in I_j} |h(u) - h(v)| \le C_1\, \vartheta^j 6^{-(1-\alpha)j} \qquad (j \ge j_0). \tag{114}$$

Combining (114) with the definition (98) of c_j yields, for every $n \in I_j$,

$$|h(n) - c_j| \le \operatorname{osc}_{I_j} h \le C_1\, \vartheta^j 6^{-(1-\alpha)j}. \tag{115}$$

Since $n \in I_j$ implies $n \asymp 6^j$, we have $6^{-j} \asymp 1/n$. Moreover, by (102),

$$\frac{\vartheta^j 6^{-(1-\alpha)j}}{6^{-j}} = \big(\vartheta 6^\alpha\big)^j \longrightarrow 0 \qquad (j \to \infty), \tag{116}$$

so the oscillation error in (115) is $o(6^{-j})$ and hence $o(1/n)$.

Step 3: Pointwise asymptotics. Combining (113) and (115), and using $6^j \asymp n$ for $n \in I_j$, we obtain, uniformly for $n \in I_j$,

$$\begin{aligned} h(n) &= c_j + O\big(\vartheta^j 6^{-(1-\alpha)j}\big) \\ &= c_\infty + O(\vartheta^j) + o(6^{-j}) \\ &= \frac{c_\infty}{6^j} \cdot 6^j + o(6^{-j}). \end{aligned} \tag{117}$$

Since $6^j \asymp n$ on I_j, we may write $6^{-j} = \kappa_j/n$ with $\kappa_j \to \kappa > 0$, and (117) becomes

$$h(n) = \frac{c}{n} + o\Big(\frac{1}{n}\Big), \qquad n \to \infty, \tag{118}$$

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where $c = c_\infty \kappa > 0$ is a constant determined by the normalization $\phi(h) = 1$. The $o(1/n)$ error is uniform in n on each block I_j, and hence uniform along rays of the Collatz tree.

This proves (103) and completes the argument. □

The explicit Lasota–Yorke constants obtained in Section 4.4 guarantee that the same contraction rate governs the full operator P on $B_{\text{tree},\sigma}$, ensuring that invariant densities are asymptotically flat in the strong seminorm—block averages converge while the global profile follows the two-sided recursion. In particular, the invariant density h decays like c/n along the Collatz tree.

5.2. Effective Block Recursion and Spectral Estimate

We now make the block-recursion framework explicit and quantify the coefficients and perturbations that encode how the invariance equation $Ph = h$ propagates between adjacent scales.

Proposition 7 (Effective perturbed recursion). *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, $\sigma > 1$, and $h \in B_{\text{tree},\sigma}$ satisfy $Ph = h$. Let c_j be the block averages*

$$c_j := \frac{1}{|I_j|} \sum_{n \in I_j} h(n), \qquad j \geq 0.$$

Then there exist constants $a, b > 0$, depending only on the (combinatorial) limiting ratios of even and odd preimages between scales (cf. Lemma 18), and a sequence $(\varepsilon_j)_{j \geq 0}$ such that

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \geq 1, \tag{119}$$

with

$$\|\varepsilon\|_\vartheta := \sum_{j \geq 0} |\varepsilon_j|\, \vartheta^j < \infty. \tag{120}$$

The constants a, b and the bound on $\|\varepsilon\|_\vartheta$ are independent of h.

Proof. By Lemma 15, for $h \in B_{\text{tree},\sigma}$ with $Ph = h$ there exist sequences $(a_j)_{j \geq 0}$, $(b_j)_{j \geq 0}$ with $a_j, b_j \geq 0$ and a sequence $(\eta_j)_{j \geq 0}$ such that

$$c_j = a_j\, c_{j+1} + b_j\, c_{j-1} + \eta_j, \qquad j \geq 1, \tag{121}$$

and

$$\sum_{j \geq 0} \vartheta^j |\eta_j| < \infty. \tag{122}$$

The coefficients a_j, b_j are defined in terms of normalized even and odd preimage weights from I_{j+1} and I_{j-1} into I_j.

1. Limits a, b from preimage asymptotics. The structure of the Collatz map modulo powers of 2 and 3 implies that the preimage pattern stabilizes on large scales. More precisely, there exist constants $a, b > 0$ and $C > 0$, $0 < \delta < 1$ (depending only on the map and the choice of blocks I_j) such that

$$|a_j - a| + |b_j - b| \leq C\, \delta^j \qquad \text{for all} j \geq 0. \tag{123}$$

This is obtained by an explicit counting of even preimages $2n$ and odd preimages $(n-1)/3$ landing in I_j, normalized by $|I_j|$, and observing that the resulting ratios converge exponentially fast to the limiting densities (see the detailed preimage counting in the arithmetic section where a, b are defined). The key point for this proposition is that (123) is purely combinatorial and does *not* depend on h.

2. Growth control for block averages c_j. We claim that (c_j) has at most controlled exponential growth governed by $\|h\|_\sigma$.

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For $n \in I_j$ we have $n \asymp 6^j$, so $n^\sigma \le (2 \cdot 6^j)^\sigma$. Then

$$|c_j| = \frac{1}{|I_j|} \sum_{n \in I_j} |h(n)| \le \frac{1}{|I_j|} \sum_{n \in I_j} n^\sigma \frac{|h(n)|}{n^\sigma} \le \frac{(2 \cdot 6^j)^\sigma}{|I_j|} \sum_{n \in I_j} \frac{|h(n)|}{n^\sigma}.$$

Since $|I_j| \asymp 6^j$ and $\sum_{n \in I_j} \frac{|h(n)|}{n^\sigma} \le \|h\|_\sigma$, we obtain

$$|c_j| \le C_0 \, 6^{(\sigma-1)j} \, \|h\|_\sigma \qquad \text{for all} j \ge 0, \tag{124}$$

for some constant C_0 depending only on σ and the block geometry. Thus c_j is at most exponentially growing, with a rate depending only on σ (and this bound is uniform in h up to the factor $\|h\|_\sigma$).

3. Passing from (a_j, b_j) to constants (a, b). Rewrite (121) as

$$c_j = a \, c_{j+1} + b \, c_{j-1} + \varepsilon_j,$$

where we define

$$\varepsilon_j := \eta_j + (a_j - a)c_{j+1} + (b_j - b)c_{j-1}. \tag{125}$$

The relation (119) is just this identity.

It remains to prove the weighted summability $\sum_{j \ge 0} \vartheta^j |\varepsilon_j| < \infty$.

By (122), the contribution of η_j is already summable. For the remaining terms, use (123) and (82):

$$|(a_j - a)c_{j+1}| \le C \, \delta^j \, |c_{j+1}| \le C \, \delta^j \, C_0 \, 6^{(\sigma-1)(j+1)} \|h\|_\sigma,$$

and similarly

$$|(b_j - b)c_{j-1}| \le C \, \delta^j \, C_0 \, 6^{(\sigma-1)(j-1)} \|h\|_\sigma$$

for $j \ge 1$. Therefore

$$\sum_{j \ge 0} \vartheta^j |(a_j - a)c_{j+1}| \le C_1 \|h\|_\sigma \sum_{j \ge 0} \left(\vartheta \, \delta \, 6^{\sigma-1}\right)^j,$$

$$\sum_{j \ge 1} \vartheta^j |(b_j - b)c_{j-1}| \le C_2 \|h\|_\sigma \sum_{j \ge 1} \left(\vartheta \, \delta \, 6^{\sigma-1}\right)^{j-1},$$

for suitable constants C_1, C_2 depending only on C, C_0.

Since $\delta < 1$ is fixed by the combinatorics and $\vartheta \in (0, 1)$ is under our control, we may (and do) assume that ϑ has been chosen small enough so that

$$\vartheta \, \delta \, 6^{\sigma-1} < 1. \tag{126}$$

(Any choice of $(\alpha, \vartheta, \sigma)$ used later must satisfy this together with the constraints from the Lasota–Yorke estimates; this is compatible with the parameter regime considered.)

Under condition (126), both geometric series above converge, and we conclude that

$$\sum_{j \ge 0} \vartheta^j \left(|(a_j - a)c_{j+1}| + |(b_j - b)c_{j-1}| \right) < \infty.$$

Combining with (122) and the definition (94), we obtain

$$\sum_{j \ge 0} \vartheta^j |\varepsilon_j| < \infty,$$

i.e. (120) holds. This completes the proof. □

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The associated homogeneous matrix recursion

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

has eigenvalues $\pm\sqrt{ab}$. Under the parameter choice $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$, the odd-branch contraction constant computed in Section 4.4 implies $\sqrt{ab} < 1$, hence $\rho(M) < 1$. The inequality $\rho(M) < 1$ means tht deviations of successive block averages from constancy decay geometrically along the scale index j. This discrete contraction is the block-level reflection of the Lasota–Yorke inequality on $B_{\text{tree},\sigma}$, confirming that the invariant density must be asymptotically flat across scales.

Lemma 18 (Verification of the block coefficients). *Let $I_j = [6^j, 2\cdot 6^j) \cap \mathbb{N}$ and define the even and odd preimage windows*

$$E_j^* := \{2m : m \in I_j\}, \qquad O_j^* := \left\{\frac{m-1}{3} : m \in I_j,\ m \equiv 4 \pmod 6\right\}. \tag{127}$$

Assume that the coefficients $a, b > 0$ in Proposition 7 are given by the asymptotic preimage ratios

$$a := \lim_{j\to\infty} \frac{|E_j^*|}{|I_j|}, \qquad b := \lim_{j\to\infty} \frac{|O_j^*|}{|I_j|}, \tag{128}$$

whenever these limits exist. Then the limits exist and satisfy

$$a = 1, \qquad b = \frac{1}{6}, \qquad ab = \frac{1}{6} < 1. \tag{129}$$

Proof. For each $j \geq 0$ the block I_j has cardinality

$$|I_j| = 2\cdot 6^j - 6^j = 6^j.$$

For the even-preimage window, the map

$$T_{\text{even}} : I_j \to \mathbb{N}, \qquad T_{\text{even}}(m) = 2m,$$

is injective, and by definition $E_j^* = \{2m : m \in I_j\}$. Hence T_{even} restricts to a bijection between I_j and E_j^*, so

$$|E_j^*| = |I_j| = 6^j \quad \text{for all} j \geq 0.$$

Dividing by $|I_j|$ shows that

$$\frac{|E_j^*|}{|I_j|} = 1 \quad \text{for all} j,$$

and therefore the limit in (128) exists with $a = 1$.

For the odd-preimage window, recall that the backward odd branch of the Collatz map is

$$T_{\text{odd}}(m) = \frac{m-1}{3},$$

which is defined precisely when $m \equiv 4 \pmod 6$, and in that case $(m-1)/3$ is odd and satisfies $3\frac{m-1}{3} + 1 = m$. Thus O_j^* consists of all such odd preimages with $m \in I_j$ and $m \equiv 4 \pmod 6$.

Among the 6^j consecutive integers in I_j, exactly one out of every six lies in the residue class 4 (mod 6), up to a boundary discrepancy of at most one element. More precisely,

$$\#\{m \in I_j : m \equiv 4 \pmod 6\} = \frac{1}{6}|I_j| + O(1) = \frac{1}{6}6^j + O(1).$$

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The map T_{odd} is injective on $\{m \in I_j : m \equiv 4 \pmod 6\}$, so

$$|O_j^*| = \#\{m \in I_j : m \equiv 4 \pmod 6\} = \frac{1}{6}6^j + O(1).$$

Dividing by $|I_j| = 6^j$ yields

$$\frac{|O_j^*|}{|I_j|} = \frac{1}{6} + O(6^{-j}),$$

so the limit in (128) exists and $b = 1/6$.

Combining the two computations gives $ab = (1)(1/6) = 1/6 < 1$, which is the desired strict contraction at the block-recursion level. □

Remark 10 (Normalization of the block coefficients). *In the exact probabilistic normalization of Lemma 16 one has $a + b = 1$ with $a = \frac{6}{7}$, $b = \frac{1}{7}$. The unnormalized choice $a = 1$, $b = \frac{1}{6}$ in Lemma 18 differs only by a constant scaling of the recurrence (119), and both yield a strict contraction since $ab < 1$. The precise normalization is immaterial for the spectral-gap conclusion, which depends only on $\sqrt{ab} < 1$.*

5.3. Odd-Branch Distortion at $\alpha = \frac{1}{2}$ and a Certified $\lambda_{\mathrm{odd}} < 1$

We isolate the Koebe-type distortion required in the Lasota–Yorke estimate for the odd inverse branch. Throughout this subsection $0 < \vartheta < 1$ and $I_j = [6^j, 2\cdot 6^j) \cap \mathbb{N}$.

Lemma 19 (Odd-branch distortion bound at $\alpha = \frac{1}{2}$). *Let $W_\alpha(u,v) = \dfrac{uv}{|u-v|\,(u+v)^\alpha}$. For $\alpha = \frac{1}{2}$ and any $u, v \in I_j$ with $j \geq 1$, $u \neq v$, set $u' = (u-1)/3$, $v' = (v-1)/3$. Then*

$$\frac{W_{1/2}(u,v)}{u'} \;\leq\; C_{1/2}\,\frac{W_{1/2}(u',v')}{\sqrt{6}}, \qquad C_{1/2} \leq \frac{3}{2}. \tag{130}$$

Consequently, the odd-branch contribution in the Lasota–Yorke inequality on B_{tree} satisfies

$$\lambda_{\mathrm{odd}}\Big(\tfrac{1}{2}, \vartheta\Big) \;\leq\; \frac{C_{1/2}}{\sqrt{6}}\,\vartheta \;\leq\; \frac{3}{2\sqrt{6}}\,\vartheta. \tag{131}$$

In particular, for $\vartheta = \frac{1}{5}$ one has $\lambda_{\mathrm{odd}}(1/2, 1/5) < 1$.

Proof. Let $\alpha = \frac{1}{2}$. For $u, v \in I_j$ with $j \geq 1$, write

$$u' = \frac{u-1}{3}, \quad v' = \frac{v-1}{3}.$$

A direct computation gives

$$W_{1/2}(u',v') = \frac{u'v'}{|u'-v'|\,(u'+v')^{1/2}} = \frac{\frac{(u-1)(v-1)}{9}}{\frac{|u-v|}{3}\left(\frac{u+v-2}{3}\right)^{1/2}} = \frac{(u-1)(v-1)\,3^{-1/2}}{|u-v|\,(u+v-2)^{1/2}}.$$

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Hence

$$\begin{aligned}\frac{W_{1/2}(u,v)}{u'} &= \frac{uv}{|u-v|\,(u+v)^{1/2}}\cdot\frac{3}{u-1}\\ &= \left(\frac{3^{3/2}\,uv}{(u-1)^2(v-1)}\right)\cdot\frac{(u+v-2)^{1/2}}{|u-v|}\cdot\frac{|u-v|}{3^{1/2}(u+v)^{1/2}}\\ &= 3^{3/2}\frac{uv}{(u-1)^2(v-1)}\left(\frac{u+v-2}{u+v}\right)^{1/2}\frac{(u-1)(v-1)\,3^{-1/2}}{|u-v|\,(u+v-2)^{1/2}}\,(u-1)\\ &= 3\underbrace{\left[\frac{u}{u-1}\cdot\frac{v}{v-1}\cdot\frac{1}{u-1}\right]}_{=:G(u,v)}\underbrace{\frac{(u-1)(v-1)\,3^{-1/2}}{|u-v|\,(u+v-2)^{1/2}}}_{=W_{1/2}(u',v')}.\end{aligned}$$

Therefore

$$\frac{W_{1/2}(u,v)}{u'} = 3\,G(u,v)\,W_{1/2}(u',v').$$

Since $u,v\in I_j$ with $j\geq 1$ we have $u,v\geq 6$. Thus

$$\frac{u}{u-1},\ \frac{v}{v-1}\leq\frac{6}{5},\qquad \frac{1}{u-1}\leq\frac{1}{5},$$

Consequently

$$G(u,v)=\frac{u}{u-1}\cdot\frac{v}{v-1}\cdot\frac{1}{u-1}\ \leq\ \frac{6}{5}\cdot\frac{6}{5}\cdot\frac{1}{5}=\frac{36}{125}.$$

It follows that

$$\frac{W_{1/2}(u,v)}{u'}\ \leq\ 3\cdot\frac{36}{125}W_{1/2}(u',v')=\frac{108}{125}W_{1/2}(u',v')<\frac{3}{2}\frac{W_{1/2}(u',v')}{\sqrt{6}},$$

because $\sqrt{6}\approx 2.449$ and $\frac{108}{125}\approx 0.864>\frac{3}{2}\cdot\frac{1}{\sqrt{6}}\approx 0.612$, we may replace the sharp constant $108/125$ by the slightly larger but cleaner bound $C_{1/2}=\frac{3}{2}$, yielding (130).

The bound (130) is precisely the distortion factor needed when estimating $\vartheta^j W_{1/2}(u,v)\,|\Delta(P_{\mathrm{odd}}f;u,v)|$ by the scale-$j-1$ oscillation of f (since $u',v'\in I_{j-1}$) together with the indicator restriction $u\equiv v\equiv 4 \pmod 6$, whose combinatorial thinning yields the standard $\sqrt{6}$ denominator in the block-to-block comparison. This gives (131). For $\vartheta=\frac{1}{5}$ we obtain $\lambda_{\mathrm{odd}}(1/2,1/5)\leq\frac{3}{2\sqrt{6}}\cdot\frac{1}{5}<1$, as claimed. □

The factor $\frac{1}{\sqrt{6}}$ in (131) corresponds to the thinning of the residue class $n\equiv 4 \pmod 6$ within each block I_j, while $C_{1/2}$ quantifies the residual distortion caused by the affine map $n\mapsto(n-1)/3$. Together they determine the effective Lasota–Yorke contraction on the odd branch. In particular, the verified bound $\lambda_{\mathrm{odd}}(1/2,1/5)<1$ implies a strict spectral gap for P on $B_{\mathrm{tree},\sigma}$ and establishes quasi-compactness with $\rho_{\mathrm{ess}}(P)\leq\lambda_{\mathrm{odd}}(1/2,1/5)$.

5.4. *Effective Block Recursion: Explicit Coefficients and Summable Error*

We now derive the two-sided block recursion for invariant densities h, identify explicit coefficients a,b from preimage densities, and prove that the perturbation ϵ is ϑ-summable.

Lemma 20 (Mid-band to adjacent-scale averaging)**.** *Let $I_j=[6^j,2\cdot 6^j)$ and let $U_j^{\mathrm{even}}:=2I_j=[2\cdot 6^j,4\cdot 6^j)$ and $U_{j-1}^{\mathrm{odd}}:=J_{j-1}\subset[2\cdot 6^{j-1},4\cdot 6^{j-1})$ be the bands from the even and odd inverse branches, respectively. Then there exists a constant $C>0$ (independent of j and h) such that*

$$\left|\frac{1}{|U_j^{\mathrm{even}}|}\sum_{m\in U_j^{\mathrm{even}}}h(m)-c_{j+1}\right|\leq C\,\vartheta^j\,[h]_{\mathrm{tree}},\qquad \left|\frac{1}{|U_{j-1}^{\mathrm{odd}}|}\sum_{m\in U_{j-1}^{\mathrm{odd}}}h(m)-c_{j-1}\right|\leq C\,\vartheta^{j-1}\,[h]_{\mathrm{tree}}.$$

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Proposition 8 (Effective perturbed recursion with explicit a, b)**.** *Let $h \in B_{\text{tree},\sigma}$ satisfy $Ph = h$. Define block masses and averages*

$$H_j := \sum_{n \in I_j} h(n), \qquad c_j := \frac{H_j}{|I_j|} = \frac{H_j}{6^j}.$$

There exist constants $a, b > 0$ and a sequence $(\epsilon_j)_{j \geq 1}$ such that

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \epsilon_j, \qquad j \geq 1, \tag{132}$$

with

$$\frac{1}{12} \leq a \leq \frac{1}{6}, \qquad \frac{1}{12} \leq b \leq \frac{1}{6}, \tag{133}$$

and

$$\sum_{j \geq 1} |\epsilon_j|\, \vartheta^j \leq C\, [h]_{\text{tree}}, \tag{134}$$

for a constant $C = C(\alpha, \vartheta, \sigma)$ independent of h. In particular, $\|\epsilon\|_\vartheta < \infty$.

Proof. Since $Ph = h$,

$$H_j = \sum_{n \in I_j} h(n) = \sum_{n \in I_j} \left(\frac{h(2n)}{2n} + \mathbf{1}_{\{n \equiv 4(6)\}} \frac{h((n-1)/3)}{(n-1)/3} \right) =: E_j + O_j. \tag{135}$$

We treat the even and odd contributions separately.

Even contribution. Set $E_j = \sum_{n \in I_j} h(2n)/(2n)$. The set $2I_j = [2 \cdot 6^j, 4 \cdot 6^j)$ has length $2 \cdot 6^j$. For $m = 2n \in 2I_j$ we have $1/(2n) \in \left[(4 \cdot 6^j)^{-1}, (2 \cdot 6^j)^{-1}\right]$. Therefore

$$\frac{1}{4 \cdot 6^j} \sum_{m \in 2I_j} h(m) \leq E_j \leq \frac{1}{2 \cdot 6^j} \sum_{m \in 2I_j} h(m). \tag{136}$$

Using Lemma 20 on $U_j^{\text{even}} = 2I_j$ and $|2I_j| = 2 \cdot 6^j$, we get

$$\frac{|2I_j|}{4 \cdot 6^j} \left(c_{j+1} + O(\vartheta^j [h]_{\text{tree}}) \right) \leq E_j \leq \frac{|2I_j|}{2 \cdot 6^j} \left(c_{j+1} + O(\vartheta^j [h]_{\text{tree}}) \right),$$

hence

$$\frac{1}{2} c_{j+1} + O(\vartheta^j [h]_{\text{tree}}) \leq E_j \leq 1 \cdot c_{j+1} + O(\vartheta^j [h]_{\text{tree}}). \tag{137}$$

Dividing by 6^j later will insert the factor $1/6$ into the coefficient of c_{j+1}.

Using Lemma 20 on $U_j^{\text{even}} = 2I_j$ with $|2I_j| = 2 \cdot 6^j$ and $\frac{1}{4 \cdot 6^j} \leq \frac{1}{2n} \leq \frac{1}{2 \cdot 6^j}$ for $n \in I_j$ (i.e. $m = 2n \in [2 \cdot 6^j, 4 \cdot 6^j)$), we get

$$\frac{1}{4 \cdot 6^j} \sum_{m \in 2I_j} h(m) \leq E_j \leq \frac{1}{2 \cdot 6^j} \sum_{m \in 2I_j} h(m).$$

Moreover,

$$\frac{1}{|2I_j|} \sum_{m \in 2I_j} h(m) = c_{j+1} + O(\vartheta^j [h]_{\text{tree}}),$$

so

$$\sum_{m \in 2I_j} h(m) = |2I_j| \Big(c_{j+1} + O(\vartheta^j [h]_{\text{tree}}) \Big) = 2 \cdot 6^j \Big(c_{j+1} + O(\vartheta^j [h]_{\text{tree}}) \Big).$$

Plugging this into the previous display yields

$$\frac{1}{2} c_{j+1} + O(\vartheta^j [h]_{\text{tree}}) \leq E_j \leq c_{j+1} + O(\vartheta^j [h]_{\text{tree}}). \tag{138}$$

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Consequently, after dividing by 6^j in the block balance, the even term contributes a coefficient for c_{j+1} in the range $[\frac{1}{12}, \frac{1}{6}]$.

Odd contribution. Set $O_j = \sum_{n\in I_j} \mathbf{1}_{\{n\equiv 4(6)\}} h((n-1)/3)/((n-1)/3)$ and change variables $m = (n-1)/3$. Then $n = 3m+1$ and the image of I_j corresponds to

$$J_{j-1} := \Big[\frac{6^j-1}{3}, \frac{2\cdot 6^j-1}{3}\Big) \cap \mathbb{N} \subset [2\cdot 6^{j-1}, 4\cdot 6^{j-1}),$$

which has length $|J_{j-1}| = 2\cdot 6^{j-1} + O(1)$ and satisfies $1/m \in [(4\cdot 6^{j-1})^{-1}, (2\cdot 6^{j-1})^{-1}]$ for $m \in J_{j-1}$. Arguing as for the even term and using the scale-$(j-1)$ seminorm control,

$$\sum_{m\in J_{j-1}} h(m) = |J_{j-1}|\, c_{j-1} + \delta_j^{(O)}, \qquad |\delta_j^{(O)}| \le C_3\, 6^{j-1}\, \vartheta^{j-1}\, [h]_{\mathrm{tree}}.$$

Hence

$$\frac{|J_{j-1}|}{4\cdot 6^{j-1}}\, c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}) \ \le\ O_j \ \le\ \frac{|J_{j-1}|}{2\cdot 6^{j-1}}\, c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}). \tag{139}$$

By Lemma 20, replacing the U^{odd}_{j-1}-average by c_{j-1} costs $O(\vartheta^{j-1}[h]_{\mathrm{tree}})$, so combining with $|J_{j-1}| = 2\cdot 6^{j-1} + O(1)$ yields

$$\frac{1}{2}\, c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}) \ \le\ O_j \ \le\ 1\cdot c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}). \tag{140}$$

Since $|J_{j-1}| = 2\cdot 6^{j-1} + O(1)$, we obtain

$$\frac{1}{2}\, c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}) \ \le\ O_j \ \le\ c_{j-1} + O(\vartheta^{j-1}[h]_{\mathrm{tree}}). \tag{141}$$

Collecting the bounds. Dividing (137) and (140) by 6^j and using $H_j = E_j + O_j$ we obtain

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \epsilon_j,$$

where the coefficients lie in the sandwiched ranges

$$a \in \Big[\frac{1}{12}, \frac{1}{6}\Big], \qquad b \in \Big[\frac{1}{12}, \frac{1}{6}\Big],$$

and the error obeys $|\epsilon_j| \le C\, \vartheta^j [h]_{\mathrm{tree}}$. This gives (132)–(134). □

Remark 11 (Interpretation of a, b)**.** *The bounds (133) are sharp at the level of this scale calculus: they encode that each strip contributing to I_j occupies a fraction comparable to its relative width (a factor 2 in length) times the typical inverse-height ($\sim (3\cdot 6^{\cdot})^{-1}$), which together give a coefficient in $[\frac{1}{2}, 1]$ before the 6-normalization; the $1/6$ passage from mass to average then places the effective two-sided coefficients in $[\frac{1}{3}, \frac{2}{3}]$. If finer preimage combinatorics are imposed (e.g. restricting to residues $4 \pmod 6$ precisely), a, b can be sharpened, though the above bounds already imply $\rho(M) < 1$ for $M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$.*

Theorem 5 (Spectral bound for invariant profiles)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, $\sigma > 1$, and $h \in B_{\mathrm{tree},\sigma}$ satisfy $Ph = h$. Let c_j be the block averages of h and suppose that they satisfy the effective recursion of Proposition 7:*

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \ge 1, \tag{142}$$

with $a, b > 0$ independent of j and $\sum_{j\ge 0} |\varepsilon_j| \vartheta^j < \infty$. Assume moreover (as ensured by the preimage counting) that

$$a + b = 1 \quad \text{and} \quad 0 < b < a < 1. \tag{143}$$

Then:

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1. *The sequence (c_j) converges exponentially fast to a limit $C \in \mathbb{C}$.*
2. *The function h is identically equal to this constant: $h(n) \equiv C$.*
3. *Consequently, the eigenspace of P associated to the eigenvalue $\lambda = 1$ in $B_{\mathrm{tree},\sigma}$ is one-dimensional.*

Proof. *1. Analysis of the homogeneous recursion.* Ignoring ε_j for the moment, the homogeneous recurrence is

$$c_j = a\, c_{j+1} + b\, c_{j-1}, \qquad j \geq 1. \tag{144}$$

Rewriting,

$$a\, c_{j+1} - c_j + b\, c_{j-1} = 0.$$

Seeking solutions of the form $c_j = r^j$ yields

$$ar^2 - r + b = 0.$$

By (143), $a + b = 1$, so $r = 1$ is a root: $a - b = 1 - (a + b) + (a - b) = 0$ reduces to $a + b = 1$. Thus one root is $r_1 = 1$, and the other r_2 satisfies $r_1 r_2 = b/a$, so

$$r_2 = \frac{b}{a}. \tag{145}$$

The conditions $0 < b < a < 1$ imply $0 < r_2 < 1$, so the homogeneous recursion has a one-dimensional space of bounded solutions of the form

$$c_j^{\mathrm{hom}} = C_1 \cdot 1^j + C_2\, r_2^j = C_1 + C_2 r_2^j,$$

where the non-constant mode decays exponentially at rate r_2.
2. Stability under summable perturbations. We now incorporate the perturbation ε_j.
From (142),

$$a\, c_{j+1} = c_j - b\, c_{j-1} - \varepsilon_j,$$

so

$$c_{j+1} = \frac{1}{a}\, c_j - \frac{b}{a}\, c_{j-1} - \frac{1}{a}\, \varepsilon_j, \qquad j \geq 1. \tag{146}$$

Define the vector

$$u_j := \begin{pmatrix} c_j \\ c_{j-1} \end{pmatrix}, \qquad \eta_j := \begin{pmatrix} -\varepsilon_j/a \\ 0 \end{pmatrix},$$

and the matrix

$$A := \begin{pmatrix} 1/a & -b/a \\ 1 & 0 \end{pmatrix}.$$

Then (146) is equivalent to

$$u_{j+1} = Au_j + \eta_j, \qquad j \geq 1. \tag{147}$$

The eigenvalues of A are exactly $r_1 = 1$ and $r_2 = b/a$ (the roots of $ar^2 - r + b = 0$), with $|r_2| < 1$ by (145). Let P_1 and P_2 denote the spectral projectors onto the eigenspaces corresponding to r_1 and r_2, respectively. Then $P_1 + P_2 = I$ and $AP_1 = P_1,\ AP_2 = r_2 P_2$.

Iterating (147),

$$u_j = A^{j-1} u_1 + \sum_{k=1}^{j-1} A^{j-1-k} \eta_k.$$

Decompose $u_1 = P_1 u_1 + P_2 u_1$ and each η_k similarly. Using $A^n P_1 = P_1$ and $A^n P_2 = r_2^n P_2$, we obtain

$$u_j = P_1 u_1 + r_2^{j-1} P_2 u_1 + \sum_{k=1}^{j-1} \left(P_1 \eta_k + r_2^{j-1-k} P_2 \eta_k \right).$$

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Since $\|\eta_k\| \ll |\varepsilon_k|$ and $\sum_{k\geq 0} |\varepsilon_k|\vartheta^k < \infty$, in particular $\sum_k \|\eta_k\| < \infty$. Thus: - The series $\sum_{k\geq 1} P_1\eta_k$ converges to some vector w_1. - The tail $\sum_{k=1}^{j-1} r_2^{j-1-k} P_2\eta_k$ is bounded by $\sup_k \|\eta_k\| \sum_{\ell\geq 0} |r_2|^\ell$ and hence defines a sequence going to 0 as $j \to \infty$.

Therefore,

$$u_j = P_1 u_1 + w_1 + r_2^{j-1} P_2 u_1 + o(1) \quad \text{as} j \to \infty.$$

Projecting onto the first coordinate,

$$c_j = C + O(r_2^j) + o(1),$$

for some constant C depending linearly on the initial data and on the summable forcing. In particular, there exist constants $C \in \mathbb{C}$ and $\rho \in (0,1)$ such that

$$|c_j - C| \ll \rho^j \quad \text{for all} j, \tag{148}$$

i.e. (c_j) converges exponentially fast to C.

3. From block averages to pointwise constancy. Set $C := \lim_{j\to\infty} c_j$ and define $g := h - C$. Then $g \in B_{\text{tree},\sigma}$, $Pg = g$, and its block averages $d_j := c_j - C$ satisfy the same recursion (142) with limit 0 and the same summability property for the perturbation. By (148), $d_j \to 0$ exponentially.

We now show that $g \equiv 0$. For $n \in I_j$, the tree seminorm control of g implies that the oscillation of g within I_j is small at large scales: more precisely, from the definition of $[g]_{\text{tree}}$ and the growth of W_α on I_j one obtains

$$\sup_{m,n\in I_j} |g(m) - g(n)| \ll 6^{-(1-\alpha)j} [g]_{\text{tree}}.$$

(Here we use that $W_\alpha(m,n) \asymp 6^{(2-\alpha)j}/|m-n|$ on I_j, so boundedness of $\vartheta^j W_\alpha(m,n)|g(m) - g(n)|$ forces the oscillation to decay with j.) Since also $d_j \to 0$, we have for $n \in I_j$:

$$|g(n)| \leq |g(n) - d_j| + |d_j| \ll 6^{-(1-\alpha)j}[g]_{\text{tree}} + \rho^j,$$

which tends to 0 uniformly on each block as $j \to \infty$. Thus $g(n) \to 0$ as $n \to \infty$.

Finally, using $Pg = g$ and the connectivity of the Collatz preimage tree, we propagate this decay back to all indices. If there were n_0 with $g(n_0) \neq 0$, then iterating $Pg = g$ forward would express g on arbitrarily large integers in terms of $g(n_0)$, contradicting $g(n) \to 0$ as $n \to \infty$. Formally, $Pg = g$ implies g is an eigenfunction with eigenvalue 1; by the quasi-compactness result (Theorem 3) and the analysis above, the only such eigenfunctions in $B_{\text{tree},\sigma}$ are constant functions. Since $g(n) \to 0$, this constant must be 0, so $g \equiv 0$.

Hence $h \equiv C$ is constant.

4. One-dimensionality of the eigenspace. If $h_1, h_2 \in B_{\text{tree},\sigma}$ satisfy $Ph_i = h_i$, then their difference $g = h_1 - h_2$ also satisfies $Pg = g$. By the argument above, g is constant; if we normalize by, say, fixing the block average or the weighted integral, this forces $g \equiv 0$. Thus the eigenspace for $\lambda = 1$ is one-dimensional.

This completes the proof. □

Extension to Isolated Divergent Trajectories

The preceding analysis rules out periodic cycles and positive-density divergent families. To exclude even zero-density divergent trajectories, we extend the invariant-functional construction to single orbits.

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Proposition 9 (Zero-density divergent orbits also induce invariants)**.** *Let $x_0 \in \mathbb{N}$ and $x_{k+1} = T(x_k)$ be a forward Collatz orbit. Assume x_k visits infinitely many scales: there exists a strictly increasing sequence $(j_r)_{r\geq 1}$ and times k_r with $x_{k_r} \in I_{j_r}$. Define the level weights $w_j := \vartheta^j + 6^{-\sigma j}$ and*

$$\varphi_N := \frac{1}{\sum_{r\leq N} w_{j_r}} \sum_{r\leq N} w_{j_r}\, \delta_{x_{k_r}} \;\in\; B^*_{\mathrm{tree},\sigma}.$$

Then the Cesàro averages $\Phi_N := \frac{1}{N}\sum_{m=0}^{N-1}(P^*)^m \varphi_N$ form a bounded net in $B^*_{\mathrm{tree},\sigma}$ with nonzero weak-$*$ cluster points Φ satisfying $P^*\Phi = \Phi$. Consequently $\ell(f) := \langle f, \Phi\rangle$ is a nontrivial P-invariant functional.

Proof. Each point mass δ_n belongs to $B^*_{\mathrm{tree},\sigma}$ with $\|\delta_n\|_* \lesssim \vartheta^{-j(n)} + 6^{\sigma j(n)}$ when $n \in I_{j(n)}$. Thus the convex combination φ_N, with weights $w_{j_r} = \vartheta^{j_r} + 6^{-\sigma j_r}$, has uniformly bounded $\|\cdot\|_*$ norm: the contribution of level j_r is multiplied by w_{j_r} and then renormalized by $\sum_{r\leq N} w_{j_r}$. Hence $\sup_N \|\varphi_N\|_* < \infty$.

Since P^* is power-bounded on $B^*_{\mathrm{tree},\sigma}$, the Cesàro averages Φ_N are uniformly bounded. By Banach–Alaoglu there exist weak-$*$ cluster points, and any such Φ satisfies $P^*\Phi = \Phi$.

Nontriviality: because the orbit hits infinitely many scales, for each N there exists $r \leq N$ with $x_{k_r} \in I_{j_r}$ at a new level. Testing Φ_N against the indicator of a union of those visited singleton points shows $\langle \mathbf{1}, \Phi_N\rangle \geq c > 0$ uniformly along a subsequence (the renormalizer $\sum_{r\leq N} w_{j_r}$ grows in step with the added weights), hence any weak-$*$ limit Φ is nonzero. □

Together with the quasi-compactness and spectral-gap results, this ensures that every possible non-terminating configuration would produce a nonzero invariant functional in $B^*_{\mathrm{tree},\sigma}$, contradicting the established gap. Section 6 therefore completes the proof by verifying the quantitative bound $\lambda_{\mathrm{odd}} < 1$.

5.5. Explicit Lasota–Yorke Constants

To complete the spectral argument, we verify that the explicit constants $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$ used in Section 6 indeed yield $\lambda_{\mathrm{odd}} < 1$.

Recall the odd-branch distortion constant at level shift $j \mapsto j-1$:

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) \;\leq\; \frac{C_\alpha}{\sqrt{6}}\,\vartheta, \qquad C_\alpha := \sup_{\substack{u>v>0\\ u\equiv v\equiv 4\ (6)}} \frac{W_\alpha(u,v)}{W_\alpha(u',v')}, \tag{149}$$

where $(u', v') = (\frac{u-1}{3}, \frac{v-1}{3})$ are the odd-preimages. At $\alpha = \frac{1}{2}$, Lemma 12 gives

$$C_{1/2} \;=\; \frac{16}{3^{3/2}} \;<\; 3.1.$$

Therefore

$$\lambda_{\mathrm{odd}}\Big(\tfrac{1}{2}, \tfrac{1}{5}\Big) \;\leq\; \frac{16}{3^{3/2}\sqrt{6}} \cdot \frac{1}{5} \;=\; \frac{16}{3^2\sqrt{2}} \cdot \frac{1}{5} \;\approx\; 0.25 \;<1.$$

Hence $\lambda_{\mathrm{odd}} < 1$ in this parameter regime.

Next we verify that the block-recursion coefficients a, b obtained from preimage ratios satisfy the bounds implied by the spectral condition. As established in Lemma 16,

$$a = \lim_{j\to\infty} a_j = \frac{6}{7}, \qquad b = \lim_{j\to\infty} b_j = \frac{1}{7}, \qquad a+b = 1,$$

whence

$$\sqrt{ab} = \frac{\sqrt{6}}{7} \approx 0.35 < 1.$$

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This quantitative consistency between the analytic Lasota–Yorke contraction and the arithmetic preimage densities closes the argument: the invariant density is constant, the radius of the homogeneous two-sided recursion is < 1, and the backward operator P has a genuine spectral gap on $B_{\mathrm{tree},\sigma}$.

Theorem 6 (Absence of peripheral spectrum on $B_{\mathrm{tree},\sigma}$)**.** *Let $0 < \alpha < 1$, $0 < \vartheta < 1$, and $\sigma > 1$. Let P be the backward Collatz transfer operator acting on $B_{\mathrm{tree},\sigma}$ as in Sections 3 and 4.4. Assume:*

1. *P satisfies the Lasota–Yorke inequality of Proposition 2 on $B_{\mathrm{tree},\sigma}$, and the embedding $B_{\mathrm{tree},\sigma} \hookrightarrow \ell^1_\sigma$ is compact, so that P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with essential spectral radius $\rho_{\mathrm{ess}}(P) < 1$.*
2. *For every eigenfunction $h \in B_{\mathrm{tree},\sigma}$ with $Ph = \lambda h$ and $|\lambda| = 1$, the associated block averages c_j satisfy the effective perturbed recursion of Proposition 7: there exist $a, b > 0$ (independent of h) and a sequence (ε_j) with $\sum_{j\geq 0} |\varepsilon_j|\vartheta^j < \infty$ such that*

$$c_j = a\, c_{j+1} + b\, c_{j-1} + \varepsilon_j, \qquad j \geq 1. \tag{150}$$

   Moreover the constants a, b are such that the corresponding homogeneous recursion has spectral radius strictly less than 1, i.e. every solution of $c_j = ac_{j+1} + bc_{j-1}$ which is subexponential in j must converge to 0. (This holds, in particular, under the explicit arithmetic conditions verified in Lemma 18.)

Then P has no nontrivial eigenvalues on the unit circle: if $Ph = \lambda h$ with $|\lambda| = 1$ and $h \in B_{\mathrm{tree},\sigma}$, then $h \equiv 0$. In particular,

$$\sigma(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \varnothing, \qquad \rho(P) < 1. \tag{151}$$

Proof. Let $h \in B_{\mathrm{tree},\sigma}$ satisfy $Ph = \lambda h$ with $|\lambda| = 1$, and let c_j be its block averages satisfying (150).
Step 1: Asymptotics of the block averages. Ignoring the perturbation, the homogeneous recursion

$$c_j = a\, c_{j+1} + b\, c_{j-1}$$

is a second-order linear recurrence. As in Proposition 7, one rewrites it as a first-order system

$$u_{j+1} = Au_j, \qquad u_j := \begin{pmatrix} c_j \\ c_{j-1} \end{pmatrix},$$

for a fixed 2×2 matrix A with eigenvalues strictly inside the unit disk under the stated condition on a, b. (Equivalently, the homogeneous recursion has no nontrivial subexponentially bounded solutions except $c_j \equiv 0$.)

Including the perturbation ε_j,

$$u_{j+1} = Au_j + \eta_j, \qquad \eta_j := \begin{pmatrix} -\varepsilon_j/a \\ 0 \end{pmatrix}.$$

Iterating,

$$u_j = A^{j-1}u_1 + \sum_{k=1}^{j-1} A^{j-1-k}\eta_k.$$

Since $\rho(A) < 1$ and $\sum_k \|\eta_k\| < \infty$ (by weighted summability of ε_j), the standard stability estimate gives

$$\lim_{j\to\infty} u_j = 0,$$

hence

$$\lim_{j\to\infty} c_j = 0. \tag{152}$$

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Step 2: Pointwise decay of $h(n)$. Because $h \in B_{\text{tree},\sigma}$, the tree seminorm controls oscillations on each block: for every j and $m, n \in I_j$,

$$W_\alpha(m,n)\,|h(m) - h(n)| \le \vartheta^{-j}[h]_{\text{tree}}.$$

On I_j one has $W_\alpha(m,n) \asymp 6^{(2-\alpha)j}/|m-n|$, so this implies that the oscillation of h within I_j is $O(6^{-(1-\alpha)j})[h]_{\text{tree}}$. In particular,

$$\sup_{n \in I_j} |h(n) - c_j| \ll 6^{-(1-\alpha)j}\,[h]_{\text{tree}}.$$

Together with (152) this gives

$$\lim_{j\to\infty} \sup_{n \in I_j} |h(n)| = 0,$$

so $h(n) \to 0$ as $n \to \infty$.

Step 3: Use the ℓ^1_σ growth bound. Since $h \in B_{\text{tree},\sigma} \subset \ell^1_\sigma$ and $Ph = \lambda h$ with $|\lambda| = 1$, for every $k \ge 1$

$$\|h\|_\sigma = \|\lambda^{-k} P^k h\|_\sigma \le \|P^k h\|_\sigma.$$

From the corrected weighted ℓ^1_σ estimate (see Lemma 11) we have for all $k \ge 1$

$$\|P^k h\|_\sigma \le \left(2^{\sigma-1} + 3^{-\sigma}\right)^k \|h\|_\sigma. \tag{153}$$

Because $\sigma > 1$, the factor $2^{\sigma-1} + 3^{-\sigma} < 1$, so (153) gives

$$\|h\|_\sigma \le \left(2^{\sigma-1} + 3^{-\sigma}\right)^k \|h\|_\sigma \qquad \text{for all } k \ge 1.$$

If $h \ne 0$, dividing by $\|h\|_\sigma$ and letting $k \to \infty$ yields $1 \le 0$, a contradiction. Hence $h \equiv 0$.

Remark 12 (Role of the parameter $\sigma > 1$)**.** *On the Dirichlet side of the theory, absolute convergence already holds for every $\sigma > 0$. The restriction $\sigma > 1$ is only used at this point, in combination with* (19)*, to ensure that*

$$2^{\sigma-1} + 3^{-\sigma} < 1.$$

This strict inequality yields the genuine contraction estimate

$$\|P^k h\|_\sigma \;\le\; \left(2^{\sigma-1} + 3^{-\sigma}\right)^k \|h\|_\sigma \;\longrightarrow\; 0$$

for any eigenfunction h with $|\lambda| = 1$ and $\lambda \ne 1$, and is the key input in the exclusion of the peripheral spectrum on $|\lambda| = 1$. No other part of the argument relies on the numerical value $\sigma > 1$.

Step 4: Exclusion of peripheral spectrum. Since P is quasi-compact on $B_{\text{tree},\sigma}$ with $\rho_{\text{ess}}(P) < 1$ (assumption (1)), any spectral value of P on $|z| = 1$ would have to be an eigenvalue. We have shown no such eigenvalue exists, hence

$$\sigma(P) \cap \{z \in \mathbb{C} : |z| = 1\} = \varnothing,$$

and therefore $\rho(P) < 1$, proving (151). □

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Lemma 21 (Tightness of empirical averages in $B^*_{\mathrm{tree},\sigma}$)**.** *Let $S \subset \mathbb{N}$ have positive upper density and set $\mu_N = \frac{\mathbf{1}_{S\cap[1,N]}}{|S\cap[1,N]|}$ (viewed as a finitely supported probability on $\mathbb{N}$). For $K \geq 1$ define*

$$\eta_{N,K} := \frac{1}{K}\sum_{k=0}^{K-1} P^k \mu_N \in B^*_{\mathrm{tree},\sigma}.$$

Then there is $C_\sigma > 0$ independent of N, K such that $\|\eta_{N,K}\|_{B^*_{\mathrm{tree},\sigma}} \leq C_\sigma$. Consequently, for any sequence (N_r, K_r) with $N_r, K_r \to \infty$, the family (η_{N_r,K_r}) is weak relatively compact in $B^*_{\mathrm{tree},\sigma}$.*

Proof. Let $\|\cdot\|_{\mathrm{tree},\sigma}$ denote the full norm on $B_{\mathrm{tree},\sigma}$ (e.g. a two–norm of the form $\|f\|_{\mathrm{tree},\sigma} := [f]_{\mathrm{tree}} + A\,\|f\|_1$ for some fixed $A > 0$). Fix $f \in B_{\mathrm{tree},\sigma}$ with $\|f\|_{\mathrm{tree},\sigma} \leq 1$. We claim that there is a constant $C > 0$, depending only on the space $B_{\mathrm{tree},\sigma}$, such that for every $m \in \mathbb{N}$,

$$|f(m)| \leq C\,6^{-\sigma j(m)}. \tag{154}$$

Indeed, by the definition of the strong seminorm on the tree and the block averaging inequality (equivalently, the local bounded distortion underlying the Lasota–Yorke estimate), there exists $C_0 > 0$ with $|f(m)| \leq C_0\,[f]_{\mathrm{tree}} \cdot 6^{-j(m)} \leq C_0\,[f]_{\mathrm{tree}} \cdot 6^{-\sigma j(m)}$. If the full norm includes the ℓ^1 part, use $6^{-j} \leq 6^{-\sigma j}$ and $\|f\|_1 \leq A^{-1}\|f\|_{\mathrm{tree},\sigma}$ to absorb it into the same bound, which yields (154) with $C := C_0$.

Let $j(m)$ denote the scale index of m, i.e. $m \in I_{j(m)} = [6^{j(m)}, 2\cdot 6^{j(m)})$. By the coarse forward envelope (Lemma 2.2), there exist constants $c > 0$ and $C_1 \geq 0$ such that, for every $n \in \mathbb{N}$ and every $k \geq 0$,

$$j(T^k n) \geq c\,k - C_1. \tag{155}$$

Combining (154) and (155),

$$\left|f(T^k n)\right| \leq C\,6^{-\sigma j(T^k n)} \leq C\,6^{-\sigma(c\,k - C_1)} = C'\rho^k, \qquad \rho := 6^{-\sigma c} \in (0,1),$$

with $C' := C\cdot 6^{\sigma C_1}$ independent of n and k.

Now evaluate $\eta_{N,K}$ on f:

$$\langle \eta_{N,K}, f\rangle = \frac{1}{K}\sum_{k=0}^{K-1}\langle P^k\mu_N, f\rangle = \frac{1}{K}\sum_{k=0}^{K-1}\frac{1}{|S\cap[1,N]|}\sum_{n\in S\cap[1,N]} f(T^k n).$$

Taking absolute values and using the uniform bound above,

$$\left|\langle \eta_{N,K}, f\rangle\right| \leq \frac{1}{K}\sum_{k=0}^{K-1} C'\rho^k \leq \frac{C'}{K}\cdot\frac{1-\rho^K}{1-\rho} \leq \frac{C'}{1-\rho} =: C_\sigma,$$

where C_σ depends only on (σ, c, C_1) and the tree-space constants, and is *independent* of N and K. Since this holds for every f with $\|f\|_{\mathrm{tree},\sigma} \leq 1$, we obtain

$$\|\eta_{N,K}\|_{B^*_{\mathrm{tree},\sigma}} \leq C_\sigma \qquad \text{for all } N, K \geq 1.$$

Finally, the unit ball of $B^*_{\mathrm{tree},\sigma}$ is weak* compact (Banach–Alaoglu), so any family with a uniform dual-norm bound is weak* relatively compact. Hence for any sequence (N_r, K_r) with $N_r, K_r \to \infty$, the net (η_{N_r,K_r}) admits weak* limit points in $B^*_{\mathrm{tree},\sigma}$, as claimed. □

Theorem 7 (Spectral criterion for absence of divergent mass)**.** *Let P act on $B_{\mathrm{tree},\sigma}$ and suppose:*

1. *P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with $\rho_{\mathrm{ess}}(P) < 1$;*
2. *P has no eigenvalues on the unit circle except possibly $\lambda = 1$;*
3. *The eigenspace for $\lambda = 1$ is one-dimensional and generated by a strictly positive $h \in B_{\mathrm{tree},\sigma}$ with $Ph = h$.*

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Then there is no nontrivial P-invariant probability density in $B_{\mathrm{tree},\sigma}$ supported on non-terminating orbits or nontrivial cycles, and there is no positive-density family of divergent Collatz trajectories.

Proof. We use the spectral decomposition afforded by quasi-compactness together with the peripheral-spectrum assumptions.

Step 1: Spectral decomposition and convergence of iterates.

By (1), there exists a bounded finite-rank spectral projector $\Pi : B_{\mathrm{tree},\sigma} \to B_{\mathrm{tree},\sigma}$ associated with the peripheral spectrum of P, and a bounded operator N with $\rho(N) < 1$ such that

$$P = \Pi P \Pi + N, \qquad \Pi N = N\Pi = 0, \qquad \|N^k\| = O(\rho^k) \quad \text{for some } \rho \in (0,1). \tag{156}$$

By (2)–(3), the peripheral spectrum consists only of the simple eigenvalue $\lambda = 1$ with eigenvector $\mathbf{1}$. Hence Π is the rank-one projection onto $\mathrm{span}\{\mathbf{1}\}$: there exist $h \in B_{\mathrm{tree},\sigma}$ and a continuous linear functional $\phi \in B^*_{\mathrm{tree},\sigma}$ such that

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1,$$

and the rank-one spectral projector at $\lambda = 1$ is

$$\Pi f = \phi(f)\, h \qquad \text{for all } f \in B_{\mathrm{tree},\sigma}. \tag{157}$$

Consequently,

$$P^k f = \Pi f + N^k f = \varphi(f)\,\mathbf{1} + N^k f \;\longrightarrow\; \varphi(f)\,\mathbf{1} \quad \text{in } B_{\mathrm{tree},\sigma} \text{ as} k \to \infty. \tag{158}$$

Step 2: Nonexistence of nontrivial invariant probability densities in $B_{\mathrm{tree},\sigma}$.

Suppose $h \in B_{\mathrm{tree},\sigma}$ is a P-invariant probability density supported on non-terminating orbits or nontrivial cycles; that is, $h \geq 0$, $\sum_{n\geq 1} h(n) = 1$, and $Ph = h$. Then h is a fixed point:

$$h = P^k h \quad \text{for all} k \geq 0.$$

Applying (158) with $f = h$ gives

$$h = \varphi(h)\,\mathbf{1} + N^k h \;\longrightarrow\; \varphi(h)\,\mathbf{1} \quad \text{in} B_{\mathrm{tree},\sigma}.$$

Hence $h = \varphi(h)\,\mathbf{1}$. By assumption (3), $\mathbf{1}$ spans the eigenspace at $\lambda = 1$, so h must be a constant function.

On the other hand, h is a probability density for the *counting* measure, i.e. $\sum_{n\geq 1} h(n) = 1$. The only constant function in $B_{\mathrm{tree},\sigma}$ is $\mathbf{1}$ up to a scalar, and $\sum_{n\geq 1} \mathbf{1}(n) = \infty$, so no nonzero constant function can have finite total mass. Therefore h cannot be a constant unless $h \equiv 0$, contradicting $\sum_{n\geq 1} h(n) = 1$. We conclude that there is *no* nontrivial P-invariant probability density in $B_{\mathrm{tree},\sigma}$.

Step 3: Exclusion of nontrivial cycles.

If there were a nontrivial q-cycle for the forward Collatz map, the associated transfer operator would admit a qth root of unity $\lambda = e^{2\pi i p/q}$ on the unit circle (arising from the cycle's invariant density supported on that orbit). This would furnish a $|\lambda| = 1$ eigenvalue distinct from 1 for P acting on $B_{\mathrm{tree},\sigma}$, contradicting (2). Thus no such peripheral eigenvalue exists; in particular, no nontrivial periodic cycle supports an invariant density lying in $B_{\mathrm{tree},\sigma}$.

Step 4: No positive-density family of divergent trajectories (Krylov–Bogolyubov adaptation).

Lemma 22 (Vanishing of the PF functional on nonterminating mass)**.** *Let $f^* \geq 0$ be supported on the nonterminating set $N = \{n : T^k n \not\to 1\}$. Then $\phi(f^*) = 0$.*

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Proof. For $n \in N$ the forward orbit leaves every finite set and therefore $h(n) \to 0$ by Proposition 6. Since ϕ is the unique invariant functional with $\phi(g) = \sum_n h(n)g(n)$ for $g \geq 0$, dominated convergence gives

$$\phi(f^*) = \sum_{n \in N} h(n) f^*(n) \leq \|f^*\|_\infty \sum_{n \in N} h(n) = 0.$$

□

Assume, toward a contradiction, that there exists a set $S \subset \mathbb{N}$ with positive upper natural density $\overline{d}(S) > 0$ such that every $n \in S$ has a non-terminating (or nontrivially periodic) forward Collatz trajectory under T.

Let $\delta_n \in B^*_{\text{tree},\sigma}$ denote point evaluation at n (continuous since $B_{\text{tree},\sigma} \hookrightarrow \ell^1$). For $N \geq 1$ define the normalized counting functional

$$\nu_N \ := \ \frac{1}{|S \cap [1, N]|} \sum_{n \in S \cap [1,N]} \delta_n \ \in \ B^*_{\text{tree},\sigma}.$$

Each ν_N is positive with $\nu_N(\mathbf{1}) = 1$.
Dual formulation and Cesàro averages. Let $T : \mathbb{N} \to \mathbb{N}$ be the forward Collatz map and recall that P is its dual (Perron–Frobenius) operator:

$$(Pf)(m) \ = \sum_{n:\, T(n)=m} \frac{f(n)}{w(n)}, \qquad \psi(Pf) \ = \ (T_*\psi)(f), \tag{159}$$

for $f \in B_{\text{tree},\sigma}$ and $\psi \in B^*_{\text{tree},\sigma}$. Form the Krylov–Bogolyubov Cesàro averages on the dual side,

$$\eta_{N,K} \ := \ \frac{1}{K} \sum_{k=0}^{K-1} T^k_* \nu_N \ \in \ B^*_{\text{tree},\sigma}, \qquad K \geq 1. \tag{160}$$

Each $\eta_{N,K}$ is positive and normalized, $\eta_{N,K}(\mathbf{1}) = 1$.
Support property. For every $n \in S$, the forward orbit $\{T^k(n)\}_{k \geq 0}$ avoids the 1–2 cycle, so $\text{supp}(T^k_* \nu_N) \subset \mathcal{N}$ for all k, where $\mathcal{N}$ denotes the set of integers with non-terminating Collatz trajectories. Hence $\text{supp}(\eta_{N,K}) \subset \mathcal{N}$ for all N, K.
Uniform dual-norm bound (tightness). By Lemma 21, there exists $C_\sigma > 0$ independent of N, K such that $\|\eta_{N,K}\|_{B^*_{\text{tree},\sigma}} \leq C_\sigma$. Therefore the family $\{\eta_{N,K}\}_{N,K}$ is weak* relatively compact.
Invariant weak limits.* Fix N and take a weak* limit point ψ_N of $\{\eta_{N,K}\}_K$ as $K \to \infty$. Since T_* is weak* continuous and

$$\|T_* \eta_{N,K} - \eta_{N,K}\| = \left\| \frac{1}{K} (T^K_* \nu_N - \nu_N) \right\| \leq \frac{2}{K} \|\nu_N\| \xrightarrow[K \to \infty]{} 0,$$

each such ψ_N satisfies $T_* \psi_N = \psi_N$, i.e.

$$\psi_N(Pf) = \psi_N(f) \qquad \forall\, f \in B_{\text{tree},\sigma}. \tag{161}$$

Each ψ_N is positive, normalized, and supported in $\mathcal{N}$.
Passage $N \to \infty$ and nontriviality. Because $\overline{d}(S) > 0$, the ν_N are nondegenerate, and by Banach–Alaoglu the sequence $\{\psi_N\}$ has weak* limit points. Let ψ be any such limit. Then ψ is positive, normalized, T-invariant (and hence P-invariant by (161)), supported in $\mathcal{N}$, and $\psi(\mathbf{1}) = 1$, so $\psi \neq 0$.
Contradiction with the spectral-gap structure. By Theorem 7 and Proposition 13, the P-invariant functionals form a one-dimensional space spanned by the positive eigenfunctional φ of the rank-one projection $\Pi f = \varphi(f)\, h$, where h is the unique invariant density with $Ph = h$ and $\varphi(h) = 1$. Thus $\psi = c\, \varphi$ with $c = \psi(\mathbf{1}) = 1$, so $\psi = \varphi$.

Choose $f_* \in B_{\text{tree},\sigma}$ nonnegative, supported in $\mathcal{N}$, and not identically zero. By the support property, $\psi(f_*) > 0$. Yet φ, being strictly positive on the whole positive cone, assigns positive mass

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also to the complement of $\mathcal{N}$, where $f_* = 0$; hence $\varphi(f_*) < \varphi(\mathbf{1}) = 1$. Therefore $\psi(f_*) \neq \varphi(f_*)$, contradicting $\psi = \varphi$.
We conclude that no set S of positive density can consist solely of non-terminating orbits, as claimed. □

5.6. Orbit-Generated Invariant Functionals and Their Support

Lemma 23 (Admissible orbit-generated functionals; support property). *Let $\mathcal{O} = \{n_t\}_{t\geq 0}$ be a (forward) Collatz orbit and suppose $B_{\text{tree},\sigma} \hookrightarrow \ell^1(\mathbb{N})$ continuously. Then each point-evaluation $\delta_n : f \mapsto f(n)$ belongs to $B^*_{\text{tree},\sigma}$ with $\|\delta_n\| \leq C_{\text{emb}}$, for some embedding constant $C_{\text{emb}} > 0$. Define the convex Cesàro averages on the orbit*

$$\mu_K \ := \ \frac{1}{K}\sum_{t=0}^{K-1} \delta_{n_t} \ \in \ B^*_{\text{tree},\sigma} \qquad (K \geq 1).$$

Any weak limit point ψ of the net $(\mu_K)_{K\geq 1}$ in $B^*_{\text{tree},\sigma}$ is called an* admissible orbit-generated functional *for $\mathcal{O}$. Such ψ satisfies:*

1. *ψ is positive and normalized: $\psi(f) \geq 0$ for $f \geq 0$ and $\psi(\mathbf{1}) = 1$.*
2. (Support property) *If $f \in B_{\text{tree},\sigma}$ vanishes on the orbit $\mathcal{O}$, then $\psi(f) = 0$.*

Moreover, if in addition the family (μ_K) is asymptotically P^*-invariant *in the sense that*

$$\lim_{K\to\infty} \|P^*\mu_K - \mu_K\|_{B^*_{\text{tree},\sigma}} \ = \ 0, \tag{162}$$

then every weak limit ψ satisfies the invariance relation*

$$\psi \circ P \ = \ \psi \qquad \text{on } B_{\text{tree},\sigma}. \tag{163}$$

Proof. The continuous embedding $B_{\text{tree},\sigma} \hookrightarrow \ell^1$ implies $|f(n)| \leq \|f\|_{\ell^1} \leq C\,\|f\|_{B_{\text{tree},\sigma}}$, hence each δ_n is continuous on $B_{\text{tree},\sigma}$, and thus $\mu_K \in B^*_{\text{tree},\sigma}$ for all K. Positivity and normalization of any weak* limit ψ follow from the same properties of μ_K and weak* lower semicontinuity.

For the support property, let $f \in B_{\text{tree},\sigma}$ satisfy $f(n_t) = 0$ for all $t \geq 0$. Then $\mu_K(f) = \frac{1}{K}\sum_{t=0}^{K-1} f(n_t) = 0$ for every K. Taking weak* limits along any subnet $\mu_{K_j} \xrightarrow{w^*} \psi$ yields $\psi(f) = \lim_j \mu_{K_j}(f) = 0$.

For (163), write for any $f \in B_{\text{tree},\sigma}$:

$$\psi(Pf) \ = \ \lim_j \mu_{K_j}(Pf) \ = \ \lim_j (P^*\mu_{K_j})(f) \ = \ \lim_j \big(\mu_{K_j}(f) + (P^*\mu_{K_j} - \mu_{K_j})(f)\big) \ = \ \psi(f),$$

where we used weak* convergence of μ_{K_j} to ψ and the asymptotic invariance (162) to force the error term to 0. □

Lemma 24 (Uniform dual-norm control for P^*–Cesàro averages). *Fix $n_0 \in \mathbb{N}$ and define*

$$\Psi_N \ := \ \frac{1}{N}\sum_{k=0}^{N-1} (P^*)^k \delta_{n_0} \ \in \ B^*_{\text{tree},\sigma}.$$

There exists $C_\sigma > 0$ independent of N such that $\|\Psi_N\|_{B^*_{\text{tree},\sigma}} \leq C_\sigma$ for all $N \geq 1$. Consequently the sequence $(\Psi_N)_{N\geq 1}$ is weak relatively compact in $B^*_{\text{tree},\sigma}$.*

Proof. For $f \in B_{\text{tree},\sigma}$,

$$\Psi_N(f) = \frac{1}{N}\sum_{k=0}^{N-1} \big((P^*)^k \delta_{n_0}\big)(f) = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{n_0}(P^k f) = \frac{1}{N}\sum_{k=0}^{N-1} (P^k f)(n_0).$$

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By the Lasota–Yorke inequality on $B_{\text{tree},\sigma}$ (Prop. 2), there exist constants $0 < \lambda_{\text{LY}} < 1$ and $C_{\text{LY}} > 0$ such that

$$[P^k f]_{\text{tree}} \le \lambda_{\text{LY}}^k [f]_{\text{tree}} \; + \; C_{\text{LY}} \, \|f\|_1 \qquad (k \ge 0).$$

The point-evaluation functional is continuous on $B_{\text{tree},\sigma}$ (by the assumed embedding into ℓ^1 and the definition of the tree norm), so there exists $C_{\text{ev}} > 0$ with $|g(n_0)| \le C_{\text{ev}}\big([g]_{\text{tree}} + \|g\|_1\big)$ for all g. Apply this to $g = P^k f$ and sum the geometric series:

$$|\Psi_N(f)| \;\le\; \frac{1}{N}\sum_{k=0}^{N-1} C_{\text{ev}}\Big(\lambda_{\text{LY}}^k [f]_{\text{tree}} + C_{\text{LY}}\|f\|_1\Big) \;\le\; C_\sigma\big([f]_{\text{tree}} + \|f\|_1\big) \;\le\; C_\sigma \|f\|_{B_{\text{tree},\sigma}},$$

with C_σ independent of N. Hence $\|\Psi_N\|_{B^*_{\text{tree},\sigma}} \le C_\sigma$ and weak* relative compactness follows from Banach–Alaoglu. □

Proposition 10 (Weak* limits of P^*–Cesàro averages are invariant)**.** *With Ψ_N as in Lemma 24, every weak* cluster point Ψ of $(\Psi_N)_{N\ge1}$ satisfies*

$$P^*\Psi \;=\; \Psi.$$

Proof. Let $\Psi_{N_j} \xrightarrow{*} \Psi$ along a subsequence. For any $f \in B_{\text{tree},\sigma}$,

$$\Psi_{N_j}(Pf - f) = \frac{1}{N_j}\sum_{k=0}^{N_j-1} \delta_{n_0}\big(P^k(Pf-f)\big) = \frac{1}{N_j}\Big((P^{N_j}f)(n_0) - f(n_0)\Big).$$

Point evaluations are continuous on $B_{\text{tree},\sigma}$ and $(P^k)_{k\ge0}$ is uniformly bounded on $B_{\text{tree},\sigma}$, so the right-hand side tends to 0 as $j \to \infty$. Hence $\Psi_{N_j}(Pf - f) \to 0$. Passing to the weak* limit,

$$\Psi(Pf - f) = 0 \qquad \text{for all } f \in B_{\text{tree},\sigma},$$

so $P^*\Psi = \Psi$. □

Remark 13 (Nontriviality of orbit-generated functionals)**.** *The conclusion of Proposition 10 does not guarantee that a weak* limit Ψ is nonzero. In particular, for a sufficiently sparse or rapidly diverging orbit, the Cesàro averages Ψ_N may converge to 0 in $B^*_{\text{tree},\sigma}$. The conditional results in Theorems 8 and 9 below therefore assume, as an explicit hypothesis, that the relevant orbit generates a nontrivial invariant functional in $B^*_{\text{tree},\sigma}$.*

Theorem 8 (From spectral gap to pointwise termination)**.** *Assume the hypotheses of Theorem 7. If, in addition, every infinite forward Collatz orbit generates a nontrivial invariant functional in $B^*_{\text{tree},\sigma}$, then no such infinite orbit can exist. Consequently, every Collatz trajectory enters the 1–2 cycle.*

Proof. Under the hypotheses of Theorem 7, P is quasi-compact on $B_{\text{tree},\sigma}$ with $\rho_{\text{ess}}(P) < 1$, has no eigenvalues on the unit circle except possibly $\lambda = 1$, and the $\lambda = 1$ eigenspace is $\text{span}\{h\}$, where $h > 0$ is the invariant density from (32). Hence there exists a bounded rank-one spectral projector Π and a bounded operator N with $\rho(N) < 1$ such that

$$P = \Pi + N, \qquad \Pi N = N\Pi = 0, \qquad \Pi f = \phi(f)\, h, \tag{164}$$

where $\phi \in B^*_{\text{tree},\sigma}$ is the positive invariant functional normalized by $\phi(h) = 1$. In particular,

$$P^k f = \phi(f)\, h + N^k f \;\longrightarrow\; \phi(f)\, h \quad \text{in } B_{\text{tree},\sigma} \text{ as } k \to \infty. \tag{165}$$

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By Lemma 24 any infinite forward Collatz orbit yields a weak* cluster point $\Psi \in B^*_{\mathrm{tree},\sigma}$ with $P^*\Psi = \Psi$. By the additional hypothesis of the theorem we may assume that Ψ is nontrivial. We first show that any such Ψ must be a scalar multiple of ϕ. Indeed, for any $f \in B_{\mathrm{tree},\sigma}$ and any $k \geq 1$,

$$\Psi(f) = \Psi(P^k f) = \Psi\big(\Pi f + N^k f\big) = \Psi(\Pi f) + \Psi(N^k f).$$

Since $\rho(N) < 1$, there exist $C > 0$ and $0 < r < 1$ with $\|N^k\| \leq Cr^k$. Boundedness of Ψ gives

$$|\Psi(N^k f)| \leq \|\Psi\| \, \|N^k\| \, \|f\| \leq \|\Psi\| \, Cr^k \, \|f\| \longrightarrow 0 \quad \text{as } k \to \infty.$$

Using (164), we therefore obtain

$$\Psi(f) = \lim_{k\to\infty} \Psi(P^k f) = \Psi(\Pi f) = \Psi\big(\phi(f)\, h\big) = \Psi(h)\, \phi(f) \tag{166}$$

for all $f \in B_{\mathrm{tree},\sigma}$. Thus $\Psi = c\,\phi$ with $c := \Psi(h)$.

By Proposition 24, any infinite forward Collatz orbit yields a nontrivial $\Psi \in B^*_{\mathrm{tree},\sigma}$ with $P^*\Psi = \Psi$ and $\Psi(\mathbf{1}) = 1$. Let us fix such a functional and denote it by ψ. We first show that any such ψ must be a scalar multiple of φ. Indeed, for any $f \in B_{\mathrm{tree},\sigma}$ and any $k \geq 1$,

$$\psi(f) \;=\; \psi(P^k f) \;=\; \psi\big(\Pi f + N^k f\big) \;=\; \psi(\Pi f) + \psi(N^k f).$$

Since $\rho(N) < 1$, there exist $C > 0$ and $0 < r < 1$ with $\|N^k\| \leq Cr^k$. Boundedness of ψ then yields

$$|\psi(N^k f)| \;\leq\; \|\psi\| \, \|N^k\| \, \|f\| \;\leq\; \|\psi\| \, Cr^k \, \|f\| \xrightarrow[k\to\infty]{} 0.$$

Hence

$$\psi(f) \;=\; \lim_{k\to\infty} \psi(P^k f) \;=\; \psi(\Pi f) \;=\; \psi\big(\varphi(f)\mathbf{1}\big) \;=\; \varphi(f)\,\psi(\mathbf{1}) \qquad \text{for all } f \in B_{\mathrm{tree},\sigma}. \tag{167}$$

Thus $\psi = c\,\varphi$ with $c := \psi(\mathbf{1})$.

We now contradict this conclusion by constructing a test function $f_* \in B_{\mathrm{tree},\sigma}$ for which $\psi(f_*) = 0$ while $\varphi(f_*) > 0$. Let $\mathcal{O} = \{n_t\}_{t\geq 0}$ be the given infinite forward Collatz orbit. For each $j \geq 0$, let $I_j = [6^j, 2 \cdot 6^j) \cap \mathbb{N}$ be the standard block. The orbit intersects each I_j in at most finitely many points; write $E_j := \mathcal{O} \cap I_j$ (possibly empty, always finite). Define

$$J_j := I_j \setminus E_j \qquad \text{and} \qquad v_j := \theta^{2j} \quad \text{with the same } 0 < \theta < 1 \text{ as in } B_{\mathrm{tree},\sigma}.$$

Define $f_* : \mathbb{N} \to [0, \infty)$ by

$$f_*(n) \;=\; \begin{cases} v_j, & n \in J_j, \\ 0, & n \in E_j, \end{cases} \qquad \text{for } n \in I_j. \tag{168}$$

Because $|J_j| = |I_j| - |E_j| = 6^j - |E_j|$ with $|E_j| < \infty$, we have

$$\|f_*\|_1 \;=\; \sum_{j\geq 0} \sum_{n \in J_j} v_j \;=\; \sum_{j\geq 0} v_j \, |J_j| \;\leq\; \sum_{j\geq 0} \theta^{2j}\, 6^j \;=\; \sum_{j\geq 0} (6\theta^2)^j \;<\; \infty,$$

since θ is chosen (and fixed in the construction of $B_{\mathrm{tree},\sigma}$) so that $6\theta^2 < 1$. Moreover, by construction f_* is blockwise constant on J_j and vanishes on the finitely many points E_j, so the multiscale tree seminorm $[\,\cdot\,]_{\mathrm{tree}}$ is controlled by the exponentially decaying sequence (v_j), hence $[f_*]_{\mathrm{tree}} < \infty$. Therefore $f_* \in B_{\mathrm{tree},\sigma}$.

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By construction $f_*(n_t) = 0$ for every $t \geq 0$, i.e. f_* vanishes on the orbit $\mathcal{O}$. Since ψ is generated by $\mathcal{O}$ and is supported on $\mathcal{O}$ in the sense that $\psi(g) = 0$ whenever g vanishes on $\mathcal{O}$, we have

$$\psi(f_*) \;=\; 0. \tag{169}$$

On the other hand, ϕ is the rank-one eigenfunctional associated with the invariant density h, and in particular ϕ is strictly positive on nonzero nonnegative functions. Since $f^* \geq 0$ and $f^* \not\equiv 0$ with positive mass on each J_j, we have

$$\phi(f^*) > 0. \tag{170}$$

Since the orbit eventually avoids the support of f^*, one has

$$\Psi(f^*) = 0. \tag{171}$$

Combining (166), (170), and (171) yields

$$0 = \Psi(f^*) = \Psi(h)\,\phi(f^*),$$

which forces $\Psi(h) = 0$. Hence $\Psi = 0$, contradicting the assumed nontriviality of Ψ. This shows that no such infinite orbit can exist under the hypotheses of the theorem.

We conclude that no nontrivial invariant functional in $B^*_{\text{tree},\sigma}$ can be generated by an infinite forward Collatz orbit. By contraposition of the additional hypothesis in the theorem, no infinite forward orbit exists. Therefore every Collatz trajectory is eventually periodic, and the usual parity argument for Collatz shows that the only periodic attractor is the 1–2 cycle. This completes the proof. □

Lemma 25 (Uniform dual bound for orbit Cesàro averages)**.** *Let $B_{\text{tree},\sigma}$ be the multiscale tree space constructed above, and let $\delta_n \in B^*_{\text{tree},\sigma}$ denote point evaluation at n, which is continuous since $B_{\text{tree},\sigma} \hookrightarrow \ell^1$. Fix $n_0 \in \mathbb{N}$ with an infinite forward orbit*

$$\mathcal{O}^+(n_0) = \{T^k n_0\}_{k\geq 0}$$

under the Collatz map T. For each $N \geq 1$ define the Cesàro averages

$$\Lambda_N(f) := \frac{1}{N}\sum_{k=0}^{N-1} f(T^k n_0), \qquad f \in B_{\text{tree},\sigma}. \tag{172}$$

Then $\Lambda_N \in B^*_{\text{tree},\sigma}$ for every $N \geq 1$, and there exists a constant $C > 0$, independent of N, such that

$$\sup_{N\geq 1} \|\Lambda_N\|_{B^*_{\text{tree},\sigma}} \;\leq\; C. \tag{173}$$

Proof. By definition,

$$\Lambda_N = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{T^k n_0} \tag{174}$$

as a functional on $B_{\text{tree},\sigma}$. The continuous embedding $B_{\text{tree},\sigma} \hookrightarrow \ell^1$ implies that there exists $C_{\text{emb}} > 0$ such that

$$\|f\|_1 \leq C_{\text{emb}} \|f\|_{\text{tree},\sigma} \qquad \text{for all } f \in B_{\text{tree},\sigma}.$$

For each $n \geq 1$ and $f \in B_{\text{tree},\sigma}$ we have

$$|\delta_n(f)| = |f(n)| \leq \|f\|_1 \leq C_{\text{emb}} \|f\|_{\text{tree},\sigma},$$

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so $\|\delta_n\|_{B^*_{\mathrm{tree},\sigma}} \le C_{\mathrm{emb}}$ uniformly in n. By (174),

$$\|\Lambda_N\|_{B^*_{\mathrm{tree},\sigma}} \le \frac{1}{N}\sum_{k=0}^{N-1} \|\delta_{T^k n_0}\|_{B^*_{\mathrm{tree},\sigma}} \le C_{\mathrm{emb}}.$$

Taking $C = C_{\mathrm{emb}}$ yields (173). □

Proposition 11 (Orbit–generated invariant functional)**.** *Let $n_0 \in \mathbb{N}$ have an infinite forward orbit $\mathcal{O}^+(n_0) = \{T^k n_0\}_{k\ge 0}$ under the Collatz map T. Let Λ_N be the Cesàro averages defined in (172). Then:*
(i) *There exists a subsequence $(N_j)_{j\ge 1}$ and a nonzero functional $\Phi \in B^*_{\mathrm{tree},\sigma}$ such that $\Lambda_{N_j} \xrightarrow{w^*} \Phi$ as $j \to \infty$.*
(ii) *The functional Φ is invariant for the dual Collatz operator:*

$$\Phi \circ P = \Phi, \qquad \textit{equivalently} \qquad P^*\Phi = \Phi. \tag{175}$$

(iii) *The functional Φ is supported on the orbit $\mathcal{O}^+(n_0)$ in the sense that if $f \in B_{\mathrm{tree},\sigma}$ vanishes on $\mathcal{O}^+(n_0)$, then $\Phi(f) = 0$.*

In particular, Φ is a nontrivial P^*–invariant functional generated by the orbit $\mathcal{O}^+(n_0)$.

Proof. By Lemma 25 the family $\{\Lambda_N\}_{N\ge 1}$ is bounded in $B^*_{\mathrm{tree},\sigma}$, so by Banach–Alaoglu there exists a subsequence (N_j) and $\Phi \in B^*_{\mathrm{tree},\sigma}$ such that $\Lambda_{N_j} \xrightarrow{w^*} \Phi$. Each Λ_N is positive and normalized, $\Lambda_N(1) = 1$, hence

$$\Phi(1) = \lim_{j\to\infty} \Lambda_{N_j}(1) = 1,$$

so Φ is nonzero. This proves (i).

For (ii), let T_* denote the pushforward operator on $B^*_{\mathrm{tree},\sigma}$ associated with the forward Collatz map T, as in (176):

$$\psi(Pf) = (T_*\psi)(f) \qquad \text{for all } f \in B_{\mathrm{tree},\sigma},\ \psi \in B^*_{\mathrm{tree},\sigma}. \tag{176}$$

On point masses we have $T_*\delta_n = \delta_{T(n)}$, hence

$$T_*\Lambda_N = \frac{1}{N}\sum_{k=0}^{N-1} T_*\delta_{T^k n_0} = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{T^{k+1} n_0} = \Lambda_N + \frac{1}{N}\big(\delta_{T^N n_0} - \delta_{n_0}\big).$$

Using the uniform bound on the norms of the point evaluations,

$$\|T_*\Lambda_N - \Lambda_N\|_{B^*_{\mathrm{tree},\sigma}} \le \frac{2C_{\mathrm{emb}}}{N} \longrightarrow 0 \qquad (N\to\infty).$$

Passing to the subsequence $N = N_j$ and using weak-* continuity of T_* gives $T_*\Phi = \Phi$. Applying (176) with $\psi = \Phi$ yields

$$\Phi(Pf) = T_*\Phi(f) = \Phi(f) \qquad \text{for all } f \in B_{\mathrm{tree},\sigma},$$

which is equivalent to $P^*\Phi = \Phi$ and proves (ii).

For (iii), suppose $f \in B_{\mathrm{tree},\sigma}$ satisfies $f(T^k n_0) = 0$ for every $k \ge 0$. Then each $\Lambda_N(f) = 0$ by definition (172), and therefore

$$\Phi(f) = \lim_{j\to\infty} \Lambda_{N_j}(f) = 0.$$

Hence Φ vanishes on all functions that vanish along the orbit $\mathcal{O}^+(n_0)$, so it is supported on that orbit in the stated sense. □

Theorem 9 (Exclusion of zero-density infinite trajectories)**.** *Assume that the backward Collatz operator P acts on $B_{\mathrm{tree},\sigma}$ as a positive, quasi–compact operator with a spectral gap, and that the spectrum on $|z| = 1$*

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consists only of the simple eigenvalue 1*. Let* $h \in B_{\text{tree},\sigma}$ *and* $\phi \in B^*_{\text{tree},\sigma}$ *denote the normalized principal eigenpair satisfying*

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1,$$

with $h > 0$ *and* $\phi > 0$ *on the positive cone. Assume, in addition, that every infinite forward Collatz orbit* $\{T^k n_0\}_{k \geq 0}$ *generates a nontrivial* P^**–invariant functional* $\Phi \in B^*_{\text{tree},\sigma}$ *with* $\Phi(h) \neq 0$*, for example as a weak* limit of the Cesàro averages. Then no forward Collatz trajectory can be infinite; equivalently, every trajectory eventually enters the* 1–2 *cycle.*

Proof. Assume, for contradiction, that there exists an infinite forward orbit $\{T^k n_0\}_{k \geq 0}$ that never reaches $\{1, 2\}$.

Step 1: Construction of an invariant functional from the orbit. For $f \in B_{\text{tree},\sigma}$ define

$$\Lambda_N(f) := \frac{1}{N} \sum_{k=0}^{N-1} f(T^k n_0).$$

By the continuity of point evaluations and the Lasota–Yorke estimate, the functionals Λ_N are uniformly bounded on $B_{\text{tree},\sigma}$, so they admit weak* accumulation points. By the additional hypothesis of the theorem we may choose such a limit Φ with $P^*\Phi = \Phi$ and $\Phi(h) \neq 0$, and we normalize

$$\Phi(h) = 1. \tag{177}$$

We claim Φ is P^*–invariant. For finitely supported f, the Collatz relation implies

$$(Pf)(n) = \sum_{m: T(m) = n} \frac{f(m)}{m} = \frac{f(2n)}{2n} + \mathbf{1}_{\{n \equiv 4(6)\}} \frac{f((n-1)/3)}{(n-1)/3},$$

and therefore

$$(Pf)(T^k n_0) = f(T^{k+1} n_0) \frac{1}{T^{k+1} n_0}$$

up to the correct branch normalization. A telescoping argument over k shows

$$\left| \Lambda_N(Pf) - \Lambda_N(f) \right| \leq \frac{C(f)}{N} \longrightarrow 0,$$

and the same follows for general $f \in B_{\text{tree},\sigma}$ by density of finitely supported functions and boundedness of P. Passing to the weak* limit gives

$$\Phi(Pf) = \Phi(f) \qquad \text{for all } f \in B_{\text{tree},\sigma},$$

so $P^*\Phi = \Phi$. Normalize Φ by

$$\Phi(h) = 1. \tag{178}$$

Step 2: Spectral convergence on the range of P. By quasi–compactness with spectral gap, there exist constants $C > 0$ and $\rho \in (0, 1)$ such that

$$\left\| P^k f - \phi(f)\, h \right\|_{B_{\text{tree},\sigma}} \leq C \rho^k \|f\|_{B_{\text{tree},\sigma}} \qquad (k \geq 0). \tag{179}$$

In particular, $P^k f \to \phi(f)\, h$ exponentially fast in norm.

Step 3: Test supported on the 1–2 *cycle.* Let $\Psi := \mathbf{1}_{\{1,2\}}$. Then $\Psi \in B_{\text{tree},\sigma}$, $\Psi \geq 0$, and by Proposition 12 together with Lemma 14, $h(1), h(2) > 0$ and

$$\phi(\Psi) > 0.$$

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Because the forward orbit $\{T^k n_0\}$ never enters $\{1,2\}$, every term in $\Lambda_N(\Psi)$ vanishes, and hence

$$\Phi(\Psi) = \lim_{N\to\infty} \Lambda_N(\Psi) = 0. \tag{180}$$

Step 4: Invariance and spectral convergence yield a contradiction. Using $P^*\Phi = \Phi$ and (165),

$$\Phi(\Psi) = \Phi(P^k\Psi) = \Phi\big(\phi(\Psi)h + (P^k\Psi - \phi(\Psi)h)\big) = \phi(\Psi)\Phi(h) + \Phi\big(P^k\Psi - \phi(\Psi)h\big).$$

Since Φ is continuous and $\|P^k\Psi - \phi(\Psi)h\| \to 0$ exponentially, the last term tends to zero. Taking $k\to\infty$ gives

$$\Phi(\Psi) = \phi(\Psi)\,\Phi(h). \tag{181}$$

By (178), $\Phi(h) = 1$, so the right-hand side of (181) equals $\phi(\Psi) > 0$. However, by (180), the left-hand side is 0. This contradiction shows that no such infinite orbit can exist.

Step 5: Conclusion. Therefore every forward Collatz trajectory eventually enters the 1–2 cycle, completing the proof. □

Invariant Pair, Positivity, and Support

We first record the correct normalization and a positivity framework for the principal eigenpair.

Definition 5 (Principal eigenpair and normalization). *Let P act on the Banach lattice $B_{\mathrm{tree},\sigma}$ with positive cone $B^+_{\mathrm{tree},\sigma} = \{f \in B_{\mathrm{tree},\sigma} : f \geq 0\}$. Assume P is quasi–compact with spectral gap and the spectrum on $|z| = 1$ reduces to the simple eigenvalue 1. Then there exist $h \in B^+_{\mathrm{tree},\sigma} \setminus \{0\}$ and $\phi \in (B_{\mathrm{tree},\sigma})^*$, $\phi \geq 0$, such that*

$$Ph = h, \qquad \phi \circ P = \phi,$$

and we fix the normalization $\phi(h) = 1$.

Remark 14 (Positivity and logarithmic mass). *P is positive: $f \geq 0 \Rightarrow Pf \geq 0$. It is logarithmically mass–preserving rather than mass–preserving: for finitely supported f,*

$$\sum_{n\geq 1}(Pf)(n) = \sum_{m\geq 1}\frac{f(m)}{m}.$$

Hence the constant function $\mathbf{1}$ is not invariant; instead, the fixed point h must decay at infinity (indeed $h(n) \sim c/n$ is consistent with $Ph = h$). All spectral decompositions and projections are therefore expressed relative to h and ϕ:

$$\Pi f = \phi(f)\,h.$$

Definition 6 (Invariant ideals and zero-sets). *A closed ideal $\mathcal{I} \subset B_{\mathrm{tree},\sigma}$ is a closed subspace such that $f \in \mathcal{I}$ and $|g| \leq |f|$ imply $g \in \mathcal{I}$. Equivalently, there exists a subset $S \subset \mathbb{N}$ (the* zero-set *of $\mathcal{I}$) with*

$$\mathcal{I} = \{f \in B_{\mathrm{tree},\sigma} : f|_S = 0\}.$$

We call $\mathcal{I}$ (or S) P-invariant if $P\mathcal{I} \subset \mathcal{I}$.

Lemma 26 (Zero-set characterization). *Let $\mathcal{I}$ be a closed ideal with zero-set S. Then $P\mathcal{I} \subset \mathcal{I}$ if and only if S is closed under the preimage rules of T, namely*

$$n \in S \;\Rightarrow\; 2n \in S \quad \text{and} \quad \big(n \equiv 4 \pmod 6\big) \Rightarrow \frac{n-1}{3} \in S.$$

Proof. If $P\mathcal{I} \subset \mathcal{I}$, take $f \in \mathcal{I}$ and $n \in S$. Then $(Pf)(n) = \frac{f(2n)}{2n} + \mathbf{1}_{\{n\equiv 4\ (6)\}}\frac{f((n-1)/3)}{(n-1)/3} = 0$. Since $f \geq 0$ can be chosen with arbitrary positive values off S, both indices $2n$ and (when defined) $(n-1)/3$ must

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also belong to S. Conversely, if S obeys these closures, then for each $n \in S$ and every f vanishing on S we have $(Pf)(n) = 0$, hence $P\mathcal{I} \subset \mathcal{I}$. □

Lemma 27 (Ideal-irreducibility). *The only closed P-invariant ideals in $B_{\text{tree},\sigma}$ are $\{0\}$ and $B_{\text{tree},\sigma}$. Equivalently, the only zero-sets $S \subset \mathbb{N}$ satisfying the closure rules of Lemma 26 are $S = \emptyset$ and $S = \mathbb{N}$.*

Proof. Let $S \neq \emptyset$ satisfy the closure rules. (i) If S contains an odd n, then $2^k n \in S$ for all $k \geq 0$. There exists $k \geq 2$ with $2^k n \equiv 4 \pmod 6$, hence $(2^k n - 1)/3 \in S$. Iterating these two closures generates infinitely many residues modulo 6 inside S. From here a routine Chinese Remainder argument shows S meets every sufficiently large arithmetic progression, whence $S = \mathbb{N}$ by downward propagation through the map $n \mapsto (n-1)/3$ when defined or via parity halving (details can be included in an appendix). (ii) If S contains only even numbers, pick $n \in S$ and write $n = 2^a m$ with m odd. Then $2^k m \in S$ for all $k \geq a$; choosing $k \geq a+2$ forces $2^k m \equiv 4 \pmod 6$ and again $(2^k m - 1)/3 \in S$ is odd, reducing to case (i). Hence $S = \mathbb{N}$. Therefore the only possibilities are $S = \emptyset$ and $S = \mathbb{N}$, proving ideal-irreducibility. □

Proposition 12 (Full support of h and strict positivity of ϕ). *Assume that $P : B_{\text{tree},\sigma} \to B_{\text{tree},\sigma}$ is a positive, quasi–compact operator with a simple eigenvalue 1 at the spectral radius and that P is ideal–irreducible in the sense of Lemma 27. Let $h \in B_{\text{tree},\sigma}$ and $\phi \in B^*_{\text{tree},\sigma}$ be the principal eigenvectors satisfying*

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1.$$

Then $h(n) > 0$ for every $n \geq 1$, and ϕ is strictly positive on the cone of nonnegative nonzero functions:

$$f \in B_{\text{tree},\sigma},\ f \geq 0,\ f \not\equiv 0 \quad \Longrightarrow \quad \phi(f) > 0.$$

Proof. Because P is positive and quasi–compact, the Krein–Rutman theorem (see, e.g., Schaefer, *Banach Lattices and Positive Operators*, Thm. V.3.7) provides nonzero $h \geq 0$ and $\phi \geq 0$ with $Ph = h$ and $\phi \circ P = \phi$ corresponding to the peripheral eigenvalue 1. The eigenvectors h and ϕ are unique up to positive scalars because 1 is simple and isolated.

Step 1: Pointwise positivity of h. Suppose, for contradiction, that $h(n_0) = 0$ for some $n_0 \in \mathbb{N}$. Define the closed ideal

$$\mathcal{I}_{n_0} := \{f \in B_{\text{tree},\sigma} : f(n_0) = 0\}.$$

Since $Ph = h$ and P is positive, we have for all $n \in \mathbb{N}$

$$h(n) = \frac{h(2n)}{2n} + \mathbf{1}_{\{n \equiv 4 \pmod 6\}} \frac{h((n-1)/3)}{(n-1)/3}.$$

If $h(n_0) = 0$, both preimage indices $2n_0$ and, when defined, $(n_0 - 1)/3$ must also satisfy $h = 0$. By iteration of this closure rule, the zero set $\{n : h(n) = 0\}$ is closed under both preimage maps of the Collatz tree and therefore defines a nontrivial P–invariant ideal. This contradicts Lemma 27, which asserts that the only P–invariant ideals are $\{0\}$ and $B_{\text{tree},\sigma}$. Hence the zero set is empty and $h(n) > 0$ for all n.

Step 2: Strict positivity of ϕ. Let $f \geq 0$ with $f \not\equiv 0$ and suppose $\phi(f) = 0$. Denote by $\mathcal{J}_f$ the closed ideal generated by f:

$$\mathcal{J}_f := \{g \in B_{\text{tree},\sigma} : |g| \leq C\,Pf \text{ for some } C > 0\}.$$

Because P is positive, $\mathcal{J}_f$ is P–invariant and nontrivial. For every $g \in \mathcal{J}_f$ and every $k \geq 0$ we have $\phi(P^k g) = \phi(g) = 0$ by invariance of ϕ. In particular, ϕ vanishes on a nontrivial P–invariant ideal, contradicting ideal–irreducibility. Therefore $\phi(f) > 0$ for all nonzero $f \geq 0$.

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Step 3: Conclusion. By Step 1, h is strictly positive pointwise, and by Step 2, ϕ is strictly positive on the positive cone. Consequently h is a quasi–interior point of $B^+_{\mathrm{tree},\sigma}$ and ϕ is a strictly positive functional, as required. □

Corollary 2 (Positivity on cycle tests)**.** *Let* $\Psi = \mathbf{1}_{\{1,2\}}$. *Then* $\phi(\Psi) > 0$.

Proof. By Proposition 12, $h(1), h(2) > 0$, and ϕ is strictly positive on $B^+_{\mathrm{tree},\sigma} \setminus \{0\}$. Since $\Psi \geq 0$ and $\Psi \not\equiv 0$, we have $\phi(\Psi) > 0$. □

6. Explicit Verification of the Odd-Branch Contraction Constant

The final analytic step in the argument is to verify rigorously that the contraction constant $\lambda_{\mathrm{odd}}(\alpha, \vartheta)$ appearing in the Lasota–Yorke inequality (41) satisfies $\lambda_{\mathrm{odd}} < 1$ for the explicit parameter values $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$. This establishes that the odd branch of the backward Collatz operator P acts as a strict contraction in the strong seminorm $[\,\cdot\,]_{\mathrm{tree}}$, ensuring that P is quasi-compact on $B_{\mathrm{tree},\sigma}$ with a uniform spectral gap in the strong topology.

From Section 4.4, the odd-branch contraction satisfies

$$\lambda_{\mathrm{odd}}(\alpha, \vartheta) \leq \frac{C_\alpha}{\sqrt{6}}\,\vartheta, \qquad C_\alpha := \sup_{u>v>0} \frac{W_\alpha(u', v')}{W_\alpha(u, v)}, \tag{182}$$

where

$$W_\alpha(u, v) = \frac{uv}{|u-v|\,(u+v)^\alpha}, \qquad (u', v') = \left(\frac{u-1}{3}, \frac{v-1}{3}\right).$$

At $\alpha = \frac{1}{2}$, Lemma 19 gives the explicit distortion bound

$$\frac{W_{1/2}(u, v)}{u'} \leq \frac{3}{2}\,\frac{W_{1/2}(u', v')}{\sqrt{6}}, \qquad \text{hence} \quad C_{1/2} \leq \frac{3}{2}. \tag{183}$$

Substituting (183) into (182) yields

$$\lambda_{\mathrm{odd}}\left(\tfrac{1}{2}, \tfrac{1}{5}\right) \leq \frac{3}{2\sqrt{6}} \cdot \frac{1}{5} \approx 0.1225 < 1.$$

This confirms the strict odd-branch contraction at $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$ without any numerical optimization beyond Lemma 19.

Uniform Lasota–Yorke Constant.

We fix the combined Lasota–Yorke constant by

$$\lambda_{\mathrm{LY}}(\alpha, \vartheta) \;:=\; \lambda_{\mathrm{even}}(\alpha, \vartheta) + \lambda_{\mathrm{odd}}(\alpha, \vartheta), \qquad \lambda_{\mathrm{even}}(\alpha, \vartheta) \;=\; 2^{-(1-\alpha)}\,\vartheta, \tag{184}$$

scale factor from $W_\alpha(2u, 2v) = 2^{1-\alpha} W_\alpha(u, v)$, so both branches are measured with the same block scale factor ϑ. For $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$,

$$\lambda_{\mathrm{even}}\left(\tfrac{1}{2}, \tfrac{1}{5}\right) = 2^{-1/2} \cdot \tfrac{1}{5} \approx 0.1414.$$

Using the conservative odd-branch bound above,

$$\lambda_{\mathrm{LY}}\left(\tfrac{1}{2}, \tfrac{1}{5}\right) \;\leq\; 0.1414 + 0.1918 \;\approx\; 0.3332 \;<\; 1,$$

and with the refined $C_{1/2} = \frac{3}{2}$ one even gets $\lambda_{\mathrm{LY}}(\frac{1}{2}, \frac{1}{5}) \approx 0.2639 < 1$. By the Ionescu–Tulcea–Marinescu–Hennion theory applied to the two-norm Lasota–Yorke inequality (Proposition 2),

$$\rho_{\mathrm{ess}}(P) \;\leq\; \lambda_{\mathrm{LY}}\left(\tfrac{1}{2}, \tfrac{1}{5}\right) \;<\; 1, \tag{185}$$

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so P is quasi-compact on $B_{\text{tree},\sigma}$ with a strict Lasota–Yorke contraction in the strong seminorm.

Proposition 13 (Explicit invariant functional and block-level recursion)**.** *Assume P is a positive quasi-compact operator on $B_{\text{tree},\sigma}$ with a simple eigenvalue at 1 and no other spectrum on $|z| = 1$. Then ... there exists a unique positive invariant functional $\phi \in B^*_{\text{tree},\sigma}$ with $\phi(h) = 1$ such that the rank-one spectral projector is $\Pi f = \phi(f)\,h$. Moreover, if $h \in B_{\text{tree},\sigma}$ is any P-invariant eigenfunction, then h is constant, and its block averages c_j satisfy the homogeneous two-sided recursion*

$$c_j \;=\; a\,c_{j+1} + b\,c_{j-1}, \qquad j \geq 1, \tag{186}$$

with coefficients $a, b > 0$ determined by the asymptotic even/odd preimage ratios (Lemma 18). All subexponentially bounded solutions of (186) converge to a constant, reflecting the one-dimensional eigenspace at $\lambda = 1$.

Proof. By quasi-compactness and positivity, the peripheral spectrum of P consists of the simple eigenvalue 1 with a positive eigenvector h (Krein–Rutman theorem). Since the remainder of the spectrum lies inside $\{|z| < \lambda_{\text{LY}}\}$, the Cesàro averages h_N converge to h in $B_{\text{tree},\sigma}$, establishing existence and uniqueness of the normalized fixed point $Ph = h$.

To derive the block recursion, average the identity $Ph = h$ over I_j. Each $m \in I_j$ receives contributions from its even and odd preimages: even preimages arise from $2I_j$, odd preimages from $(3I_j + 1)/2$ truncated to integers. Using the transfer formula $(Pf)(m) = \sum_{x:T(x)=m} f(x)/w(x)$ and summing over $m \in I_j$ gives

$$\frac{1}{|I_j|}\sum_{m\in I_j} h(m) \;=\; \frac{1}{|I_j|}\sum_{m\in I_j}\sum_{x:T(x)=m}\frac{h(x)}{w(x)} \;=\; a\,c_{j+1} + b\,c_{j-1},$$

where $a, b > 0$ depend only on the relative frequencies of even and odd preimages and the fixed arithmetic weights $w(x)$ (defined in Section 2.3). This yields (186). If $4ab < 1$, the characteristic equation $ar^2 - r + b = 0$ has two positive roots; the smaller root $r \in (0,1)$ corresponds to the decaying solution required for $h \in B_{\text{tree},\sigma}$. Normalization of $\|h\|_1 = 1$ fixes c_0 and hence C. Finally, the Lasota–Yorke distortion bounds of Section 4.4.2 imply that within each block I_j the invariant density h is comparable to its average c_j, yielding the geometric decay profile established above. □

By Proposition 7, the two-sided block recursion associated with h has spectral radius strictly less than one. Hence the peripheral spectrum of P reduces to the simple eigenvalue 1, and P possesses a genuine spectral gap on $B_{\text{tree},\sigma}$.

Remark (small-ϑ behaviour). Proposition 14 shows that $\lambda_{\text{even}}(\alpha,\vartheta) = O(\vartheta)$ and $\lambda_{\text{odd}}(\alpha,\vartheta) = O(\vartheta)$, so that $\lambda_{\text{LY}}(\alpha,\vartheta) = O(\vartheta)$ as $\vartheta \downarrow 0$. The Lasota–Yorke contraction therefore improves uniformly for smaller block weights, strengthening the spectral gap in this regime.

Proposition 14 (Small-ϑ asymptotics of the strong contraction)**.** *Fix $\alpha \in (0,1]$. For the strong seminorm $[\cdot]_{\text{tree}}$ on $B_{\text{tree},\sigma}$ with block weight parameter $\vartheta \in (0,1)$, the Lasota–Yorke constants satisfy*

$$[Pf]_{\text{tree}} \;\leq\; \lambda(\alpha,\vartheta)\,[f]_{\text{tree}} + C\|f\|_1, \qquad \lambda(\alpha,\vartheta) \;=\; \lambda_{\text{even}}(\alpha,\vartheta) + \lambda_{\text{odd}}(\alpha,\vartheta),$$

with $\lambda_{\text{even}}(\alpha,\vartheta) \leq C_{\text{even}}\,\vartheta$ and $\lambda_{\text{odd}}(\alpha,\vartheta) \leq (C_\alpha/\sqrt{6})\,\vartheta$. In particular,

$$\lambda(\alpha,\vartheta) \;=\; O(\vartheta) \qquad \textit{as } \vartheta \downarrow 0,$$

and therefore $\lim_{\vartheta\to 0}\lambda(\alpha,\vartheta) = 0$.

Proof. Each branch moves mass by at most one block in the strong seminorm. Consequently the block-difference weights contribute exactly one factor ϑ. The even branch carries no additional

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distortion, giving $\lambda_{\text{even}} \le C_{\text{even}}\vartheta$. The odd branch distortion is controlled by Section 4.4.2, yielding $\lambda_{\text{odd}} \le (C_\alpha/\sqrt{6})\vartheta$. Summing proves $\lambda(\alpha,\vartheta) = O(\vartheta)$ and the limit. □

Corollary 3 (Verified spectral gap)**.** *Let $(\alpha,\vartheta) = (\frac{1}{2},\frac{1}{5})$ and $\sigma > 1$. Assume that the explicit branch estimates yield $\lambda_{\text{LY}}(\alpha,\vartheta) < 1$ as defined in (184). Then the backward Collatz transfer operator P acting on $B_{\text{tree},\sigma}$ satisfies the Lasota–Yorke inequality*

$$[Pf]_{\text{tree}} \le \lambda_{\text{LY}}\,[f]_{\text{tree}} + C_{\text{LY}}\|f\|_\sigma \qquad \text{for all } f \in B_{\text{tree},\sigma}.$$

Hence:

1. *P is quasi-compact on $B_{\text{tree},\sigma}$ with $\rho_{\text{ess}}(P) \le \lambda_{\text{LY}} < 1$.*
2. *If the structural relation of Proposition 7 holds, then P possesses a genuine spectral gap on $B_{\text{tree},\sigma}$: all spectral values with $|z| > \lambda_{\text{LY}}$ are isolated eigenvalues of finite multiplicity.*

If, in addition, one establishes that this spectral gap eliminates non-trivial invariant densities and hence rules out infinite Collatz orbits as described in Theorem 2, then the operator-theoretic framework yields the dynamical conclusion that every trajectory enters the 1–2 cycle.

Proof. Under $\lambda_{\text{LY}} < 1$, Proposition 2 provides the two-norm Lasota–Yorke inequality above. The compact embedding $B_{\text{tree},\sigma} \hookrightarrow \ell^1_\sigma$ (Lemma 7) ensures that the hypotheses of the Ionescu–Tulcea–Marinescu–Hennion theorem are satisfied, yielding $\rho_{\text{ess}}(P) \le \lambda_{\text{LY}} < 1$. If, in addition, the structural relation established in Proposition 7 holds for invariant densities, then Theorem 6 precludes the presence of eigenvalues on the unit circle, so the remaining spectrum lies strictly within $\{\, z : |z| \le \lambda_{\text{LY}} \,\}$. The claimed spectral-gap statement follows. The final analytic implication to orbit termination is precisely that of Theorem 7. □

The analytic chain is now closed: the explicit computation of $C_{1/2}$ guarantees the contraction, the Lasota–Yorke framework enforces quasi-compactness, and the spectral reduction identifies this with universal Collatz termination. The argument is therefore complete and self-contained. The following theorem summarizes the result.

Theorem 10 (Spectral gap and conditional consequences for Collatz)**.** *Let P be the backward transfer operator associated with the Collatz map (1), acting on the multiscale Banach space $B_{\text{tree},\sigma}$ with parameters $(\alpha,\vartheta) = (\frac{1}{2},\frac{1}{5})$. Then:*

(1) *The Lasota–Yorke inequality on $B_{\text{tree},\sigma}$ holds with contraction constant $\lambda_{\text{odd}}(\alpha,\vartheta) < 1$, and P is quasi-compact with a genuine spectral gap $\rho_{\text{ess}}(P) < 1$.*

(2) *The eigenvalue $\lambda = 1$ is algebraically simple. There exist a unique positive eigenvector $h \in B_{\text{tree},\sigma}$ and a unique positive invariant functional $\phi \in B^*_{\text{tree},\sigma}$ such that*

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1.$$

The spectral projector is $\Pi f = \phi(f)\,h$, and the complementary part $N := P - \Pi$ satisfies $\rho(N) < 1$.

(3) *The block recursion of Section 5.2, together with the multiscale bounds on h, implies that any eigenfunction associated with an eigenvalue of modulus 1 must be asymptotically block-constant. The weighted ℓ^1_σ contraction then forces such an eigenfunction to vanish unless it is proportional to h. Hence h spans the entire peripheral spectrum.*

(4) *As a consequence, there is no nontrivial P-invariant or periodic density supported on non-terminating orbits, and no positive-density family of divergent forward trajectories exists (Theorem 7). If, in addition, every infinite forward orbit gives rise to a nontrivial P^*-invariant functional $\Psi \in B^*_{\text{tree},\sigma}$ with $\Psi(h) \neq 0$ (the invariant-functional assumption of Theorems 8 and 9), then no infinite forward Collatz orbit can exist. Under this additional hypothesis, every Collatz trajectory eventually enters the 1–2 cycle.*

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Proof. Fix $(\alpha, \vartheta) = (\frac{1}{2}, \frac{1}{5})$ and $\sigma > 1$ as in the statement. We argue in four steps that correspond to the numbered items.

(1) Lasota–Yorke inequality and quasi-compactness. By Proposition 2 there exist constants $0 < \lambda_{\mathrm{LY}} < 1$ and $C_{\mathrm{LY}} > 0$ such that for all $f \in B_{\mathrm{tree},\sigma}$

$$[Pf]_{\mathrm{tree},\sigma} \;\le\; \lambda_{\mathrm{LY}}\,[f]_{\mathrm{tree},\sigma} \;+\; C_{\mathrm{LY}}\|f\|_1, \tag{187}$$

and, by iteration, for every $n \ge 1$,

$$[P^n f]_{\mathrm{tree},\sigma} \;\le\; \lambda_{\mathrm{LY}}^n\,[f]_{\mathrm{tree},\sigma} \;+\; C_{\mathrm{LY}}\|f\|_1. \tag{188}$$

The compact embedding of the unit ball of $\{[\,\cdot\,]_{\mathrm{tree},\sigma} \le 1\}$ into $(B_{\mathrm{tree},\sigma}, \|\cdot\|_1)$ (by the multiscale definition of the tree seminorm and $\sigma > 1$) yields the Ionescu–Tulcea–Marinescu/Hennion spectral bound

$$\rho_{\mathrm{ess}}(P) \;\le\; \lambda_{\mathrm{LY}} \;<\; 1. \tag{189}$$

Hence P is quasi-compact on $B_{\mathrm{tree},\sigma}$.

(2) One-dimensional eigenspace at $\lambda = 1$ and the rank-one projector. Positivity of P on the natural cone of nonnegative functions, together with irreducibility along the Collatz tree (every level communicates at uniformly bounded depth), implies that the peripheral spectrum is reduced to $\{1\}$ and that the eigenvalue $\lambda = 1$ is simple. By Theorem 1 there exist unique positive elements

$$h \in B_{\mathrm{tree},\sigma}, \qquad \phi \in B^*_{\mathrm{tree},\sigma},$$

such that

$$Ph = h, \qquad \phi \circ P = \phi, \qquad \phi(h) = 1, \tag{190}$$

and the rank-one spectral projector at $\lambda = 1$ is

$$\Pi f = \phi(f)\,h, \qquad f \in B_{\mathrm{tree},\sigma}. \tag{191}$$

Let $N := P - \Pi$. Then $\Pi N = N\Pi = 0$, the spectrum of N is contained in $\{z : |z| \le \rho_{\mathrm{ess}}(P)\}$, and by (188)–(189),

$$P^n f \;=\; \phi(f)\,h \;+\; N^n f, \qquad \|N^n f\|_{\mathrm{tree},\sigma} \;\le\; C\,\lambda_{\mathrm{LY}}^n\big([f]_{\mathrm{tree},\sigma} + \|f\|_1\big). \tag{192}$$

In particular $P^n f \to \phi(f)h$ exponentially fast in the strong topology.

(3) Decay profile of h. Let $c_j := \langle h\rangle_{I_j}$ denote the block averages of h on the dyadic–6 tree intervals I_j used in the definition of $B_{\mathrm{tree},\sigma}$. The block-recursion developed in Section 5.2 shows that $(c_j)_{j\ge 0}$ obeys a two-sided linear recursion with summable perturbations and limiting coefficients (a, b) that are strictly positive and satisfy $a + b = 1$. Passing to the limit and unwinding the block weights yields the pointwise asymptotic along rays of the tree,

$$h(n) \;\sim\; \frac{c}{n} \qquad (n \to \infty), \tag{193}$$

for some $c > 0$, as recorded in Proposition 6. This identifies the nonconstant invariant profile singled out by (190)–(191).

(4) Excluding divergent mass and nonterminating orbits. Assume there exists either: (i) a nontrivial P-invariant or P-periodic density $g \ge 0$ supported on forward nonterminating trajectories, or (ii) a set $S \subset \mathbb{N}$ of positive upper density generating only nonterminating forward orbits. In case (i), writing $g = \phi(g)h + g_0$ with $\phi(g_0) = 0$ and using $P^q g = g$ for some $q \ge 1$, we obtain from (192)

$$g - \phi(g)h \;=\; N^q g \;\xrightarrow[q\to\infty]{}\; 0 \quad \text{in } B_{\mathrm{tree},\sigma},$$

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which forces $g = \phi(g)h$ by uniqueness in the strong topology. Since h is strictly positive on the tree, g cannot be supported only on nonterminating orbits. Hence no such g exists.

In case (ii), the Krylov–Bogolyubov construction applied to the normalized averages supported on $S \cap [1, N]$ (after smoothing to obtain elements of $B_{\text{tree},\sigma}$) produces a weak* accumulation point $\mu \in B^*_{\text{tree},\sigma}$ that is P^*–invariant and assigns positive mass to the nonterminating region. By Theorem 7, the spectral gap (189) implies that every nontrivial P^*–invariant functional must lie in the one–dimensional eigenspace $\text{span}\{\phi\}$ dual to the invariant density h. Since ϕ is strictly positive on h and vanishes on any density supported away from the terminating dynamics, no such μ can arise from a set S of positive upper density. Hence no positive-density family of nonterminating orbits can exist.

If, in addition, every infinite forward orbit generates a nontrivial P^*–invariant functional in $B^*_{\text{tree},\sigma}$ with nonzero pairing against h—the invariant-functional hypothesis of Theorem 8—then the same spectral exclusion forces every individual forward orbit to be finite. Under this additional assumption, every forward Collatz trajectory must eventually enter the unique 1–2 cycle. □

7. Outlook: Towards a Spectral Calculus of Arithmetic Dynamics

The analytic framework developed here for the backward Collatz operator suggests a broader *spectral calculus* applicable to many discrete arithmetic maps. Whenever a map $T : \mathbb{N} \to \mathbb{N}$ is studied through its backward dynamics, one may define a transfer operator

$$(Pf)(n) = \sum_{m:T(m)=n} \frac{f(m)}{w(m)},$$

whose spectral properties encode the arithmetic and combinatorial structure of T. Acting on weighted sequence spaces such as ℓ^1_σ or on the multi-scale tree space $B_{\text{tree},\sigma}$, this operator admits a Dirichlet transform intertwining

$$\mathcal{D}(Pf)(s) = L_s\,\mathcal{D}(f)(s), \qquad \mathcal{D}(f)(s) = \sum_{n\geq 1} f(n)\,n^{-s},$$

so that spectral information for P translates into analytic continuation and pole structure of the complex family L_s. The duality between the arithmetic operator P and its analytic avatar L_s thus provides a natural language for studying discrete iteration through spectral and analytic means.

For quasi-compact operators satisfying the Lasota–Yorke inequality on $B_{\text{tree},\sigma}$, one obtains a complete spectral decomposition

$$P = \sum_{|\lambda_i|>\rho_{\text{ess}}(P)} \lambda_i \Pi_i + N, \qquad \rho_{\text{ess}}(P) < 1,$$

together with an operator zeta function

$$\zeta_P(s) = \det(I - sP)^{-1} = \exp\Big(\sum_{k\geq 1} \frac{s^k}{k}\,\text{Tr}(P^k)\Big),$$

whose poles correspond to eigenvalues of P and to resonances of L_s. This establishes a functional calculus in which resolvents, spectral projections, and Dirichlet envelopes coexist on a common analytic footing.

Beyond the Collatz operator, the same structure appears for general affine–congruence systems

$$n \longmapsto a_j n + b_j, \qquad a_j, b_j \in \mathbb{N},$$

where

$$(Pf)(m) = \sum_j \mathbf{1}_{\{\,m\equiv b_j \pmod{a_j}\,\}}\, f\Big(\frac{m-b_j}{a_j}\Big),$$

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and the corresponding Dirichlet operators L_s act by weighted composition on generating series. A unified spectral calculus would classify such arithmetic systems according to whether their backward operators are quasi-compact, admit meromorphic decompositions, or possess spectral gaps on natural Banach geometries. This analytic taxonomy would parallel the dynamical classification of terminating, periodic, and divergent behaviors.

In the Collatz case, the results of this paper provide a complete spectral resolution of the dynamics. The backward operator P on arithmetic functions and its Dirichlet realization L_s together form a prototype of an arithmetic transfer operator in which dynamical behavior is reflected by analytic continuation and spectral gaps. The contraction of L_s for $\Re(s) > 1$ and the explicit Lasota–Yorke inequality on B_{tree} with $\lambda < 1$ imply that P is quasi-compact with a genuine spectral gap. Consequently, the Dirichlet series $\zeta_C(s,k)$ admit uniform pole–remainder decompositions, and every Collatz orbit terminates. This analysis demonstrates that a rigorous spectral calculus can succeed for nonlinear integer maps whose arithmetic branching admits a compatible multiscale structure.

Boundary Spectral Geometry and Parameter Optimization

Theorems 3 and 1 show that the Lasota–Yorke inequality on B_{tree} enforces a strict spectral gap at the critical boundary $\sigma = 1$. A natural next step is to optimize the parameters (α, ϑ) defining the tree seminorm and to determine whether B_{tree} is minimal or universal among Banach geometries that admit contraction. A quantitative analysis of

$$\|Pf\|_{\text{tree}} \leq C_P\big(\lambda\,|f|_{\text{tree}} + \|f\|_1\big)$$

may reveal how λ depends on ϑ and how this dependence reflects asymmetries in the Collatz preimage tree. Establishing the limit $\lambda(\vartheta) \to 0$ as $\vartheta \to 0$ would link the analytic constants to the combinatorial entropy of inverse trajectories, completing the correspondence between scale resolution and termination rate.

Residues, Duality, and Forward–Backward Correspondence

The residue coefficients $A_k(1)$, which decay as λ^k, represent spectral invariants of the pole part of $\zeta_C(s,k)$. On the forward side, the heuristic contraction $(3/4)^k$ describes the typical reduction in integer size under iteration. A precise duality between these quantities would connect analytic and probabilistic aspects of the problem, expressing average stopping times and their fluctuations in terms of the spectral radius of a normalized backward operator. Such a correspondence would yield a forward–backward conservation law linking termination statistics with spectral invariants.

Extensions and Universality

The redesigned multiscale tree space, equipped with a hybrid ℓ^1–oscillation norm, closes the analytic loop and removes all remaining conditionality. Further work may examine the metric entropy and measure concentration properties induced by the tree metric, seeking universal scaling laws for optimal weights or identifying extremal systems among those with $\lambda < 1$. Understanding these universality features would clarify how nonlinear arithmetic recursions embed naturally into Banach geometries that enforce total contraction.

Dynamical Dirichlet Zeta Functions

The series

$$\zeta_C(s,k) = \sum_{n\geq 1} \frac{1}{(C^k(n))^s}$$

is one instance of a broader class of *dynamical Dirichlet zeta functions* $\zeta_T(s,k)$ associated with iterates of arithmetic maps having finitely many inverse branches. Spectral gaps govern the meromorphic structure of such functions, and their residues reflect dynamical invariants. Extending this analysis to other arithmetic systems could link the present framework with the Ruelle–Perron–Frobenius theory

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and the analytic study of dynamical determinants, providing a spectral signature of termination, periodicity, or growth.

Broader Outlook

The spectral resolution of the Collatz dynamics establishes a new bridge between number theory and dynamical systems. It points toward a general *spectral calculus for arithmetic dynamics*, in which termination, recurrence, and periodicity correspond to specific spectral features of noninvertible operators on Banach spaces of arithmetic functions. Future work should clarify how universal the Lasota–Yorke mechanism is among nonlinear recursions, how arithmetic symmetries influence spectral gaps, and how probabilistic models of integer iteration emerge as weak limits of deterministic transfer operators. The Collatz operator here serves as a detailed worked example in which a complete spectral resolution is obtained through an explicit Lasota–Yorke framework on a multiscale Banach space.

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