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Multi-Dimensional Integral Transform with Fox Function in Kernel on Lebesgue-Type Spaces

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Abstract: This paper is devoted to the study of multi-dimensional integral transform with Fox H-function in the kernel in weighted spaces integrable functions in the domain $\mathbb{R}^n_+$ with positive coordinates. Mapping properties such as the boundedness, the range, the representations of the considered transformation are established.

Keywords: multi-dimensional integral transform; fox H-function; melling transform; weighted space; fractional integrals and derivatives

MSC: 44A30; 33C60; 35A22

1. Introduction

We consider the multi-dimensional H- integral transform ([1], formula (43)):

$$(\mathrm{H}f)(\mathbf{x}) = \int_{\mathbf{0}}^{\infty} \mathrm{H}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\mathbf{xt}\left|\begin{matrix}(\mathbf{a}_i,\overline{\alpha}_i)_{1,p}\\(\mathbf{b}_j,\overline{\beta}_j)_{1,q}\end{matrix}\right.\right] f(\mathbf{t})\mathbf{dt}, \qquad \mathbf{x}>0; \tag{1}$$

where (see [1–3], ch. 28; [4], ch. 1; [5,6]) $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$; $\mathbf{t} = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$, $\mathbb{R}^n$ be the n-dimensional Euclidean space; $\mathbf{x}\cdot\mathbf{t} = \sum\limits_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x}\cdot\mathbf{1} = \sum\limits_{n=1}^{n} x_n$ for $\mathbf{1}$= (1,1,...,1). The inequality $\mathbf{x} > \mathbf{t}$ means that $x_1 > t_1, ..., x_n > t_n$, and inequalities $\geq, <, \leq$ have similar meanings ; $\int\limits_{\mathbf{0}}^{\infty} = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\cdots\int\limits_{0}^{\infty}$; by $\mathbb{N} = \{1, 2, ...\}$ we denote the set of natural numbers, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $\mathbb{N}^n_0 = \mathrm{N}_0 \times ... \times \mathbb{N}_0$; $k = (k_1, k_2, ..., k_n) \in \mathbb{N}^n_0$ $(k_i \in \mathbb{N}_0,\ i = 1, 2, ..., n)$ is a multi-index with $k! = k_1! \cdots k_n!$ and $|k| = k_1 + ... + k_n$; $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} > 0\}$; for $\kappa = (\kappa_1, \kappa_2, ..., \kappa_n) \in \mathbb{R}^n_+$ $\mathbf{D}^\kappa = \frac{\partial^{|\kappa|}}{(\partial x_1)^{\kappa_1}\cdots(\partial x_n)^{\kappa_n}}$; $\mathbf{dt} = dt_1 \cdots dt_n$; $\mathbf{t}^\kappa = t^{\kappa_1} t^{\kappa_2} \cdots t^{\kappa_n}$; $f(\mathbf{t}) = f(t_1, t_2, ..., t_n)$; $\mathbb{C}^n$ $(n \in \mathbb{N})$ be the n-dimensional space of n complex numbers $z = (z_1, z_2, \cdots, z_n)$ $(z_j \in \mathbb{C},\ j = 1, 2, \cdots, n)$; $\overline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$; $\overline{h} = (h_1, h_2, ..., h_n) \in \mathbb{R}^n_+$; $\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} = \frac{d}{dx_1\cdot dx_2\cdots dx_n}$;

$\mathbf{m} = (m_1, m_2, ..., m_n) \in \mathbb{N}^n_0$ and $m_1 = m_2 = ... = m_n$; $\mathbf{n} = (\overline{n}_1, \overline{n}_2, ..., \overline{n}_n) \in \mathbb{N}^n_0$ and $\overline{n}_1 = \overline{n}_2 = ... = \overline{n}_n$; $\mathbf{p} = (p_1, p_2, ..., p_n) \in \mathbb{N}^n_0$ and $p_1 = p_2 = ... = p_n$; $\mathbf{q} = (q_1, q_2, ..., q_n) \in \mathbb{N}^n_0$ and $q_1 = q_2 = ... = q_n$ $(0 \leq \mathbf{m} \leq \mathbf{q}, 0 \leq \mathbf{n} \leq \mathbf{p})$;

$\mathbf{a}_i = (a_{i_1}, a_{i_2}, ..., a_{i_n})$, $1 \leq i \leq \mathbf{p}$, $a_{i_1}, a_{i_2}, ..., a_{i_n} \in \mathbb{C}$ $(i_1 = 1, 2, ..., p_1; ...; i_n = 1, 2, ..., p_n)$;
$\mathbf{b}_j = (b_{j_1}, b_{j_2}, ..., b_{j_n})$, $1 \leq j \leq \mathbf{q}$, $b_{j_1}, b_{j_2}, ..., b_{j_n} \in \mathbb{C}$ $(j_1 = 1, 2, ..., q_1; ...; j_n = 1, 2, ..., q_n)$;
$\overline{\alpha}_i = (\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n})$, $1 \leq i \leq \mathbf{p}$, $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n} \in \mathbb{R}^+_1$ $(i_1 = 1, 2, ..., p_1; ...; i_n = 1, 2, ..., p_n)$;
$\overline{\beta}_j = (\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_n})$, $1 \leq j \leq \mathbf{q}$, $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_n} \in \mathbb{R}^+_1$ $(j_1 = 1, 2, ..., q_1; ...; j_n = 1, 2, ..., q_n)$.

The function in the kernel of (1)

$$\mathrm{H}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\mathbf{xt}\left|\begin{matrix}(\mathbf{a}_i,\overline{\alpha}_i)_{1,p}\\(\mathbf{b}_j,\overline{\beta}_j)_{1,q}\end{matrix}\right.\right] = \prod_{k=1}^{n} H^{m_k,\overline{n}_k}_{p_k,q_k}\left[x_k t_k\left|\begin{matrix}(a_{i_k},\alpha_{i_k})_{1,p_k}\\(b_{j_k},\beta_{j_k})_{1,q_k}\end{matrix}\right.\right] \tag{2}$$

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is the product of H-functions $H^{m,n}_{p,q}[z]$:

$$H^{m,n}_{p,q}[z] \equiv H^{m,n}_{p,q}\left[z\left|\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right.\right] = \frac{1}{2\pi i}\int\limits_L \mathcal{H}^{m,n}_{p,q}(s)z^{-s}ds,\ z\neq 0, \tag{3}$$

where

$$\mathcal{H}^{m,n}_{p,q}(s) \equiv \mathcal{H}^{m,n}_{p,q}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right|s\right] = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod\limits_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}. \tag{4}$$

In the representation (3) L is a specially chosen infinite contour, and the empty products, if any, are taken to be one.

The H-function (3) is the most general of the known special functions and includes as special cases elementary functions, special functions of hypergeometric and bessel type, as well as the Meyer G-function. One may find its properties, for example, in the books by Mathai and Saxena ([7], Ch. 2), Srivastava, Gupta and Goyal ([8], ch. 1), Prudnikov, Brychkov and Marichev ([9], Section 8.3), Kiryakova [10] and Kilbas and Saigo ([11], Ch.1 – Ch.4).

Our paper is devoted to the study of H- transform (1) on Lebesgue-type weighted spaces $\mathfrak{L}_{\overline{\nu},\overline{2}}$ of functions $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ on $\mathbb{R}^{\mathbf{n}}_+$, such that

$$\|f\|_{\overline{\nu},\overline{2}} = \{\int_{\mathbb{R}^1_+} x_n^{2\cdot\nu_n-1}\{\cdots\{\int_{\mathbb{R}^1_+} x_2^{2\cdot\nu_2\cdot-1}\times$$

$$[\int_{\mathbb{R}^1_+} x_1^{2\cdot\nu_1-1}|f(x_1,...,x_n)|^2dx_1]dx_2\}\cdots\}dx_n\}^{1/2}<\infty,$$

$\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$, $\nu_1 = \nu_2 = ... = \nu_n$, and $\overline{2} = (2, 2, ..., 2)$.

In this paper we apply the results from [2] to obtain mapping properties such as boundedness, the rang and representations for the H- transform (1).

Research results for transformation (1) generalize those obtained earlier for the corresponding one-dimensional transformation (see [11], Ch. 3):

$$(\mathrm{H}f)(x) = \int\limits_0^\infty \mathrm{H}^{m,n}_{p,q}\left[xt\left|\begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right.\right]f(t)dt, \qquad x>0; \tag{5}$$

in the space $\mathfrak{L}_{\nu,2}$ of Lebesgue measurable functions f on $\mathbb{R}^1_+ = (0,\infty)$, such that

$$\int_0^\infty |t^\nu f(t)|^2\frac{dt}{t} < \infty \quad (\nu\in\mathbb{R}).$$

The H- transform (5) generalizing many integral transforms: transforms with the Meijer G-function, Laplace and Hankel transforms, transforms with Gauss hypergeometric function, transforms with other hypergeometric and Bessel functions in the kernels. One may find a survey of results and bibliography in this field for one-dimensional case in the monograph ([11],Sections 6–8). Note that a very important class of transforms under consideration is a class of Buschman–Erdélyi operators, they have many important properties and applications, cf. [1,12–16]. And topic of this paper is also in a very tight connection with transmutation theory, cf. [17–21].

2. Preliminaries

The properties of the H-function $H^{m,n}_{p,q}[z]$ (3) depend on the numbers ([11], formulas 1.1.7–1.1.15):

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j; \;\; \Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i; \tag{6}$$

$$\delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j}; \tag{7}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}; \tag{8}$$

$$a_1^* = \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i; \; a_2^* = \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j; \; a_1^* + a_2^* = a^*, \;\; a_1^* - a_2^* = \Delta; \tag{9}$$

$$\xi = \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j + \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i; \tag{10}$$

$$c^* = m + n - \frac{p+q}{2}. \tag{11}$$

An empty sum in (6), (8), (9), (10) and an empty product in (7), if they occur, are taken to be zero and one, respectively.

There holds the following assertions.

Lemma 1.([11], Lemma 1.2) *For σ, $t \in \mathbb{R}$, there holds the estimate*

$$|\mathcal{H}_{p,q}^{m,n}(\sigma + it)| \sim C|t|^{\Delta\sigma + \mathrm{Re}(\mu)} \exp^{-\pi[|t|a^* + \mathrm{Im}(\xi)\mathrm{sign}(t)]/2} \;\; (|t| \to \infty) \tag{12}$$

uniformly in σ on any bounded interval in $\mathbb{R}$, where

$$C = (2\pi)^{c^*} \exp^{-c^* - \Delta\sigma - \mathrm{Re}(\mu)} \delta^{\sigma} \prod_{i=1}^{p} \alpha_i^{1/2 - \mathrm{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\mathrm{Re}(b_j) - 1/2} \tag{13}$$

and ξ and c^* are defined in (10) *and* (11).

Theorem 1.([11],Theorem 3.4) *Let $\alpha < \zeta < \beta$ and either of the conditions $a^* > 0$ or $a^* = 0$ and $\Delta\zeta + \mathrm{Re}(\mu) < -1$ are hold. Then for $x > 0$, except for $x = \delta$ when $a^* = 0$ and $\Delta = 0$, the relation*

$$H_{p,q}^{m,n}\left[x \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_p, \beta_p) \end{matrix} \right] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{H}_{p,q}^{m,n}\left[\begin{matrix} (a_p, \alpha_p) \\ (b_p, \beta_p) \end{matrix} \middle| t \right] x^{-t} dt \tag{14}$$

holds and the estimate

$$|H_{p,q}^{m,n}\left[x \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_p, \beta_p) \end{matrix} \right]| \leq A_\zeta x^{-\zeta} \tag{15}$$

is valid, where A_ζ is a positive constant depending only on ζ.

A set of bounded linear operators acting from a Banach space X into a Banach space Y denote by $[X, Y]$.

Multidimensional Mellin integral transform $(\mathfrak{M}f)(\mathbf{x})$ of function $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$, $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}_+^n$, is determined by the formula

$$(\mathfrak{M}f)(\mathbf{s}) = \int_0^\infty f(\mathbf{t}) \mathbf{t}^{\mathbf{s}-1} \mathbf{dt}, \;\; \mathrm{Re}(\mathbf{s}) = \overline{\nu}, \tag{16}$$

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$\mathbf{s} = (s_1, s_2, ..., s_n) \in \mathbb{C}^n$. The inverse multidimensional Mellin transform has the form

$$(\mathfrak{M}^{-1}g)(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \cdots \int_{\gamma_n - i\infty}^{\gamma_n + i\infty} \mathbf{x}^{-\mathbf{s}} g(\mathbf{s}) d\mathbf{s}, \tag{17}$$

$\mathbf{x} \in \mathbb{R}^n_+$, $\gamma_j = \mathrm{Re}(s_j)$ $(j = 1, \cdots, n)$. The theory of multidimensional integral transformations (16) and (17) can be recognized, for example, in books ([4], Ch. 1; [22,23]).

We will need the following spaces. As usual, by $\mathrm{L}_{\overline{p}}(\mathbb{R}^n)$ we will understand the space of functions $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$, for which

$$\|f\|_{\overline{p}} = \left\{ \int_{\mathbb{R}^n} |f(\mathbf{x})|^{\overline{p}} d\mathbf{x} \right\}^{1/\overline{p}} < \infty, \ \ \overline{p} = (p_1, p_2, ..., p_n), \ 1 \le \overline{p} < \infty.$$

If $\overline{p} = \infty$, then the space $\mathrm{L}_\infty(\mathbb{R}^n)$ is defined as the collection of all measurable functions with a finite norm

$$\|f\|_{\mathrm{L}_\infty(\mathbb{R}^n)} = esssup|f(\mathbf{x})|,$$

here $esssup\,|f(\mathbf{x})|$ is the essential supremum of the function $|f(\mathbf{x})|$ [24].

We need the following properties of the Mellin transform (16).

Lemma 2. ([2], Lemma 1) *Let $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$, $\nu_1 = \nu_2 = ... = \nu_n$. The following properties of the Mellin transform (16) are valid:*

(a) Transformation (16) is a unitary mapping of the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ onto the space $\mathrm{L}_{\overline{2}}(\mathbb{R}^n)$.

(b)For $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ there holds

$$f(\mathbf{x}) = \frac{1}{(2\pi i)^n} \lim_{\mathrm{R}\to\infty} \int_{\nu_1 - i\mathrm{R}}^{\nu_1 + i\mathrm{R}} \int_{\nu_2 - i\mathrm{R}}^{\nu_2 + i\mathrm{R}} \cdots \int_{\nu_n - i\mathrm{R}}^{\nu_n + i\mathrm{R}} (\mathfrak{M}f)(\mathbf{s}) \mathbf{x}^{-\mathbf{s}} d\mathbf{s}, \tag{18}$$

where the limit is taken in the topology of the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ and where,

if $F(\overline{\nu} + i\mathbf{t}) = \prod_{i=1}^{n} F_j(\nu_j + it_j)$, $F_j(\nu_j + it_j) \in \mathrm{L}_1(-\mathrm{R}, \mathrm{R})$, $j = 1, 2, ..., n$, then

$$\int_{\nu_1 - iR}^{\nu_1 + iR} \int_{\nu_2 - iR}^{\nu_2 + iR} \cdots \int_{\nu_n - iR}^{\nu_n + iR} F(\mathbf{s}) d\mathbf{s} = i^n \int_{-R}^{R} \int_{-R}^{R} \cdots \int_{-R}^{R} F(\overline{\nu} + i\mathbf{t}) d\mathbf{t}.$$

(c) For functions $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $g \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$ the following equality holds

$$\int_{\mathbf{0}}^{\infty} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} = \frac{1}{(2\pi i)^n} \int_{\overline{\nu} - i\infty}^{\overline{\nu} + i\infty} (\mathfrak{M}f)(\mathbf{s}) (\mathfrak{M}g)(1 - \mathbf{s}) \mathbf{x}^{-\mathbf{s}} d\mathbf{s}. \tag{19}$$

In [2] we consider the general multi-dimensional integral transform ([2], formula (1)):

$$(\mathrm{K}f)(\mathbf{x}) = \overline{h} \mathbf{x}^{1 - (\overline{\lambda} + 1)/\overline{h}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \mathbf{x}^{(\overline{\lambda} + 1)/\overline{h}} \int_{\mathbf{0}}^{\infty} \mathrm{k}[\mathbf{x}\mathbf{t}] f(\mathbf{t}) d\mathbf{t} \ \ (\mathbf{x} > 0), \tag{20}$$

where the function $\mathrm{k}[\mathbf{x}\mathbf{t}]$ in the kernel of (20) is the product of some one type special functions:

$$\mathrm{k}[\mathbf{x}\mathbf{t}] = \mathrm{k}[x_1 t_1] \cdot \mathrm{k}[x_2 t_2] \cdots \mathrm{k}[x_n t_n].$$

Transformation (20) satisfies the following theorem.

Theorem 2.([2], Theorem 1) *Let $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$ $(\nu_1 = \nu_2 = ... = \nu_n)$, $\overline{h} = (h_1, h_2, ..., h_n) \in \mathbb{R}^n_+$, and $\overline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$.*

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(a) If the transformation operator (20) satisfies the condition $\mathrm{K} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$*, then the kernel on the right side of (20)* $\mathrm{k} \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$*. If we set for* $\nu_j \neq 1 - (\mathrm{Re}(\lambda_j) + 1)/h_j, j = 1, 2, ..., n$*,*

$$(\mathfrak{M}\mathrm{k})(1 - \overline{\nu} + i\mathbf{t}) = \frac{\theta(\mathbf{t})}{\overline{\lambda} + 1 - (1 - \overline{\nu} + i\mathbf{t})\overline{h}} =$$

$$= \prod_{j=1}^{n} \frac{\theta(t_j)}{\lambda_j + 1 - (1 - \nu_j + it_j)h_j} \tag{21}$$

almost everywhere, then function $\theta \in \mathrm{L}_\infty(\mathbb{R}^n)$*, and for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *there holds the relation*

$$(\mathfrak{M}\mathrm{K}f)(1 - \overline{\nu} + i\mathbf{t}) = \theta(\mathbf{t})(\mathfrak{M}f)(\overline{\nu} - i\mathbf{t}) \tag{22}$$

almost everywhere.

(b) Conversely, for given function $\theta \in \mathrm{L}_\infty(\mathbb{R}^n)$*, there is a transform* $\mathrm{K} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ *so that the equality (22) holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$*. Moreover, if* $\nu_j \neq 1 - (\mathrm{Re}(\lambda_j) + 1)/h_j$*,* $j = 1, 2, ..., n$*, then transformation* $\mathrm{K}f$ *(20) is representable in the form (20) with the kernel* k *definite by (21).*

(c) Based on statements (a) or (b) with $\theta \neq 0$*,* K *is one-to-one transformation from the space* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *into the space* $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$*, and if in addition* $1/\theta \in \mathrm{L}_\infty(\mathbb{R}^n)$*, then* K *maps* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *onto* $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$*, and for functions* $f, g \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *the relation*

$$\int_{\mathbf{0}}^{\infty} f(\mathbf{x})(\mathrm{K}g)(\mathbf{x})d\mathbf{x} = \int_{\mathbf{0}}^{\infty} (\mathrm{K}f)(\mathbf{x})g(\mathbf{x})d\mathbf{x} \tag{23}$$

is valid.

3. $\mathfrak{L}_{\overline{\nu},2}$-Theory for the Multi-Dimensional H-Transform

To formulate the results for the transform $\mathrm{H}f$ (1) we need the following constants ([1], (57)–(60)), analogical for one-dimensional case defined via the parameters of the H - function (3) ([11], (3.4.1), (3.4.2), (1.1.7), (1.1.8), (1.1.10)):

let $\widetilde{\alpha} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, ..., \widetilde{\alpha}_n)$ and $\widetilde{\beta} = (\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n)$ where

$$\widetilde{\alpha}_1 = \begin{cases} -\min\limits_{1\leq j_1 \leq m_1}\left[\frac{\mathrm{Re}(b_{j_1})}{\beta_{j_1}}\right], & m_1 > 0, \\ -\infty, & m_1 = 0, \end{cases} \quad \widetilde{\beta}_1 = \begin{cases} \min\limits_{1\leq i_1 \leq \overline{n}_1}\left[\frac{1-\mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right], & \overline{n}_1 > 0, \\ \infty, & \overline{n}_1 = 0, \end{cases}$$

$$\widetilde{\alpha}_2 = \begin{cases} -\min\limits_{1\leq j_2 \leq m_2}\left[\frac{\mathrm{Re}(b_{j_2})}{\beta_{j_2}}\right], & m_2 > 0, \\ -\infty, & m_2 = 0, \end{cases} \quad \widetilde{\beta}_2 = \begin{cases} \min\limits_{1\leq i_2 \leq \overline{n}_2}\left[\frac{1-\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right], & \overline{n}_2 > 0, \\ \infty, & \overline{n}_2 = 0, \end{cases}$$

and so on

$$\widetilde{\alpha}_n = \begin{cases} -\min\limits_{1\leq j_n \leq m_n}\left[\frac{\mathrm{Re}(b_{j_n})}{\beta_{j_n}}\right], & m_n > 0, \\ -\infty, & m_2 = 0, \end{cases} \quad \widetilde{\beta}_n = \begin{cases} \min\limits_{1\leq i_n \leq \overline{n}_n}\left[\frac{1-\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right], & \overline{n}_n > 0, \\ \infty, & \overline{n}_n = 0; \end{cases} \tag{24}$$

let $a^* = (a_1^*, a_2^*, ..., a_n^*)$, $\Delta = (\Delta_1, \Delta_2, ..., \Delta_n)$ and

$$a_1^* = \sum_{i=1}^{\overline{n}_1} \alpha_{i_1} - \sum_{i=\overline{n}_1+1}^{p_1} \alpha_{i_1} + \sum_{j=1}^{m_1} \beta_{j_1} - \sum_{j=m_1+1}^{q_1} \beta_{j_1}, \quad \Delta_1 = \sum_{j=1}^{q_1} \beta_{j_1} - \sum_{i=1}^{p_1} \alpha_{i_1},$$

$$a_2^* = \sum_{i=1}^{\overline{n}_2} \alpha_{i_2} - \sum_{i=\overline{n}_2+1}^{p_2} \alpha_{i_2} + \sum_{j=1}^{m_2} \beta_{j_2} - \sum_{j=m_1+1}^{q_2} \beta_{j_2}, \quad \Delta_2 = \sum_{j=1}^{q_2} \beta_{j_2} - \sum_{i=1}^{p_2} \alpha_{i_2},$$

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and so on

$$a_n^* = \sum_{i=1}^{\overline{n}_n} \alpha_{i_n} - \sum_{i=\overline{n}_n+1}^{p_n} \alpha_{i_n} + \sum_{j=1}^{m_n} \beta_{j_n} - \sum_{j=m_n+1}^{q_n} \beta_{j_n}; \Delta_n = \sum_{j=1}^{q_n} \beta_{j_n} - \sum_{i=1}^{p_n} \alpha_{i_n}; \tag{25}$$

let $\mu = (\mu_1, \mu_2, ..., \mu_n)$ and

$$\mu_1 = \sum_{j=1}^{q_1} b_{j_1} - \sum_{i=1}^{p_1} a_{i_1} + \frac{p_1 - q_1}{2}, \mu_2 = \sum_{j=1}^{q_2} b_{j_2} - \sum_{i=1}^{p_2} a_{i_2} + \frac{p_2 - q_2}{2}, ...,$$

$$\mu_n = \sum_{j=1}^{q_n} b_{j_n} - \sum_{i=1}^{p_n} a_{i_n} + \frac{p_n - q_n}{2}; \tag{26}$$

The exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of a function $\overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}(\mathbf{s})$:

$$\overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}(\mathbf{s}) \equiv \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\begin{array}{c} (\mathbf{a}_i, \overline{\alpha}_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \overline{\beta}_j)_{1,\mathbf{q}} \end{array} \middle| \mathbf{s}\right] = \prod_{k=1}^{n} \mathcal{H}^{m_k,\overline{n}_k}_{p_k,q_k}\left[\begin{array}{c} (a_{i_k}, \alpha_{i_k})_{1,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k} \end{array} \middle| s\right], \tag{27}$$

is called a set of vectors $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$ $(\nu_1 = \nu_2 = ... = \nu_n)$, such that $\widetilde{\alpha}_k < 1 - \nu_k < \widetilde{\beta}_k$, $k = 1, 2, ...n$, where the parameters $\widetilde{\alpha}_k$, $\widetilde{\beta}_k (k = 1, 2, ..., n)$ are defined by formulas (24), and functions $\mathcal{H}^{m_k,\overline{n}_k}_{p_k,q_k}(s_k)$ $(k = 1, 2, ..., n)$ of the view (4) have zeros on lines $\mathrm{Re}(s_k) < 1 - \nu_k$ $(k = 1, 2, ..., n)$, respectively (see [1], (61)).

Applying multidimensional Mellin transformation (16) to (1), formally we obtain:

$$(\mathfrak{M}\mathrm{H}f)(\mathbf{s}) = \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\begin{array}{c} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{array} | \mathbf{s}\right](\mathfrak{M}f)(1 - \mathbf{s}). \tag{28}$$

Theorem 3. *Suppose that*

$$\widetilde{\alpha}_k < 1 - \nu_k < \widetilde{\beta}_k; \; \nu_k = \nu_l, \; k \neq l \; (k, l = 1, 2, ..., n); \tag{29}$$

and that either of the conditions

$$a_k^* > 0 \; (k = 1, 2, ..., n); \tag{30}$$

or

$$a_k^* = 0, \Delta_k[1 - \nu_k] + \mathrm{Re}(\mu_k) \leq 0 \; (k = 1, 2, ..., n) \tag{31}$$

holds. Then we have the following results:

(a) There exists a one-to-one transform $\mathrm{H} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ *so that the relation (28) holds for* $\mathrm{Re}(\mathbf{s}) = 1 - \overline{\nu}$ *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$.

If $a_k^* = 0$, $\Delta_k[1 - \nu_k] + \mathrm{Re}(\mu_k) = 0$ $(k = 1, 2, ..., n)$, *and* $\overline{\nu}$ *does not belong to an exceptional set* $\mathcal{E}_{\overline{\mathcal{H}}}$, *then the operator* H *maps* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *onto* $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$.

(b) If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then for H there holds the relation (23):*

$$\int_0^\infty f(\mathbf{x})(\mathrm{H}g)(\mathbf{x})\mathbf{dx} = \int_0^\infty (\mathrm{H}f)(\mathbf{x})g(\mathbf{x})d\mathbf{x}. \tag{32}$$

(c) Let $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, $\overline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$, $\overline{h} = (h_1, h_2, ..., h_n) \in \mathbb{R}^n_+$. *If* $\mathrm{Re}(\overline{\lambda}) > (1 - \overline{\nu})\overline{h} - 1$, *then* $\mathrm{H}f$ *is given by formula:*

$$(\mathrm{H}f)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$$

$$\times \frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\middle| \begin{matrix} (-\overline{\lambda},\overline{h}),(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}},\ (-\overline{\lambda}-1,\overline{h}) \end{matrix}\right] f(\mathbf{t})\mathbf{dt} \tag{33}$$

When $\mathrm{Re}(\overline{\lambda}) < (1-\overline{v})\overline{h} - 1$, $\mathrm{H}f$ *is given by:*

$$(\mathrm{H}f)(\mathbf{x}) = -\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$$

$$\times \frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_0^\infty \mathrm{H}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\middle| \begin{matrix} (\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}},(-\overline{\lambda},\overline{h}) \\ (-\overline{\lambda}-1,\overline{h}),(\mathbf{b}_j,\beta_j)_{1,\mathbf{q}} \end{matrix}\right] f(\mathbf{t})\mathbf{dt}. \tag{34}$$

(d) The transform H *is independent of* $\overline{v}$ *in the sense that, for* $\overline{v}$ *and* $\widetilde{\overline{v}}$ *satisfying the assumptions (29), and either (30) or (31), and for the respective transforms* H *on* $\mathfrak{L}_{\overline{v},\overline{2}}$ *and* $\widetilde{\mathrm{H}}$ *on* $\mathfrak{L}_{\widetilde{\overline{v}},\overline{2}}$ *given in (28), then* $\mathrm{H}f = \widetilde{\mathrm{H}}f$ *for* $f \in \mathfrak{L}_{\overline{v},\overline{2}} \bigcap \mathfrak{L}_{\widetilde{\overline{v}},\overline{2}}$.

Proof. Let $\overline{\omega}(\mathbf{t}) = \overline{\mathcal{H}}(1-\overline{v}+i\mathbf{t}) = \prod_{k=1}^{n} \mathcal{H}(1-\nu_k+it_k)$. By virtue of (4), (24), and the conditions (29) the functions $\mathcal{H}^{m_1,\overline{n}_1}_{p_1,q_1}(s_1), \mathcal{H}^{m_2,\overline{n}_2}_{p_2,q_2}(s_2),...,\mathcal{H}^{m_n,\overline{n}_n}_{p_n,q_n}(s_n)$ are analytic in the strips $\widetilde{\alpha}_1 < 1-\nu_1 < \widetilde{\beta}_1, \dots, \widetilde{\alpha}_n < 1-\nu_n < \widetilde{\beta}_n, \nu_1 = \nu_2 = ... = \nu_n$, respectively. In accordance with (12) and conditions (30) or (31), $\overline{\omega}(\mathbf{t}) = O(1)$ as $|\mathbf{t}| \to \infty$. Therefore $\overline{\omega} \in \mathrm{L}_\infty(\mathbb{R}^n)$, and hence we obtain from Theorem 2 (b) that there exists a transform $\mathrm{H} \in [\mathfrak{L}_{\overline{v},\overline{2}}, \mathfrak{L}_{1-\overline{v},\overline{2}}]$ such that

$$(\mathfrak{M}\mathrm{H}f)(\mathbf{s})(1-\overline{v}+i\mathbf{t}) = \overline{\mathcal{H}}(1-\overline{v}+i\mathbf{t})(\mathfrak{M}f)(\overline{v}-i\mathbf{t})$$

for $f \in \mathfrak{L}_{\overline{v},\overline{2}}$. This means that the equality (28) holds when condition $\mathrm{Re}(\mathbf{s}) = 1-\overline{v}$ is met. Since the functions $\mathcal{H}^{m_1,\overline{n}_1}_{p_1,q_1}(s_1), \mathcal{H}^{m_2,\overline{n}_2}_{p_2,q_2}(s_2),...,\mathcal{H}^{m_n,\overline{n}_n}_{p_n,q_n}(s_n)$ are analytic in the strips $\widetilde{\alpha}_1 < 1-\nu_1 < \widetilde{\beta}_1, \dots, \widetilde{\alpha}_n < 1-\nu_n < \widetilde{\beta}_n, \nu_1 = \nu_2 = ... = \nu_n$, respectively, and have isolated zeros, then $\overline{\omega}(\mathbf{t}) \neq 0$ almost everywhere. So it follows from the Theorem 2(c) that $\mathrm{H} \in [\mathfrak{L}_{\overline{v},\overline{2}}, \mathfrak{L}_{1-\overline{v},\overline{2}}]$ is a one-to-one transform. If $a_k^* = 0$, $\Delta_k(1-\nu_k) + \mathrm{Re}(\mu_k) = 0$ $(k = 1,2,...n)$ and $\overline{v}$ is not in the exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of $\overline{\mathcal{H}}$, then $1/\overline{\omega} \in \mathrm{L}_\infty(\mathbb{R}^n)$, and from Theorem 2 (c) we have that H transforms the space $\mathfrak{L}_{\overline{v},\overline{2}}$ onto $\mathfrak{L}_{1-\overline{v},\overline{2}}$. This completed the proof of the statement (a) of the theorem.

According to the statement of the Theorem 2 (c), if $f \in \mathfrak{L}_{\overline{v},\overline{2}}$ and $g \in \mathfrak{L}_{\overline{v},\overline{2}}$, then the relation (32) is valid. Thus the assertion (b) is true.

Let us prove the validity of the representation (33). Suppose that $f \in \mathfrak{L}_{\overline{v},\overline{2}}$ and $\mathrm{Re}(\overline{\lambda}) > (1-\overline{v})\overline{h} - 1$. To show the relation (33), it is sufficient to calculate the kernel k in the transform (20) for such $\overline{\lambda}$. From (21) we get the equality

$$(\mathfrak{M}\mathrm{k})(1-\overline{v}+i\mathbf{t}) = \overline{\mathcal{H}}(1-\overline{v}+i\mathbf{t})\frac{1}{\overline{\lambda}+1-(1-\overline{v}+i\mathbf{t})\overline{h}}$$

$$= \prod_{k=1}^{n} \mathcal{H}(1-\nu_k+it_k)\frac{1}{\lambda_k+1-(1-\nu_k+it_k)h_k}$$

or, for $\mathrm{Re}(\mathbf{s}) = 1-\overline{v}$

$$(\mathfrak{M}\mathrm{k})(\mathbf{s}) = \overline{\mathcal{H}}(\mathbf{s})\frac{1}{\overline{\lambda}+1-\overline{h}\mathbf{s}} = \prod_{k=1}^{n} \mathcal{H}(s_k)\frac{1}{\lambda_k+1-h_k s_k}. \tag{35}$$

Then from (18) and (35) we obtain the expression for the kernel k

$$\mathrm{k}(\mathbf{x}) = \prod_{k=1}^{n} \mathrm{k}(x_k) = \frac{1}{(2\pi i)^n}\prod_{k=1}^{n} \lim_{\mathrm{R}\to\infty} \int_{1-\nu_k-i\mathrm{R}}^{1-\nu_k+i\mathrm{R}} (\mathfrak{M}\mathrm{k})(s_k) x_k^{-s_k} ds_k$$

$$= \frac{1}{(2\pi i)^n} \prod_{k=1}^{n} \lim_{R\to\infty} \int_{1-\nu_k - iR}^{1-\nu_k+iR} \mathcal{H}_k(s_k) \frac{1}{\lambda_k + 1 - h_k s_k} x_k^{-s_k} ds_k, \tag{36}$$

where the limits are taken in the topology of $\mathfrak{L}_{\nu,2}$.

According to (4) and (27) we have

$$\overline{\mathcal{H}}(\mathbf{s}) \frac{1}{\overline{\lambda} + 1 - \overline{h}\mathbf{s}} = \overline{\mathcal{H}}(\mathbf{s}) \frac{\Gamma(1-(-\overline{\lambda}) - \overline{h}\mathbf{s})}{\Gamma(1-(-\overline{\lambda}-1) - \overline{h}\mathbf{s})}$$

$$= \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1} \left[\begin{matrix} (-\overline{\lambda}, \overline{h}), (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}}, (-\overline{\lambda}-1, \overline{h}) \end{matrix} \Big| \mathbf{s} \right]$$

$$= \prod_{k=1}^{n} \mathcal{H}^{m_k, \overline{n}_k+1}_{p_k+1, q_k+1} \left[\begin{matrix} (-\lambda_k, h_k), (a_{i_k}, \alpha_{i_k})_{1,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k}, (-\lambda_k - 1, h_k) \end{matrix} \Big| s_k \right]. \tag{37}$$

Denote by $\hat{\alpha}_k, \hat{\beta}_k$ $(k = 1, 2, ..., n)$ the constants $\widetilde{\alpha}_k, \widetilde{\beta}_k$ $(k = 1, 2, ..., n)$ in (24) respectively; by $\widetilde{a}_k^*$ $(k = 1, 2, ..., n)$ the constants a_k^* $(k = 1, 2, ..., n)$ and by $\widetilde{\Delta}_k$ $(k = 1, 2, ..., n)$ the constants Δ_k $(k = 1, 2, ..., n)$ in (25),respectively; by $\widetilde{\mu}_k$ $(k = 1, 2, ..., n)$ the constants μ_k $(k = 1, 2, ..., n)$ in (26) respectively for $\mathcal{H}^{m_k, \overline{n}_k+1}_{p_k+1, q_k+1}$ $(k = 1, 2, ..., n)$ in (37). Then $\hat{\alpha}_k = \widetilde{\alpha}_k$ $(k = 1, 2, ..., n)$; $\hat{\beta}_k = \min[\widetilde{\beta}_k, (1 + \mathrm{Re}(\lambda_k))/h_k]$ $(k = 1, 2, ..., n)$; $\widetilde{a}_k^* = a_k^*$ $(k = 1, 2, ..., n)$; $\widetilde{\Delta}_k = \Delta_k$ $(k = 1, 2, ..., n)$; $\widetilde{\mu}_k = \mu_k - 1$ $(k = 1, 2, ..., n)$. Thus, it follows that

(a') $\hat{\alpha}_k < 1 - \nu_k < \hat{\beta}_i$ $(k = 1, 2, ..., n)$;
from $\mathrm{Re}(\overline{\lambda}) > (1 - \overline{\nu})\overline{h} - 1$, and either of the conditions:
(b') $\widetilde{a}_k^* > 0$ $(k = 1, 2, ..., n)$; or
(c') $\widetilde{a}_k^* = 0$ $(k = 1, 2, ..., n)$;
$\widetilde{\Delta}_k(1 - \nu_k) + \mathrm{Re}(\widetilde{\mu}_k) = \Delta_k(1 - \nu_k) + \mathrm{Re}(\mu_k) - 1 \leq -1$
$(k = 1, 2, ..., n)$ holds. Applying Theorem 1 for $\mathbf{x} > 0$, then the equality

$$\mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1} \left[\mathbf{xt} \Big| \begin{matrix} (-\overline{\lambda}, \overline{h}), (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}}, (-\overline{\lambda}-1, \overline{h}) \end{matrix} \right]$$

$$= \prod_{k=1}^{n} \mathrm{H}^{m_k, \overline{n}_k+1}_{p_k+1, q_k+1} \left[x_k \Big| \begin{matrix} (-\lambda_k, h_k), (a_{i_k}, \alpha_{i_k})_{1,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k}, (-\lambda_k - 1, h_k) \end{matrix} \right]$$

$$= \frac{1}{(2\pi i)^n} \prod_{k=1}^{n} \lim_{R\to\infty} \int_{1-\nu_k - iR}^{1-\nu_k+iR} \mathcal{H}_k(s_k) \frac{1}{\lambda_k + 1 - h_k s_k} x_k^{-s_k} ds_k \tag{38}$$

holds almost everywhere. Then, (36) and (38) lead to the fact that the kernel k is given by

$$\mathrm{k}(\mathbf{x}) = \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1} \left[\mathbf{x} \Big| \begin{matrix} (-\overline{\lambda}, \overline{h}), (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}}, (-\overline{\lambda}-1, \overline{h}) \end{matrix} \right],$$

and (33) is proved.

The representation (34) is proved similarly to (33). We use the equality

$$\overline{\mathcal{H}}(\mathbf{s}) \frac{1}{\overline{\lambda} + 1 - \overline{h}\mathbf{s}} = -\overline{\mathcal{H}}(\mathbf{s}) \frac{\Gamma(\overline{h}\mathbf{s} - \overline{\lambda} - 1)}{\Gamma(\overline{h}\mathbf{s} - \overline{\lambda})}$$

$$= -\overline{\mathcal{H}}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1} \left[\begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}}, (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda}-1, \overline{h}), (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix} \Big| \mathbf{s} \right]$$

$$= -\prod_{k=1}^{n} \mathcal{H}^{m_k+1,\overline{n}_k}_{p_k+1,q_k+1}\left[\begin{matrix}(a_{i_k},\alpha_{i_k})_{1,p_k},(-\lambda_k,h_k)\\(-\lambda_k-1,h_k),(b_{j_k},\beta_{j_k})_{1,q_k}\end{matrix}|s_k\right]. \quad (39)$$

instead of (37).Thus, the statement (c) is proved.

Let us prove (d). If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}} \bigcap \mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ and $\mathrm{Re}(\overline{\lambda}) > \max[(1-\overline{\nu})\overline{h}-1,(1-\widetilde{\overline{\nu}})\overline{h}-1]$ or $\mathrm{Re}(\overline{\lambda}) < \min[(1-\overline{\nu})\overline{h}-1,(1-\widetilde{\overline{\nu}})\overline{h}-1]$, then both transforms $\mathrm{H}f$ and $\widetilde{\mathrm{H}}f$ are given in (33) or (34), respectively, which shows that they are independent of $\overline{\nu}$.

Corollary 1. *Suppose that* $\widetilde{\alpha}_k < \widetilde{\beta}_k$ $(k=1,2,...,n)$, *and that one of the following conditions holds:*
(a) $a_k^* > 0$ $(k=1,2,...,n)$;
(b) $a_k^* = 0$ $(k=1,2,...,n)$; $\Delta_k > 0$ $(k=1,2,...,n)$; *and*
$\widetilde{\alpha}_k < -\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,...,n)$;
(c) $a_k^* = 0$; $\Delta_k < 0$ $(k=1,2,...,n)$; *and*
$\widetilde{\beta}_k > -\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,...,n)$;
(d) $a_k^* = 0$ $(k=1,2,...,n)$; $\Delta_k = 0$, $(k=1,2,...,n)$; *and*
$\mathrm{Re}(\mu_k) \le 0$ $(k=1,2,...,n)$.
Then the H*-transform* (1) *can be defined on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *with*
$\widetilde{\alpha}_k < \nu_k < \beta_k$ $(k=1,2,...,n)$; $\nu_1 = \nu_2 = ... = \nu_n$.

Proof. When $1-\widetilde{\beta}_k < \nu_k < 1-\widetilde{\alpha}_k$ $(k=1,2,...,n)$, by Theorem 3, if either $a_k^* > 0$ $(k=1,2,...,n)$ or $a_k^* = 0$ $(k=1,2,...,n)$; $\Delta_k(1-\nu_k)\mathrm{Re}(\mu_k) \le 0$ $(k=1,2,...,n)$ are satisfied, then the H- transform can be defined on $\mathfrak{L}_{\overline{\nu},\overline{2}}$, which is also valid when $\widetilde{\alpha}_k < \nu_k < \widetilde{\beta}_k$ $(k=1,2,...,n)$. Hence the corollary is clear in cases (a) and (d). When $\Delta_k > 0$ and $\widetilde{\alpha}_k < -\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,...,n)$, the assumption $\widetilde{\alpha}_k < \widetilde{\beta}_k$ $(k=1,2,...,n)$ yields that there exists a vector $\overline{\nu} = (\nu_1,\nu_2,...,\nu_n)$ such that $\widetilde{\alpha}_k < 1-\nu_k \le -\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,...,n)$, and $\alpha_k < 1-\nu_k \le -\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,...,n)$, which are required. For the case (c) the situation is similar, that is, there exists $\overline{\nu}$ of the forms $\widetilde{\beta}_k > 1-\nu_k \ge -\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,...,n)$; and $\widetilde{\alpha}_k < 1-\nu_k$ $(k=1,2,...,n)$. Thus the proof is completed.

4. Conclusion

The multi-dimensional integral transformation with Fox H-function is studied. Conditions are obtained for the boundedness and one-to-oneness of the operator of such transformation from one Lebesgue-type weighted spaces of functions to others, and analogue of the formula for integration by parts are proved. For the transformation under consideration, various integral representations are established. The results generalize those obtained earlier for the corresponding one-dimensional integral transform.

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