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Attack on the Riemann Hypothesis

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Abstract

Let $\xi(z)$ be the Riemann xi function. We prove the boundedness of coefficients of the power series expansion of $\xi'(1/z)/\xi(1/z)$. By an observation of Keiper this implies that the Riemann hypothesis is true.

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1. Introduction

Throughout we write $z = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Let $\zeta(z)$ be the Riemann zeta function and

$$\xi(z) = \frac{z(z-1)}{2}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

the Riemann xi function where $\Gamma(z)$ is the gamma function. It is well known that the Riemann zeta function has zeros at negative even integers which are called trivial zeros. The Riemann hypothesis asserts that all nontrivial complex zeros satisfy $\sigma = 1/2$.

In 1992 Keiper [1] studied the power series

$$\frac{\xi'(1/z)}{\xi(1/z)} = \sum_{n=0}^{\infty} \tau_n (1-z)^n, \tag{1.1}$$

$$\log(2\xi(1/z)) = \sum_{n=0}^{\infty} \lambda_n^K (1-z)^n. \tag{1.2}$$

In 1997 Li [2] defined the numbers

$$\lambda_n^L = \frac{1}{(n-1)!}\frac{d^n}{dz^n}\left[z^{n-1}\log\xi(z)\right]\Big|_{z=1}$$

and they are also the coefficients of the power series expansion of

$$\frac{\varphi'(z)}{\varphi(z)} = \sum_{n=0}^{\infty} \lambda_{n+1}^L z^n,$$

where

$$\varphi(z) = \xi\left(\frac{1}{1-z}\right).$$

Li's λ_n^L has the expression

$$\lambda_n^L = \sum_{\rho}\left[1 - \left(1 - \frac{1}{\rho}\right)^n\right] \tag{1.3}$$

where ρ runs over the nontrivial zeros of the Riemann zeta function.

Remark 1. *The relation between Keiper's λ_n^K and Li's λ_n^L is $\lambda_n^L = n\lambda_n^K$. In the following we call both λ_n^K and λ_n^L the Keiper-Li coefficients. We will use Keiper's and set $\lambda_n := \lambda_n^K$.*

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Keiper [1] noticed that the Riemann hypothesis implies $\lambda_n > 0$ for all $n > 0$ and Li [2] showed the equivalence, which is now known as Li's criterion:

$$\text{Riemann Hypothesis} \Longleftrightarrow \lambda_n > 0.$$

Given this the Keiper-Li coefficient λ_n was extensively studied, see for example [3–5] and the references therein.

Another observation of Keiper is that the Riemann hypothesis is equivalent to the radius of convergence of the series (1.1) being 1. In particular as Keiper ([1] p. 769) stated we have

Theorem 1. *If the $|\tau_n|$ in (1.1) are bounded, then the Riemann hypothesis is true.*

Indeed if $|\tau_n|$ are bounded, then

$$\limsup_{n\to\infty} |\tau_n|^{1/n} = 1$$

and thus the radius of convergence of (1.1) is 1 and therefore the Riemann hypothesis is true.

The purpose of this paper is to show that $|\tau_n|$ are bounded.

Theorem 2. *The $|\tau_n|$ in (1.1) are bounded.*

The proof involves asymptotic estimates of modulus of nontrivial zeros of the Riemann zeta function.

2. Proof of Theorem 2

For the coefficients τ_n in (1.1) Keiper ([1] p. 769) proved that

$$\tau_{n-1} = -\sum_{\rho} \frac{1}{\rho^2} \left(\frac{\rho}{\rho - 1} \right)^n \tag{2.1}$$

and that they are the second central difference of $n\lambda_n$:

$$\tau_n = (n+1)\lambda_{n+1} - 2n\lambda_n + (n-1)\lambda_{n-1}. \tag{2.2}$$

This was also showed by the author in ([6] Theorem 3.2) in terms of Li's λ_n^L.

Remark 3. We note that there is also

$$\tau_{n-1} = -\sum_{\rho} \frac{1}{\rho^2} \left(1 - \frac{1}{\rho} \right)^{n-2}. \tag{2.3}$$

Indeed by Keiper ([1] (30)-(32)) we have

$$\begin{aligned} \tau_{n-1} &= -\sum_{\rho} (\rho - 1)^{-n} \rho^{n-2} \\ &= -\sum_{\rho} (\rho - 1)^{-(n-2)} \rho^{n-2} (\rho - 1)^{-2} \\ &= -\sum_{\rho} \left(\frac{\rho}{\rho - 1} \right)^{n-2} \cdot \frac{1}{(\rho - 1)^2}. \end{aligned}$$

Now replace ρ by $1 - \rho$ we obtain (2.3).

Let $N(T)$ denote the number of zeros of $\zeta(z)$ in the region $0 < \sigma < 1, 0 < t < T$. Then it is well known ([7] Theorem 9.4) that as $T \to \infty$,

$$N(T) = \frac{1}{2\pi} T \log T - \frac{\log 2\pi e}{2\pi} T + o(T). \tag{2.4}$$

As a consequence of (2.4) we have the following estimate.

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Lemma 4 ([7] p. 214). *Let the complex zeros $\beta + i\gamma$ of $\zeta(z)$ with $\gamma > 0$ be arranged in a sequence $\rho_n = \beta_n + i\gamma_n$ so that $\gamma_{n+1} \geq \gamma_n$, then as $n \to \infty$,*

$$|\rho_n| \sim \gamma_n \sim \frac{2\pi n}{\log n}. \tag{2.5}$$

Proof. We follow the proof in ([7] p. 214). By (2.4) we have

$$N(T) \sim \frac{1}{2\pi} T \log T$$

and thus

$$2\pi N(\gamma_n \pm 1) \sim (\gamma_n \pm 1)\log(\gamma_n \pm 1) \sim \gamma_n \log \gamma_n. \tag{2.6}$$

Also

$$N(\gamma_n - 1) \leq n \leq N(\gamma_n + 1) \tag{2.7}$$

and therefore

$$2\pi n \sim \gamma_n \log \gamma_n, \tag{2.8}$$

and then taking logarithms on both sides of (2.8) we have

$$\log n \sim \log \gamma_n. \tag{2.9}$$

Combining (2.8) and (2.9) gives

$$\gamma_n \sim \frac{2\pi n}{\log n}. \tag{2.10}$$

That $|\rho_n| \sim \gamma_n$ is obvious. □

We now prove Theorem 2.

Proof of Theorem 2. By (2.3) and $|z_1 - z_2| \leq |z_1| + |z_2|$ we have

$$|\tau_n| \leq \sum_\rho \frac{1}{|\rho|^2}\left(1 + \frac{1}{|\rho|}\right)^{n-1}. \tag{2.11}$$

Together with (2.5) we have

$$\sum_\rho \frac{1}{|\rho|^2}\left(1 + \frac{1}{|\rho|}\right)^{n-1} \ll \sum_{n=1}^{\infty} \frac{\log^2 n}{(2\pi n)^2}\left(1 + \frac{\log n}{2\pi n}\right)^{n-1}. \tag{2.12}$$

Since

$$(n-1)\log\left(1 + \frac{\log n}{2\pi n}\right) \leq n \cdot \frac{\log n}{2\pi n} = \frac{\log n}{2\pi}, \tag{2.13}$$

and thus we have

$$\left(1 + \frac{\log n}{2\pi n}\right)^{n-1} \leq n^{1/2\pi} \tag{2.14}$$

and

$$\sum_\rho \frac{1}{|\rho|^2}\left(1 + \frac{1}{|\rho|}\right)^{n-1} \ll \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2-1/2\pi}} = \zeta''\left(2 - \frac{1}{2\pi}\right) = O(1), \tag{2.15}$$

which implies Theorem 2. □

3. Conclusions

In this paper we find a proof of the Riemann hypothesis that Keiper missed. In a forthcoming paper we will generalize the method to the study of the extended Riemann hypothesis for general number fields.

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