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Super-Extended Split Graphs and the 3-Sphere Regular Cellulation Conjecture

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Abstract

The sphere regular cellulation conjecture claims that every 2-connected graph is the 1-dimensional skeleton of a regular cellulation of the 3-dimensional sphere. The conjecture is obviously true for planar graphs. 2-connectivity is a necessary condition for a graph to satisfy such property. Therefore, the question whether a graph is the 1-dimensional skeleton of a regular cellulation of the 3-dimensional sphere would be equivalent to the 2-connectivity test if the conjecture were proved to be true. On the contrary, it is not even clear whether such decision problem is computationally tractable. We introduced a superclass of planar graphs, called the class of extended split graphs, and proved the conjecture for it. This is a superclass of split graphs including also Hamiltonian and complete k-partite graphs. In this paper, we further extend such class, keeping the validity of the conjecture, by introducing a positive integer parameter k and the notion of orientable homotopic disjointness, which define the class of k-extended split graphs. With this parameter, 1-extended split graphs are a superclass of extended split graphs by means of this notion. Then, k-extended split graphs are defined inductively. A graph is super-extended split if it is k-extended split for some k. Super-extended split graphs are, therefore, the new state of the art for the proof of the conjecture.

Keywords: 3-Sphere; CW-Complex; regular cellulation;split graph

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1. Introduction

Let X be a CW-complex [8] on the 3-sphere $S^3 = \{x \in R^4 : |x| = 1\}$ with its standard topology. X is also called a *cellulation* of the 3-sphere. The ascending sequence $X^0 \subset X^1 \subset X^2 \subset X^3 = X$ of closed subspaces of X satisfies the following conditions:

[1] X^0 is a discrete set of points (0-cells)

[2] For $0 < k \leq 3$, $X^k - X^{k-1}$ is the disjoint union of open subspaces, called k-cells, each of which homeomorphic to the open k-dimensional ball $U^k (= \{x \in R^k : |x| < 1\})$.

X^k is the k-dimensional skeleton of X and is a k-dimensional CW-complex for $0 \leq k \leq 3$ on a subspace of the 3-sphere. X is a *regular* CW-complex if the boundary of every k-cell is homeomorphic to the $k-1$-dimensional sphere S^{k-1}, for $1 \leq k \leq 3$. Then, X is called a regular cellulation of S^3. If X is regular, the boundary of every 1-cell is a pair of 0-cells. It follows that the 1-dimensional skeleton of a regular CW-complex represents a graph with no loops where the 0-cells correspond to the vertices and the 1-cells correspond to the edges. From now on, we will consider simple graphs (no loops and no multiple edges between two vertices). In particular we are interested in *cyclic* graphs, that is, graphs which contain at least one cycle. Since the graphs are simple, the cycles must be closed paths comprising at least three vertices.

A biconnected graph $G = (V, E)$ is 2-connected if $|V| > 2$. The *3-sphere regular cellulation conjecture* claims that every 2-connected graph is the 1-dimensional skeleton of a regular cellulation of the 3-dimensional sphere [6,10]. The conjecture is trivially true for planar graphs. Indeed, the embedding of a planar graph into the 2-dimensional sphere provides a regular cellulation of the 3-dimensional

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sphere with two 3-cells. 2-connectivity is a necessary condition for a graph to satisfy such property. Therefore, the question whether a graph is the 1-dimensional skeleton of a regular cellulation of the 3-dimensional sphere would be equivalent to the 2-connectivity test if the conjecture were proved to be true. On the contrary, it is not even clear whether such decision problem is computationally tractable. In [5], we introduced the class of weakly split graphs and proved the conjecture is true for such class. Hamiltonian, split, complete k-partite and matrogenic cyclic graphs are weakly split. Matrogenic graphs include matroidal graphs. Split matrogenic graphs include threshold graphs. Several characterizations of these classes are given in [9]. Hamiltonian graphs include complete graphs. Over all the graphs with n vertices, the complete graph is an obvious case where the genus is maximized. On the other hand, when the genus of the graph is 0 the regular cellulation of the 3-sphere is provided by the graph embedding into the 2-sphere (planar case). This consideration suggested the conjecture that every 2-connected graph is the 1-dimensional skeleton of a regular cellulation of the 3-sphere since this property might hold when the graph lies, as far as embeddability into surfaces is concerned, in between a planar one and a complete one. We also want to point out that such extremal results were obtained for k-partite graphs since complete k-partite graphs are weakly split for every k. Finally, a superclass of planar graphs and weakly split graphs verifying the conjecture was introduced in [7] and called the class of *extended split* graphs. Extended split graphs were, therefore, the state of the art for the proof of this open problem.

In this paper, we further extend such class, keeping the validity of the conjecture, by introducing a positive integer parameter k and the notion of orientable homotopic disjointness, which define the class of k-extended split graphs. With this parameter, 1-extended split graphs are a superclass of extended split graphs by means of this notion. Then, k-extended split graphs are defined inductively. A graph is super-extended split if it is k-extended split for some k. So, super-extended split graphs are the new state of the art for the proof of the conjecture.

In Section 2 we describe the previous work on the conjecture. In section 3, we introduce the notions of homotopic disjointness of an embedding and orientable homotopic disjointness of a graph. Then, we show that graphs with orientable homotopic disjointness greater than 1 verify the 3-sphere regular cellulation conjecture. Section 4 introduces the class of super-extended split graphs and proves the conjecture for it. Conclusions and future work are given in Section 5.

2. Previous Work

The first subsection shows the proof of the conjecture for hamiltonian graphs and, therefore, for complete graphs [2]. Then, the second subsection extends the result to complete k-partite graphs and to split graphs. These results are a corollary to the proof of the conjecture for the class of crownless weakly split graphs which is a superclass of all the classes previously mentioned [5]. In the third subsection weakly split graphs are presented to include matrogenic graphs and extend further the validity of the conjecture [5]. The theorems in [2] and [5] with their proofs are presented again to play a role as lemmas in this paper in order to prove the theorem on extended split graphs in the fourth subsection [7]. All these theorems and their proofs are necessary for the proof of the conjecture on super-extended split graphs in section 4.

2.1. Hamiltonian Graphs

In [2], the 3-sphere regular cellulaton conjecture has been proved true for hamiltonian graphs as it follows:

Theorem 1. *Every hamiltonian graph $G = (V, E)$ is the 1-dimensional skeleton of a regular cellulation of S^3.*

Proof. We embed V into the 3-sphere. Let $v_1, v_2, ...v_n, v_1$ be the sequence of vertices (0-cells) ordered by a hamiltonian cycle h of G, where $|V| = n$. We embed the edges of h (1-cells) into the 3-sphere so that we have a 1-dimensional complex X. Then, we add to X a 2-cell with boundary h. If G is a simple cycle, another 2-cell with boundary h is added to X. At this point, by adding two 3-cells to X we obtain

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a regular cellulation of the 3-sphere. If G is not a simple cycle, let us consider any edge, say (v_i, v_j), which does not belong to h, with $i < j$. We add to X the edge (v_i, v_j) as a 1-cell and two 2-cells with the cycles $v_1, ..., v_i, v_j, ...v_n, v_1$ and $v_i, v_j, v_{j-1}, ..., v_i$ as boundaries, respectively. These 2-cells are added so that the intersection of their closures is the edge (v_i, v_j) to satisfy the property of a CW-complex on the disjointness of cells. Then, we add one 3-cell bounded by these 2-cells and by the 2-cell with h as boundary. Since we added only one 3-cell, we can embed the remaining edges of G and, similarly, the corresponding two 2-cells and one 3-cell for each edge. Differently from the first 3-cell we added, the boundaries of these additional 3-cells comprise four 2-cells instead of three. Finally, we add to X one more 3-cell to obtain the regular cellulation of the 3-sphere with G as 1-dimensional skeleton. □

Since complete graphs with at least three vertices are hamiltonian, theorem 1 provides an extremal result for the 3-sphere regular cellulation conjecture. as far as embeddability of graphs into surfaces is concerned, as the one for planar graphs mentioned in the introduction. Such extremal results hold for k-partite graphs as we will see in the next subsection.

2.2. Crownless Weakly Split Graphs

We define a superclass of cyclic split graphs and hamiltonian graphs which also includes complete k-partite graphs, as shown in [5].

A connected graph $G = (V, E)$ is *crownless weakly split* if V is the union of two disjoint sets I and H such that:

- I is empty or a stable set in G;
- H is non-empty and the subgraph induced by H is hamiltonian.

If the subgraph induced by H is complete, G is *split*. If I is empty, G is hamiltonian. Furthermore, a complete k-partite graph $K_{m_1, m_2, \cdots, m_k}$ is crownless weakly split (with $m_1, m_2 > 1$ if $k = 2$) [4]. In [5], the 3-sphere regular cellulaton conjecture has been proved true for crownless weakly split graphs graphs as it follows:

Theorem 2. *Every 2-connected crownless weakly split graph $G = (V, E)$ is the 1-dimensional skeleton of a regular cellulation of S^3.*

Proof. Since G is crownless weakly split, V is the union of two disjoint sets I and H such that I is stable and the subgraph induced by H is hamiltonian. We embed H into the 3-sphere. Let $w_1, w_2, ...w_k, w_1$ be the sequence of vertices ordered by the hamiltonian cycle h of the subgraph induced by H. We embed the edges of h into the 3-sphere so that we have a one-dimesional complex X and we add to X a 2-cell with boundary h. Then, we can apply to X the procedure of theorem 1 to produce a regular cellulation of a proper subspace B_1 of S^3. B_1 is a proper subspace of S^3 because we do not add to X the last 3-cell produced by the procedure of theorem 1. Therefore, B_1 is homeomorphic to a closed 3-dimensional ball while the complement B_2 of B_1 in S^3 is an open 3-dimensional ball where we embed the vertices $u_1, u_2, ...u_i$ of I. For each vertex u_j, $1 \leq j \leq i$, first we add the edges connecting u_j to the adjacent vertices in h to X. Since G is 2-connected, there are at least two such vertices for each u_j. Then, for each pair of vertices w and w' adjacent to u_j and consecutive in h, we add to X a 2-cell with boundary the cycle defined by u_j, w, w' and the vertices in h between w and w' (which, obviously, are not adjacent to u_j). These 2-cells can be added so that they are disjoint and a 3-cell bounded by these 2-cells and the 2-cells determined by u_{j-1} (if $j = 1$, the 2-cell with boundary h) is added as well. The homeomorphism of such boundary to the 2-sphere follows from the disjointness of the 2-cells. Then, we add to X one more 3-cell to obtain the regular cellulation of the 3-sphere with G as 1-dimensional skeleton. □

Theorem 2 strengthens the 3-sphere regular cellulation conjecture since the extremal results of the previous subsection are extended to k-partite graphs, for $2 \leq k \leq n$, where n is the number of vertices. In the next subsection, we extend the validity of the conjecture to a superclass of the crownless weakly split graphs by adding a "crown" which is a linear forest.

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2.3. Weakly Split Graphs

A connected graph $G = (V, E)$ is *weakly split* if V is the union of three disjoint sets I, H and C such that:

- I is empty or a stable set in G;
- H is non-empty and the subgraph induced by K is hamiltonian;
- C is either empty or none of its vertices is adjacent to a vertex in I and C induces a subgraph such that each connected component is a simple path where each vertex in it is adjacent either to at least two vertices in H or to none.

We call the subgraph induced by C the *crown* of G.

Theorem 3. *Every 2-connected weakly split graph $G = (V, E)$ is the 1-dimensional skeleton of a regular cellulation of S^3.*

Proof. It follows from theorem 1 that the subgraph of G induced by $I \cup H$ is the 1-dimensional skeleton of a regular cellulation X of a subspace Σ^3 of S^3. If C is empty G is crownless weakly split and the statement of the theorem follows from theorem 1. Otherwise, the vertices in C are embedded into $S^3 - \Sigma^3$. C induces a graph with p connected components where each connected component is a simple path. Let $C_1, \cdots C_p$ be the partition of C such that each element of the partition induces one of the p connected components. Let $t_1, \cdots t_c$ be the vertices of C_1 in one of the two orders induced by the corresponding simple path. Then, for $1 \leq j \leq c$ we add to X the edges (if any) connecting t_j to the adjacent vertices in h and, for each pair of vertices w and w' adjacent to t_j and consecutive in h, we add to X a 2-cell with boundary the cycle defined by t_j, w, w' and the vertices in h between w and w' (which are not adjacent to t_j since w and w' are consecutive in h). As for the vertices in I, these 2-cells can be added so that they are disjoint. Let $j_1 \cdots j_\ell$ be the subsequence of $1 \cdots c$ such that $t_{j_1} \cdots t_{j_\ell}$ are the vertices of C_1 adjacent to at least two vertices in K. Since G is 2-connected, we have $j_1 = 1$ and $j_\ell = c$. Then, for $1 \leq r \leq \ell$, we add to X the edges of the path from t_{j_r} to $t_{j_{r+1}}$. It follows from the definition of weakly split graph that we can select in h two vertices adjacent to t_{j_r} and two vertices adjacent to $t_{j_{r+1}}$. These selections define a set S of vertices in h of cardinality between two and four, depending on whether two, one or none of the selected vertices adjacent to t_{j_r} coincide with the two selected vertices adjacent to $t_{j_{r+1}}$. Then, we add two 2-cells with boundaries the cycles defined by the vertices of the path from t_{j_r} to $t_{j_{r+1}}$, two vertices of S respectively adjacent to t_{j_r} and $t_{j_{r+1}}$ which are consecutive (unless they coincide) in h with respect to S and the vertices in h (if any) between them (which do not belong to S since the two vertices of S are consecutive). It follows that these two 2-cells can be added to X so that they are disjoint. Therefore, two disjoint 3-cells can be added to X bounded by these two 2-cells and complementary subsets of the 2-cells determined by $t_{j_{r+1}}$ and by t_{j_r}. Moreover, we add one 3-cell bounded by the 2-cells determined by t_{j_1} and the ones determined by u_i, the vertex in I on the boundary of Σ^3. Again, the boundaries of these 3-cells are homeomorphic to the 2-sphere. Such embedding procedure is repeated for each connected component $C_2, \cdots C_p$ of the crown (for each of these components, the last 3-cell added to X is partially bounded by 2-cells of the previous component). Finally, we add to X one more 3-cell to obtain the regular cellulation of the 3-sphere with G as 1-dimensional skeleton. □

Weakly split graphs are a superclass of cyclic matrogenic graphs [5]. Matrogenic graphs include matroidal and threshold graphs. Differently from threshold graphs, matrogenic and matroidal graphs are not always split. As mentioned in the introduction, several characterizations of these classes can be found in [9]. Since a 2-connected graph is always cyclic, theorem 3 validates the 3-sphere regular cellulation conjecture for matrogenic graphs.

Among all the classes we mentioned so far, the class of planar graphs is the only one for which the class of weakly split graphs is not a superclass. In the next subsection, we introduce a superclass

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of planar graphs and weakly split graphs for which the 3-sphere regular cellulation conjecture is validated [7].

2.4. Extended Split Graphs

A graph $G = (V, E)$ is called *extended split* if V is the union of two disjoint sets H and C, such that:

- the subgraph induced by H is hamiltonian or H is empty;
- the subgraph induced by C is planar or C is empty;
- a connected component of the subgraph induced by C is connected to the subgraph induced by H only if it is a single vertex, a simple path or 2-connected;
- if a connected component of the subgraph induced by C is a simple path, each vertex in it is adjacent to at least two vertices in H or to none (*first linking rule*);
- if a connected component of the subgraph induced by C is hamiltonian then it is connected to the subgraph induced by H by at most three edges with at least two disjoint edges (*second linking rule);*
- if a connected component of the subgraph induced by C is non-hamiltonian 2-connected then it is connected to the subgraph induced by H by exactly two disjoint edges (*third linking rule*).

The subgraphs induced by H and C are called the *head* and the *crown*, respectively. Planar graphs are extended split since H may be empty. The class of weakly split graphs is the subclass of extended split graphs where only the first linking rule applies. Therefore, the next theprem will prove the conjecture when the second or third linking rule is applied since the other cases have already been considered by the previous theorems.

Theorem 4. *A 2-connected extended split graph $G = (V, E)$ is the 1-dimensional skeleton of a regular cellulation of S^3.*

Proof. Let H and C be the head and the crown of G, respectively. If H is empty, G is planar and the statement of the theorem is trivially true. If C is empty, G is hamiltonian and the statement follows from theorem 1. If the subgraph induced by C is a planar graph where each connected component is either a single vertex or a simple path, G is weakly split and the statement of the theorem follows from theorem 3. Finally, since G is 2-connected the only case left by the definition of extended split graph is that there is a subset C' of C such that the connected components of the subgraph induced by C' are 2-connected. It follows from theorem 3 that the subgraph of G induced by $H \cup (C - C')$ is the 1-dimensional skeleton of a regular cellulation X of a subspace Σ^3 of S^3. We know from theorem 3 that one of the 2-cells of X on the boundary of $S^3 - \Sigma^3$ is bounded by a hamiltonian cycle h of the subgraph induced by H. Let $C'_1, \cdots C'_q$ be the partition of C' such that each element of the partition induces one of the connected components of the subgraph induced by C'. Each of the p components is connected to the subgraph induced by H by at least two disjoint edges. Let us consider, first, the case of exactly two disjoint edges.

Without loss of generality, let $C'_1, \cdots C'_{q'}$ induce the components connected to the subgraph induced by H by exactly two disjoint edges, with $q' \leq q$. The subgraph induced by C'_1 is embedded into a subspace Σ^2 of $S^3 - \Sigma^3$ homeomorphic to S^2. With such embedding, we obtain a regular cellulation of Σ^2. Let (v_1, w_1) and (v_2, w_2) be the two disjoint edges connecting the subgraph induced by C'_1 to the subgraph induced by H with $v_1, v_2 \in C'_1$. Since the subgraph induced by C'_1 is 2-connected, there is in it a simple cycle including v_1 and v_2. Such simple cycle is the boundary of two open disks in Σ^2 and comprises two simple paths p'_1 and p_1'' from v_1 to v_2. On the other hand, h comprises two simple paths h'_1 and h_1'' between w_1 and w_2. We call c'_1 and c_1'' the simple cycles that (v_1, w_1) and (v_2, w_2) form with p'_1, h'_1 and p_1'', h_1'', respectively. Then, we add to X two 3-cells with their boundaries. One 3-cell is bounded by Σ^2 with its regular cellulation. The other 3-cell is bounded by the 2-cells on the boundary

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of $S^3 - \Sigma^3$ except the one bounded by h, the 2-cells on one of the two open disks bounded by the simple cycle including v_1 and v_2 plus a couple of 2-cells bounded by c_1' and c_1'', respectively. It is easy to see that this can be done preserving the property of a regular cellulation for X. The subgraphs induced by C_i' for $2 \leq i \leq q'$ can be embedded into S^3, similarly as the one induced by C_1', to extend the regular cellulation X.

According to the definition of extended split graph, $C'_{q'+1}, \cdots C'_q$ induce hamiltonian components. Then, for each of these components there might be a third edge connecting it to the subgraph induced by H besides the two disjoint edges required for 2-connected graphs by the second linking rule. Let (v_1', w_1') and (v_2', w_2') be the two disjoint edges connecting the subgraph induced by $C'_{q'+1}$ to the subgraph induced by H with $v_1', v_2' \in C'_{q'+1}$. Then, we can extend the regular cellulation X in a similar way as for the components induced by $C_1', \cdots C'_{q'}$. Therefore, there is a simple cycle including v_1' and v_2' in the subgraph induced by $C'_{q'+1}$ and comprising two simple paths $p'_{q'+1}$ and $p_{q'+1}''$ from v_1' to v_2' involved with the extension of X. On the other hand, h comprises two simple paths $h'_{q'+1}$ and $h_{q'+1}''$ from w_1' to w_2'. Let (v_3', w_3') be the third edge with $v_3' \in C'_{q'+1}$. Without loss of generality, we assume that $h'_{q'+1}$ and $p'_{q'+1}$ are the paths including w_3' and v_3', respectively. Moreover, since vertices in H are the only ones to which vertices in $C'_{q'+1}$ may be adjacent in $V - C'_{q'+1}$, we assume that $h'_{q'+1}$ and $p'_{q'+1}$ have the same orientation. Then, the third edge can be drawn on the 2-cell with the boundary including the two paths (obviously, dividing such cell into two cells). The subgraphs induced by C_i' for $q' + 2 \leq i \leq q$ can be embedded into S^3, similarly as the one induced by $C'_{q'+1}$, to extend the regular cellulation X. Finally, a 3-cell covering the complement of X completes the regular cellulation of S^3. □

Before defining the class of super-extended split graphs, we introduce the notion of homotopic disjointness in the next section.

3. Homotopic Disjointnes

Given a closed surface Σ topologically distinct from the 2-sphere and a 2-cell embedding of a graph G in Σ, a cycle in the 2-cell embedding of G is *non-contractible* in Σ if it cannot be shrunk to a vertex by edge contraction. If g is the genus (orientable or non-orientable) of Σ, the fundamental group of the surface has $2g$ (orientable case) or g (non-orientable case) standard generators (simple closed curves) drawn in an obvious manner [1]. Let Γ be the set of standard generators and, for each $\gamma \in \Gamma$, let c_γ be the greatest number of disjoint cycles homotopic to it in a graph embedding. Then, the *embedding homotopic disjointness* is $\min\{c_\gamma : \gamma \in \Gamma\}$ for a given embedding. If Σ is a 2-sphere and G is planar, the homotopic disjointness of the embedding of G in Σ is defined to be infinity. So,an embedding is a closed 2-cell embedding if its homotopic disjointness is strictly greater than 1.

We can define the *graph homotopic disjointness* of a graph G as the greatest homotopic disjointness over all the possible 2-cell embeddings of G. Analogously, we can define the *orientable* homotopic disjointness of G as the greatest homotopic disjointness over all possible 2-cell embeddings of G into orientable closed surfaces. 2-connectivity is implied if homotopic disjointness (orientable or not) is finite and greater than 1. We prove the following theorem:

Theorem 5. *Every graph G with finite orientable homotopic disjointness greater than 1 is the 1-dimensional skeleton of a regular cellulation of S^3.*

Proof. If G has finite orientable homotopic dijointness strictly greater than 1, there is an orientable closed surface Σ where G has a closed 2-cell embedding. The orientable homotopic dijointness strictly greater than 1 implies, for each of the standard generators of the fundamental group of Σ, the existence of two disjoint simple cycles of G homotopic to it. If we embed Σ into S^3, $S^3 - \Sigma$ comprises two connected components which are not homeomorphic to the open 3-ball. We extend Σ by adding, for each of the standard generators of the fundamental group, a pair of disjoint 2-cells so that each one is bounded by one of the two homotopic simple cycles. The self intersecting surface Ψ, obtained by adding $4g$ pairwise disjoint 2-cells in such a way, is such that $S^3 - \Psi$ comprises $2g + 2$ connected components homeomorphic to the open 3-ball with boundary homeomorphic to the bidimensional sphere. Therefore, we can add $2g + 2$ 3-cells to obtain a regular cellulation of the 3-sphere. □

We are, now, ready to show the new results on classes related to split graphs in the next sevtion.

4. Super-Extended Split Graphs

A graph $G = (V, E)$ is called 1-*extended split* if V is the union of two disjoint sets H (the head) and C (the crown), such that:

- H is empty or the subgraph induced by H is hamiltonian;
- C is empty or a connected component of the subgraph induced by C is either hamiltonian or with orientable homotopic disjointness greater than 1;
- a connected component of the subgraph induced by C is connected to the subgraph induced by H only if it is a single vertex, a simple path or 2-connected;
- if a connected component of the subgraph induced by C is a simple path, each vertex in it is adjacent to at least two vertices in H or to none (*first linking rule*);
- if a connected component of the subgraph induced by C is hamiltonian then it is connected to the subgraph induced by H by at most three edges with at least two disjoint edges (*second linking rule);*
- if a connected component of the subgraph induced by C has orientable homotopic disjointness greater than 1 and is 2-connected non-hamiltonian then it is connected to the subgraph induced by H by exactly two disjoint edges (*third linking rule*).

Observe that the 2-connectivity requirement in the third linking rule is needed only when the orientable homotopic disjointness is infinity (the planar case) since finite orientable homotopic disjointness greater than 1 implies 2-connectivity.1-extended split graphs are, obviously, a superclass of extended split graphs. From theorem 4, we derive the following theorem:

Theorem 6. *Every 2-connected 1-extended split graph $G = (V, E)$ is the 1-dimensional skeleton of a regular cellulation of S^3.*

Proof. As far as the subgraph induced by H and the single vertex connected components of C are concerned, the proof of the theorem is exactly the same as the one of theorem 4. This is true for the first linking rule as well, when we add the simple path connected components of the crown. As far as the second linking rule is concerned,it is enough to observe that planarity is not a requirement in the proof of theorem 4. Therefore, such proof works when we add the hamiltonian connected components of C. For the third linking rule, the proof of theorem 4 works for connected components with finite orientable homotopic disjointness greater than 1 as it does for the planar case. Therefore, the statement of the theorem follows. □

Now, we can define inductively the class of k-extended split graphs for any positive integer k. A graph $G = (V, E)$ is *k-extended split*, for $k > 1$, if V is the union of two disjoint non-empty sets T and S, such that:

- the subgraph induced by T is $k - 1$-extended split;
- the subgraph induced by S is 1-extended split;
- the head of the subgraph induced by S is *a head* of the k-extended split graph and it is connected to one of the $k - 1$ heads of the subgraph induced by T by at most three edges with at least two disjoint edges.

A graph $G = (V, E)$ is called *super-extended split* if it is k-extended split for some positive integer k. We prove the following theorem.

Theorem 7. *Every 2-connected super-extended split graph is the 1-dimensional skeleton of a regular cellulation of S^3.*

Proof. Since a super-extended split graph is k-extended split for some positive integer k, we prove the theorem by induction on k. The statement follows from the previous theorem for $k = 1$. Moreover, we know there is a 2-cell of the regular cellulation whose boundary is a hamiltonian cycle of the head of its 1-dimensional skeleton. Therefore, we make the induction hypothesis that every 2-connected k-extended split graph is the 1-dimensional skeleton of a regular cellulation of S^3, with k 2-cells whose boundaries are hamiltonian cycles of the k graph heads, and prove it for $k + 1$.

So, let $G = (V, E)$ be a 2-connected $k + 1$-extended split graph where V is the union of two disjoint non-empty sets T and S, such that the subgraph induced by T is k-extended split and the subgraph induced by S is 1-extended split. Then, in virtue of the induction hypothesis we can assume that the subgraph of G induced by T is the 1-dimensional skeleton of a regular cellulation X of a subspace Σ^3 of S^3, such that one of the 2-cells of X on the boundary of $S^3 - \Sigma^3$ is bounded by a hamiltonian cycle h of one of the k subgraph heads. Let χ be a hamiltonian cycle of the $k + 1$-th head of G, that is, the head of the subgraph induced by S. Assume, without loss of generality, that χ is connected to h. Then, if h and χ are connected by exactly two disjoint edges we add to X a 3-cell bounded by four 2-cells in an obvious way. If there is a third edge, the 2-cell bounded by χ is added to X and is connected to the 2-cell bounded by h so that the third edge can be drawn on one of the other 2-cells of the 3-cell boundary. So, the boundary of the new 3-cell comprises five 2-cells in this case and, in both cases, we have a 2-cell bounded by χ while the other k heads have still hamiltonian cycles which are boundaries of 2-cells. From theorem 4, we know how to extend X to provide a regular cellulation of the space covered by this 3-cell with the subgraph induced by S as one-dimensional skeleton. Finally, a 3-cell covering the complement of X completes the regular cellulation of S^3 and proves the hypothesis for $k + 1$. □

We wish to point out that the definition of a k-extended split graph was given by connecting heads of a $k - 1$-extended split graph and a 1-extended split graph in order to have it independent from its representation as a complex skeleton.

5. Conclusions

A fundamental question for 2-connected graphs has been faced, that is: is a 2-connected graph always the 1-dimensional skeleton of a regular cellulation of the 3-dimensional sphere? We presented the partial positive results and argued there is enough evidence to conjecture an affirmative answer to the question. Super-extended split graphs are the new state of the art for the proof of the conjecture and their definition suggests further extensions. However, it might be more interesting to search for other properties besides planarity, hamiltonicity and splittability that could provide more positive results.

The 3-sphere regular cellulation conjecture, as it was called in [6] was given for graphs with at least two cycles in [4] because we assumed that two 2-cells cannot share the same boundary in order to relate it to the concept of spatiality degree. The *spatiality degree* of a connected graph G is the maximum number of 3-cells that the cellulation of a 3-sphere can have with G as a 1-dimensional skeleton, assuming that two distinct 2-cells of the complex cannot share the same boundary and the 2-dimensional skeleton is regular. In [2,3], it is shown that the 3-sphere regular cellulation conjecture is true if and only if the spatiality degree of a 2-connected graph $G = (V, E)$ with at least two cycles is equal to $2(|E| - |V|)$. We denote the spatiality degree of a connected graph G with $s(G)$. In [2], it is also shown that for any connected graph G

$$s(G) = \sum_{i=1}^{k} s(B_i) - k + 1$$

where $B_1 \cdots B_k$ are the biconnected components of G. It follows that computing the spatiality degree of a connected graph could be an interesting combinatorial optimization problem only if the conjecture were proved to be false.

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