Fractional Modified Bessel Function of the First Kind of Integer Order

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the date of receipt and acceptance should be inserted later

Abstract The modified Bessel function (MBF) of the first kind is a fundamental special function in mathematics with applications in a large number of areas. When the order of this function is integer, it has an integral representation which includes the exponential of the cosine function. Here we generalize this MBF to include a fractional parameter, such that the exponential in the previously mentioned integral is replaced by a Mittag-Leffler function. The necessity for this generalization arises from a problem of communication in networks. We find the power series representation of the fractional MBF of the first kind, as well as some differential properties. We give some examples of its utility in graph/networks analysis and mention some fundamental open problems for further investigation.

Keywords Modified Bessel functions; Communicability in graphs; Estrada index; Power-series; Fractional calculus; Caputo derivative; Riemann-Liouville integral; Paths; Cycles

MSC 33C10; 33E20; 05C38; 40H05

1 Introduction

The modified Bessel function (MBF) of the first kind of order ν, denoted by $I_\nu(z)$ is defined by [1,2,3]

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k}, \tag{1}$$

for an unrestricted real (or complex) number ν. The MBF of the first kind is one of the linearly independent solutions of the differential equation

$$u'' + \frac{1}{z}u' - \left(1 - \frac{\nu^2}{z^2}\right) u = 0, \tag{2}$$

which frequently appears in mathematical physics. Then, it is common to find this function related to a large variety of applications, such as elasticity [4], imaging [5,6], sport data [7], statistics [8], to mention just a few. In particular, when $\nu \in \mathbb{Z}$, the MBF of the first kind has the following integral representation:

$$I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z\cos\theta} \cos(\nu\theta) d\theta, \tag{3}$$

which appears in the analytical expression of the so-called Estrada index of an infinite linear and of an infinite circular graph [9]. The importance of the MBF of the first kind is also reflected by the fact that a

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large number of publications in mathematics deal with the properties of these functions (see [10,11,12,13, 14,15] for some recent examples).

Here we propose a fractional generalization of the MBF of the first kind. This function, which we will call fractional MBF (FMBF) of the first kind is found in the analytical calculations of communicability [16, 17] and related indices [18] in certain classes of infinite graphs. Therefore, it is not proposed in an ad hoc way as in other previous cases [19], but in the context of the study of graphs and networks. We introduce the context in which the FMBF of the first kind arises and then study some of its properties, such as its power series, convergence and recurrence relations. We hope these functions and other related ones play a fundamental role in the study of fractional analogous of Bessel functions and their applications.

2 Preliminaries

It is known that the MBF of the first kind satisfies the following recurrence:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z). \tag{4}$$

Also, the derivative of the MBF of the first kind has the following well-known recurrence relations:

$$\frac{d}{dz}\left(z^{\nu} I_\nu\left(z\right)\right) = z^{\nu} I_{\nu-1}\left(z\right), \tag{5}$$

$$\frac{d}{dz}\left(z^{-\nu} I_\nu\left(z\right)\right) = z^{-\nu} I_{\nu+1}\left(z\right), \tag{6}$$

$$\frac{d}{dz} I_\nu\left(z\right) = \frac{1}{2}\left(I_{\nu-1}\left(z\right) + I_{\nu+1}\left(z\right)\right). \tag{7}$$

First, we will rewrite these recurrence formulas for the MBF of the first kind in the following way. We apply the classical rule of derivation to Eq. (5) and obtain

$$\frac{d}{dz}\left(z^{\nu} I_\nu\left(z\right)\right) = \nu z^{\nu-1} I_\nu\left(z\right) + z^{\nu}\frac{d}{dz} I_\nu\left(z\right), \tag{8}$$

such that we can write

$$\frac{d}{dz} I_\nu\left(z\right) = I_{\nu-1}\left(z\right) - \frac{\nu}{z} I_\nu\left(z\right). \tag{9}$$

Similarly, we have

$$\frac{d}{dz} I_\nu\left(z\right) = I_{\nu+1}\left(z\right) + \frac{\nu}{z} I_\nu\left(z\right). \tag{10}$$

We now define the Caputo fractional derivative which we will use in this work. We start by defining the Riemann-Liouville integral, which for a parameter γ representing the fractional order, it is written as [20]

$$\mathcal{I}_{0,t}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} d\tau \qquad \gamma > 0, \tag{11}$$

where it can be easily proved that $\text{lím}_{\gamma\to 0}\mathcal{I}_{0,t}^{\gamma} f(t) = f(t)$. Now, let $\alpha \in (0, +\infty)$ be the fractional ordering parameter and let $m := \lceil \alpha \rceil$, the Caputo derivative of f is defined as [20]:

$$D_{0,t}^{\alpha} f\left(t\right) = \mathcal{I}_{0,t}^{(m-\alpha)}\frac{d^m}{dt^m} f\left(t\right) = \frac{1}{\Gamma\left(m-\alpha\right)}\int_0^t \frac{f^{(m)}\left(\tau\right)}{\left(t-\tau\right)^{\alpha+1-m}} d\tau. \tag{12}$$

Here we consider only the cases where $\alpha \in (0, 1]$ and $\gamma \in [0, 1)$ for the Caputo derivative and Riemann-Liouville integral, respectively. Hereafter we will consider (12) for $m = 1$, which we will write with the

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following notation:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau; \qquad \alpha \in (0,1). \tag{13}$$

The following results are possibly proved elsewhere and we show them here for the sake of self-containment of our work. Let $\alpha \in (0,1)$ and let $f(t) = \sum_{n=0}^{\infty} a_n t^n$ be a power series with convergence radius $R \in (0,+\infty]$. Then,

$$D_x^{\alpha} f(x) = D_x^{\alpha} \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n D_x^{\alpha} x^n, \tag{14}$$

$$\mathcal{I}^{\alpha} f(x) = \mathcal{I}^{\alpha} \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n \mathcal{I}^{\alpha} x^n. \tag{15}$$

We also provide a few other properties of the Caputo derivative and of the Riemann-Liouville integral which are useful in the proof of our results.

1. Let $\alpha \in (0,1]$ and $C \in \mathbb{R}$, then: $D_t^{\alpha}(C) = 0$;
2. Let $\alpha \in (0,1]$ and $r > 0$, then $D_t^{\alpha}(t^r) = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}$;
3. Let $\gamma \in [0,1)$ and $r > -1$, then $\mathcal{I}^{\gamma}(t^r) = \frac{\Gamma(r+1)}{\Gamma(r+\gamma+1)} t^{r+\gamma}$;
4. Let $\gamma \in (0,1)$ and $r > 0$, then $D_t^{\gamma}\left(\mathcal{I}^{\gamma}(t^r)\right) = \mathcal{I}^{\gamma}\left(D_t^{\gamma}(t^r)\right) = t^r$.

3 Fractional communicability in graphs

Let $G = (V,E)$ be a simple, connected graph where V is the set of vertices and E is the set of edges. Let A be the adjacency matrix of G. Let $v,w \in V$ be two nodes of G. Then, the communicability function $\Gamma_{v,w}(\zeta,G)$ of the graph with parameter $\zeta \in \mathbb{R}$ is known to be

$$\Gamma_{v,w}(\zeta,G) = \sum_{k=0}^{\infty} \frac{\zeta^k \left(A^k\right)_{vw}}{\Gamma(k+1)} = \left(\exp(\zeta A)\right)_{vw}, \tag{16}$$

where $\Gamma(\cdot)$ is the Euler gamma function. When $v = w$ the self-communicability of the node v is known as the subgraph centrality of that node and the sum of all subgraph centralities in the graph is the so-called Estrada index: $EE(\zeta,G) = tr \exp(\zeta A)$, where tr is the trace of the matrix. The communicability function and the Estrada index can be derived in different theoretical contexts studying dynamical systems on graphs. This includes, for instance, a linearized yet stable susceptible-infected epidemiological model, tight-binding models in quantum mechanics, thermal Green's function of system of quantum harmonic oscillators as well as in synchronization of networks. In all of them, the parameter $\zeta \in \mathbb{R}$ adquires different "physical" meanings.

Let us make the following generalization of the communicability function:

$$\left(E_{\alpha,\beta}(\zeta,G)\right)_{v,w} := \sum_{k=0}^{\infty} \frac{\zeta^k \left(A^k\right)_{vw}}{\Gamma(\alpha k+\beta)}, \tag{17}$$

which is equal to the standard communicability function when $\alpha = \beta = 1$. Evidently, this function corresponds to the v,w entry of the Mittag-Leffler matrix function of ζA. This function, particularly $E_{\alpha,1}(\zeta,G)$ has been found in the analytical solution of the fractional version of the linearized yet stable susceptible-infected epidemiological model, where the standard time derivative has been replaced by the Caputo fractional one. That is, let x_i be the probability that a node $i \in V$ in G gets infected from a contagious disease circulating the graph. If the birth rate of the disease is β, then the fractional Susceptible-Infected model developed in [21] is given by

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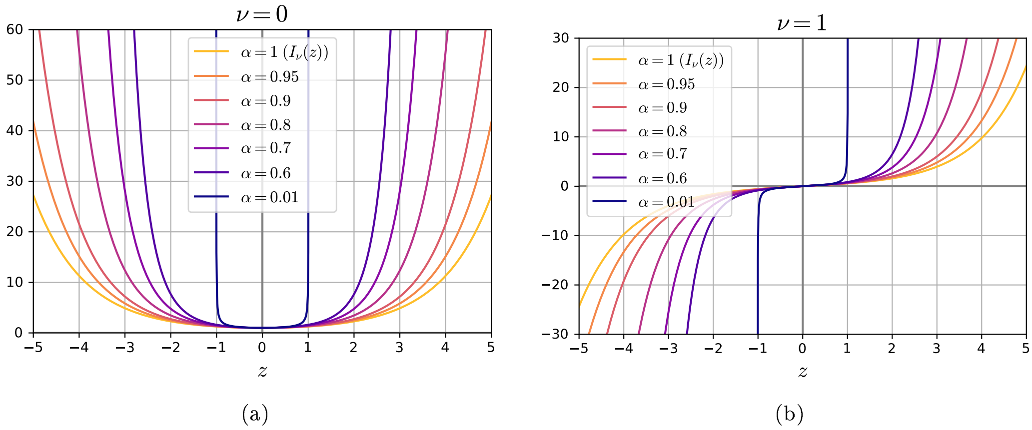

Fig. 1: Plot of the functions $\mathcal{E}_{\nu,\alpha}(z)$ for $z \in \mathbb{Z}$ and for $\nu = 0$ (panel a) and $\nu = 1$ (panel b) as well as for different values of the fractional parameter α. The functions were computed using numerical integration on the basis of the Eq. (20).

$$\begin{cases} D_t^{\alpha}\left(-\log(1-x_i)\right)(t) = \beta^{\alpha}\sum_j A_{ij}x_j(t), \\ x_i(0) = x_{0i} \in [0,1). \end{cases} \tag{18}$$

Then, under certain plausible initial conditions $\gamma = 1 - c/n$ (c is a constant and n is the number of vertrices) on the graph the vector of solutions of this model is given by:

$$x = \left(\frac{1-\gamma}{\gamma}\right) E_{\alpha,1}\left(t^{\alpha}\beta^{\alpha}\gamma A\right)\mathbf{1} - \left(\frac{1-\gamma}{\gamma} + \log\gamma\right)\mathbf{1}, \tag{19}$$

where $\mathbf{1}$ is a vector of ones.

Other more ad hoc encounters with these functions have recently appeared in the literature by Arrigo and Durastante [22], and reviewed by Estrada [23] in the context of the so-called "Estrada indices". Hereafter, we will call $\left(E_{\alpha,\beta}(\zeta, G)\right)_{v,w}$ the fractional communicability between the corresponding nodes. When $v = w$ we will call it the fractional subgraph centrality of the node, and the index defined by $tr\left(E_{\alpha,\beta}(\zeta, G)\right)$ the Estrada-Mittag-Leffler index of the graph [23]. From now on we will focus only on the cases where $\beta = 1$ and we will use the notation $E_{\alpha}(\zeta, G)$ for $E_{\alpha,\beta=1}(\zeta, G)$.

4 Fractional communicabilities in linear and circular chains

Let us start with the following.

Definition 1 Let $E_{\alpha}(z)$ be the Mittag-Leffler function of z. Then, we define the following integral:

$$\mathcal{E}_{\nu,\alpha}(z) := \frac{1}{\pi}\int_0^{\pi} \cos(\nu\theta)\, E_{\alpha}(z\cos\theta)\, d\theta,\ \nu \in \mathbb{Z}, \tag{20}$$

Remark 1 Notice that $\mathcal{E}_{\nu,\alpha=1}(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(\nu\theta)\, d\theta =: I_{\nu}(z)$ which is the modified Bessel function of the first kind. Therefore, $\mathcal{E}_{\nu,\alpha}(z)$ is the fractional analogous of $I_{\nu}(z)$ and will be named here as the fractional modified Bessel function of the first kind (see plots in Fig. 1).

Remark 2 Notice that the FMBF of the first kind also satisfies a similar recurrence relations as the not fractional one. That is,

$$\mathcal{E}_{\nu-1,\alpha}(z^{\alpha}) - \mathcal{E}_{\nu+1,\alpha}(z^{\alpha}) = 2\alpha\nu \cdot \mathcal{I}^{1-\alpha}\left(z^{-1}\cdot\mathcal{E}_{\nu,\alpha}(z^{\alpha})\right). \tag{21}$$

Let G be a path graph (linear chain) on n nodes P_n, i.e., the graph in which all nodes have degree two but two nodes that have degree one. Without any loss of generality we will take $\zeta = 1$ from now on. Then, we can now prove the following result.

Theorem 1 *Let v and w be two nodes of the linear chain of n nodes P_n and let $(E_\alpha(P_n))_{vw}$ be the Mittag-Leffler communicability between nodes v and w. Let,*

$$\left(\hat{E}_\alpha(P_n)\right)_{vw} = \begin{cases} \mathcal{E}_{v-w,\alpha}(2) - \mathcal{E}_{v+w,\alpha}(2) & v \neq w \\ \mathcal{E}_{0,\alpha}(2) - \mathcal{E}_{2r(v),\alpha}(2) & v = w, \end{cases} \tag{22}$$

where

$$r(p) = \begin{cases} v & \text{if } v \leq n/2 \ (n \text{ even}) \text{ or } v \leq (n+1)/2 \ (n \text{ odd}) \\ n - v + 1 & \text{if } v > n/2 \ (n \text{ even}) \text{ or } v > (n+1)/2 \ (n \text{ odd}). \end{cases} \tag{23}$$

Then, $\lim_{n\to\infty} (E_\alpha(P_n))_{vw} / \left(\hat{E}_\alpha(P_n)\right)_{vw} = 1.$

Proof Let us recall that in a linear chain of n nodes, the eigenvalues of the adjacency matrix are given by

$$\lambda_j(P_n) = 2\cos\left(\frac{j\pi}{n+1}\right), \tag{24}$$

and the pth entry of the jth eigenvector is

$$\psi_j(p) = \sqrt{\frac{2}{n+1}} \sin\left(\frac{jp\pi}{n+1}\right). \tag{25}$$

By replacing these spectral values on the expression of $(E_\alpha(A))_{vw}$ we get

$$(E_\alpha(P_n))_{vw} = \frac{1}{n+1} \sum_{j=1}^{n} \left(\cos\left(\frac{j\pi(v-w)}{n+1}\right) - \cos\left(\frac{j\pi(v+w)}{n+1}\right)\right) E_\alpha\left(2\cos\left(\frac{j\pi}{n+1}\right)\right). \tag{26}$$

Let

$$\begin{aligned} \left(\hat{E}_\alpha(P_n)\right)_{vw} &:= \frac{1}{\pi}\int_0^\pi \cos(\theta(v-w)) E_\alpha(2\cos\theta)\, d\theta - \frac{1}{\pi}\int_0^\pi \cos(\theta(v+w)) E_\alpha(2\cos\theta)\, d\theta \\ &= \mathcal{E}_{v-w,\alpha}(2) - \mathcal{E}_{v+w,\alpha}(2) \end{aligned} \tag{27}$$

where $\theta := \frac{j\pi}{n+1}$.
When $v = w$, let

$$\begin{aligned} \left(\hat{E}_\alpha(P_n)\right)_{vv} &:= \frac{1}{\pi}\int_0^\pi [1 - \cos(2v\theta)] E_\alpha(2\cos(\theta)) \\ &= \mathcal{E}_{0,\alpha}(2) - \mathcal{E}_{2r(v),\alpha}(2), \end{aligned} \tag{28}$$

where the term $r(p)$ arises due to the equivalence between the nodes labeled as i and $n - i + 1$. The relationship between $\left(\hat{E}_\alpha(P_n)\right)_{vv}$ and $(E_\alpha(P_n))_{vw}$ can be seen as the one existing in numerical methods like the trapezium or Simpson's rules in which an integral is approximated by a summation. Here the analogous of the number of strips used in those numerical methods of integration is the number of nodes in the path. Then, it is easy to realize that $\lim_{n\to\infty} (E_\alpha(P_n))_{vw} / \left(\hat{E}_\alpha(P_n)\right)_{vw} = 1.$ □

Following the same scheme of proof we can show the following.

Theorem 2 *Let v and w be two nodes of the circular chain of n nodes C_n and let $(E_\alpha(C_n))_{vw}$ be the Mittag-Leffler communicability between nodes v and w. Let,*

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v	$(E_\alpha(P_{20}))_{v,v}$			$(\hat{E}_\alpha(P_{20}))_{v,v}$		
	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$
1	14.0351209	3.1971466	2.0259468	14.0351209	3.1971466	2.0259468
2	40.2363816	6.2883718	3.2379302	40.2363816	6.2883718	3.2379302
3	61.4983128	7.3852388	3.4402925	61.4983128	7.3852388	3.4402925
v	$(E_\alpha(C_{40}))_{v,v}$			$(\hat{E}_\alpha(C_{40}))_{v,v}$		
	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$
1	80.9762993	7.6594588	3.4584489	80.9762993	7.6594588	3.4584489

Table 1: Computational results of values of $(E_\alpha(P_{20}))_{v,v}$ and $(E_\alpha(C_{40}))_{v,v}$ computed using the Mittag-Leffler matrix function and of its approximate values using the FMBF of the first kind, $(\hat{E}_\alpha(P_n))_{v,v}$ and $\hat{E}_\alpha(C_{40})$ for which we have used numerical integration.

$$\left(\hat{E}_\alpha(C_n)\right)_{vw} = \mathcal{E}_{d_{vw},\alpha}(2), \tag{29}$$

where d_{vw} is the shortest path distance between the two nodes, (notice that $d_{vv}=0$). Then,

$$\lim_{n\to\infty} (E_\alpha(P_n))_{vw} / \left(\hat{E}_\alpha(P_n)\right)_{vw} = 1. \tag{30}$$

Example 1 We calculate the values of $(E_\alpha(P_n))_{v,v}$ and $(E_\alpha(C_n))_{v,v}$ as well as their approximation based on the FMBF of the first kind $(\hat{E}_\alpha(P_n))_{v,v}$ and $(\hat{E}_\alpha(C_n))_{v,v}$ for $n=5,20,40,60,80,100$. In Table 1 we give the values of these indexes for the first three nodes of the linear chain (labeled 1, 2 and 3, respectively, starting from one end) with $n=20$ and for any node of the circular chain (all are equivalent). The results for $n>20$ in P_n and those for $n>40$ in C_n do not differ significantly (up to seveth decimal place) from those for $n=20$ or $n=40$, respectively . The important result here is that for relatively large n the approximate values based on the FMBF of the first kind are indistinguishable from those based on the Mittag-Leffler matrix function.

The previous results show that the fractional modified Bessel function (FMBF) of the first kind appears in the expression of the fractional communicability between any pair of nodes in a linear or circular chain when the number of nodes is very large. Therefore, we will focus now on some of the most important properties of this new function.

5 On the Estrada-Mittag-Leffler indices of P_n and C_n

By the definition of the Estrada index we have that the Estrada-Mittag-Leffler index (see [22, 23]) is defined as

$$EE_\alpha(G) = \sum_{v=1}^{n} (E_\alpha(G))_{vv} = \sum_{j=1}^{n} E_\alpha(\lambda_j). \tag{31}$$

Then, for a cycle C_n we have:

$$\begin{aligned} EE_\alpha(C_n) &= \sum_{j=1}^{n} E_\alpha\left(2\cos\left(\frac{2j\pi}{n}\right)\right) \\ &= n\left(\frac{1}{n}\sum_{j=1}^{n} E_\alpha\left(2\cos\left(\frac{2j\pi}{n}\right)\right)\right). \end{aligned} \tag{32}$$

Let

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$$\begin{aligned}\widehat{EE_\alpha}(C_n) &:= n\left(\frac{1}{2\pi}\int_0^{2\pi} E_\alpha\left(2\cos z\right)dz\right)\\ &= n\left(\frac{1}{\pi}\int_0^{\pi} E_\alpha\left(2\cos z\right)dz\right)\\ & n\cdot\mathcal{E}_{\alpha,0}\left(2\right).\end{aligned} \tag{33}$$

Then, $\lim\limits_{n\to\infty} EE_\alpha\left(C_n\right)/\widehat{EE_\alpha}(C_n) = 1$.
For the case of the linear chain we have:

$$\begin{aligned}EE_\alpha\left(P_n\right) &= \sum_{k=1}^{n} E_\alpha\left(2\cos\left(\frac{k\pi}{n+1}\right)\right)\\ &= \frac{1}{2}\sum_{k=0}^{n} E_\alpha\left(2\cos\left(\frac{k\pi}{n+1}\right)\right) + \frac{1}{2}\sum_{k=1}^{n+1} E_\alpha\left(2\cos\left(\frac{k\pi}{n+1}\right)\right) -\\ &- \frac{1}{2}\left(E_\alpha(2) + E_\alpha(-2)\right)\\ &= \frac{n+1}{2}\left(\frac{1}{n+1}\sum_{k=0}^{n} E_\alpha\left(2\cos\left(\frac{k\pi}{n+1}\right)\right)\right) + \frac{n+1}{2}\left(\frac{1}{n+1}\sum_{k=1}^{n+1} E_\alpha\left(2\cos\left(\frac{k\pi}{n+1}\right)\right)\right)\\ &- \frac{1}{2}\left(E_\alpha(2) + E_\alpha(-2)\right).\end{aligned} \tag{34}$$

Let

$$\begin{aligned}\widehat{EE_\alpha}\left(P_n\right) &:= \frac{n+1}{2}\left(\frac{1}{\pi}\int_0^{\pi} E_\alpha\left(2\cos z\right)dz\right) + \frac{n+1}{2}\left(\frac{1}{\pi}\int_0^{\pi} E_\alpha\left(2\cos z\right)dz\right)\\ &- \frac{1}{2}\left(E_\alpha\left(2\right) + E_\alpha\left(-2\right)\right)\\ &= \left(n+1\right)\mathcal{E}_{\alpha,0}\left(2\right) - \frac{1}{2}\left(E_\alpha\left(2\right) + E_\alpha\left(-2\right)\right).\end{aligned} \tag{35}$$

Then, $\lim\limits_{n\to\infty} EE_\alpha\left(P_n\right)/\widehat{EE_\alpha}\left(P_n\right) = 1$.

6 Power series of the the FMBF of the first kind

We start by expressing the FMBF of the first kind as a power series.

Lemma 1 *Let $\mathcal{E}_{\nu,\alpha}(z)$ be the fractional modified Bessel function of the first kind of z with fractional parameter α and $\nu \in \mathbb{Z}$. Then,*

$$\mathcal{E}_{\nu,\alpha}(z) = \sum_{k=0}^{\infty} \frac{(2k+\nu)!}{\Gamma\left(\alpha\left(2k+\nu\right)+1\right)k!\left(k+\nu\right)!}\left(\frac{z}{2}\right)^{2k+\nu}. \tag{36}$$

Proof First we use the Taylor series expression of the Mittag-Leffler function on the definition of $\mathcal{E}_{\nu,\alpha}$:

$$\begin{aligned}\mathcal{E}_{\nu,\alpha}\left(z\right) &= \frac{1}{\pi}\int_0^{\pi}\sum_{j=0}^{\infty}\frac{z^j\cos^j\left(\theta\right)}{\Gamma\left(\alpha j+1\right)}\cos\left(\nu\theta\right)d\theta\\ &= \sum_{j=0}^{\infty}\left[\frac{z^j}{\pi\Gamma\left(\alpha k+1\right)}\int_0^{\pi}\cos^{\,j}\left(\theta\right)\cos\left(\nu\theta\right)d\theta\right]\end{aligned} \tag{37}$$

To obtain the value of the integral we equalize $\mathcal{E}_{\nu,\alpha}\left(z\right)$ to $I_\nu\left(z\right)$ for $\alpha = 1$, such that we obtain:

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$$\int_0^\pi \cos(\theta)^j \cos(\nu\theta)d\theta = \begin{cases} \dfrac{\pi \cdot j!}{\left(\frac{j-\nu}{2}\right)!\left(\frac{j+\nu}{2}\right)!2^j} & \text{if } j \geq \nu \text{ and } j+\nu \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

Finally, we replace the solution of this integral into $\mathcal{E}_{\nu,\alpha}(z)$ and we are done. □

Remark 3 When $\alpha = 1$ the term $\Gamma(\alpha k + 1) = k!$ and $\mathcal{E}_{\nu,\alpha=1}(z) = \mathcal{I}_\nu(z)$.

Lemma 2 *The power series*

$$\sum_{k=0}^{\infty} \frac{(\nu+2k)!}{\Gamma(\alpha(\nu+2k)+1)} \frac{1}{k!(\nu+k)!} \left(\frac{z}{2}\right)^{2k+\nu} \tag{39}$$

converges $\forall z$ *if* $\alpha \in (0,1]$ *and in the limiting case when* $\alpha = 0$ *it converges if* $|z| < 1$ *and diverges if* $|z| > 1$.

Proof Let us consider the coefficients $a_j(\nu,\alpha)$, which are given by:

$$a_j(\nu,\alpha) = \begin{cases} \dfrac{(\nu+2k)!}{\Gamma(\alpha(\nu+2k)+1)} \dfrac{1}{k!(\nu+k)!} \dfrac{1}{2^{2k+\nu}} & \text{if } j = 2k+\nu,\ k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

We apply the Cauchy-Hadamard theorem to calculate the convergence radius $R(\nu,\alpha)$ of the series, which is defined as

$$R(\nu,\alpha) = \frac{1}{\limsup_{j\to\infty}\left(|a_j(\nu,\alpha)|^{1/j}\right)} \tag{41}$$

On the upper limit there is only influence of the values of $j = 2k+\nu$. Thus, when $k \longrightarrow \infty$ we have

$$\limsup_{j\to\infty}\left(|a_j(\nu,\alpha)|^{1/j}\right) = \limsup_{k\to\infty}\left(|a_{2k+\nu}(\nu,\alpha)|^{1/(2k+\nu)}\right) = \lim_{k\to\infty}\left(|a_{2k+\nu}(\nu,\alpha)|^{1/(2k+\nu)}\right) \tag{42}$$

Using Stirling approximation, the following limit can be obtained for $\alpha \in (0,1]$ (see Remark 4):

$$\lim_{k\to\infty} |a_{2k+\nu}(\nu,\alpha)|^{1/(2k+\nu)} = \lim_{k\to\infty}\left(\frac{e}{\alpha(\nu+2k)}\right)^\alpha = 0 \tag{43}$$

Therefore, when $\alpha \in (0,1]$ we get that $R(\nu,\alpha) = +\infty$, indicating global convergence of the series. Now, when $\alpha = 0$, we have that

$$a_{2k+\nu}(\nu,0) = \frac{(\nu+2k)!}{k!(\nu+k)!}\frac{1}{2^{2k+\nu}} \tag{44}$$

From this equation, it can be proved that (see Remark 5):

$$\lim_{k\to\infty} |a_{2k+\nu}(\nu,\alpha)|^{1/(2k+\nu)} = \lim_{k\to\infty}\left(\sqrt{\frac{\nu+2k}{k(\nu+k)}}\right)^{\frac{1}{2k+\nu}} = 1 \tag{45}$$

Therefore, $R(\nu,0) = 1$, such that (39) converges for z if $|z| < 1$ and does not converge if $|z| > 1$. □

Remark 4 In the proof of Lemma 2 we take into account Stirling approximation for both the factorial and the gamma function Γ. This approximation allows us to replace terms of the form $n!$ and $\Gamma(z+1)$ by $n^n e^{-n}\sqrt{2\pi n}$ and $z^z e^{-z}\sqrt{2\pi z}$ respectively when both n and z tend to $+\infty$.

 doi:10.20944/preprints202302.0468.v1

$$\lim_{k\to\infty}|a_{2k+\nu}(\nu,\alpha)|^{\frac{1}{2k+\nu}} = \lim_{k\to\infty}\left(\frac{(\nu+2k)!}{\Gamma\left(\alpha\left(\nu+2k\right)+1\right)}\frac{1}{k!\,(\nu+k)!}\frac{1}{2^{2k+\nu}}\right)^{\frac{1}{2k+\nu}} =$$
$$= \lim_{k\to\infty}\left(\frac{(\nu+2k)^{\nu+2k}\,e^{-(\nu+2k)}\sqrt{2\pi(\nu+2k)}}{(\alpha(\nu+2k))^{\alpha(\nu+2k)}\,e^{-\alpha(\nu+2k)}\sqrt{2\pi\alpha(\nu+2k)}}\frac{e^k e^{\nu+k}}{k^k\,(\nu+k)^{\nu+k}\,2\pi\sqrt{k(\nu+k)}}\frac{1}{2^{2k+\nu}}\right)^{\frac{1}{2k+\nu}} =$$
$$= \lim_{k\to\infty}\left(\frac{(\nu+2k)^{\nu+2k}}{k^k\,(\nu+k)^{\nu+k}\,2^{2k+\nu}}\cdot\frac{1}{2\pi\sqrt{\alpha k(\nu+k)}}\cdot e^{\alpha(\nu+2k)}\cdot\frac{1}{(\alpha(\nu+2k))^{\alpha(\nu+2k)}}\right)^{\frac{1}{2k+\nu}} \tag{46}$$

It is easy to study the correspondent limit for the first and second factor of the previous expression:

$$\lim_{k\to\infty}\left(\frac{(\nu+2k)^{\nu+2k}}{k^k\,(\nu+k)^{\nu+k}\,2^{2k+\nu}}\right)^{\frac{1}{2k+\nu}} = \frac{1}{2}\lim_{k\to\infty}\frac{\nu+2k}{\nu+k}\left(\frac{\nu+k}{k}\right)^{\frac{k}{2k+\nu}} = \lim_{k\to\infty}\left(1+\frac{\nu}{k}\right)^{\frac{k}{2k+\nu}} = 1 \tag{47}$$

$$\lim_{k\to\infty}\left(\frac{1}{2\pi\sqrt{\alpha k(\nu+k)}}\right)^{\frac{1}{2k+\nu}} = \lim_{k\to\infty}\left(\sqrt{\alpha k(\nu+k)}\right)^{-\frac{1}{2k+\nu}} = 1 \tag{48}$$

Therefore:

$$\lim_{k\to\infty}|a_{2k+\nu}(\nu,\alpha)|^{\frac{1}{2k+\nu}} = \lim_{k\to\infty}\left(\frac{e^{\alpha(\nu+2k)}}{(\alpha(\nu+2k))^{\alpha(\nu+2k)}}\right)^{\frac{1}{2k+\nu}} = \lim_{k\to\infty}\left(\frac{e}{\alpha\,(\nu+2k)}\right)^{\alpha} = 0. \tag{49}$$

Remark 5 In the proof of Lemma 2 we also considered the following

$$\lim_{k\to\infty}|a_{2k+\nu}(\nu,0)|^{\frac{1}{2k+\nu}} = \lim_{k\to\infty}\left(\frac{(\nu+2k)!}{k!\,(\nu+k)!}\frac{1}{2^{2k+\nu}}\right)^{\frac{1}{2k+\nu}}$$
$$= \lim_{k\to\infty}\left(\frac{(\nu+2k)^{\nu+2k}\,e^{-(\nu+2k)}\sqrt{2\pi(\nu+2k)}}{k^k e^{-k}\sqrt{2\pi k}\,(\nu+k)^{\nu+k}\,e^{-(\nu+k)}\sqrt{2\pi(\nu+k)}}\frac{1}{2^{2k+\nu}}\right)^{\frac{1}{2k+\nu}} \tag{50}$$
$$= \lim_{k\to\infty}\left(\frac{(\nu+2k)^{\nu+2k}}{k^k\,(\nu+k)^{\nu+k}\,2^{2k+\nu}}\sqrt{\frac{(\nu+2k)}{2\pi k(\nu+k)}}\right)^{\frac{1}{2k+\nu}}.$$

As it was shown in the Remark 4, the first of the two previous factors tends to one as $k\to\infty$. Therefore:

$$\lim_{k\to\infty}|a_{2k+\nu}(\nu,0)|^{\frac{1}{2k+\nu}} = \lim_{k\to\infty}\left(\sqrt{\frac{(\nu+2k)}{k(\nu+k)}}\right)^{\frac{1}{2k+\nu}} = 1. \tag{51}$$

7 Differential properties of the FMBF of the first kind

We now obtain recurrence relations for the Caputo derivatives of $\mathcal{E}_{\nu,\alpha}\left(z^{\alpha}\right)$.

Theorem 3 *The following equalities hold*

1.
$$D_z^{\alpha}\left(\mathcal{E}_{\nu,\alpha}\left(z^{\alpha}\right)\right) = \mathcal{E}_{\nu-1,\alpha}\left(z^{\alpha}\right) - \alpha\nu\cdot\mathcal{I}^{1-\alpha}\left(z^{-1}\cdot\mathcal{E}_{\nu,\alpha}\left(z^{\alpha}\right)\right) \tag{52}$$

2.
$$D_z^{\alpha}\left(\mathcal{E}_{\nu,\alpha}\left(z^{\alpha}\right)\right) = \mathcal{E}_{\nu+1,\alpha}\left(z^{\alpha}\right) + \alpha\nu\cdot\mathcal{I}^{1-\alpha}\left(z^{-1}\cdot\mathcal{E}_{\nu,\alpha}\left(z^{\alpha}\right)\right) \tag{53}$$

3.
$$D_z^{\alpha}\left(\mathcal{E}_{\nu,\alpha}\left(z^{\alpha}\right)\right) = \frac{1}{2}\left(\mathcal{E}_{\nu-1,\alpha}\left(z^{\alpha}\right) + \mathcal{E}_{\nu+1,\alpha}\left(z^{\alpha}\right)\right). \tag{54}$$

Before proceeding with the proof we need to state the following Auxiliary result.

Lema 3 *Let* $\alpha \in (0,1]$ *and* $\nu \in \mathbb{N}$*, then:*

$$\alpha \cdot \mathcal{I}^{1-\alpha}\left(z^{-1}\cdot \mathcal{E}_{\nu,\alpha}(z^{\alpha})\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \tag{55}$$

Proof The expression 15 allows us to calculate $\mathcal{I}^{1-\alpha}\left(z^{-1}\cdot \mathcal{E}_{\nu,\alpha}(z^{\alpha})\right)$ by integrating partially each term of the power series. This operation can be applied to powers of z with exponent larger than -1. Then, because we are considering only the cases where $\gamma \in \mathbb{N}$ it is always true that $\alpha(2k+\nu)-1 > -1$ for all possible k and ν, such that:

$$\alpha \cdot \mathcal{I}^{1-\alpha}\left(z^{-1}\cdot \mathcal{E}_{\nu,\alpha}(z^{\alpha})\right) = \alpha \cdot \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu)+1)} \frac{(2k+\nu)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} \mathcal{I}^{1-\alpha}\left(z^{\alpha(2k+\nu)-1}\right). \tag{56}$$

Using properties of the Riemann-Liouville integral when $r = \alpha(2k+\nu)-1$ and $\gamma = 1-\alpha \in (0,1)$ we have:

$$\begin{aligned}\mathcal{I}^{1-\alpha}\left(z^{\alpha(2k+\nu)-1}\right) &= \frac{\Gamma(\alpha(2k+\nu)-1+1)}{\Gamma(\alpha(2k+\nu)-1+2-\alpha)} z^{\alpha(2k+\nu)-1+1-\alpha} \\ &= \frac{\Gamma(\alpha(2k+\nu))}{\Gamma(\alpha(2k+\nu-1)+1)} z^{\alpha(2k+\nu-1)}\end{aligned} \tag{57}$$

Plugging (57) in (56), we obtain the result. □

We now proceed with the proof of Theorem 3.

Proof We start by proving the recurrence (1). For that we take into account the property (2) with $r = \alpha(2k+\nu)$ to show that:

$$\begin{aligned}D_z^{\alpha}(\mathcal{E}_{\nu,\alpha}(z^{\alpha})) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu)+1)} \frac{(2k+\nu)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} D_z^{\alpha}\left(z^{\alpha(2k+\nu)}\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu)+1)} \frac{(2k+\nu)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} \frac{\Gamma(\alpha(2k+\nu)+1)}{\Gamma(\alpha(2k+\nu)-\alpha+1)} z^{\alpha(2k+\nu)-\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\ &= \sum_{k=0}^{\infty} \left(\frac{2k+\nu}{k+\nu}\right) \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu-1)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\ &= \sum_{k=0}^{\infty} \left(1-\frac{1}{2}\frac{\nu}{k+\nu}\right) \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu-1)!} \frac{1}{2^{2k+\nu-1}} z^{\alpha(2k+\nu-1)} \\ &= \mathcal{E}_{\nu-1,\alpha}(z^{\alpha}) - \sum_{k=0}^{\infty} \frac{\nu}{k+\nu} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu-1)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\ &= \mathcal{E}_{\nu-1,\alpha}(z^{\alpha}) - \nu \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\ &= \mathcal{E}_{\nu-1,\alpha}(z^{\alpha}) - \Xi.\end{aligned} \tag{58}$$

We now consider the term Ξ, which by using the Auxiliary result (3) gives the following:

$$\begin{aligned}\Xi &= \nu \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\ &= \alpha\nu \cdot \mathcal{I}^{1-\alpha}\left(z^{-1}\cdot \mathcal{E}_{\nu,\alpha}(z^{\alpha})\right),\end{aligned} \tag{59}$$

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which proves (1).

Let us now prove (2). Let $\nu \in \mathbb{N}$ and using again the property (2) with $r = \alpha(2k+\nu)$ we write

$$\begin{aligned}
D_z^\alpha(\mathcal{E}_{\nu,\alpha}(z^\alpha)) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu)+1)} \frac{(2k+\nu)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} D_z^\alpha\left(z^{\alpha(2k+\nu)}\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu)+1)} \frac{(2k+\nu)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} \frac{\Gamma(\alpha(2k+\nu)+1)}{\Gamma(\alpha(2k+\nu)-\alpha+1)} z^{\alpha(2k+\nu)-\alpha} \\
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu)!}{k!\,(k+\nu)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\
&= \sum_{k=0}^{\infty} \binom{2k+\nu}{k+\nu} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu-1)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\
&= \underbrace{\sum_{k=0}^{\infty} \frac{2k}{k+\nu} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu-1)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)}}_{A} \\
&\quad + \underbrace{\sum_{k=0}^{\infty} \frac{\nu}{k+\nu} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu-1)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)}}_{B}
\end{aligned} \tag{60}$$

Let us rewrite now the expression for A by observing that the contribution for $k = 0$ is zero, so that we start the summation by $k = 1$:

$$\begin{aligned}
A &= \sum_{k=1}^{\infty} \frac{2k}{k+\nu} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{k!\,(k+\nu-1)!} \frac{1}{2^{2k+\nu}} z^{\alpha(2k+\nu-1)} \\
&= \sum_{k=1}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu-1)+1)} \frac{(2k+\nu-1)!}{(k-1)!\,(k+\nu)!} \frac{1}{2^{2k+\nu-1}} z^{\alpha(2k+\nu-1)} \\
&= \sum_{k=1}^{\infty} \frac{1}{\Gamma(\alpha(2(k-1)+\nu+1)+1)} \frac{(2(k-1)+\nu+1)!}{(k-1)!\,(k-1+\nu+1)!} \frac{1}{2^{2(k-1)+\nu+1}} z^{\alpha(2(k-1)+\nu+1)}.
\end{aligned} \tag{61}$$

We now identify the index $k-1$ as the initial point in the summation such that we can start it at 0, so that

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha(2k+\nu+1)+1)} \frac{(2k+\nu+1)!}{k!\,(k+\nu+1)!} \frac{1}{2^{2k+\nu+1}} z^{\alpha(2k+\nu+1)} = \mathcal{E}_{\nu+1,\alpha}(z^\alpha) \tag{62}$$

It is easy to check that the term B is just the term Ξ used in the proof of the previous recurrence. Thus, we have that

$$B = \alpha\nu \cdot \mathcal{I}^{1-\alpha}\left(z^{-1} \cdot \mathcal{E}_{\nu,\alpha}(z^\alpha)\right), \tag{63}$$

which gives us the result for recurrence (2).

Finally, for recursion (3) we have

$$\begin{aligned}
2D_z^\alpha\left(\mathcal{E}_{\nu,\alpha}\left(z^\alpha\right)\right) &= \mathcal{E}_{\nu+1,\alpha}\left(z^\alpha\right) + \alpha\nu \cdot \mathcal{I}^{1-\alpha}\left(z^{-1} \cdot \mathcal{E}_{\nu,\alpha}\left(z^\alpha\right)\right) + \mathcal{E}_{\nu-1,\alpha}\left(z^\alpha\right) - \alpha\nu \cdot \mathcal{I}^{1-\alpha}\left(z^{-1} \cdot \mathcal{E}_{\nu,\alpha}\left(z^\alpha\right)\right) \\
&= \mathcal{E}_{\nu+1,\alpha}\left(z^\alpha\right) + \mathcal{E}_{\nu-1,\alpha}\left(z^\alpha\right),
\end{aligned} \tag{64}$$

which finally proves the result. □

Remark 6 The Theorem 3 generalizes the recurrence formulae obtained for the standard Bessel function of the first kind. That is, when $\alpha = 1$ the expressions in Theorem 3 transform into the recurrence formulas for the MBF of the first kind given in the Introduction.

8 Open problems

Here we have proposed a generalization of MBF of the first kind $I_\nu(z)$ when $\nu \in \mathbb{Z}$, which transforms the integral representation

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)d\theta, \tag{65}$$

into $\mathcal{E}_{\nu,\alpha}(z)$, which for $\nu \in \mathbb{Z}$ has the following representation:

$$\mathcal{E}_{\nu,\alpha}(z) = \frac{1}{\pi}\int_0^\pi E_{\alpha,1}\left(z\cos\theta\right)\cos(\nu\theta)d\theta. \tag{66}$$

Therefore, the first extension of the current work is to generalize $\mathcal{E}_{\nu,\alpha}(z)$ for $\nu \in \mathbb{R}$. This extension can be obtained by starting from the power series expression of the FMBF of the first kind obtained here. The focus in the current work has been in the restricted domain of $\nu \in \mathbb{Z}$ which is the one that naturally emerges from the problem of considering fractional analogous of the communicability functions in simple graphs like the path and cycle of n nodes.
A second generalization is obviously to consider the more general form of the Mittag-Leffler function, such that:

$$\mathcal{E}_{\nu,\alpha,\beta}(z) = \frac{1}{\pi}\int_0^\pi E_{\alpha,\beta}\left(z\cos\theta\right)\cos(\nu\theta)d\theta. \tag{67}$$

Acknowledgements EE acknowledges funding from project OLGRA (PID2019-107603GB-I00) funded by Spanish Ministry of Science and Innovation as well as the Maria de Maeztu project CEX2021-001164-M funded by the MCIN/AEI/10.13039/501100011033. Both authors thank J. Gómez-Gardeñes for useful discussions.

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