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Randomized Nonuniform Sampling for Random Signals Bandlimited in the Special Affine Fourier Transform Domain

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Abstract: The special affine Fourier transform (SAFT) is a useful and powerful analyzing tool in signal processing, optics and communications. In this paper, we mainly discuss the randomized nonuniform sampling and reconstruction for random signals bandlimited in the SAFT domain. First, we show that the nonuniform sampling is identical to the uniform sampling after a pre-filter in the sense of second order statistic characters. Then, we propose an approximate reconstruction based on sinc interpolation for the nonuniform sampling of random signals bandlimited in the SAFT domain. Finally, we give the mean square error estimate for the proposed approximate recovery approach.

Keywords: special affine Fourier transform; nonuniform sampling; random signals; error estimate; approximate recovery

MSC: 46E22, 94A20.

1. Introduction

The special affine Fourier transform (SAFT) was firstly proposed in [1] to model optical systems. It offers a unified viewpoint of known signal processing transforms, such as Fourier transform (FT), fractional Fourier transform (FrFT), linear canonical transform (LCT), Laplace transform (LT) and so on. It can also include some optical operations on light waves, such as rotation, magnification, hyperbolic transformation, free space propagation, Lens transformation and so on. The SAFT is a six-parameters linear integral transform which is defined by offsetting two extra parameters on the basis of the LCT, so SAFT is also known as the offset linear canonical transform (OLCT). It has been proved that the SAFT is a useful tool for signal processing, communications, quantum mechanics and optics [12,16,23,26]. Many classical results such as Zak transform, Poisson summation formula and convolution theorems are established in the SAFT domain [6,24,33].

Let

$$A = \left[\begin{array}{cc|c} a & b & u_0 \\ c & d & \omega_0 \end{array}\right]$$

be a matrix with six real parameters satisfying $ad - bc = 1$. The continuous-time SAFT associated with the parameter matrix A of a signal $f(t)$ is defined as in [1],

$$F_A(u) = SAFT[f](u) = \begin{cases} \int_{-\infty}^{+\infty} f(t) K_A(t,u) dt, & b \neq 0, \\ \sqrt{d} e^{\frac{jcd(u-u_0)^2}{2} + jw_0 u} f\left[d\left(u - u_0\right)\right], & b = 0, \end{cases} \tag{1.1}$$

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where the kernel function $K_A(t,u)$ is given by

$$K_A(t,u) = \sqrt{\frac{1}{2\pi jb}} e^{\frac{jdu_0^2}{2b}} e^{\frac{j}{2b}\left[at^2+2t(u_0-u)-2u(du_0-bw_0)+du^2\right]}. \tag{1.2}$$

It is noted that when $b=0$, the SAFT of a signal is essentially a chirp multiplication. Therefore, we shall confine our attention to the case of $b \neq 0$. The inverse SAFT is expressed as

$$f(t) = C\int_{-\infty}^{+\infty} F_A(u) K_{A^{-1}}(u,t)du, \tag{1.3}$$

where $C = e^{\frac{j}{2}\left(cdu_0^2 - 2adu_0w_0 + abw_0^2\right)}$ and

$$A^{-1} := \left[\begin{array}{cc|c} d & -b & b\omega_0 - du_0 \\ -c & a & cu_0 - a\omega_0 \end{array}\right].$$

Sampling is one of the most fundamental process in digital signal processing which provides a bridge between the continuous physical signals and the discrete digital signals. Beginning with the Shannon's sampling theorem of bandlimited signals [15], various sampling such as nonuniform sampling, average sampling, dynamic sampling, random sampling, mobile sampling, timing sampling and multi-channel sampling have been generally studied for signals bandlimited in the FT domain [2,3,5,9]. With the appearance and developments of the more general transforms, the corresponding sampling theories are extended to the signals bandlimited in the FrFT, LCT and SAFT domains [6,12,14,18,19,21–23,25–27,30,32].

Signals in the real world often presents random characteristics and sampling for random signals bandlimited in the FT domain has been generally studied [5,7,8,17]. In recent years, there have existed many researches for sampling of random signals bandlimited in the FrFT and LCT domains [10,11,20,28, 31]. The uniform sampling theorems in [10] was extended to the SAFT domain as in [29]. Nonuniform sampling is a more realistic sampling scheme due to the limitations of data acquisition and processing ability. In fact, the nonuniform sampling theories including the periodic nonuniform sampling model, N-order recurrent nonuniform sampling model, nonuniform sampling due to migration of a finite number of uniform samples and the general nonuniform sampling have been given for signals bandlimited in the LCT domain [31] and signals bandlimited in the SAFT domain [4,30], respectively. In particular, the nonuniform sampling problem was also considered in [11] for random signals bandlimited in the LCT domain, where a randomized nonuniform sampling method and a class of approximate recovery approaches by using sinc interpolation functions were studied. In this paper, we will further study the randomized nonuniform sampling for random signals bandlimited in the SAFT domain and also give an approximate recovery method based on the sinc interpolation.

The paper is organized as follows. In section 2, we give the definition of the power spectral density in the SAFT domain. In section 3, we study the nonuniform sampling scheme and propose an approximate recovery approach. In section 4, the mean square error estimate for the proposed approximate recovery method is demonstrated.

2. Power Spectral Density in the SAFT Domain

Given a probability space $(\Omega, \mathcal{F}, p)$, a stochastic process $x(t)$ is said to be wide sense stationary if it has zero mean and its auto-correlation function

$$R_{xx}(t+\tau,t) = E\left[x(t+\tau)x^*(t)\right] \tag{2.1}$$

is independent of $t \in \mathbb{R}$, i.e., $R_{xx}(t+\tau, t) = R_{xx}(\tau)$, where $E[\cdot]$ denotes mathematical expectation and the superscript $*$ stands for the complex conjugate. Two stochastic processes $x(t)$ and $y(t)$ are said to be jointly stationary, if $x(t)$ and $y(t)$ are both stationary and their cross-correlation function

$$R_{xy}(t+\tau, t) = E\left[x(t+\tau)y^*(t)\right] \tag{2.2}$$

is independent of $t \in \mathbb{R}$, i.e., $R_{xy}(t+\tau, t) = R_{xy}(\tau)$.

We next introduce the SAFT auto-correlation function, the SAFT cross-correlation function, the SAFT auto-power spectral density and the SAFT cross-power spectral density as in [29]. For two random signals $x(t)$ and $y(t)$, the SAFT auto-correlation function of $x(t)$ is defined as

$$R^A_{xx}(\tau) = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} R_{xx}(t+\tau, t)e^{j\frac{a}{b}t\tau}dt. \tag{2.3}$$

Similarly, the SAFT cross-correlation function of $x(t)$ and $y(t)$ is defined as

$$R^A_{xy}(\tau) = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} R_{xy}(t+\tau, t)e^{j\frac{a}{b}t\tau}dt. \tag{2.4}$$

Remark 2.1. *If the random signal $\tilde{x}(t) = x(t)e^{j\frac{a}{2b}t^2}$ is stationary, then $x_1(t) = \tilde{x}(t)e^{j\frac{u_0}{b}t}$ is also stationary. In fact,*

$$R_{x_1x_1}(t+\tau, t) = e^{\frac{ju_0\tau}{b}} R_{\tilde{x}\tilde{x}}(t+\tau, t).$$

Moreover, one has

$$\begin{aligned} R_{\tilde{x}\tilde{x}}(t+\tau, t) &= E\left[\tilde{x}(t+\tau)\tilde{x}^*(t)\right] \\ &= E\left[x(t+\tau)e^{j\frac{a}{2b}(t+\tau)^2}x^*(t)e^{-j\frac{a}{2b}t^2}\right] \\ &= E\left[x(t+\tau)x^*(t)e^{j\frac{a}{2b}\tau^2}e^{j\frac{at\tau}{b}}\right] \\ &= R_{xx}(t+\tau, t)e^{j\frac{at\tau}{b}}e^{j\frac{a}{2b}\tau^2}. \end{aligned}$$

Therefore, $R_{xx}(t+\tau, t)e^{\frac{jat\tau}{b}}$ must be independent of t. In such case, we have

$$R^A_{xx}(\tau) = R_{x_1x_1}(\tau)e^{-j\frac{a}{2b}\tau^2}e^{-j\frac{u_0}{b}\tau}. \tag{2.5}$$

Define the SAFT auto-power spectral density of the random signal $x(t)$ by

$$P^A_{xx}(u) = \sqrt{\frac{1}{-j2\pi b}}e^{-j\frac{d}{2b}u^2}e^{-j\frac{du_0^2}{2b}}e^{j\frac{u}{b}(du_0-bw_0)}F_A\left\{R^A_{xx}(\tau)\right\}(u) \tag{2.6}$$

and the SAFT cross-power spectral density of the random signals $x(t)$ and $y(t)$ as

$$P^A_{xy}(u) = \sqrt{\frac{1}{-j2\pi b}}e^{-j\frac{d}{2b}u^2}e^{-j\frac{du_0^2}{2b}}e^{j\frac{u}{b}(du_0-bw_0)}F_A\left\{R^A_{xy}(\tau)\right\}(u). \tag{2.7}$$

It follows from (1.1) and (2.6) that

$$
\begin{aligned}
R_{xx}^{A}(\tau) &= C\cdot\int_{-\infty}^{+\infty} P_{xx}^{A}(u)\,1/\sqrt{\frac{1}{-j2\pi b}}e^{j\frac{d}{2b}u^2}e^{j\frac{d}{2b}u_0^2}e^{-j\frac{u}{b}(du_0-bw_0)}\sqrt{\frac{1}{-j2\pi b}} \\
&\quad\times e^{-j\frac{a}{2b}(bw_0-du_0)^2}e^{-j\frac{d}{2b}u^2}e^{-j\frac{u}{b}(bw_0-du_0-\tau)}e^{-j\frac{u_0}{b}\tau}e^{-j\frac{a}{2b}\tau^2}du \\
&= C\cdot\int_{-\infty}^{+\infty} P_{xx}^{A}(u)\,e^{j\frac{d}{2b}u_0^2}e^{-j\frac{a}{2b}(bw_0-du_0)^2}e^{j\frac{u}{b}\tau}e^{-j\frac{u_0}{b}\tau}e^{-j\frac{a}{2b}\tau^2}du \\
&= e^{\frac{j}{2}\left(cdu_0^2-2adu_0w_0+abw_0^2\right)}\int_{-\infty}^{+\infty} P_{xx}^{A}(u)\,e^{j\frac{d}{2b}u_0^2}e^{-j\frac{a}{2b}(bw_0-du_0)^2}e^{j\frac{u}{b}\tau}e^{-j\frac{u_0}{b}\tau}e^{-j\frac{a}{2b}\tau^2}du \\
&= \int_{-\infty}^{+\infty} P_{xx}^{A}(u)e^{-j\frac{a}{2b}\tau^2}e^{j\frac{1}{b}(u-u_0)\tau}du. \qquad (2.8)
\end{aligned}
$$

Multiplicative filtering in the SAFT domain is showed in Figure 1, which has been introduced in [29]. More specifically, we first obtain the SAFT of the input signal $f_1(t)$ and apply the multiplicative filter $H(u)$ in the SAFT domain. Then the output signal $f_2(t)$ in the time domain is obtained by the inverse SAFT. Mathematically, the output $f_2(t)$ is given by

$$f_2(t) = F_A^{-1}\{F_2(u)\}(t) = F_A^{-1}\{F_1(u)H(u)\}(t), \qquad (2.9)$$

where $F_1(u) = F_A\{f_1(t)\}(u)$ and $F_2(u) = F_1(u)H(u)$.

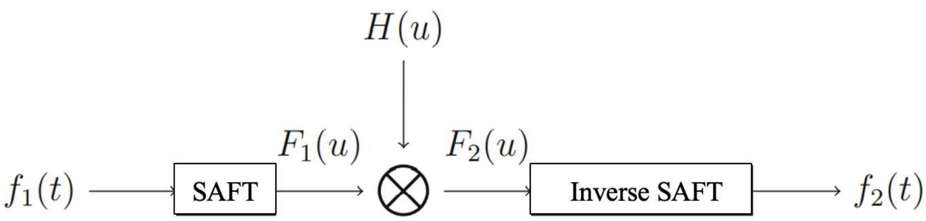

Figure 1. Multiplicative filtering in the SAFT

Define normalized convolution

$$(f\Theta g)(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)g(t-x)e^{-j\frac{a}{2b}(t^2-x^2)}dx \qquad (2.10)$$

for $f, g \in L^2(\mathbb{R})$ [23]. Then we have the following conclusion.

Proposition 2.2. *Let*

$$H(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} h(t)e^{-j\frac{(u-u_0)}{b}t}dt. \qquad (2.11)$$

Then the multiplicative filter in Figure 1 is equivalent to

$$f_2(t) = (f_1\Theta h)(t). \qquad (2.12)$$

Proof we only need to prove

$$F_A\{(f_1\Theta h)(t)\}(u) = F_1(u)H(u).$$

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It follows from the definition of the SAFT that

$$
\begin{aligned}
& F_A\left\{\left(f_1 \Theta h\right)(t)\right\}(u) \\
& =\int_{\mathbb{R}}\left(f_1 \Theta h\right)(t) \sqrt{\frac{1}{2 \pi j b}} e^{\frac{j d u_0^2}{2 b}} e^{\frac{j}{2 b}\left[a t^2+2 t\left(u_0-u\right)-2 u\left(d u_0-b w_0\right)+d u^2\right]} d t \\
& =\frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(x) h(t-x) e^{-j \frac{a}{2 b}\left(t^2-x^2\right)} \sqrt{\frac{1}{2 \pi j b}} e^{\frac{j d u_0^2}{2 b}} e^{\frac{j}{2 b}\left[a t^2+2 t\left(u_0-u\right)-2 u\left(d u_0-b w_0\right)+d u^2\right]} d x d t \\
& =\int_{\mathbb{R}} f_1(x) \sqrt{\frac{1}{2 \pi j b}} e^{\frac{j d u_0^2}{2 b}} e^{\frac{j}{2 b}\left[a x^2+2 x\left(u_0-u\right)-2 u\left(d u_0-b w_0\right)+d u^2\right]} d x \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} h(t-x) e^{\frac{j}{b} t\left(u_0-u\right)} e^{-\frac{j}{2 b}\left[2 x\left(u_0-u\right)\right]} d t \\
& =\int_{\mathbb{R}} f_1(x) \sqrt{\frac{1}{2 \pi j b}} e^{\frac{j d u_0^2}{2 b}} e^{\frac{j}{2 b}\left[a x^2+2 x\left(u_0-u\right)-2 u\left(d u_0-b w_0\right)+d u^2\right]} d x \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} h(t) e^{-j \frac{\left(u-u_0\right)}{b} t} d t \\
& =F_1(u) H(u) .
\end{aligned}
$$

Lemma 2.3. *[29] Suppose that the random signals $x(t)$ and $y(t)$ are the input and the output in Figure 1, then*

$$
P_{xy}^A(u) = H(u) P_{xx}^A(u) \tag{2.13}
$$

and

$$
P_{yy}^A(u) = |H(u)|^2 P_{xx}^A(u). \tag{2.14}
$$

3. Nonuniform Sampling and Approximate Recovery

In this section, we will study the nonuniform sampling and reconstruction of random signals which are bandlimited in the SAFT domain.

Definition 3.1. *[29] We say that a random signal $x(t)$ is bandlimited in the SAFT domain if its SAFT power spectral density $P_{xx}^A(u)$ satisfies*

$$
P_{xx}^A(u) = 0, \quad |u| > u_r, \tag{3.1}
$$

where u_r is called the bandwidth of the random signal $x(t)$ in the SAFT domain.

Lemma 3.2. *Assume that a random signal $x(t)$ is bandlimited in the SAFT domain with bandwidth u_r and $\tilde{x}(t) = x(t)e^{j\frac{a}{2b}t^2}$ is stationary. Then $x_1(t)$ is bandlimited in the FT domain with bandwidth $\frac{u_r}{b}$ and the power spectral density satisfies supp$\{P_{x_1x_1}(u)\} \subseteq [-\frac{u_r}{b}, \frac{u_r}{b}]$.*

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Proof Since $x_1(t)$ is stationary, it follows from (2.5) and (2.6) that

$$\begin{aligned} P_{xx}^{A}(u) &= \sqrt{\frac{1}{-j2\pi b}} e^{-j\frac{d}{2b}u^2} e^{-j\frac{d}{2b}u_0^2} e^{j\frac{u}{b}(du_0-bw_0)} F_A\left\{R_{xx}^{A}(\tau)\right\}(u) \\ &= \sqrt{\frac{1}{-j2\pi b}} e^{-j\frac{d}{2b}u^2} e^{-j\frac{d}{2b}u_0^2} e^{j\frac{u}{b}(du_0-bw_0)} F_A\left\{R_{x_1x_1}(\tau)\, e^{-j\frac{a}{2b}\tau^2} e^{-j\frac{u_0}{b}\tau}\right\}(u) \\ &= \sqrt{\frac{1}{-j2\pi b}} e^{-j\frac{d}{2b}(u^2+u_0^2)} e^{j\frac{u}{b}(du_0-bw_0)} \int_{-\infty}^{+\infty} R_{x_1x_1}(\tau)\, e^{-j\frac{a}{2b}\tau^2} e^{-j\frac{u_0}{b}\tau} \\ &\times \sqrt{\frac{1}{2\pi jb}} e^{\frac{jdu_0^2}{2b}} e^{\frac{j}{2b}\left[a\tau^2+2\tau(u_0-u)-2u(du_0-bw_0)+du^2\right]} d\tau \\ &= \frac{1}{2\pi b}\int_{-\infty}^{+\infty} R_{x_1x_1}(\tau)\, e^{-j\frac{u}{b}\tau} d\tau \\ &= \frac{1}{2\pi b} P_{x_1x_1}\left(\frac{u}{b}\right). \end{aligned} \tag{3.2}$$

Note that $P_{xx}^{A}(u) = 0, |u| > u_r$. Then the desired result is proved.

First, we will show that the nonuniform sampling is identical to uniform sampling after a pre-filter in the sense of second order statistic characters.

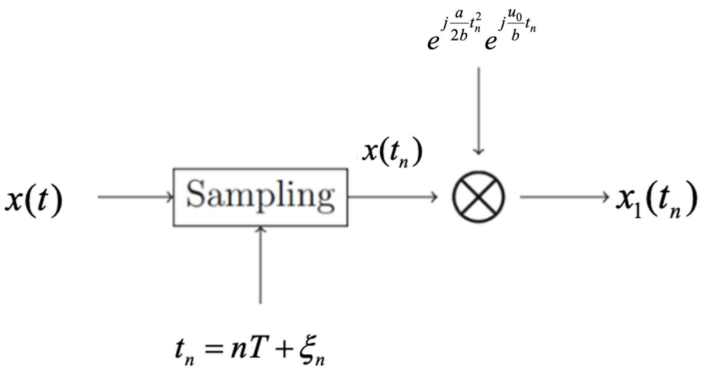

Figure 2. The nonuniform sampling process

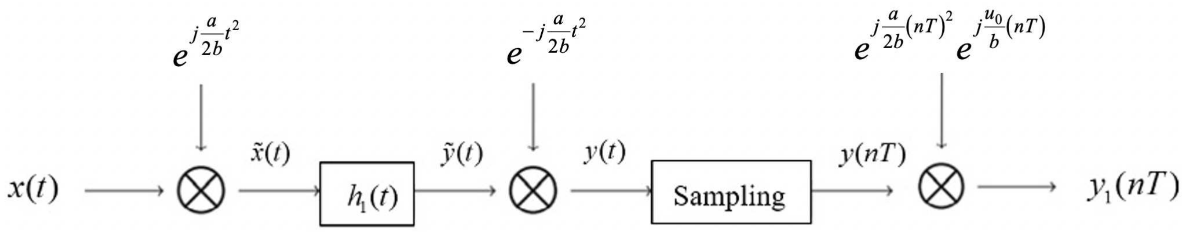

Figure 3. The equivalent system of the nonuniform sampling, where the filtering through filter $h_1(t)$ means that $\tilde{y}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{x}(s) h_1(t-s) ds$

Theorem 3.3. *Suppose that the random signal $x(t)$ is bandlimited in the SAFT domain with bandwidth u_r and $\tilde{x}(t) = x(t)e^{j\frac{a}{2b}t^2}$ is stationary. Then in the sense of second order statistic characters, the nonuniform sampling of $x(t)$ at the sampling points $t_n = nT + \xi_n$ (Figure 2) is identical to the uniform sampling after a SAFT filter $h_1(t)$ as in Figure 3, where $T \le T_N = \frac{\pi b}{u_r}$, $\{\xi_n\}$ is a sequence of independent identically distributed random variables with zero mean in the interval $(-T/2, T/2)$. Moreover,*

$$H_1(u) = \phi_\xi\left(\frac{u}{b}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_1(t) e^{-j\frac{(u-u_0)}{b}t} dt$$

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and $\phi_{\xi}(u)$ denotes the characteristic function of ξ_n.

Proof Note that $y(t) = (x\Theta h_1)(t)$. Then it follows from Lemma 2.3 that

$$P_{yy}^A(u) = |H_1(u)|^2 P_{xx}^A(u). \tag{3.3}$$

Moreover, one has

$$\begin{aligned} y_1(t) &= y(t)e^{j\frac{a}{2b}t^2}e^{j\frac{u_0}{b}t} \\ &= \frac{1}{\sqrt{2\pi}}e^{j\frac{u_0}{b}t}\int_{\mathbb{R}} x(s)h_1(t-s)e^{j\frac{a}{2b}s^2}ds \\ &= \frac{1}{\sqrt{2\pi}}e^{j\frac{u_0}{b}t}\int_{\mathbb{R}} \tilde{x}(s)h_1(t-s)ds. \end{aligned} \tag{3.4}$$

Hence, we have

$$R_{y_1y_1}(t+\tau,t) = \frac{1}{2\pi}e^{j\frac{u_0}{b}\tau}\int_{\mathbb{R}}\int_{\mathbb{R}} h_1^*(s)R_{\tilde{x}\tilde{x}}\left(\tau - s' + s\right)h_1\left(s'\right)ds'ds,$$

which is independent of t and $y_1(t)$ is stationary. It follows from (2.8) and (3.3) that

$$\begin{aligned} R_{yy}^A(kT) &= \int_{-u_r}^{u_r} P_{yy}^A(u)e^{-j\frac{a}{2b}(kT)^2+j\frac{1}{b}(u-u_0)kT}du \\ &= \int_{-u_r}^{u_r} |H_1(u)|^2 P_{xx}^A(u)e^{-j\frac{a}{2b}(kT)^2+j\frac{1}{b}(u-u_0)kT}du. \end{aligned} \tag{3.5}$$

This together with (2.5) obtains

$$\begin{aligned} R_{y_1y_1}(nT,(n-k)T) &= R_{y_1y_1}(kT) \\ &= R_{yy}^A(kT)e^{j\frac{a}{2b}(kT)^2}e^{j\frac{u_0}{b}kT} \\ &= e^{j\frac{a}{2b}(kT)^2}e^{j\frac{u_0}{b}(kT)}\int_{-u_r}^{u_r} |H_1(u)|^2 P_{xx}^A(u)e^{-j\frac{a}{2b}(kT)^2+j\frac{1}{b}(u-u_0)kT}du \\ &= \int_{-u_r}^{u_r} |H_1(u)|^2 P_{xx}^A(u)e^{j\frac{u}{b}kT}du. \end{aligned} \tag{3.6}$$

Combining (2.5) and (2.8), we have

$$\begin{aligned} & E\left[R_{x_1x_1}\left(kT+\xi_n-\xi_{n-k}\right)\right] \\ &= E\left[R_{xx}^A\left(kT+\xi_n-\xi_{n-k}\right)e^{j\frac{a}{2b}(kT+\xi_n-\xi_{n-k})^2}e^{j\frac{u_0}{b}(kT+\xi_n-\xi_{n-k})}\right] \\ &= E\left[\int_{-u_r}^{u_r} P_{xx}^A(u)e^{-j\frac{a}{2b}(kT+\xi_n-\xi_{n-k})^2+j\frac{1}{b}(u-u_0)(kT+\xi_n-\xi_{n-k})}du \cdot e^{j\frac{a}{2b}(kT+\xi_n-\xi_{n-k})^2}e^{j\frac{u_0}{b}(kT+\xi_n-\xi_{n-k})}\right] \\ &= E\left[\int_{-u_r}^{u_r} P_{xx}^A(u)e^{j\frac{u}{b}(kT+\xi_n-\xi_{n-k})}du\right] \\ &= \int_{-u_r}^{u_r} P_{xx}^A(u)e^{j\frac{u}{b}kT}E\left[e^{j\frac{u}{b}(\xi_n-\xi_{n-k})}\right]du. \end{aligned} \tag{3.7}$$

Let $Z = \xi_n - \xi_{n-k}$ and $f_Z(\eta)$ be the probability density function of Z. Note that ξ_n and ξ_{n-k} are independent and have identical distributions. Let $f_{\xi}(\eta)$ be their common probability density function. Then we have

$$f_Z(\eta) = \left[f_{\xi}(\cdot) * f_{\xi}(-\cdot)\right](\eta), \tag{3.8}$$

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where $*$ denotes the convolution operator. Moreover, one has

$$
\begin{aligned}
E\left[e^{j\frac{u}{b}(\xi_n-\xi_{n-k})}\right] &= \int_{-\infty}^{+\infty} f_Z(\eta)e^{j\frac{u}{b}\eta}d\eta \\
&= \int_{-\infty}^{+\infty} \left[f_\xi(\cdot) * f_\xi(-\cdot)\right](\eta)e^{j\frac{u}{b}\eta}d\eta \\
&= \int_{-\infty}^{+\infty} f_\xi(\eta)e^{j\frac{u}{b}\eta}d\eta \cdot \int_{-\infty}^{+\infty} f_\xi(-\eta)e^{j\frac{u}{b}\eta}d\eta \\
&= \left|\phi_\xi\left(\frac{u}{b}\right)\right|^2,
\end{aligned} \tag{3.9}
$$

where

$$\phi_\xi(u) = \int_{-\infty}^{+\infty} f_\xi(\eta)e^{ju\eta}d\eta.$$

Substituting (3.9) into (3.7) obtains

$$E\left[R_{x_1x_1}\left(kT+\xi_n-\xi_{n-k}\right)\right] = \int_{-u_r}^{u_r} \left|\phi_\xi\left(\frac{u}{b}\right)\right|^2 P_{xx}^A(u)e^{j\frac{u}{b}kT}du. \tag{3.10}$$

This together with $H_1(u) = \phi_\xi\left(\frac{u}{b}\right)$ and (3.6) proves the desired result.

In the following, we will give an approximate recovery method for bandlimited signals in the SAFT domain based on randomized nonuniform samples.

Lemma 3.4. *[13] Suppose that the random signal $x(t)$ is bandlimited in the Fourier transform domain with bandwidth $\frac{u_r}{b}$, $\{\xi_n\}$ and $\{\zeta_n\}$ are two sequences of independent identically distributed random variables with zero mean. Then an approximate recovery formula of nonuniform sampling for the random signal $x(t)$ can be represented by*

$$x''(t) = \frac{T}{T_N}\sum_{n=-\infty}^{+\infty} x\left(t_n\right)h_2\left(t-\tilde{t}_n\right), \tag{3.11}$$

where $h_2(t) = sinc\left(\frac{u_rt}{b}\right)$, $sinc(x) \triangleq \frac{\sin x}{x}$, $t_n = nT+\xi_n$ and $\tilde{t}_n = nT+\zeta_n$.

Theorem 3.5. *Suppose that the random signal $x(t)$ is bandlimited in the SAFT domain with bandwidth u_r and $\tilde{x}(t) = x(t)e^{j\frac{a}{2b}t^2}$ is stationary. Then $x(t)$ can be approximated from its nonuniform samples by utilizing the sinc interpolation function as*

$$\hat{x}(t) = \frac{T}{T_N}e^{-j\frac{u_0}{b}t}e^{-j\frac{a}{2b}t^2}\sum_{n=-\infty}^{+\infty} x\left(t_n\right)e^{j\frac{a}{2b}t_n^2}e^{j\frac{u_0}{b}t_n}h_2\left(t-\tilde{t}_n\right), \tag{3.12}$$

where t_n and $\tilde{t}_n$ are as in Lemma 3.4.

Proof It follows from Lemma 3.2 that $x_1(t)$ is bandlimited in the FT domain with bandwidth $\frac{u_r}{b}$. By (3.11), we know that

$$x_1''(t) = \frac{T}{T_N}\sum_{n=-\infty}^{+\infty} x_1(t_n)h_2\left(t-\tilde{t}_n\right) = \frac{T}{T_N}\sum_{n=-\infty}^{+\infty} x\left(t_n\right)e^{j\frac{a}{2b}t_n^2}e^{j\frac{u_0}{b}t_n}h_2\left(t-\tilde{t}_n\right) \tag{3.13}$$

is an approximation of $x_1(t)$. Note that $x(t) = e^{-j\frac{u_0}{b}t}e^{-j\frac{a}{2b}t^2}x_1(t)$. Then $\hat{x}(t)$ in (3.12) is an approximate recovery approach of $x(t)$ and the proof is completed.

From Theorem 3.5, one can see that the approximate recovery approach using the sinc interpolation for a random signal that is bandlimited in the SAFT domain can be expressed in Figure 4.

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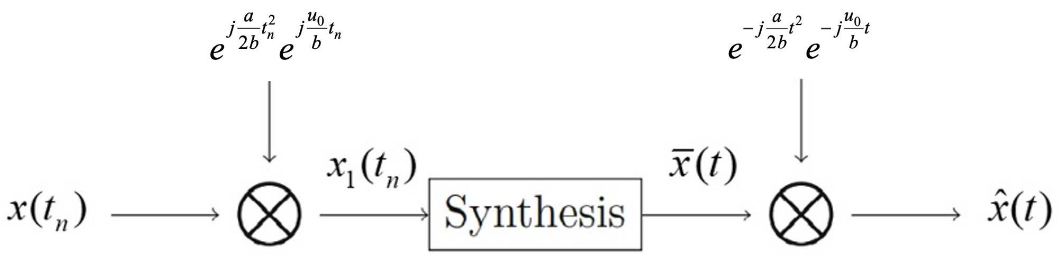

Figure 4. The approximate reconstruction with sinc interpolation function, where $\bar{x}(t) = \frac{T}{T_N}\sum_{n=-\infty}^{+\infty} x_1(t_n)h_2(t-\tilde{t}_n)$

4. Error estimate for Nonuniform Sampling

Since the reconstruction with randomized sinc interpolation is an approximate method, we will estimate the approximation error in this section.

Lemma 4.1. *Let random signals $x_1(t)$ and $y_1(t)$ be the input and output of the FT multiplicative filter as in Figure 5. Then*

$$P_{y_1y_1}(u) = \left|\widehat{h}_3(u)\right|^2 P_{x_1x_1}(u),$$

where $\widehat{h}_3(u)$ is the FT of $h_3(t)$, that is,

$$\widehat{h}_3(u) = \int_{\mathbb{R}} h_3(t)e^{-jut}dt.$$

Proof Note that $y_1(t) = \int_{-\infty}^{+\infty} x_1(t-u)h_3(u)du$. Then

$$R_{y_1x_1}(t+\tau,t) = E\left[y_1(t+\tau)x_1^*(t)\right] = \int_{-\infty}^{+\infty} R_{x_1x_1}(\tau-u)h_3(u)du, \tag{4.1}$$

which is independent of t. Moreover, one has

$$R_{y_1y_1}(t+\tau,t) = E\left[y_1(t+\tau)y_1^*(t)\right] = \int_{-\infty}^{+\infty} R_{y_1x_1}(\tau+u)h_3^*(u)du. \tag{4.2}$$

Taking FT on both sides of (4.1) and (4.2) obtains

$$P_{y_1x_1}(u) = \widehat{h}_3(u)P_{x_1x_1}(u) \tag{4.3}$$

and

$$P_{y_1y_1}(u) = \widehat{h}_3^*(u)P_{y_1x_1}(u). \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$P_{y_1y_1}(u) = \left|\widehat{h}_3(u)\right|^2 P_{x_1x_1}(u).$$

Theorem 4.2. *Suppose that the random signal $x(t)$ is bandlimited in the SAFT domain with bandwidth u_r and $\tilde{x}(t) = x(t)e^{j\frac{a}{2b}t^2}$ is stationary. Let $v(t)$ be an additive noise with zero mean, which is stationary, uncorrelated with $x(t)$ and has the power spectral density*

$$P_{vv}(u) = T\int_{-u_r}^{u_r} P_{xx}^A(u_1)\left[1-\left|\phi_{\xi\zeta}\left(\frac{u_1}{b},-u\right)\right|^2\right]du_1,\ |u| \le \frac{u_r}{b}, \tag{4.5}$$

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where $\phi_{\xi\zeta}(s,t)$ is the joint characteristic function of the random variables ξ_n and ζ_n. If $\phi_{\xi\zeta}(u,-u)$ is the frequency response of the filter $h_3(t)$, then the model described in Figure 5 is identical to the procedure represented in Figure 4 in the sense of second order statistic characters. Moreover, we have

$$E\left[|\hat{x}(t)-x(t)|^2\right] = \int_{-u_r}^{u_r} P_{xx}^A(u)\left|1-\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u}{b}\right)\right|^2 du + \frac{T}{2\pi b}\int_{-u_r}^{u_r} P_{xx}^A(u)\int_{-u_r}^{u_r}\left[1-\left|\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u_1}{b}\right)\right|^2\right] du_1 du.$$

Proof It follows from Theorem 3.5 that

$$\bar{x}(t) = \hat{x}(t)e^{j\frac{u_0}{b}t}e^{j\frac{a}{2b}t^2} = \frac{T}{T_N}\sum_{n=-\infty}^{+\infty} x(t_n)e^{j\frac{a}{2b}t_n^2}e^{\frac{ju_0 t_n}{b}}h_2(t-\tilde{t}_n). \tag{4.6}$$

Then one has

$$\begin{aligned}R_{\bar{x}\bar{x}}(t,t-\tau) &= \left(\frac{T}{T_N}\right)^2 E\left[\left(\sum_{n=-\infty}^{+\infty} x(nT+\xi_n)e^{j\frac{a}{2b}(nT+\xi_n)^2}e^{\frac{ju_0(nT+\xi_n)}{b}}h_2(t-nT-\zeta_n)\right)\cdot\right.\\ &\left.\left(\sum_{k=-\infty}^{+\infty} x^*(kT+\xi_k)e^{-j\frac{a}{2b}(kT+\xi_k)^2}e^{\frac{-ju_0(kT+\xi_k)}{b}}h_2^*(t-\tau-kT-\zeta_k)\right)\right]\\ &= \left(\frac{T}{T_N}\right)^2 \sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} E\Big[R_{x_1x_1}(nT-kT+\xi_n-\xi_k)h_2(t-nT-\zeta_n)h_2^*(t-\tau-kT-\zeta_k)\Big].\end{aligned}$$

Moreover, it can be represented by two terms as

$$\begin{aligned}R_{\bar{x}\bar{x}}(t,t-\tau) &= \left(\frac{T}{T_N}\right)^2 R_{x_1x_1}(0)\sum_{n=-\infty}^{+\infty} E\Big[h_2(t-nT-\zeta_n)h_2^*(t-\tau-nT-\zeta_n)\Big]\\ &+ \left(\frac{T}{T_N}\right)^2 \sum_{n\neq k} E\Big[R_{x_1x_1}(nT-kT+\xi_n-\xi_k)h_2(t-nT-\zeta_n)h_2^*(t-\tau-kT-\zeta_k)\Big]\\ &\triangleq I+II.\end{aligned} \tag{4.7}$$

Note that $\sum_n e^{j(u_2-u_1)nT} = 2\pi\sum_k \delta\big((u_2-u_1)T-2\pi k\big)$ and

$$h_2(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} H_2(u)e^{j\frac{u}{b}t}du = \frac{b}{\sqrt{2\pi}}\int_{\mathbb{R}} H_2(ub)e^{jut}du. \tag{4.8}$$

These together with the fact that $H_2(u) = \frac{\pi}{\sqrt{2\pi}u_r}\chi_{[-u_r,u_r]}(u)$ show that

$$\begin{aligned}I &= \frac{1}{2\pi}\left(\frac{Tb}{T_N}\right)^2 R_{x_1x_1}(0)\int_{\mathbb{R}}\int_{\mathbb{R}} H_2(bu_1)H_2^*(bu_2)e^{j(u_1-u_2)t}e^{ju_2\tau}\sum_{n=-\infty}^{+\infty} e^{j(u_2-u_1)nT}E\left[e^{j(u_2-u_1)\zeta_n}\right]du_1du_2\\ &= \left(\frac{Tb}{T_N}\right)^2 R_{x_1x_1}(0)\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}}\frac{1}{T}|H_2(bu)|^2 e^{ju\tau}du\\ &= T\left(\frac{b}{T_N}\right)^2\frac{1}{2\pi}\left[\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)du_1\right]\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}}|H_2(bu)|^2 e^{ju\tau}du\\ &= \frac{T}{4\pi^2}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} e^{ju\tau}\left[\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)du_1\right]du.\end{aligned} \tag{4.9}$$

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Moreover, we have

$$
\begin{aligned}
II =& \left(\frac{bT}{2\pi T_N}\right)^2 \sum_{n\neq k} E\left[\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u)e^{ju(nT-kT+\xi_n-\xi_k)}du \int_{\mathbb{R}} H_2(bu_1)e^{ju_1(t-nT-\zeta_n)}du_1 \cdot \right.\\
& \left.\int_{\mathbb{R}} H_2^*(bu_2)e^{-ju_2(t-\tau-kT-\zeta_k)}du_2\right]\\
=& \left(\frac{bT}{2\pi T_N}\right)^2 \sum_{n\neq k} \int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} \int_{\mathbb{R}}\int_{\mathbb{R}} P_{x_1x_1}(u)H_2(bu_1)H_2^*(bu_2)e^{ju_2\tau}e^{j(u_1-u_2)t}e^{j(u-u_1)nT}e^{-j(u-u_2)kT}\\
& \cdot E\left[e^{ju\xi_n}e^{-ju\xi_k}e^{-ju_1\zeta_n}e^{ju_2\zeta_k}\right] du_1du_2du\\
=& \left(\frac{bT}{2\pi T_N}\right)^2 \int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} \int_{\mathbb{R}}\int_{\mathbb{R}} P_{x_1x_1}(u)H_2(bu_1)H_2^*(bu_2)\phi_{\xi\zeta}(u,-u_1)\,\phi_{\xi\zeta}^*(u,-u_2)\,e^{ju_2\tau}e^{j(u_1-u_2)t}\\
& \cdot \Big(\sum_n e^{j(u-u_1)nT}\Big)\Big(\sum_k e^{-j(u-u_2)kT}\Big)du_1du_2du - \left(\frac{bT}{2\pi T_N}\right)^2 \int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} \int_{\mathbb{R}}\int_{\mathbb{R}} P_{x_1x_1}(u)H_2(bu_1)H_2^*(bu_2)\\
& \cdot \phi_{\xi\zeta}(u,-u_1)\,\phi_{\xi\zeta}^*(u,-u_2)\,e^{ju_2\tau}e^{j(u_1-u_2)t}\Big(\sum_n e^{j(u_2-u_1)nT}\Big)du_1du_2du\\
=& \left(\frac{b}{T_N}\right)^2 \int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u)\left|\phi_{\xi\zeta}(u,-u)\right|^2|H_2(bu)|^2e^{ju\tau}du-\\
& \frac{T}{2\pi}\left(\frac{b}{T_N}\right)^2 \int_{-\frac{u_r}{b}}^{\frac{u_r}{b}}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)\left|\phi_{\xi\zeta}(u_1,-u)\right|^2|H_2(bu)|^2e^{ju\tau}du_1du\\
=& \left(\frac{b}{T_N}\right)^2 \int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} |H_2(bu)|^2e^{ju\tau}\left[P_{x_1x_1}(u)|\phi_{\xi\zeta}(u,-u)|^2 - \frac{T}{2\pi}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)|\phi_{\xi\zeta}(u_1,-u)|^2du_1\right]\mathrm{du}\\
=& \frac{1}{2\pi}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} e^{ju\tau}\left[P_{x_1x_1}(u)|\phi_{\xi\zeta}(u,-u)|^2 - \frac{T}{2\pi}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)|\phi_{\xi\zeta}(u_1,-u)|^2du_1\right]du.
\end{aligned}
\tag{4.10}
$$

Substituting (4.9) and (4.10) into (4.7) obtains

$$
\begin{aligned}
R_{\bar{x}\bar{x}}(t,t-\tau) =& \frac{T}{4\pi^2}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}}\left(\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)\left[1-\left|\phi_{\xi\zeta}(u_1,-u)\right|^2\right]du_1\right)e^{ju\tau}du\\
&+\frac{1}{2\pi}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} e^{ju\tau}P_{x_1x_1}(u)\left|\phi_{\xi\zeta}(u,-u)\right|^2du.
\end{aligned}
\tag{4.11}
$$

Similarly, we can obtain

$$
R_{\bar{x}x_1}(t,t-\tau) = \frac{1}{2\pi}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u)e^{ju\tau}\phi_{\xi\zeta}(u,-u)du. \tag{4.12}
$$

Therefore, we have

$$
P_{\bar{x}\bar{x}}(u) = P_{x_1x_1}(u)\left|\phi_{\xi\zeta}(u,-u)\right|^2 + \frac{T}{2\pi}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)\left[1-\left|\phi_{\xi\zeta}(u_1,-u)\right|^2\right]du_1 \tag{4.13}
$$

and

$$
P_{\bar{x}x_1}(u) = P_{x_1x_1}(u)\phi_{\xi\zeta}(u,-u). \tag{4.14}
$$

It follows from Lemma 4.1 that the first term $P_{x_1x_1}(u)\left|\phi_{\xi\zeta}(u,-u)\right|^2$ in (4.13) is the FT power spectral density of $y_1(t)$ in Figure 5. Furthermore, since $\bar{x}(t) = y_1(t) + v(t)$ and $v(t)$ is uncorrelated with $x(t)$, then

$$
\begin{aligned}
R_{\bar{x}\bar{x}}(t+\tau,t) &= E\left[(y_1(t+\tau)+v(t+\tau))\,(y_1(t)+v(t))^*\right]\\
&= R_{y_1y_1}(t+\tau,t) + R_{y_1v}(t+\tau,t) + R_{vy_1}(t+\tau,t) + R_{vv}(t+\tau,t)\\
&= R_{y_1y_1}(t+\tau,t) + R_{vv}(t+\tau,t).
\end{aligned}
$$

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Moreover, one has

$$P_{\tilde{x}\tilde{x}}(u) = P_{y_1y_1}(u) + P_{vv}(u),$$

which shows that the second term in (4.13) is just the power spectral density of $v(t)$, that is,

$$\begin{aligned} P_{vv}(u) &= \frac{T}{2\pi}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)\left[1 - \left|\phi_{\xi\zeta}(u_1, -u)\right|^2\right] du_1 \\ &= T\int_{-u_r}^{u_r} P^A_{xx}(u_1)\left[1 - \left|\phi_{\xi\zeta}\left(\frac{u_1}{b}, -u\right)\right|^2\right] du_1,\ |u| \le \frac{u_r}{b}. \end{aligned}$$

Therefore, the model described in Figure 5 is identical to the procedure represented in Figure 4 in the sense of second order statistic characters.

Next, we will estimate the error $E\left[|\hat{x}(t) - x(t)|^2\right]$. Let $\varepsilon(t) = \hat{x}(t) - x(t)$. Combining (3.2) and (4.13), we get

$$\begin{aligned} P^A_{\hat{x}\hat{x}}(u) &= \frac{1}{2\pi b}P_{\tilde{x}\tilde{x}}(\frac{u}{b}) \\ &= \frac{1}{2\pi b}P_{x_1x_1}(\frac{u}{b})\left|\phi_{\xi\zeta}\left(\frac{u}{b}, -\frac{u}{b}\right)\right|^2 + \frac{T}{4\pi^2 b}\int_{-\frac{u_r}{b}}^{\frac{u_r}{b}} P_{x_1x_1}(u_1)\left[1 - \left|\phi_{\xi\zeta}\left(u_1, -\frac{u}{b}\right)\right|^2\right] du_1 \\ &= P^A_{xx}(u)\left|\phi_{\xi\zeta}\left(\frac{u}{b}, -\frac{u}{b}\right)\right|^2 + \frac{T}{2\pi b}\int_{-u_r}^{u_r} P^A_{xx}(u_1)\left[1 - \left|\phi_{\xi\zeta}\left(\frac{u_1}{b}, -\frac{u}{b}\right)\right|^2\right] du_1. \end{aligned} \tag{4.15}$$

Similarly, we can obtain

$$P^A_{\hat{x}x}(u) = P^A_{xx}(u)\,\phi_{\xi\zeta}\left(\frac{u}{b}, -\frac{u}{b}\right). \tag{4.16}$$

In fact, it is easy to see that

$$\begin{aligned} R_{\tilde{x}x_1}(t+\tau, t) &= E\left[\hat{x}(t+\tau)e^{j\frac{u_0}{b}(t+\tau)}e^{j\frac{a}{2b}(t+\tau)^2}x^*(t)e^{-j\frac{u_0}{b}t}e^{-j\frac{a}{2b}t^2}\right] \\ &= R_{\hat{x}x}(t+\tau, t)\,e^{j\frac{u_0}{b}\tau}e^{j\frac{a}{b}t\tau}e^{j\frac{a}{2b}\tau^2}. \end{aligned}$$

Therefore, $R_{\hat{x}x}(t+\tau, t)\,e^{j\frac{a}{b}t\tau}$ is independent of t due to (4.12). Then

$$\begin{aligned} R^A_{\hat{x}x}(\tau) &= R_{\hat{x}x}(t+\tau, t)\,e^{j\frac{a}{b}t\tau} \\ &= R_{\tilde{x}x_1}(t+\tau, t)\,e^{-j\frac{u_0}{b}\tau}e^{-j\frac{a}{2b}\tau^2}. \end{aligned}$$

Moreover, it follows from (2.7) that

$$\begin{aligned} P^A_{\hat{x}x}(u) &= \sqrt{\frac{1}{-j2\pi b}}e^{-j\frac{d}{2b}u^2}e^{-j\frac{d}{2b}u_0^2}e^{j\frac{u}{b}(du_0 - bw_0)}F_A\left\{R_{\tilde{x}x_1}(\tau)\,e^{-j\frac{u_0}{b}\tau}e^{-j\frac{a}{2b}\tau^2}\right\}(u) \\ &= \sqrt{\frac{1}{-j2\pi b}}e^{-j\frac{d}{2b}(u^2+u_0^2)}e^{j\frac{u}{b}(du_0 - bw_0)}\int_{-\infty}^{+\infty} R_{\tilde{x}x_1}(\tau)\,e^{-j\frac{u_0}{b}\tau}e^{-j\frac{a}{2b}\tau^2}\sqrt{\frac{1}{2\pi jb}}e^{\frac{jdu_0^2}{2b}} \\ &\quad\cdot e^{\frac{j}{2b}\left[a\tau^2 + 2\tau(u_0 - u) - 2u(du_0 - bw_0) + du^2\right]}d\tau \\ &= \frac{1}{2\pi b}\int_{-\infty}^{+\infty} R_{\tilde{x}x_1}(\tau)e^{-j\frac{u}{b}\tau}d\tau \\ &= \frac{1}{2\pi b}P_{\tilde{x}x_1}\left(\frac{u}{b}\right) \\ &= \frac{1}{2\pi b}P_{x_1x_1}\left(\frac{u}{b}\right)\phi_{\xi\zeta}\left(\frac{u}{b}, -\frac{u}{b}\right) \\ &= P^A_{xx}(u)\,\phi_{\xi\zeta}\left(\frac{u}{b}, -\frac{u}{b}\right). \end{aligned} \tag{4.17}$$

Hence, the SAFT auto-power spectral density of the reconstruction error $\varepsilon(t)$ is

$$\begin{aligned}
P^A_{\varepsilon\varepsilon}(u) &= P^A_{\hat{x}\hat{x}}(u) - P^A_{\hat{x}x}(u) - P^A_{x\hat{x}}(u) + P^A_{xx}(u) \\
&= P^A_{xx}(u)\left|\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u}{b}\right)\right|^2 + \frac{T}{2\pi b}\int_{-u_r}^{u_r} P^A_{xx}(u_1)\left[1-\left|\phi_{\xi\zeta}\left(\frac{u_1}{b},-\frac{u}{b}\right)\right|^2\right]du_1 \\
&\quad - P^A_{xx}(u)\,\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u}{b}\right) - \left[P^A_{xx}(u)\,\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u}{b}\right)\right]^* + P^A_{xx}(u) \\
&= P^A_{xx}(u)\left|1-\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u}{b}\right)\right|^2 + \frac{T}{2\pi b}\int_{-u_r}^{u_r} P^A_{xx}(u_1)\left[1-\left|\phi_{\xi\zeta}\left(\frac{u_1}{b},-\frac{u}{b}\right)\right|^2\right]du_1, \qquad (4.18)
\end{aligned}$$

where we have used the fact that $P^A_{xx}(u)$ is real due to (3.2). Note that

$$\varepsilon_1(t) = \varepsilon(t)e^{j\frac{u_0}{b}t}e^{j\frac{a}{2b}t^2} = (\hat{x}(t)-x(t))e^{j\frac{u_0}{b}t}e^{j\frac{a}{2b}t^2} = \bar{x}(t) - x_1(t).$$

Then $\varepsilon_1(t)$ is stationary. Moreover, it follows from (2.5) and (2.8) that

$$\begin{aligned}
&E\left[|\varepsilon(t)|^2\right] = R_{\varepsilon_1\varepsilon_1}(0) = R^A_{\varepsilon\varepsilon}(0) = \int_{-u_r}^{u_r} P^A_{\varepsilon\varepsilon}(u)\,du \\
&= \int_{-u_r}^{u_r} P^A_{xx}(u)\left|1-\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u}{b}\right)\right|^2 du + \frac{T}{2\pi b}\int_{-u_r}^{u_r} P^A_{xx}(u)\int_{-u_r}^{u_r}\left[1-\left|\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u_1}{b}\right)\right|^2\right]du_1du.
\end{aligned}$$

This completes the proof.

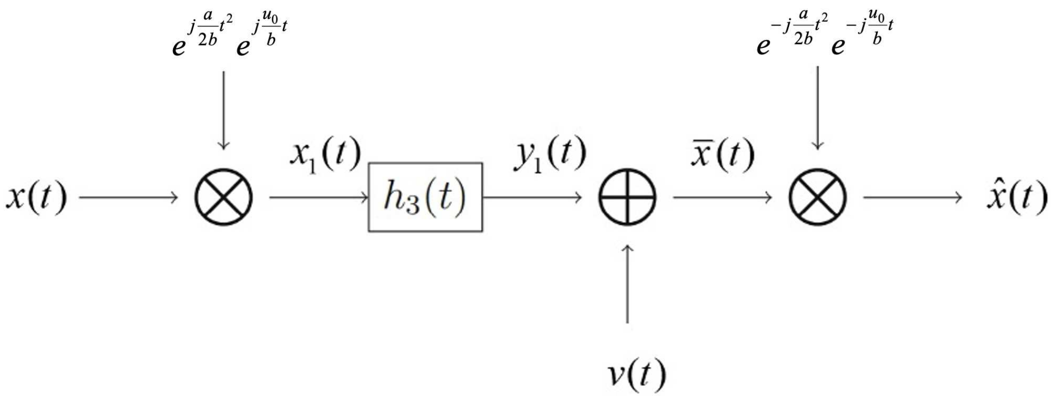

Figure 5. An equivalent nonuniform sampling and reconstruction system to Figure 4, where $v(t)$ is an additive noise with zero mean which is uncorrelated with $x(t)$ and has power spectral density as (4.5)

Remark 4.3. *If ξ_n and ζ_n equal to zero, then the nonuniform sampling studied in this paper reduces to the uniform sampling. In such case, $\phi_{\xi\zeta}(s,t) \equiv 1$. Then it follows from Theorem 4.2 that*

$$\begin{aligned}
E\left[|\hat{x}(t)-x(t)|^2\right] &= \int_{-u_r}^{u_r} P^A_{xx}(u)\left|1-\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u}{b}\right)\right|^2 du \\
&\quad + \frac{T}{2\pi b}\int_{-u_r}^{u_r} P^A_{xx}(u)\int_{-u_r}^{u_r}\left[1-\left|\phi_{\xi\zeta}\left(\frac{u}{b},-\frac{u_1}{b}\right)\right|^2\right]du_1du \\
&= 0.
\end{aligned}$$

That is to say, $x(t)$ is equal to its approximation $\hat{x}(t)$ in the mean square sense. From Theorem 3.5, one can see that for $T = T_N = \frac{\pi b}{u_r}$, the approximation of $x(t)$ obtained in (3.12) becomes

$$\hat{x}(t) = e^{-j\frac{a}{2b}t^2}\sum_{n=-\infty}^{+\infty} x(nT)e^{j\frac{a}{2b}(nT)^2}e^{j\frac{u_0}{b}(nT-t)}sinc\left(\frac{u_r(t-nT)}{b}\right), \qquad (4.19)$$

which coincides with Theorem 3 in [29]. Therefore, the result of uniform sampling proposed in [29] is a special case of Theorems 3.5 and 4.2 in this paper.

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