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Article

A Geometric and Visual Perspective on the Four Color Map Theorem and K5 Non-Planarity and Their Connection

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Abstract: This paper presents a unique approach to the Four Color and K5 non planarity and draws a conclusion that both are basically the same problem and their connection goes to geometry of interaction properties of a group of circles(Kissing number problem). Using a geometric representation of planar graphs with circles and tangents. We explore the implications of the kissing number problem in two dimensions to provide a new perspective on the theorem.

Keywords: geometry; graph theory; four color map theorem; k5 non-planarity; kissing number problem; planar graphs; visual representation; contradiction proofs; coloring; mathematical connections

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Introduction

In this paper, I have depicted a visual and geometric proof of the Four Color Map Theorem and K_5 non-planarity, and shown how both theorems are connected. They fundamentally represent the same constraint — a geometric limitation that arises from circle interactions in the plane.

This novel approach sheds new light on these mathematical problems, making them more accessible and intuitive. We demonstrate that their common ground lies in the kissing number in two dimensions, a classical geometric limit stating that no more than four equal-sized circles can be simultaneously tangent to a given circle in $\mathbb{R}2$.

We build upon the Koebe–Andreev–Thurston Circle Packing Theorem, which states that every planar graph can be represented as a circle packing — a configuration where vertices map to circles and adjacencies map to tangencies. This bridges graph theory with geometry directly.

Furthermore, by invoking Wagner's Theorem, we connect high chromatic number claims with the necessity of a K_5 minor — which we then demonstrate cannot exist in planar circle packings due to the kissing number constraint.

We also acknowledge Descartes' Circle Theorem, used in the appendix, to illustrate precise mutual tangency configurations among four circles — reinforcing the geometric feasibility of K_4 but impossibility of K_5.

Methods

Our method uses geometric representations of graphs, specifically mapping each vertex to a circle in 2D space, and each edge to a tangency between circles.

1. Circle Packing Framework

   By applying the Koebe–Andreev–Thurston Circle Packing Theorem, we ensure that any planar graph can be visualized as a collection of tangent circles. This forms the geometric stage for analyzing chromatic bounds.

2. Tangency and Chromatic Bound via Kissing Number

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The kissing number in ℝ2 establishes that at most four circles can be mutually tangent — directly limiting the possible degrees of adjacency in a planar embedding. This acts as a hard upper bound on the chromatic number of any planar graph.

3. Obstruction from K_5 via Wagner's Theorem

We utilize Wagner's Theorem to justify that any planar graph requiring more than four colors must contain a K_5 minor. We then show such a K_5 cannot be embedded due to the tangency limitation from the kissing number.

4. Validation through Descartes' Circle Theorem

In the appendix, we show that for three mutually tangent circles, exactly two more circles (inner and outer Soddy circles) can be constructed. This precisely models K_4 but mathematically excludes a configuration equivalent to K_5.

Results

From our constructions and proofs, we derive:

- Theorem 1 (Four Color Theorem via Circle Packing):

  Any planar graph G has chromatic number $\chi(G) \leq 4$, by showing that a 5-chromatic configuration requires 5 mutually tangent circles, violating the 2D kissing number.

- Corollary 1 (Non-Planarity of K_5):

  K_5 cannot be embedded in ℝ2 using mutually tangent circles due to geometric infeasibility, and it also violates Euler's formula for planar graphs.

- Lemma (Chromatic-Kissing Bound):

  $\chi(G) \leq$ maximum kissing number in circle packing P
  $\Rightarrow \chi(G) \leq 4$

These results are obtained by synthesizing graph minor theory (Wagner's Theorem), circle packing (Koebe–Andreev–Thurston), and geometric tangency limits (kissing number and Descartes' Theorem).

Discussion

This work reframes two of graph theory's most well-known results — the Four Color Theorem and K_5 non-planarity — not as separate facts but as manifestations of a common geometric limitation.

Our main insight is that the 2D kissing number implicitly governs the allowable structure of planar graphs. While classical approaches focus on colorings and embeddings, our model visually and mathematically shows why you can't exceed four mutually adjacent regions: five mutually tangent circles simply don't fit in the plane.

Using the Koebe–Andreev–Thurston theorem, we transformed abstract graph problems into visual geometry. Using Wagner's Theorem, we tied chromatic number to forbidden minors. And with Descartes' Circle Theorem, we grounded K_4 as the maximal mutually tangent structure in 2D.

This makes the entire framework not only non-computational, but also teachable, intuitive, and provably minimal.

Key Theorem: Descartes' Circle Theorem (1643)

Given three mutually tangent circles (with curvatures), there are exactly two possible fourth circles that can be tangent to all three:

- A **small inner circle** (solution with) that fits snugly in the gap.This is also referred to as the **internal Soddy circle(C4)**

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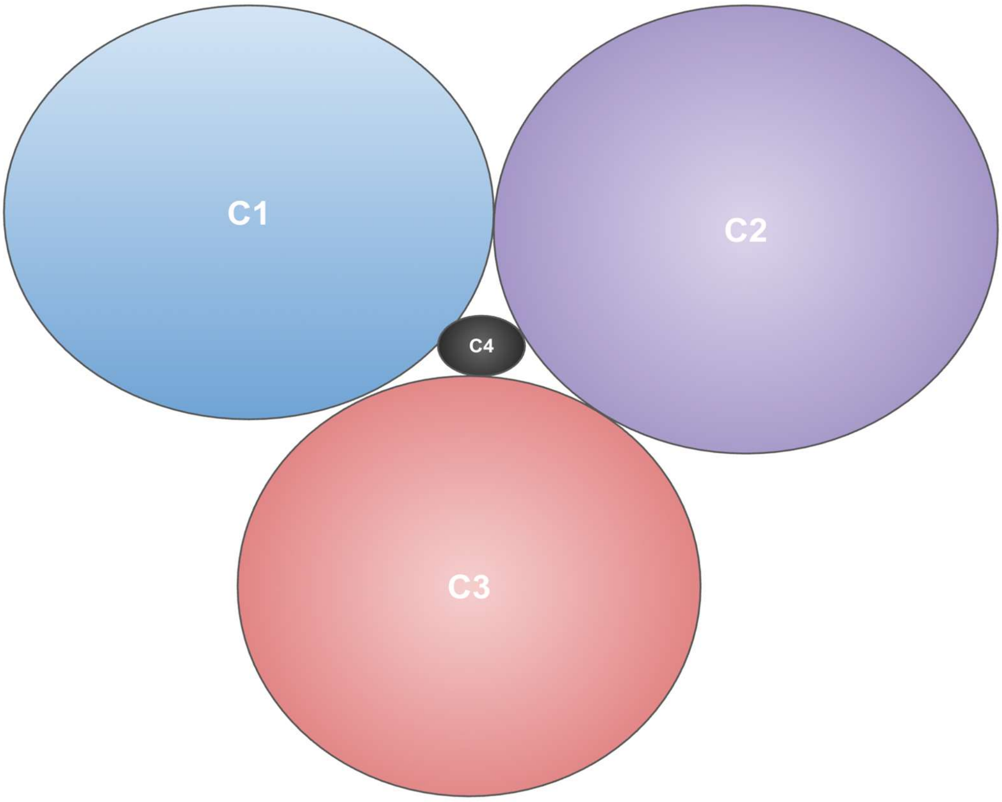
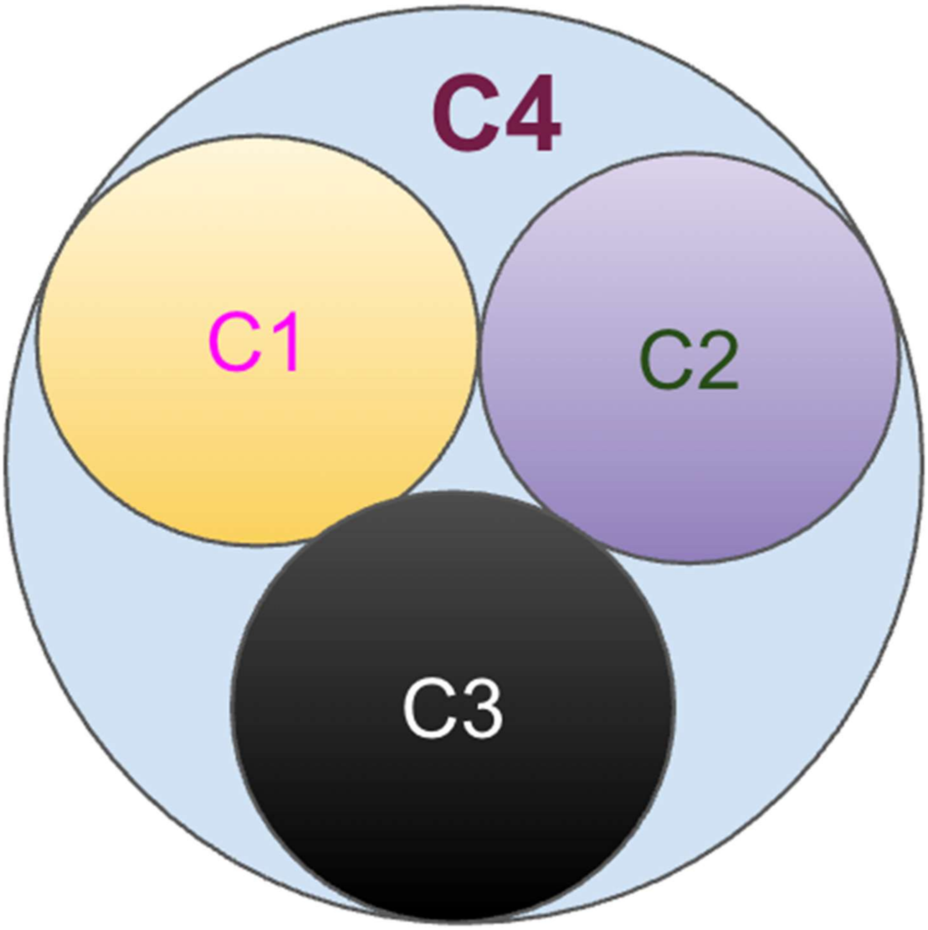


- A **larger outer enclosing circle** (solution with) that wraps around the three. This is also referred to as the external Soddy circle(C4)



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Given three mutually tangent circles with curvatures k_1, k_2, k_3 , there are exactly two possible fourth circles that can be tangent to all three:

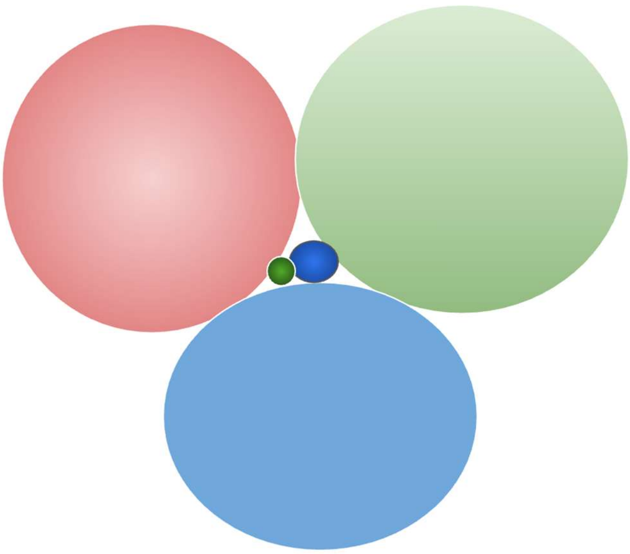
$$(k_1 + k_2 + k_3 + k4)2 = 2(k_12 + k_22 + k_32 + k42)$$

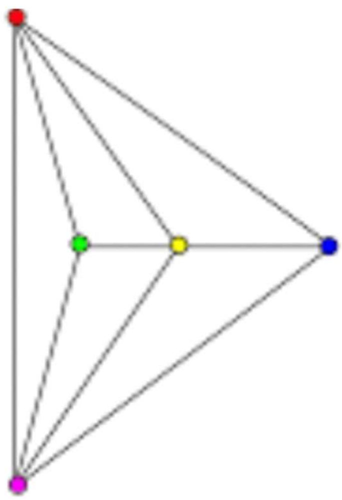
This result confirms that for any trio of mutually tangent circles, only two distinct fourth circles can satisfy full tangency — a fact elegantly visualized by the "inner gap method" and the "outer encasing method" in our K4 constructions.

** The geometric representation introduced in this paper aligns with the principles of circle packing. Each vertex as a circle, and each edge as a tangent, mirrors the circle packing representations used in the proof of the Circle Packing Theorem. The Koebe–Andreev–Thurston Circle Packing Theorem asserts that every planar graph can be represented as a packing of circles in the plane, where tangents between circles correspond to graph edges. This further supports the equivalence of geometric representations and combinatorial colorability.

Historically, Paul Koebe introduced the foundational result in 1936. Later, William Thurston in the 1980s revived and extended the implications of this theorem, connecting it to conformal mappings and the Four Color Theorem itself. Thus, this paper's visual model is naturally grounded in circle packing theory.

Connection to Circle Packing : The geometric representation introduced in this paper aligns with the principles of circle packing. Each vertex as a circle, and each edge as a tangent, mirrors the circle packing representations used in the proof of the Circle Packing Theorem. The Koebe–Andreev–Thurston Circle Packing Theorem asserts that every planar graph can be represented as a packing of circles in the plane, where tangents between circles correspond to graph edges. This further supports the equivalence of geometric representations and combinatorial colorability.





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1. One Vertex N1 or K1:

N1(Represents C1 as circle**)**

If we represent it as circle:

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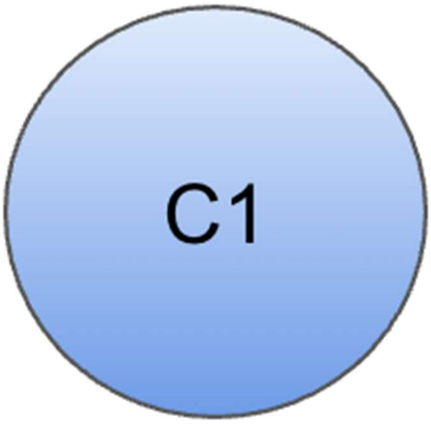
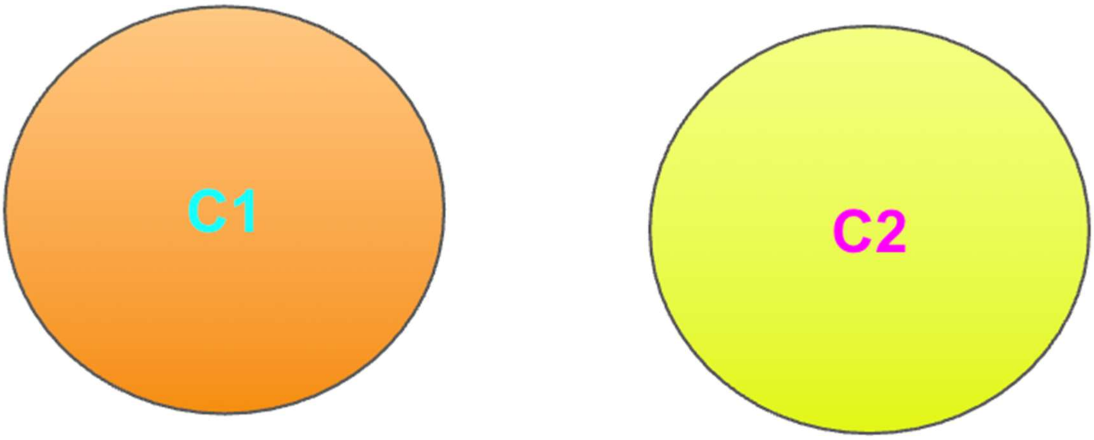


2. Two Vertex isolated N1 and N2:

(N1 and N2 represent C1 and C2**)**
If we represent it as circle:



2.1. Two Vertices Adjacent to Each Other K2(One Tangent)

N1  N2

If we represent it as circle(N1 as C1 and N2 as C2):

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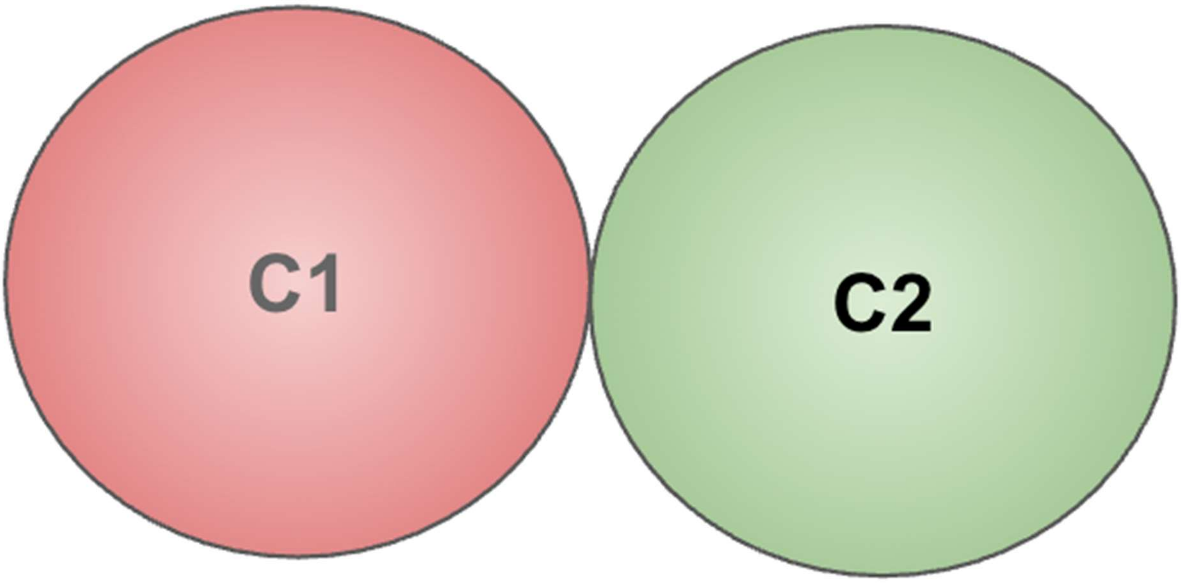

3.1. Three Vertices Adjacent to Each Other K3

N1

N2 N3

If we represent it as circle(N1 as C1, N2 as C2 and N3 as C3, three tangents):

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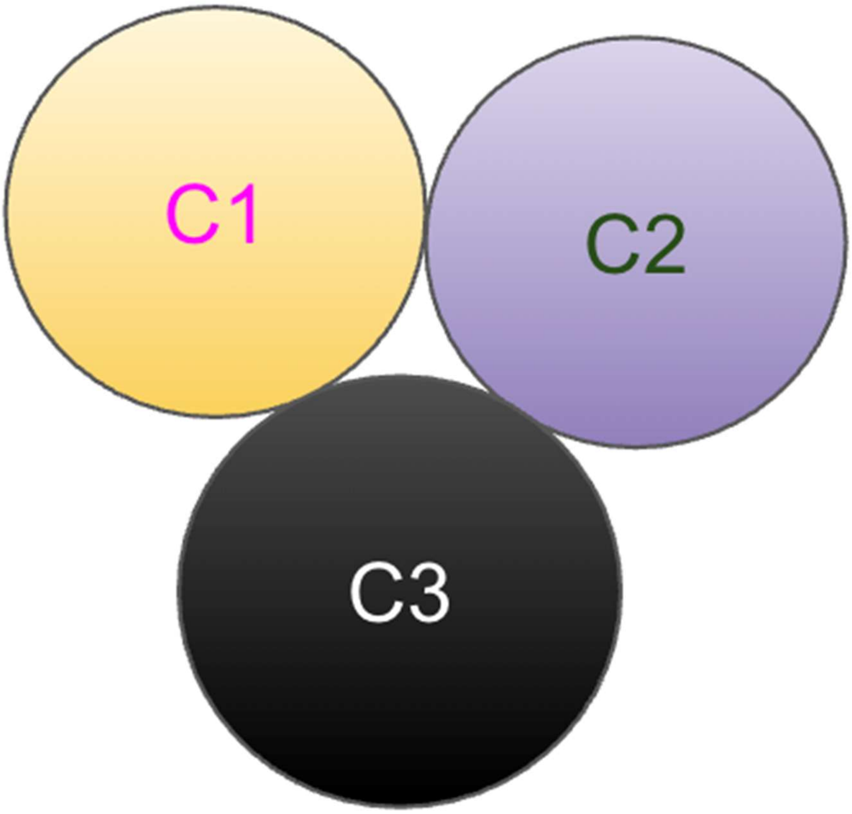

3.2. Also Another Variety of 3 Vertices Where Only Middle Vertex Connect Two Other Two Vertices

If we represent it as circle(two tangents):

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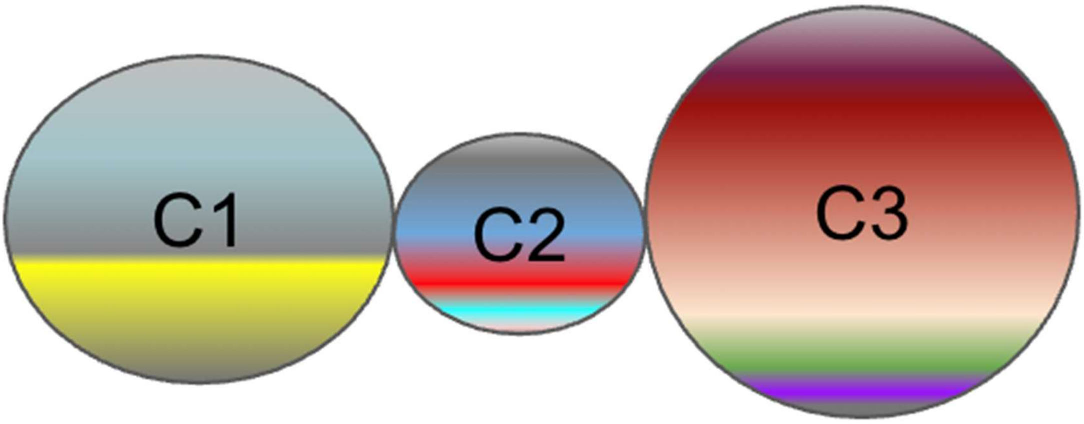
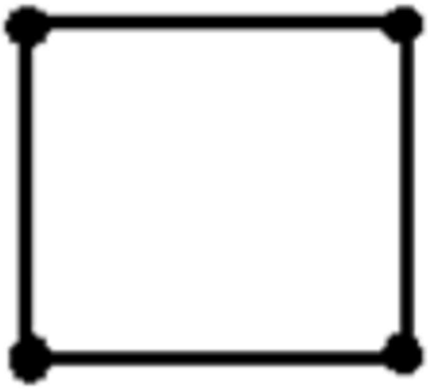
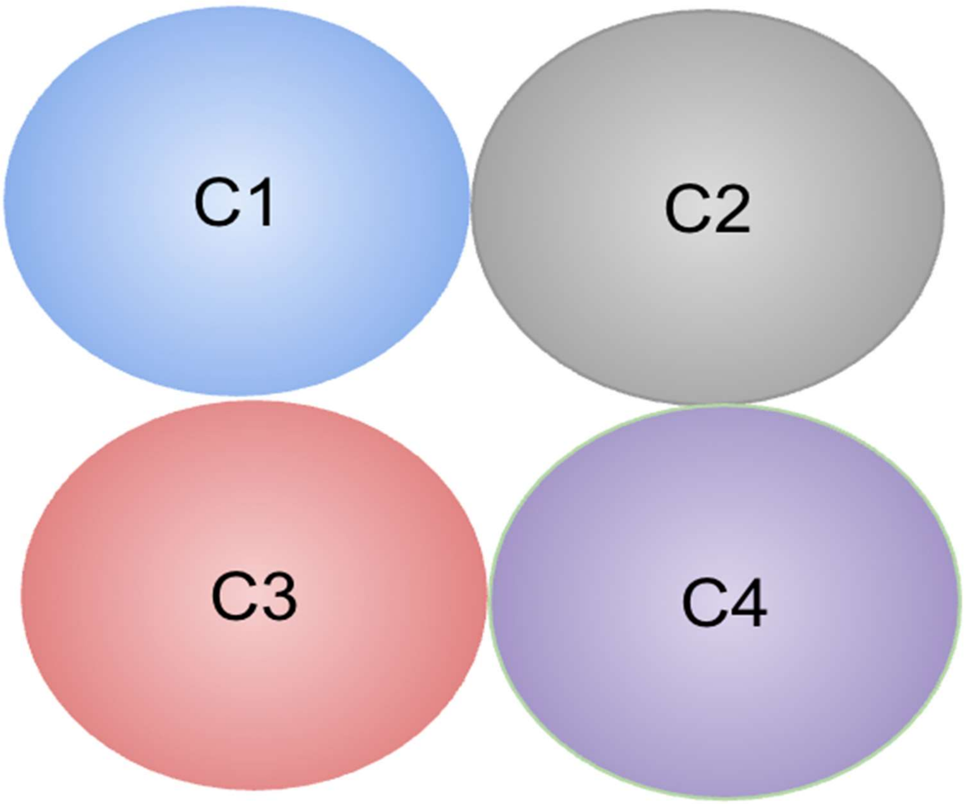


4.1. Four Vertices in a Simple Rectangular Arrangement



If we represent it as circle:



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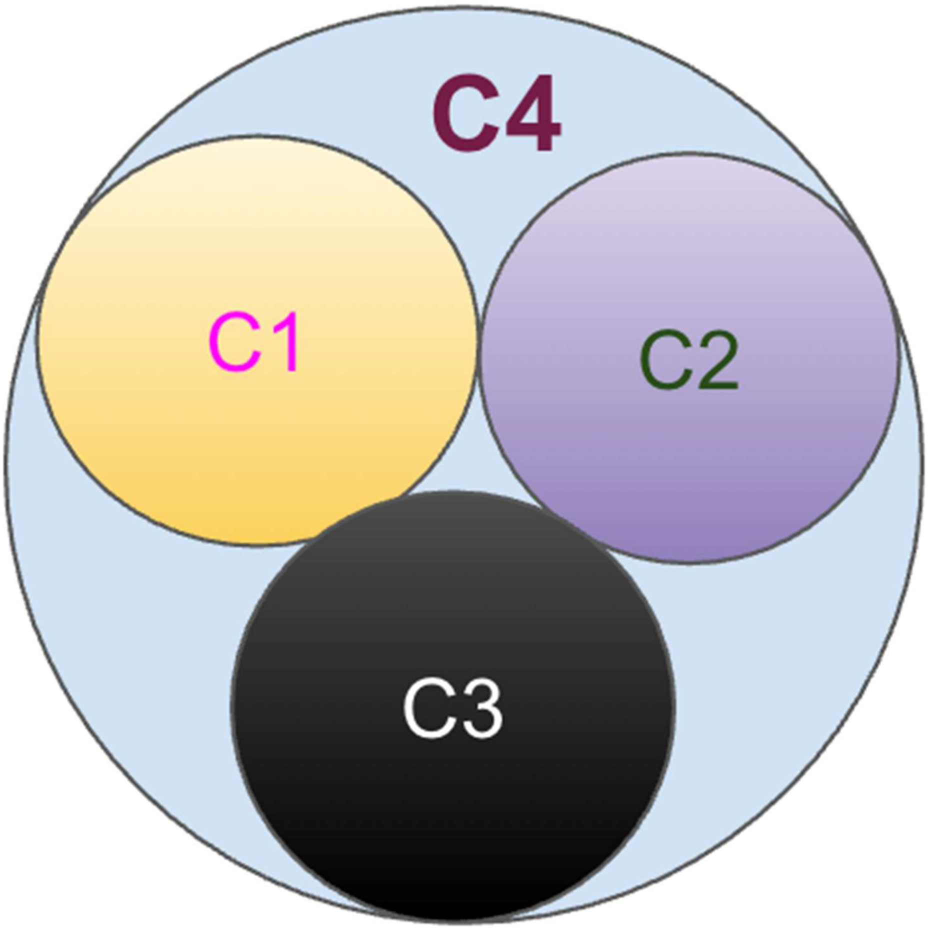
4.2. Four Vertices in K4 Rectangular Arrangement

If we represent it as circle:



OR

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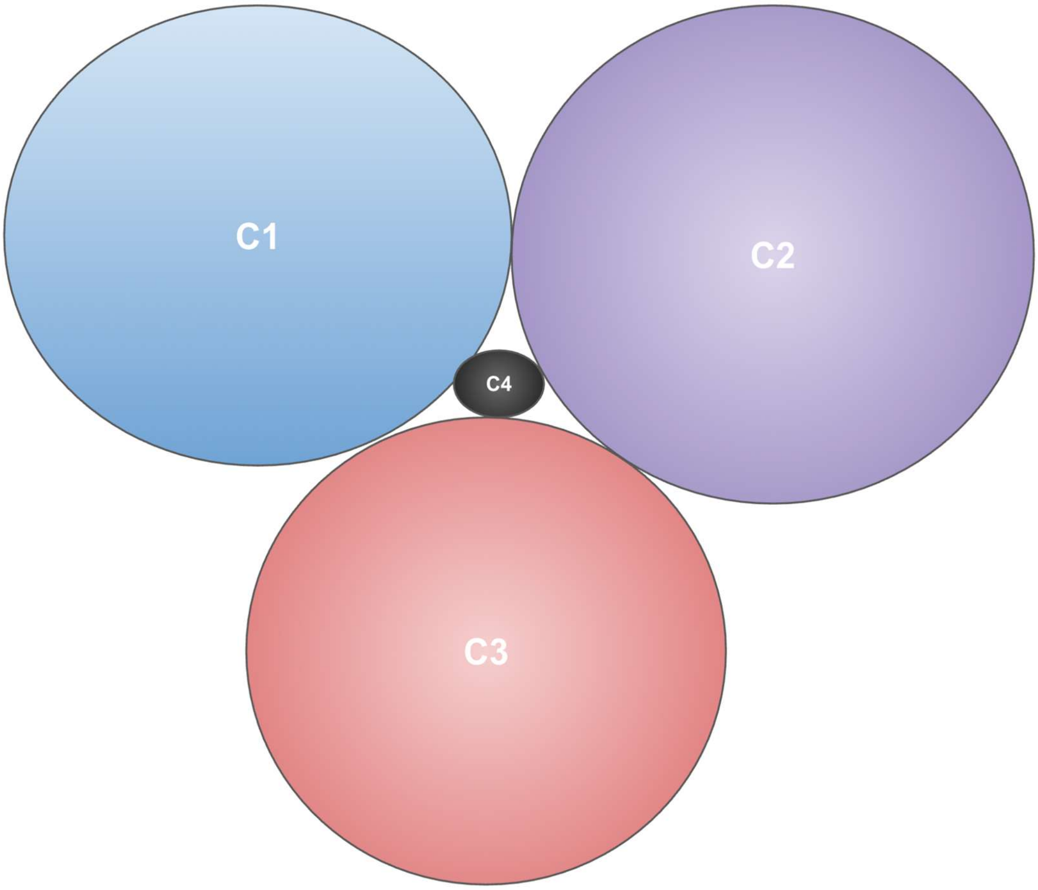

Four Vertices in K4 Rectangular Arrangement: If we represent it as circles:
We can construct K4 circle-packed in two valid geometric styles:

1. Inner Circle Gap Method: Start with three mutually tangent circles forming a triangle, and place the fourth smaller circle in the gap — this circle will be tangent to all three others. This demonstrates a compact form of K4 realization in 2D.
2. Outer Encasing Method: Place three mutually tangent circles in contact, and use a larger circle to enclose and touch all three — the outer circle acts as the fourth vertex. This layout maintains the K4 adjacency through tangency from one outer circle.

(Refer to figure showing both these styles visually. These forms emphasize the flexibility of circle packing to realize full mutual adjacency of four vertices — i.e., K4.)

5.1. Five Vertices in K5

Upon closer examination, we observe that it is impossible to arrange five circles in the same plane in such a way that they all touch each other simultaneously. This fundamental insight serves as a crucial foundation for our proof, leveraging a well-established geometric property governing the interactions of multiple circles.

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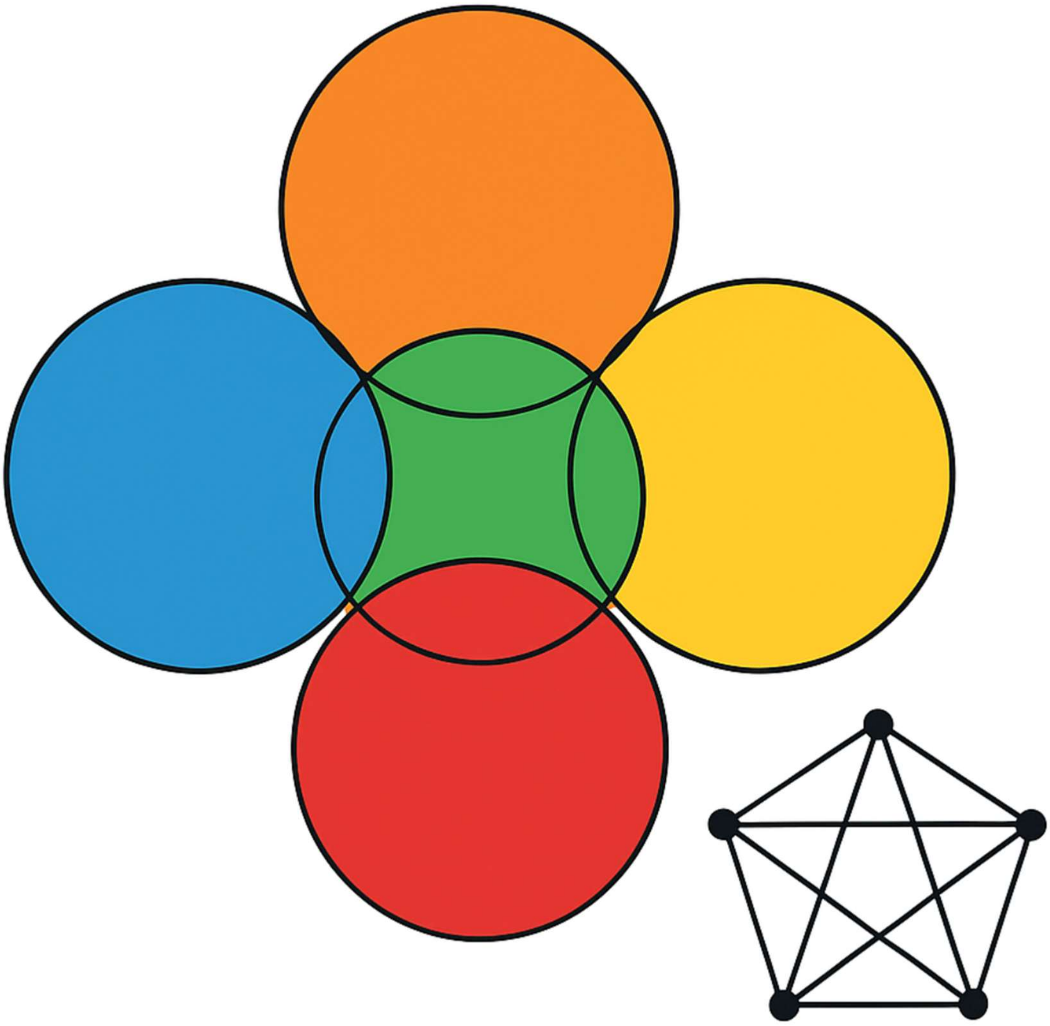

Background:

The Four Color Theorem has been a fundamental problem in graph theory for over a century. It posits that any planar graph can be colored with no more than four colors such that no two adjacent regions share the same color.

The Four Color Theorem in This Context

The theorem states that no more than four colors (or planes, in your interpretation) are needed to ensure that no two adjacent circles (vertices) share the same color (or are on the same plane).

Lemma: Kissing Number in 2D Plane

In a 2D plane, the maximum number of equal-sized circles that can touch another circle without any of them overlapping is 4. The four color map theorem is deeply connected with this geometrical nature of interaction of circles or any closed graphs in a 2d plane.

Lemma (Chromatic-Kissing Bound)

Let **G** be a planar graph with a circle packing **P**, where each vertex **v ∈ V(G)** is represented by a circle $\mathbf{C}_v$, and edges correspond to tangencies. Then:

$\chi(G) \leq$ maximum kissing number of $P \leq 4$

Theorem 1 (Four Color Theorem via Circle Packing)

Let $G = (V, E)$ be a planar graph. Then its chromatic number satisfies:

$\boldsymbol{\chi(G) \leq 4}$

Proof:

Assume, for contradiction, that there exists a planar graph G with:

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$\chi(G) \geq 5$
Graph Minor Implication
By Wagner's Theorem, since $\chi(G) \geq 5$, G must contain K_5 as a minor:
$K_5 \preceq G$
Circle Packing Representation
By the Koebe–Andreev–Thurston Theorem, G admits a circle packing P in $\mathbb{R}2$ where:

- Each vertex $v_i \in V$ maps to a circle C_i

Each edge $(v_i, v_j) \in E$ implies C_i and C_j are tangent
This extends to minors, so the K_5 minor corresponds to 5 circles $\{C_1, C_2, C_3, C_4, C_5\}$.
Tangency Requirement for K_5
Because K_5 is complete, in P:
For all $i \neq j$, $C_i \cap C_j \neq \emptyset$ (i.e., they must be tangent)
$\Rightarrow$ Each C_i must be tangent to 4 others
Kissing Number Contradiction
The maximum number of mutually tangent circles in $\mathbb{R}2$ is 4 (kissing number $k_2 = 4$).
Thus:
There does not exist a configuration where every circle C_i is tangent to four others.
This contradicts the K_5 packing requirement.
$\therefore$ The assumption $\chi(G) \geq 5$ is false, and
$\chi(G) \leq 4$ □
Corollary 1 (Non-Planarity of K_5)
The complete graph K_5 is non-planar.
Proof:
Combinatorial Argument:
For K_5 ($|V| = 5$, $|E| = 10$):
$|E| = 10 > 9 = 3|V| - 6$ (violates Euler's formula for planar graphs)
Geometric Argument:
A planar K_5 would require a circle packing with 5 mutually tangent circles, but:
$k_2 = 4 \Rightarrow$ No such packing exists.
$\therefore$ K_5 is non-planar. ■
$\therefore$ K_5 is non-planar both geometrically and topologically.

Theorem (4CT and K_5 Non-Planarity as the Same Problem)

The Four Color Theorem and the non-planarity of K_5 are dual consequences of the same geometric limitation:

"In the 2D Euclidean plane, it is impossible to arrange five mutually tangent circles."

- Combinatorially: This implies no planar graph requires more than four colors (→ Four Color Theorem).

Topologically: This prevents K_5 from being planar (→ non-planarity of K_5).

- Geometrically: This arises from the 2D kissing number being exactly 4.

Closing and Conclusion

In this paper, we have explored a fascinating intersection between geometry and graph theory, shedding light on the intriguing concept of the "kissing number" of circles in a 2D plane. We began by introducing the fundamental idea of representing planar graphs using groups of circles, where circles symbolize vertices, and tangents between them represent edges. This visual representation not only simplifies complex graph structures but also offers new insights into long-standing problems.

Our journey led us to the heart of the "kissing number" problem, a classic question in mathematics. We established the essential theorem that in a 2D plane, the maximum number of circles that can touch another circle without overlapping is 4. This simple yet profound result has far-reaching implications in various fields, from geometry to network design.

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By providing a lemma and proof, we have contributed to the comprehensive understanding of this intriguing problem. We have clarified that the size of the circles need not be uniform; it is their relative positions and non-overlapping nature that define the kissing

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