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Posted Date: 6 May 2024

doi: 10.20944/preprints202405.0191.v1

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Article

Discovering Bohr's Yin-Yang Diagram in Quantum Tunneling Dynamics

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Abstract: On 17 October 1947, Niels Bohr was made a knight of the Order of the Elephant by the King of Denmark in view of his outstanding achievements and contributions to science. Bohr designed his own coat of arms that featured a pattern of Yin and Yang (Tai Chi symbol) to symbolize the wave-particle complementarity. However, Bohr's Yin-Yang diagram (YYD) was neither drawn based on the principles of quantum mechanics, nor did it originate from the traditional Taoist YYD. Scientists still have doubts about the legitimacy of using YYD as the icon of the wave-particle complementarity, because the YYD belonging to quantum mechanics itself is unknown so far. This article reports the YYDs existing in quantum mechanics and justifies the role of YYD in the wave-particle duality by showing that any system, whether classical or quantum, has an ideal YYD as long as it satisfies Bohr's principle of complementarity (BPC). The deviation of a deformed YYD from the ideal YYD indicates the extent to which a real system satisfies BPC. This article constructs the quantum YYD by the complex quantum trajectory of a particle tunneling via a step barrier, which displays the continuous transition between the wave behavior and the particle behavior. It appears that the YYD designed by Bohr in his coat of arms resembles the YYD generated by tunneling motion, not only in appearance but also in the governing equation.

Keywords: Bohr's coat of arms; principle of complementarity; Yin-Yang diagram; quantum tunneling dynamics; complex quantum trajectory

1. Introduction

The Yin-Yang diagram (YYD), which is known as the Tai Chi diagram in China [1,2], is a graphical demonstration of a system's two states, called Yin and Yang, by means of areas in black and red in order to display their interlaced changes within a period of time (see Figure 1a). Why Bohr used this special symbol to represent his principle of complementarity [3] has remained a mystery for nearly eighty years. Except for philosophical or even religious connections, physicists have so far found no scientific clues to substantiate Bohr's original ideas [4]. The key reason why this mystery has not been solved for many years is because physicists and even Bohr himself have no idea about how to create the YYD from the principle of complementarity. The Taoist YYD is actually a graphic representation of the ancient Chinese version of the principle of complementarity [5]. As shown in Figure 1, the areas occupied by red and black in the Taoist YYD serve as the quantitative indices p_A and p_B for the state A (Yang) and state B (Yin), respectively. Using the two indices p_A and p_B , the two properties of Bohr's principle of complementarity (BPC), i.e., joint completeness and mutual exclusion, can be expressed quantitatively.

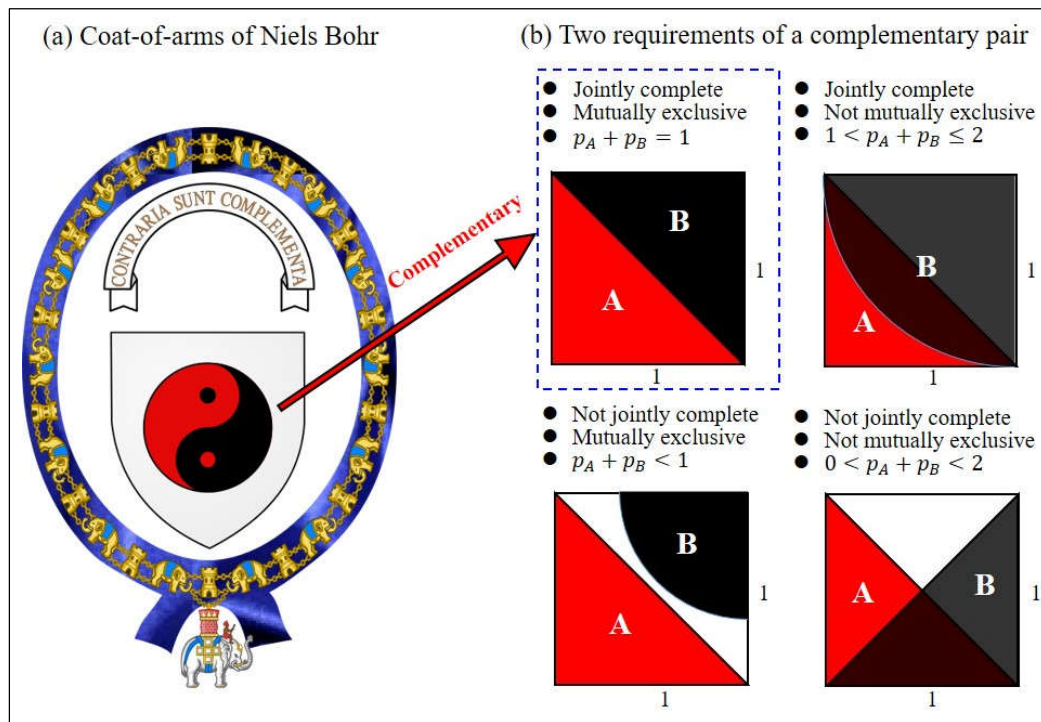


Figure 1. The Yin-Yang diagram in the Bohr's coat of arms and the quantification of the principle of complementarity. (a) Bohr designed his own coat of arms that featured a symbol of yin and yang, and a motto in Latin: *contraria sunt complementa*, "opposites are complementary". (Niels Bohr, Wikipedia, The Free Encyclopedia) (b) Two states A and B of a system are said to be complementary, if they are jointly complete and mutually exclusive, or equivalently, $p_A + p_B = 1$, where p_A and p_B are the quantitative measures for A and B, respectively.

YYD is a graphical indicator that reflects the degree to which a system satisfies the principle of complementarity, and it is present in almost every system in nature, whether classical or quantum. What Bohr could not have imagined was that he did not need to borrow the Taoist YYD to convey his principle of complementarity, because the principle of complementarity is the theoretical foundation of the Taoist YYD. This article reverses the Bohr's idea by applying the definition of YYD to reproduce the Taoist YYD, which is an ideal YYD satisfying the principle of complementarity, and using the same definition to construct the YYDs belonging to quantum mechanics itself. Our results reveal that the YYD designed by Bohr in his coat of arms is different from the Taoist YYD, but is close to the quantum YYD based on tunneling effect.

The Taoist YYD contains two states: Yin and Yang, whose magnitudes wax and wane over time, but their sum remains constant, hence forming a pattern like two fish rotating around each other within a circle. The Yin-Yang states in the Taoist YYD are a complementary pair possessing three properties. The first property is that they are jointly complete, that is, the two states constitute the entire system. The second property is that they are mutually exclusive, that is, there is no intersection or interference between the two states. The combination of the above two properties indicates that the Yin-Yang states satisfy BPC [6], which can be expressed as $p_A + p_B = 1$ in terms of the quantitative measures p_A and p_B (see Figure 1b). The third property of the Yin-Yang pair is that the pair remains complementary over a complete period of time, i.e., $p_A(t) + p_B(t) = 1$, $0 \leq t \leq T$. The YYD composed of the Yin-Yang states having the above three properties is said to be an ideal YYD. The traditional Taoist YYD is a typical example of ideal YYDs, and in this paper we will reveal two ideal YYDs emerging from quantum mechanics, one from two-path interference experiments and the other from quantum tunneling dynamics.

The complementary pair A and B in the Bohr's YYD correspond to the particle nature and the wave nature of a quanton, whose relationship with BPC has been tested by many experiments in recent decades. In these experiments, the quantitative measures for the complementary pair are given

by $(p_A, p_B) = (D^2, V^2)$, where D is the distinguishability of the which-way information and V is visibility of wave fringe measured by the Mach-Zehnder interferometer (ZMI) [7,8]. The manifested wave and/or particle property of the tested quantons depend on the detecting devices used in the experiments. In the all-or-nothing cases [9–11], quantons behave as a particle ($p_A = 1$ and $p_B = 0$) or as a wave ($p_A = 0$ and $p_B = 1$), while in the intermediate cases [12–15], quantons behave simultaneously as a particle and as a wave, whose quantitative measures satisfy the inequality $p_A + p_B \leq 1$. It was further verified [16] that the equality $p_A + p_B = 1$ holds if the tested quantons have no internal degree of freedoms, for which the conditions of joint completeness and mutual exclusion are both met.

In the recent years, the advancement of quantum beam splitter [17] allows the MZI to be set in a state of quantum superposition of “on” and “off” such that the wave and particle behaviors of a single photon can coexist simultaneously, with a continuous morphing between them. Unlike the classical ZMI where the wave nature and the particle nature are treated independently, the wave and particle natures of a single quanton in the setup of quantum MZI become quantum states [18,19] so that there could be interference between them. The interference between the wave state and the particle state in the quantum ZMI implies that they are not mutually exclusive and cannot form a qualified complementary pair (A, B) , for which the violation of the complementarity, $1 < p_A + p_B \leq 2$, has been experimentally observed [20,21].

Although BPC has been tested by numerous experiments, these experiments still cannot tell us why Bohr used the YYD to symbolize his principle. The first task of this paper is to construct the ideal YYD and the deformed YYD from the experimental data $(p_A, p_B) = (D^2, V^2)$. According to the difference between the ideal YYD and the deformed YYD, we can judge the extent to which the experimental results satisfy the principle of complementarity. Interestingly, we find that the ideal YYD constructed by the two-path interference experiment is closer to the traditional Taoist YYD, but different from the YYD designed by Bohr in his coat of arms. This discovery prompts us to explore another complementary pair in order to more perfectly represent Bohr's YYD.

The finding of a more suitable complementary pair is inspired by the quantum tunneling motion of a quanton via a step barrier, wherein we find that the changes in the quanton's real velocity $\dot{x}_R$ and imaginary velocity $\dot{x}_I$ synchronously reflect the alternating changes in its particle nature and wave nature. It was pointed out that the wave-particle duality is a direct result of quantum motion occurring in complex space [22,23]. The validity of the complex-trajectory interpretation of wave-particle duality is further supported by the recent findings of weak measurements, which reveal that every physical observable has a real part as well as an imaginary part [24,25]. Like entanglement, imaginarity has been recognized as an indispensable resource responsible for quantum advantages [26–28], which ruled out the real-valued standard formalism of quantum theory [29]. When a quanton exhibits full particle behavior, the real component x_R alone is sufficient to describe its motion, as treated in classical mechanics. But when a quanton's motion exhibits wave behavior, we need the imaginary component x_I together to describe its motion. Our results reveal that the quantum YYD constructed by the complementary pair $(p_A, p_B) = (\dot{x}_R^2, \dot{x}_I^2)$ is closer to the Bohr's YYD than the YYD constructed by the complementary pair $(p_A, p_B) = (D^2, V^2)$.

In the following discussions, we will first define the general YYD and the ideal YYD in Section 2, and show that the traditional Taoist YYD is the oldest example of the ideal YYD satisfying BPC. With $(p_A, p_B) = (D^2, V^2)$, Section 3 constructs the quantum YYD by the data of the two-path interference experiments, from which the similarity between the Taoist YYD and the interference-based YYD is shown. Section 4 introduces the mapping between the complex tunneling trajectory and the quantum YYD for a particle's tunneling motion via a step barrier. The particle's tunneling trajectories are solved by quantum Hamilton mechanics [30] in Section 5 to yield the particle's complex position $x_R + ix_I$ and complex velocity $\dot{x}_R + i\dot{x}_I$, by which the tunneling-based YYD is then generated. The obtained tunneling-based YYD is actually a graphical recorder of the particle's tunneling dynamics, which moves together with the particle, recording the velocity of the particle, and presenting it graphically, as demonstrated in Section 6. The tunneling-based YYD is not always an ideal YYD and how its shape evolves with the intensity of the tunneling effect is discussed in

Section 7. The last section compares the four different YYDs mentioned in this paper and clarifies their relationship with BPC, from which we draw the following conclusions: Bohr's YYD as a symbolization of BPC and the tunneling-based YYD as a visualization of BPC belong to the same category, while Taoist YYD as an ancient realization of BPC and the interference-based YYD as an experimental test of BPC belong to another category.

2. Definitions for General YYD and Ideal YYD

Traditional Taoist YYD is an ideal YYD satisfying BPC, which is a special case of the general YYD. Below we first define the general YYD, and then explain under what conditions the ideal YYD can be induced. A general YYD is defined for any periodic system, which is composed of state A (Yang) and state B (Yin) with quantitative measures p_A and p_B normalized between 0 and 1. The growth and decline of the Yin-Yang states are conveyed through the interlacing changes of p_A (in red) and p_B (in black). As shown in Figure 2, the shape of YYD is uniquely determined by its inner curve and outer curve.

The radial coordinates of the inner curve and the outer curves are called inner radius and outer radius, respectively. In terms of the quantitative measures $p_A(t)$ and $p_B(t)$ recorded at each moment t , the inner radius of a general YYD is defined by

$$r = \begin{cases} p_A(t), & 0 \leq t \leq T/2 \\ p_B(t), & T/2 \leq t \leq T \end{cases} = \begin{cases} p_A(\theta), & 0 \leq \theta \leq \pi \\ p_B(\theta), & \pi \leq \theta \leq 2\pi \end{cases} \quad (1)$$

and its outer radius is defined by $r = p_A(\theta) + p_B(\theta)$, where the azimuth angle $\theta = 2\pi t/T$ is measured clockwise from the vertical axis and T is the period of time, as shown in Figure 2. It is noted that the periodic property of the system implies $p_A(0) = p_A(2\pi)$ and $p_B(0) = p_B(2\pi)$.

The ideal YYD is a special case of the general YYD, which further meets the continuity condition and the complementarity condition. The continuity condition requires that the two sections of the inner curve defined by Eq. (1) must be continuous at the origin, i.e., $p_A(\pi) = p_B(2\pi) = 0$, or equivalently, the state A must occupy the whole system at $\theta = 2\pi$ ($\theta = 0$) and the state B must occupy the whole system at $\theta = \pi$. The complementarity condition requires that the states A and B must satisfy BPC: $p_A(\theta) + p_B(\theta) = 1$ over the entire interval, or equivalently, the outer curve has to be a unit circle, $r = p_A(\theta) + p_B(\theta) = 1$, $0 \leq \theta \leq 2\pi$.

In ancient China, the most representative complementary pair (A, B) was day and night. In this case, p_A represents the proportion of daytime in a day, and p_B represents the proportion of night. These two proportions vary on different days of the year, but they always meet the complementarity condition $p_A + p_B = 1$. By recording the proportions of day and night for each day of the year in the polar coordinates, with white area denoting the daytime and black area denoting the nighttime, the original YYD in ancient China was then formed [5]. However, the original YYD determined by sunshine time is not an ideal YYD, because it does not necessarily satisfy the continuity condition. The continuity condition requires that in the year there be exactly one day with 24hr daytime (at the summer solstice, $\theta = 0$) and exactly one another day with 24hr nighttime (at the winter solstice, $\theta = \pi$). Except for those who live at a certain latitude on the earth, people cannot actually draw an ideal YYD based on local sunshine hours.

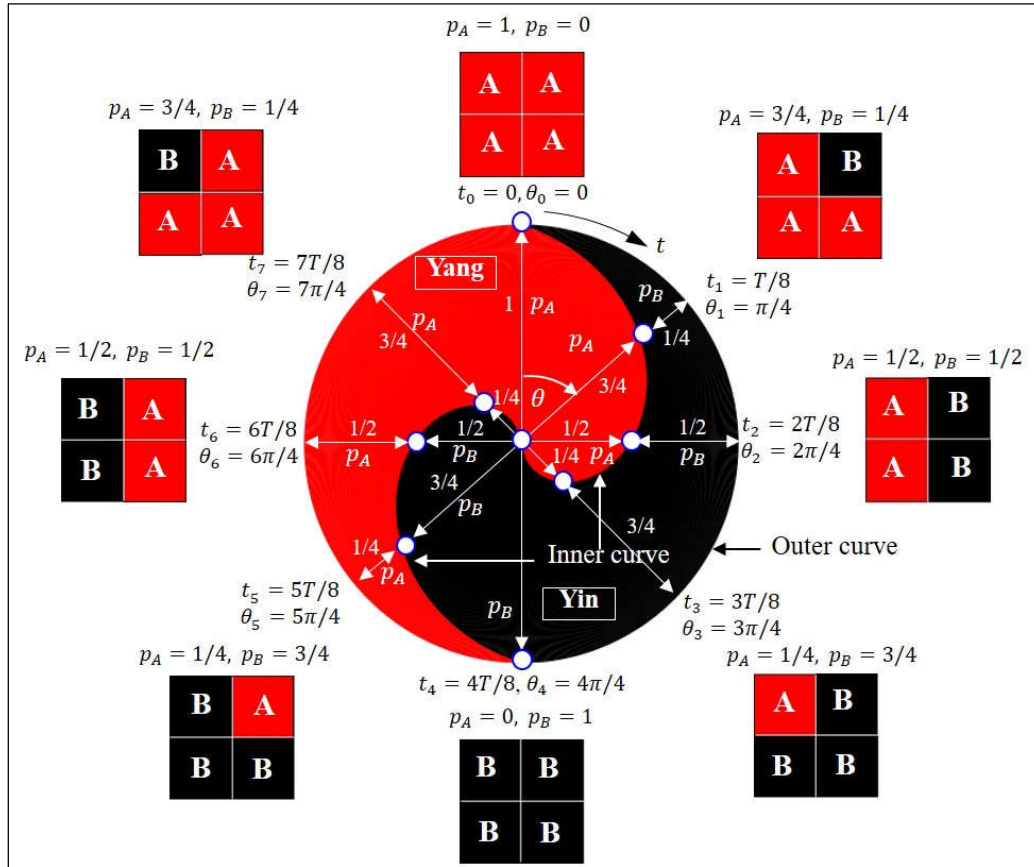


Figure 2. A schematic diagram of constructing the Taoist YYD according to the principle of complementarity. (A,B) is a qualified complementary pair of a periodic system satisfying the condition $p_A(t_k) + p_B(t_k) = 1$ at every moment t_k within the period $0 \leq t_k \leq T$. The inner curve of the Taoist YYD is defined by the polar coordinates (r_k, θ_k) , where $r_k = p_A(t_k)$ for $0 \leq t_k \leq T/2$, and $r_k = p_B(t_k)$, for $T/2 \leq t_k \leq T$, and the outer curve is defined by $r_k = p_A(t_k) + p_B(t_k)$. The quantitative measures $p_A(t_k)$ and $p_B(t_k)$ used in the traditional Taoist YYD are the proportions of the states A and B in the entire system at time t_k , just like the proportions of day and night on a certain day t_k of the year.

The first ideal YYD, called Taoist YYD, was constructed by Chen Tuan (871AC - 989AC), a philosopher in the early Song Dynasty of China. He drew the YYD according to the sequence of the sixty-four hexagrams appeared in the Book of Changes [9] (also called *Yijing* or *I Ching*), which is the oldest Chinese classics formed around the ninth century BC. Here we will reproduce the Taoist YYD by BPC. Figure 2 shows a schematic diagram of constructing the Taoist YYD from the measured $p_A(t)$ and $p_B(t)$ under the situation that the system is divided into four parts, and one period of time is divided into eight intervals. In the Taoist YYD, the quantitative measures $p_A(t_k)$ and $p_B(t_k)$, $k = 0, 1, \dots, 8$, represent the proportions of the states A and B in the entire system at time t_k . For example, at time t_1 , state A occupies three-quarters of the system, and state B occupies one-quarter of the system, so we get $p_A = 3/4$ and $p_B = 1/4$. The situation considered in Figure 2 can be extended to the general case, where the system has $N/2$ partitions and the period of time has N partitions. Now, the measure p_A (p_B) represents the ratio of the number of partitions containing A (B) state to the total $N/2$ partitions of the system. Therefore, we have

$$p_A(t_k) = \begin{cases} 1 - k/(N/2), & k = 0, 1, \dots, N/2 \\ k/(N/2) - 1, & k = N/2, \dots, N \end{cases} \quad (2)$$

In order to satisfy the complementarity condition, the measure of B is automatically set to $p_B(t_k) = 1 - p_A(t_k)$. Substituting the above $p_A(t_k)$ and $p_B(t_k)$ into Eq. (1) yields the polar coordinates (r_k, θ_k) describing the inner curve of the Taoist YYD as

$$r_k = \begin{cases} 1 - k/(N/2), & k = 0, 1, \dots, N/2 \\ 2 - k/(N/2), & k = N/2, \dots, N \end{cases} = \begin{cases} 1 - \theta_k/\pi, & 0 \leq \theta_k \leq \pi \\ 2 - \theta_k/\pi, & \pi \leq \theta_k \leq 2\pi \end{cases} \quad (3)$$

where the index k and the azimuth angle θ_k has the linear relation $k/(N/2) = \theta_k/\pi$ by noting that when the integer k increases sequentially from 0 to $N/2$, the azimuth angle θ_k increases linearly from 0 to π . It is noted that the continuity condition $p_A(\pi) = p_B(2\pi) = 0$ is also satisfied by the above definition of p_A and p_B . When N is set to 8, the eight points (r_k, θ_k) defined by Eq. (3) are just those shown in Figure 2. When N is set to 64, the curve formed by connecting the 64 coordinate points (r_k, θ_k) defined in Eq. (3) recovers the traditional Taoist YYD, which was first constructed according to the order of the sixty-four hexagrams in the Book of Changes. Here we have reproduced the ancient Taoist YYD by the principle of complementarity, i.e., by requiring that the complementary pair (A, B) must be jointly complete and mutually exclusive.

As the number of partitions N approaches infinity, the discrete points defined by Eq. (3) become a continuous polar curve

$$r = \begin{cases} 1 - \theta/\pi, & 0 \leq \theta \leq \pi \\ 2 - \theta/\pi, & \pi \leq \theta \leq 2\pi \end{cases} \quad (4)$$

This is the inner curve of the ideal Taoist YYD formed by two sections of Archimedean spiral, which has a general expression $r = a + b\theta$. If Bohr's original intention was to use the Taoist YYD to convey his principle of complementarity, his correct choice should be the pattern in Figure 2, which is the Taoist YYD drawn exactly according to his principle of complementarity. However, the YYD designed by Bohr in his coat of arms, as shown in Figure 3, is based on a popular drawing method, which is composed of a unit circle and two inscribed circles with polar equation given by

$$r = \begin{cases} \cos \theta, & 0 \leq \theta \leq \pi/2 \\ -\cos \theta, & \pi \leq \theta \leq 3\pi/2 \end{cases} \quad (5)$$

Apparently, the inner curve of the Bohr's YYD only passes through the first and third quadrants, and is significantly different from the Taoist YYD defined by Eq. (4). In the following, we will apply the above definition of the general YYD to derive the YYDs belonging to quantum mechanics itself.

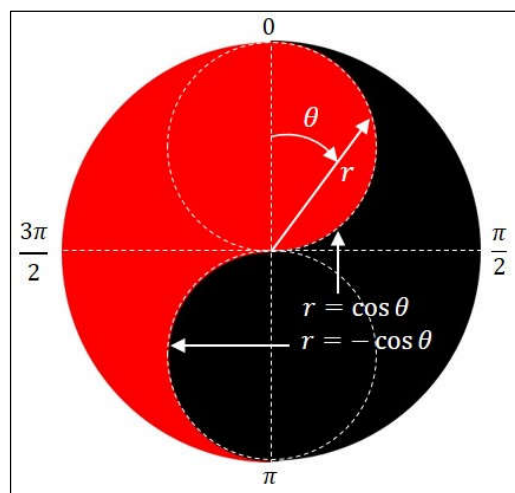


Figure 3. The Bohr's YYD designed in his coat of arms is composed of a unit circle and two inscribed circles. The polar equation of the inner curve is given by $r = \pm \cos \theta$.

3. Constructing Quantum YYD by Two-Path Interference Experiments

Among various natural phenomena, wave-particle duality best embodies the complementary relationship. In the past several decades, many experiments have been proposed to test whether the wave-particle duality satisfies BPC. However, all experiments on wave-particle duality so far still cannot explain the relationship between Bohr's YYD and wave-particle duality. In this section, we will show that regardless of the test results of BPC, a YYD can always be constructed according to Eq. (1) by the experimental data $(p_A, p_B) = (D^2, V^2)$ from the Mach-Zehnder interferometer (ZMI), where $p_A = D^2$ is a measure of the particle nature regarding the distinguishability of the which-way information, and $p_B = V^2$ is a measure of the wave nature regarding the visibility of wave fringe.

An experimental setup to construct the YYD for wave-particle duality is shown in Figure 4a [15,31], wherein a single photon pulse is split by a variable beam splitter (VBS) with adjustable reflectivity in the range $0 \leq R_1 \leq 1$, and a piezoelectric transducer (PZT) is employed to adjust the relative phase ϕ between the two paths. The two paths of photon are recombined at the beam merger (BM), which is a fixed 50/50 beam splitter, and finally the photon is detected by the detectors D_1 and D_2 . The measured values of D and V depend on the reflectivity R_1 of the VBS. If $R_1 = 0$ or $R_1 = 1$, the which-path information is known deterministically and the photon manifests the property of a pure particle with $D = 1$ and $V = 0$. If $0 < R_1 < 1$, the interference fringe appears behind the beam merger and yields a nonzero wave visibility V , which achieves its maximum value $V = 1$ at $R_1 = 1/2$.

Following Englert [8], the distinguishability D has two different notions: *a priori* distinguishability and *a posteriori* distinguishability, where the former is also called predictability P , which refers to a which-way information obtained by using an unbalanced interferometer with different particle flux along the two paths. The *a priori* distinguishability and visibility are especially useful in the construction of the ideal YYD, because it requires analytical expressions for the distinguishability and the visibility. In the case of a MZI without path loss, the *a priori* distinguishability P and the *a priori* visibility V can be expressed analytically as a function of R_1 as [31]

$$P(R_1) = |1 - 2R_1|, \quad V(R_1) = 2\sqrt{R_1(1 - R_1)}, \quad 0 \leq R_1 \leq 1. \quad (6)$$

Substituting $p_A = P^2$ and $p_B = V^2$ into Eq. (1), we obtain the polar equation of the YYD corresponding to the two-path interference experiment as

$$r = \begin{cases} (1 - 2R_1)^2, & 0 \leq R_1 \leq 1/2 \\ 4R_1(1 - R_1), & 1/2 \leq R_1 \leq 1 \end{cases} = \begin{cases} (1 - \theta/\pi)^2, & 0 \leq \theta \leq \pi \\ (\theta/\pi)(2 - \theta/\pi), & \pi \leq \theta \leq 2\pi \end{cases} \quad (7)$$

where the azimuth angle θ is related to the reflectivity R_1 as $\theta = 2\pi R_1$. The continuity condition $p_A(\pi) = p_B(2\pi) = 0$ and the complementarity condition $p_A(\theta) + p_B(\theta) = P^2 + V^2 = 1$ are satisfied so that the resulting YYD is an ideal YYD as shown in Figure 4b.

It is surprising to find that although the systems considered in Eq. (4) and Eq. (7) are totally different, their accompanying YYDs are found to be similar by comparing Figure 2 with Figure 4b. The slight difference between Eq. (4) and Eq. (7) is that the former is a linear function of θ/π , while the latter is a quadratic function of θ/π . Nevertheless, the Taoist YYD described by Eq. (4) and the interference-based YYD described by Eq. (7) are different kinds of YYD from the Bohr's YYD described by Eq. (5).

The ideal YYD shown in Figure 4b is constructed from a perfect MZI without path loss; however, path loss in a MZI is inevitable, which will cause the actual YYD to deviate from the ideal YYD. In the following, two kinds of nonideal YYDs will be considered, one from the violation of the continuity condition and the other from the violation of the complementarity condition. When path loss occurs, the measured values of P and V are different from their ideal values given by Eq. (6) and can be estimated by the following formula [31]

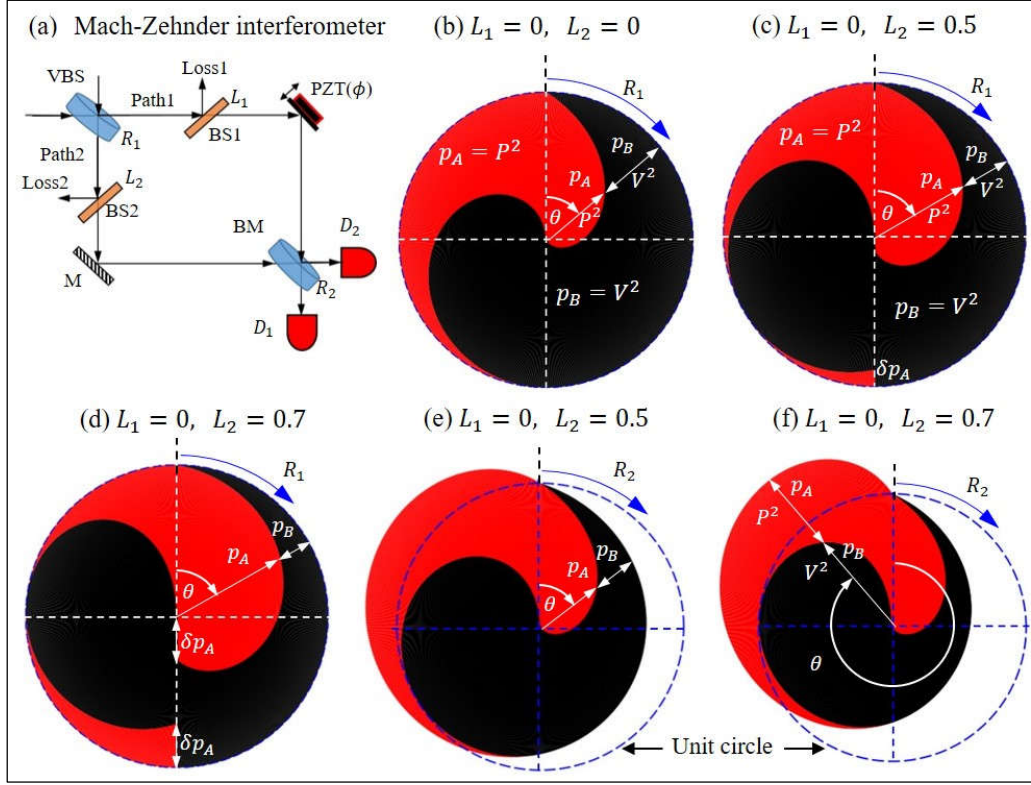


Figure 4. The quantum YYDs obtained from the two-path interference experiments. Part (a) is the schematic diagram of the Mach-Zehnder interferometer used to measure the predictability P and the visibility V . Part (b) is the ideal quantum YYD constructed by using the quantitative measures $(p_A, p_B) = (P^2, V^2)$ without path loss in the interferometer. Part (c) and (d) show the deformed YYDs caused by the path loss L_2 , whose outer curves remain the same as the ideal YYD, but whose inner curves have a discontinuity δp_A at $\theta = \pi$. Part (e) and (f) show the deformed YYDs with (p_A, p_B) measured from the other configuration of the experiment by interchanging the input port and the output port in part (a). The resulting YYD has a deformation in the outer curve without causing the discontinuity in the inner curve.

$$V(R_1) = \frac{2\sqrt{R_1(1-R_1)(1-L_1)(1-L_2)}}{(1-R_1)(1-L_1) + R_1(1-L_2)}, \quad P(R_1) = \frac{|(1-R_1)(1-L_1) - R_1(1-L_2)|}{(1-R_1)(1-L_1) + R_1(1-L_2)}, \quad (8)$$

where L_1 and L_2 with values between 0 and 1 are the losses of the two paths and are simulated in the experiment by the two beam splitters BS1 and BS2 with reflectivity equal to L_1 and L_2 , as shown in Figure 4a. Again by substituting $p_A = P^2$ and $p_B = V^2$ given by Eq. (8) into Eq. (1), we obtain the deformed YYD caused by the path loss as shown in Figure 4c and Figure 4d. It can be seen that this type of path loss does not produce the deformation of the outer curve of the YYD, that is, the complementarity condition $p_A + p_B = P^2 + V^2 = 1$ is still satisfied, but the inner curve of the YYD is not continuous. The continuity condition requires $p_A(\pi) = p_B(2\pi) = 0$, but for this case, we have $p_A(\pi) = P^2(1/2) = (L_2/2)^2/(1-L_2/2)^2 \neq 0$ and $p_B(2\pi) = V^2(1) = 0$. The discrepancy between $p_A(\pi)$ and $p_B(2\pi)$ denoted by δp_A increases with the path loss L_2 as shown in Figure 4c and Figure 4d.

Figure 4e and Figure 4f demonstrates the YYD deformation caused by the path loss under another configuration of the experiment, where the setups in the input and output ports are interchanged such that the input port is a fixed 50/50 beam splitter, while the output port is a variable beam splitter with adjustable reflectivity in the range $0 \leq R_2 \leq 1$. The predictability P and the visibility V for this configuration can be expressed by [31]

$$P(R_1) = |1 - 2R_2|, \quad V(R_1) = \frac{2\sqrt{R_2(1-R_2)(1-L_1)(1-L_2)}}{(1-R_2)(1-L_1) + R_2(1-L_2)}, \quad 0 \leq R_2 \leq 1. \quad (9)$$

Opposite to the previous case, we can see from Figure 4e and Figure 4f that this type of path loss causes the outer curve of the YYD to deviate from the unit circle, i.e., the complementarity condition $p_A + p_B = 1$ is not satisfied over the entire interval, but the continuity conditions $p_A(\pi) = P^2(1/2) = 0$ and $p_B(2\pi) = V^2(1) = 0$ still hold. As the path loss increases, the deformation of YYD intensifies and deviates further from the ideal YYD. In short, the violation of the continuity condition (Figure 4c and Figure 4d) and the violation of complementarity condition (Figure 4e and Figure 4f) both can deform the ideal YYD (Figure 4b) and the deformation tells us graphically the extent to which the two conditions are violated.

4. The Mapping between Quantum YYD and Complex Tunneling Velocity

The YYD generated by the two-path interference experiments shows us a static picture of YYD, which is similar to the Taoist YYD but different from the Bohr's YYD. In this section we will consider a complementary pair (A, B) , whose quantitative measures $(p_A(t), p_B(t))$ are time dependent so that the pattern of YYD changes with time. This dynamic picture of the YYD can be visualized by the quantum tunneling process, where the particle's real and imaginary velocity components $(\dot{x}_R(t), \dot{x}_I(t))$ constitute a time-dependent complementary pair. The concepts of trajectory and velocity given to a tunneling particle are derived from the complex trajectory interpretation of quantum mechanics [30,32–38], which is different from the probability interpretation, as compared in Figure 5, but they have a common source: the wavefunction $\psi(x)$. Probability interpretation treats the particle's position $x \in \mathbb{R}$ as a real variable and employs $|\psi(x)|^2$ as the probability of finding the particle at the position x . On the other hand, the complex trajectory interpretation uses the same wavefunction $\psi(x)$ but with $x \in \mathbb{C}$ to generate the complex quantum trajectory $x(t) = x_R(t) + ix_I(t)$.

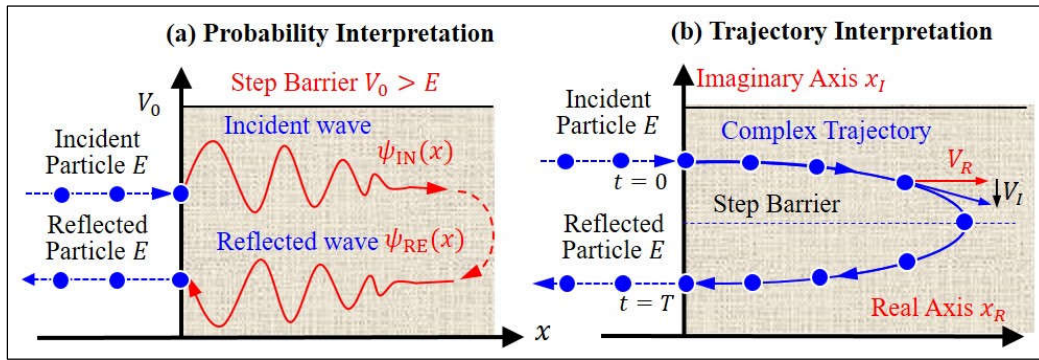


Figure 5. Comparing two quantum interpretations of tunneling phenomenon. Probability interpretation uses the wavefunction $\psi(x)$ with $x \in \mathbb{R}$ to describe the particle's wave behavior within the step barrier and treats $|\psi(x)|^2$ as the probability of finding the particle at position x . Trajectory interpretation uses the same wavefunction $\psi(x)$ with $x \in \mathbb{C}$ to generate the complex trajectory $x(t) = x_R(t) + ix_I(t)$, which is a continuous trajectory traversing the barrier.

Here we will consider the wavefunction describing a particle's tunneling motion via a step barrier with height V_0 greater than the particle's energy E . Such a wavefunction $\psi(x)$ can be solved analytically from the Schrödinger equation, and then used in the quantum Hamilton mechanics [30] to produce the particle's complex position $x(t) = x_R(t) + ix_I(t)$ and complex velocity $\dot{x}(t) = \dot{x}_R(t) + i\dot{x}_I(t)$ during the tunneling process. Once $x(t)$ and $\dot{x}(t)$ are determined, we can use the quantitative measures $(p_A, p_B) = (x_R^2, x_I^2)$ and $(p_A, p_B) = (\dot{x}_R^2, \dot{x}_I^2)$ in Eq. (1) to construct the tunneling-based YYDs.

Figure 6 shows how the complex tunneling velocities are mapped into the quantum YYD. The left side of Figure 6 shows the positions P_i of the particle on the tunneling trajectory at nine moments

t_i , $i = 0, 1, \dots, 8$. A trajectory point P_i with velocity $\dot{x}_R(t_i) + i\dot{x}_I(t_i)$ is mapped into the point P'_i on the YYD with radial coordinate $r_i = \dot{x}_R^2(t_i) + \dot{x}_I^2(t_i)$ and azimuth coordinate $\theta_i = 2\pi t_i/T$, where T is the total tunneling time. As time t_i increases from 0 to T , the area swept by the radial line segment $\dot{x}_R^2$ forms the Yang region (in red) of the YYD, while the area swept by the radial line segment $\dot{x}_I^2$ forms the Yin region (in black) of the YYD.

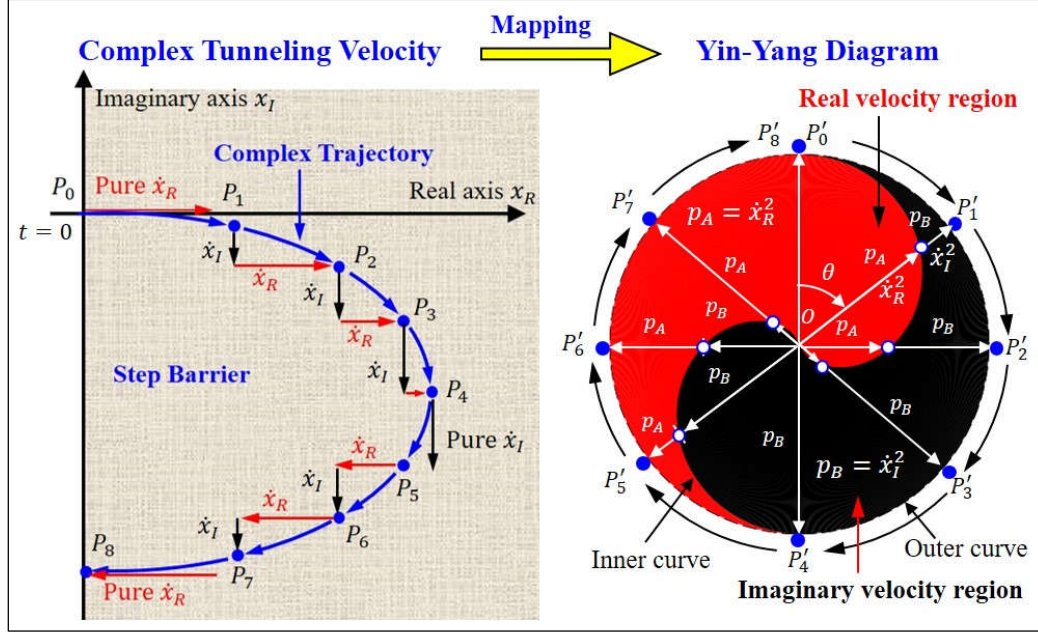


Figure 6. The mapping of the complex tunneling velocity into the quantum YYD. A trajectory point P_i with real velocity $\dot{x}_R$ and imaginary velocity $\dot{x}_I$ is mapped into a point P'_i on the YYD in such a way that the Yang (in red) region is formed by the time evolution of the squared real velocity $\dot{x}_R^2$, while the Yin (in black) region is formed by the squared imaginary velocity $\dot{x}_I^2$.

Among the nine trajectory points, P_0 is the entry point whose velocity $\dot{x}_0$ is pure real, P_4 is the turning point whose velocity $\dot{x}_4$ is pure imaginary, and P_8 is the exit point whose velocity is opposite to the entrance velocity $\dot{x}_8 = -\dot{x}_0$. Given the trajectory point P_i with the accompanying velocity $x(t_i) = x_R(t_i) + ix_I(t_i)$, we can determine the inner curve and the outer curve of the YYD as shown in the right side of Figure 6. As time t evolves continuously, the tunneling particle follows a continuous complex trajectory $x(t) = x_R(t) + ix_I(t)$, and according to Eq. (1), the inner curve of the velocity-based quantum YYD can be expressed in polar coordinates as

$$r = \begin{cases} \dot{x}_R^2(t), & 0 \leq t \leq T/2 \\ \dot{x}_I^2(t), & T/2 \leq t \leq T \end{cases} = \begin{cases} p_A(\theta), & 0 \leq \theta \leq \pi \\ p_B(\theta), & \pi \leq \theta \leq 2\pi \end{cases} \quad (10)$$

and the outer curve is given by $r = p_A + p_B = \dot{x}_R^2(t) + \dot{x}_I^2(t)$. The azimuth coordinate θ is related to time t as $\theta = 2\pi t/T$, $0 \leq t \leq T$. In the time range of $0 \leq t < T/2$, the inner radius of the YYD is given by the squared real velocity $r = p_A = \dot{x}_R^2$, while in the time range of $T/2 \leq t < T$, the inner radius is given by the squared imaginary velocity $r = p_B = \dot{x}_I^2$.

5. Computing Complex Tunneling Trajectory

To construct the YYD by Eq. (10), we need the particle's tunneling velocity $\dot{x}(t) = \dot{x}_R(t) + i\dot{x}_I(t)$, which can be derived from the complex-extended Schrödinger equation

$$i\hbar \frac{\partial \Psi(t, x)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(t, x)}{\partial x^2} + V(x)\Psi(t, x), \quad (11)$$

where the coordinate x is defined on the complex plane $x = x_R + ix_I \in \mathbb{C}$. Using the transformation $\Psi(t, x) = e^{iS(t, x)/\hbar}$, we can transform the Schrödinger equation (11) into the quantum Hamilton-Jacobi equation (QHJE)

$$\frac{\partial S(t, x)}{\partial t} + \left[\frac{1}{2m} p^2 + V(t, x) + \frac{1}{2mi} \frac{\partial p}{\partial x} \right]_{p=\partial S/\partial x} = \frac{\partial S(t, x)}{\partial t} + H(t, x, p)|_{p=\partial S/\partial x} = 0, \quad (12)$$

where $p = \partial S/\partial x$ is the canonical momentum conjugate to x , $S(t, x)$ is the quantum action function and $H(t, x, p)$ is the quantum Hamiltonian defined as

$$H(t, x, p) = \frac{1}{2m} p^2 + V(t, x) + Q(t, x). \quad (13)$$

The Quantum Hamiltonian $H(t, x, p)$ contains three items: (1) the particle's kinetic energy $p^2/2m$, (2) the applied potential $V(t, x)$, and (3) the internal quantum potential $Q(t, x)$ defined as:

$$Q(t, x) = \frac{1}{2mi} \frac{\partial p}{\partial x} \Big|_{p=\partial S/\partial x} = \frac{\hbar}{2mi} \frac{\partial^2 S(t, x)}{\partial x^2} = -\frac{\hbar^2}{2m} \frac{\partial^2 \ln \Psi(t, x)}{\partial x^2}. \quad (14)$$

It can be seen that the quantum potential is state-dependent and is uniquely determined by the wavefunction $\Psi(t, x)$. If the applied potential $V(t, x)$ is independent of time, that is, $V(t, x) = V(x)$, then the wavefunction $\Psi(t, x)$ can be further separated as:

$$\Psi(t, x) = \psi(x) e^{-i(E/\hbar)t} \Rightarrow S(t, x) = -i\hbar \ln \psi(x) - Et. \quad (15)$$

The particle's momentum p according to its definition in QHJE is then given by

$$p = \frac{\partial S}{\partial x} = \frac{\hbar}{i} \frac{d \ln \psi(x)}{dx} = \frac{\hbar}{i} \frac{\psi'(x)}{\psi(x)}. \quad (16)$$

Substituting the separated form of $S(t, x)$ into QHJE (12) yields

$$H(t, x, p) = \frac{p^2}{2m} + V(x) + Q(x) = -\frac{\partial S}{\partial t} = E = \text{constant}. \quad (17)$$

This formula expresses the law of energy conservation, indicating that the total energy of the particle represented by the Hamiltonian $H(t, x, p)$ is a constant. The energy conservation law (17) is actually an alternative expression of the time-independent Schrödinger equation

$$\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + (E - V(x))\psi = 0, \quad (18)$$

as can be shown by substituting the momentum p from Eq. (16) and the quantum potential Q from Eq. (14) into Eq. (17).

According to the Hamiltonian $H(t, x, p)$ given by Eq. (17), the quantum Hamilton equations of motion can be derived as follows:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad x(t_0) = x_0 \in \mathbb{C}, \quad (19a)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{d}{dx}(V + Q), \quad p(t_0) = p_0 \in \mathbb{C}, \quad (19b)$$

where the sum of V and Q is defined as the total potential

$$V_T = V + Q = E - \frac{p^2}{2m} = E + \frac{\hbar^2}{2m} \left(\frac{\psi'(x)}{\psi(x)} \right)^2. \quad (20)$$

From Eq. (20) we can see that except for a constant bias E , the magnitude of the total potential $|V_T|$ is inversely proportional to the probability density function $|\psi(x)|^2$. Therefore, the smaller the probability density function $|\psi(x)|^2$, the higher the corresponding total potential. Where $\psi(x)$ is

equal to zero, the total potential is infinitely high, so a particle with limited energy cannot reach it, and the probability of finding the particle there is zero.

For a given quantum state $\psi(x)$, the velocity and acceleration of a particle moving in this state can be computed by Eq. (19a) and Eq. (19b) respectively as follows:

$$\dot{x} = \frac{p}{m} = \frac{\hbar}{im} \frac{1}{\psi(x)} \frac{d\psi(x)}{dx}, \quad x(t_0) = x_0 \in \mathbb{C}, \quad (21a)$$

$$\ddot{x} = -\frac{1}{m} \frac{dV_{\text{Total}}}{dx} = -\frac{1}{2} \frac{d}{dx} \left(\frac{\hbar \psi'(x)}{m \psi(x)} \right)^2, \quad \dot{x}(t_0) = \dot{x}_0 \in \mathbb{C}. \quad (21b)$$

The particle's trajectory $x(t)$ in the state $\psi(x)$ can be obtained by integrating Eq. (21a), and the quantum force exerted on the particle at different positions is given by Eq. (21b).

The applied potential $V(x)$ in Eq. (18) to be considered here is the step barrier

$$V(x) = \begin{cases} V_0, & x_R > 0 \\ 0 & x_R < 0 \end{cases}, \quad (22)$$

where the height V_0 of the step barrier is greater than the energy E of the incident particle in order to explore quantum tunneling motion. The solution of the Schrödinger equation (18) with $V(x)$ given by Eq. (22) can be obtained readily as

$$\psi(x) = \begin{cases} Ce^{-k_2x} + De^{k_2x}, & x_R \geq 0 \\ Ae^{ik_1x} + Be^{-ik_1x}, & x_R < 0 \end{cases}, \quad (23)$$

where $k_1 = \sqrt{2mE}/\hbar$ and $k_2 = \sqrt{2m(V_0 - E)}/\hbar$. The range $x_R < 0$ belongs to the region of free motion, while $x_R \geq 0$ is the region where quantum tunneling motion takes place to generate the associated YYD. With the wavefunction $\psi(x)$ given by Eq. (23), the particle's velocity in the region $x_R \geq 0$ can be obtained by Eq. (21a) as

$$\frac{dx}{dt} = \frac{\hbar}{im} \frac{d \ln \psi}{dx} = \frac{\hbar k_2}{im} \frac{-Ce^{-k_2x} + De^{k_2x}}{Ce^{-k_2x} + De^{k_2x}}, \quad \text{Re}(x) \geq 0, \quad (24)$$

where the two constants C and D are to be determined from the given initial conditions. To avoid the influence of physical units on the shape of the quantum YYD, we adopt the dimensionless variables $(\sqrt{2mE}/\hbar)x \rightarrow x$ and $(2E/\hbar)t \rightarrow t$ to rewrite Eq. (24) as

$$\frac{dx}{dt} = \frac{n}{i} \frac{\alpha e^{nx} - e^{-nx}}{\alpha e^{nx} + e^{-nx}}, \quad \text{Re}(x) \geq 0, \quad (25)$$

where $\alpha = C/D$ and $n = \sqrt{V_0/E - 1}$. Above equation can be solved analytically as

$$\alpha e^{nx} - e^{-nx} = \beta e^{-in^2t}, \quad (26)$$

where the two constants α and β are determined from the given initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$ as

$$\alpha = \frac{n+i}{n-i}, \quad \beta = \frac{2i}{n-i}. \quad (27)$$

Due to $|\alpha| = 1$, it can be expressed as $\alpha = e^{i\theta_\alpha}$ with $\theta_\alpha = 2\tan^{-1}(1/n)$. In terms of θ_α and n , the particle's complex trajectory $x(t)$ can be solved from Eq. (26) as an explicit function of time

$$x(t) = \frac{1}{n} \left[-i \frac{\theta_\alpha}{2} + \sinh^{-1} \left(\frac{\beta}{2} e^{-i(n^2t + \theta_\alpha/2)} \right) \right] = x_R(t) + ix_I(t). \quad (28)$$

Regarding the particle's velocity, we substitute Eq. (26) into Eq. (25) to yield an analytical expression for $\dot{x}(t)$

$$\frac{dx}{dt} = \pm \frac{n}{i} \frac{\beta e^{-in^2t}}{\sqrt{\beta^2 e^{-2in^2t} + 4\alpha}} = \dot{x}_R(t) + i\dot{x}_I(t), \quad (29)$$

where the plus sign is taken for the time interval $0 \leq t < T/2$ and the minus sign is for $T/2 \leq t < T$. The quantum YYD shown in Figure 6 is drawn by using $\dot{x}_R(t)$ and $\dot{x}_I(t)$ computed from Eq. (29). The remaining task is to determine the tunneling time $T = t_{\text{out}}$, the turning point x_{tp} , and the exit point x_{out} of the tunneling trajectory, as shown in Figure 7.

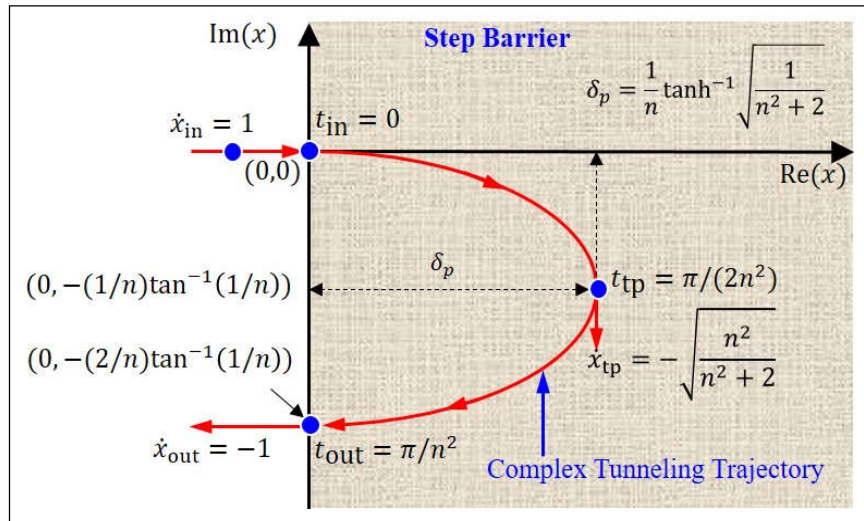


Figure 7. The analytical expression of the tunneling trajectory. The time, position, and velocity of the three trajectory points (entry point x_{in} , turning point x_{tp} , and exit point x_{out}) are expressed analytically in terms of the parameter $n = \sqrt{V_0/E - 1}$.

The turning point x_{tp} is the position where the particle reaches the maximum depth of penetration. Taking the absolute values of both sides of Eq. (26), the relationship between x_R and x_I is obtained as follows:

$$\cosh(2nx_R) = \cos(\theta_\alpha + 2nx_I) + |\beta|^2/2. \quad (30)$$

The maximum value of x_R appears at $\cos(\theta_\alpha + 2nx_I) = 1$ and the related coordinate x_I can be solved as

$$x_I = -\frac{\theta_\alpha}{2n} = -\frac{1}{n} \tan^{-1}\left(\frac{1}{n}\right). \quad (31)$$

Substituting the above x_I into Eq. (30) yields the maximum penetration depth

$$\delta_p = \frac{1}{2n} \cosh^{-1}\left(10 + \frac{2}{n^2+1}\right) = \frac{1}{n} \tanh^{-1}\left(\frac{1}{n^2+2}\right)^{1/2}. \quad (32)$$

When the particle reaches the maximum penetration depth, its position is the turning point x_{tp} , so the complex coordinates of x_{tp} can be expressed as

$$x_{\text{tp}} = \delta_p + ix_I = \delta_p - i\frac{\theta_\alpha}{2n} = \frac{1}{n} \tanh^{-1}\left(\frac{1}{n^2+2}\right)^{1/2} - \frac{i}{n} \tan^{-1}\left(\frac{1}{n}\right). \quad (33)$$

Next we compute the time t_{tp} and the velocity $\dot{x}_{\text{tp}}$ when the particle reaches the turning point t_{tp} . Squaring both sides of Eq. (26) and using the turning condition, $\theta_\alpha + 2nx_I = 0$, we get t_{tp} as

$$t_{\text{tp}} = \pi/(2n^2). \quad (34)$$

The velocity $\dot{x}_{\text{tp}}$ at the turning point is then determined by substituting the position x_{tp} into Eq. (25) to give

$$\dot{x}_{\text{tp}} = \frac{n e^{2n\delta_p} - 1}{i e^{2n\delta_p} + 1} = \frac{n}{i} \tanh(n\delta_p) = -i \left(\frac{n^2}{n^2+2}\right)^{1/2}. \quad (35)$$

This result verifies what Figure 5 and Figure 7 show, that the particle has only imaginary velocity component at the turning point.

Finally we calculate the position x_{out} and time t_{out} at the exit point. From Figure 7 we observe that the real coordinates x_R of the entry point and exit point are both zero. By using this condition in Eq. (30), we get

$$\cos(\theta_\alpha + 2nx_I) = \frac{n^2 - 1}{n^2 + 1} = \cos(\theta_\alpha), \quad (36)$$

from which we find $x_I = 0$ or $x_I = -\theta_\alpha/n$, where the former corresponds to the position of the entrance, and the latter corresponds to the position of the exit. Therefore, the position coordinate of the particle at the exit is

$$x_{\text{out}} = 0 - \frac{i\theta_\alpha}{n} = -i\frac{2}{n}\tan^{-1}\left(\frac{1}{n}\right). \quad (37)$$

Comparing Eq. (37) with Eq. (33), we have $\text{Im}(x_{\text{out}}) = 2\text{Im}(x_{\text{tp}}) = -\theta_\alpha/n$, that is, the imaginary coordinate of the exit point is twice that of the turning point. Furthermore, the time for the particle to reach the exit point can be obtained by substituting x_{out} from Eq. (37) into Eq. (26) as

$$T = t_{\text{out}} = \pi/n^2. \quad (38)$$

Comparing T with t_{tp} in Eq. (34), we find that the time for the particle to reach the turning point is exactly half of the overall tunneling T . This result explains why in Figure 6, when the YYD is drawn halfway at P'_4 , the particle just reaches the turning point P_4 . The particle's velocity at the exit point is found by substituting $x_{\text{out}} = i\theta_\alpha/n$ into Eq. (25) as $\dot{x}_{\text{out}} = -1$. Hence, the reflected velocity $\dot{x}_{\text{out}}$ from the step barrier is the same as the incident velocity $\dot{x}_{\text{in}}$, but in the opposite direction. The characteristics of the entire tunneling trajectory are summarized in Figure 7.

6. Time Evolution of Quantum YYD

The obtained tunneling-based YYD in Figure 6 is actually a graphical recorder of the particle's tunneling dynamics, which stores the particle's velocity at every moment so far. Figure 8 shows the time evolution of the YYD at nine moments, $t_0, t_1, \dots, t_8$, corresponding to nine trajectory points in Figure 6. When time increases from 0 to $T/2$, the real velocity of the particle (in red) decreases from the maximum value at P'_0 to zero at P'_4 , while at the same time, the imaginary velocity of the particle (in black) increases from zero to the maximum value. In the time range of $T/2 \leq t < T$, we see the opposite trend that the real velocity of the particle increases from zero at P'_4 to the maximum value at P'_8 , and at the same time, the imaginary velocity decreases from the maximum value to zero. Only when the quantum tunneling process is completed can we see the complete quantum YYD, wherein the intertwined red and black pattern expresses the alternating changes of the real velocity and the imaginary velocity during the tunneling process.

Figure 9 shows the velocity distributions at nine trajectory points along the tunneling trajectory. It appears that the quantum YYD moves together with the particle, recording the velocity of the particle, and then presenting it graphically. The quantum YYD allows us to examine how the particle nature and the wave nature interchanges during the tunneling process. The imaginary velocity of a particle is a measure of its wave nature such that the greater the imaginary velocity of a particle, the higher the uncertainty of its motion in the real space and the greater the wave nature it has [22]. As shown in Figure 9, when the particle continues to tunnel toward the barrier, the particle's real velocity (in red) gradually decreases while its imaginary velocity (in black) increases. This means that as the particle tunnels deeper, its wave nature becomes larger. When the particle reaches the turning point P_4 , its imaginary velocity reaches the maximum, while its real velocity is zero, which means that the particle behaves completely like a wave at the turning point. In the process of returning from point P_4 to the exit point P_8 , the real velocity increases while the imaginary velocity decreases, indicating that the particle nature gradually recovers toward the exit, at which the imaginary velocity of the particle is zero and the wave nature of the particle disappears completely.

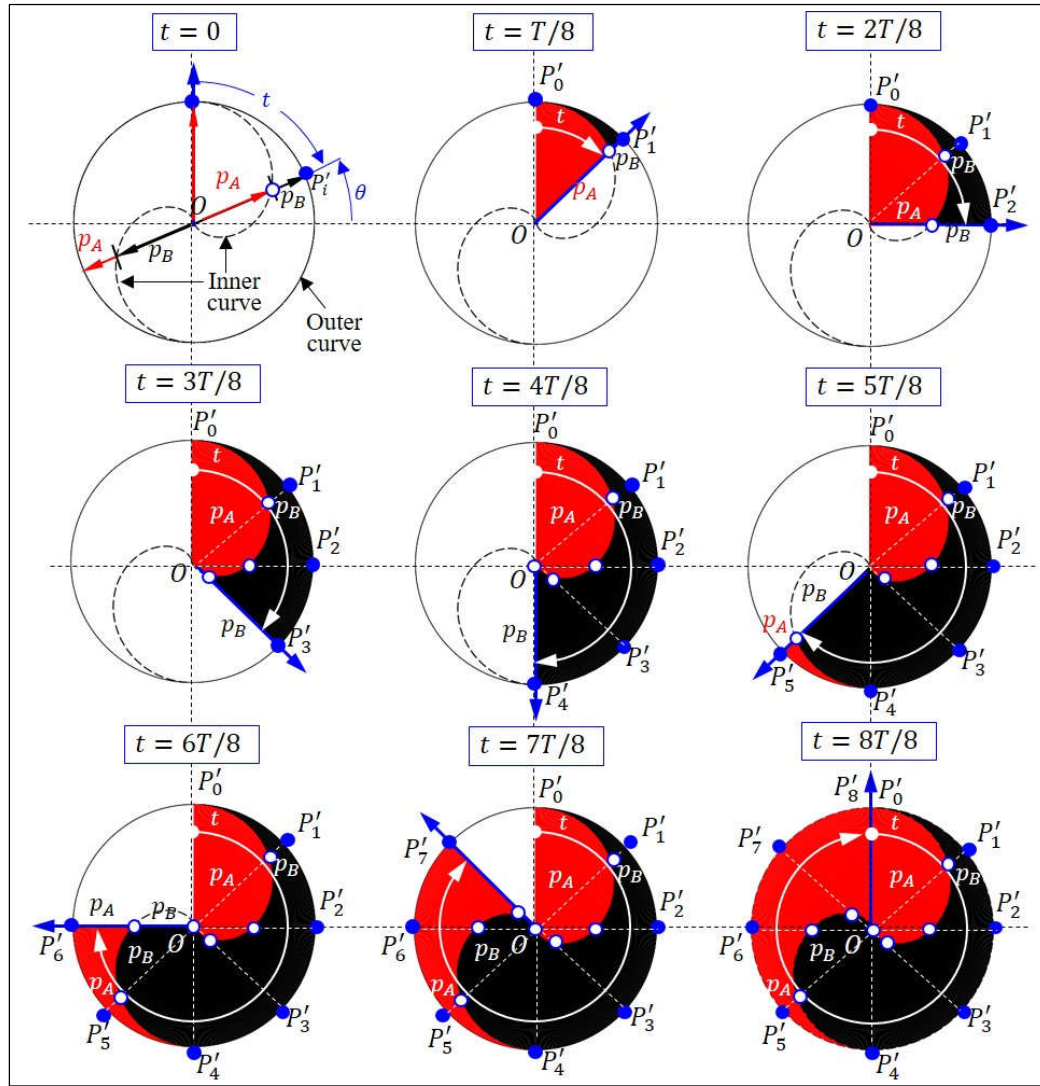


Figure 8. The time evolution of the YYD in the quantum tunneling process. The areas swept by the radius $r = p_A = \dot{x}_R^2$ and $r = p_B = \dot{x}_I^2$ are plotted at nine instances $t = kT/8$, $k = 0, 1, \dots, 8$. The Yang (in red) region is the area swept by $r = p_A(\theta)$, showing the distribution of particle's real velocity, while the Yin (in black) region is the area swept $r = p_B(\theta)$, showing the distribution of particle's imaginary velocity.

The driving force behind the tunneling dynamics comes from the total potential $V_T = V_0 + Q$, as shown by Eq. (21b). Since the applied potential V_0 is a constant, the quantum potential Q is the dominant potential governing the particle's tunneling motion. It is precisely because of the effect of quantum potential Q that the particle is forced to turn back after penetrating the step barrier for a certain distance. The deeper the particle penetrates the step barrier, the greater resistance it encounters. The maximum depth to which a particle penetrates is where it tolerates the greatest resistance.

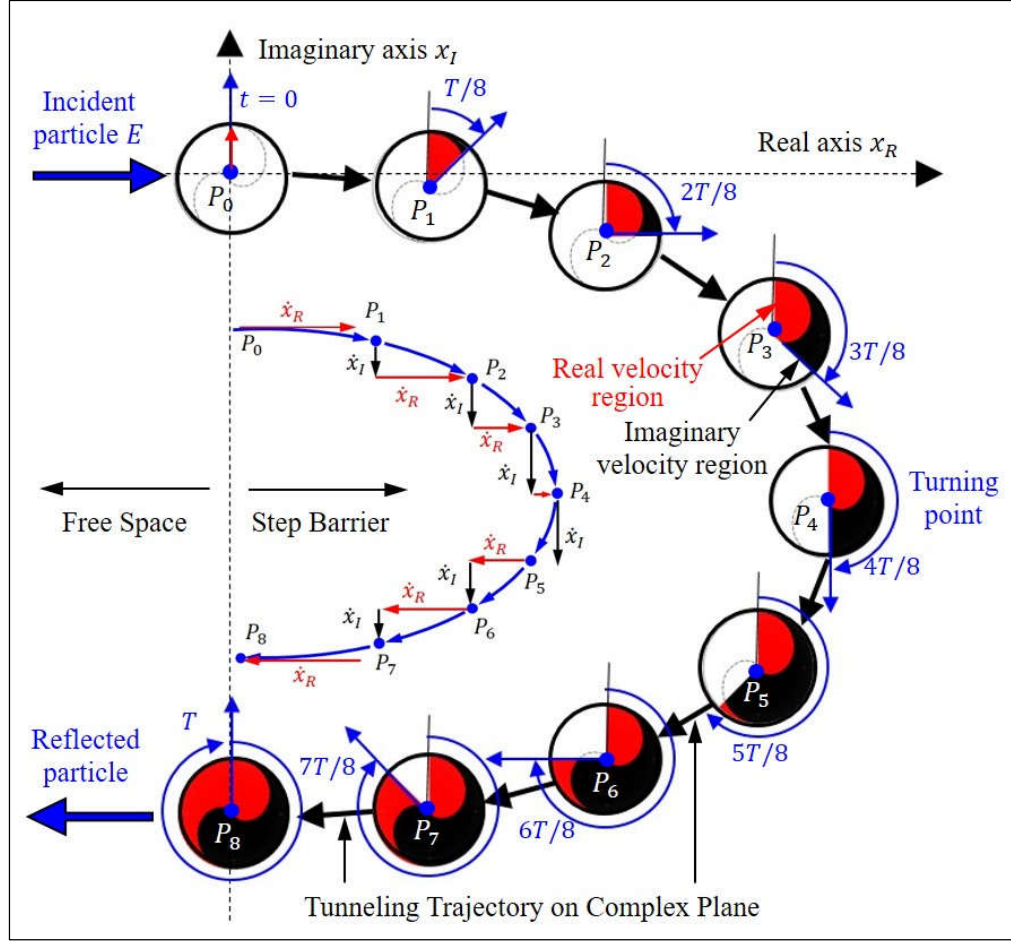


Figure 9. The tunneling-based quantum YYD moves with the particle and records the particle's velocity. From the entry point to the turning point, the real velocity component (in red) decreases while the imaginary velocity component (in black) increases, indicating that the particle nature decreases and the wave nature increases. When reaching the turning point, the particle has only an imaginary velocity component (pure wave nature). From the turning point to the exit point, the red area increases while the black area decreases, indicating that the particle nature increases and the wave nature decreases. When the particle reaches the exit, the particle is left with only real velocity component and exhibits pure particle nature.

We can determine the total potential V_T by substituting the wavefunction $\psi(x)$ from Eq. (23) into Eq. (20) to obtain the following result

$$\bar{V}_T = \frac{Q(x) + V_0}{E} = 1 + \left(\frac{d \ln \psi(x)}{dx} \right)^2 = 1 + n^2 \left(\frac{\alpha e^{nx} - e^{-nx}}{\alpha e^{nx} + e^{-nx}} \right)^2, \quad (39)$$

where V_T has been expressed in a dimensionless form. Once the total potential is obtained, the quantum force exerted on the particle during the tunneling process can be found as

$$\bar{F}_Q = -\frac{d\bar{V}_T}{dx} = -2n^3 \left(\frac{\alpha e^{nx} - e^{-nx}}{\alpha e^{nx} + e^{-nx}} \right) \left[1 - \left(\frac{\alpha e^{nx} - e^{-nx}}{\alpha e^{nx} + e^{-nx}} \right)^2 \right]. \quad (40)$$

If expressed in terms of the particle's velocity $\dot{x}$, $\bar{F}_Q$ can be rewritten in a simple form as

$$\bar{F}_Q = -2i\dot{x}(n^2 + \dot{x}^2). \quad (41)$$

With the velocity $\dot{x}_{tp}$ given by Eq. (35), we can determine the quantum force exerted on the particle at the turning point as

$$(\bar{F}_Q)_{\text{tp}} = -\frac{2n^3(n^2 + 1)}{(n^2 + 2)^{3/2}}. \quad (42)$$

As expected, the quantum force experienced by the particle at the turning point is in the negative x_R direction, preventing the particle from continuing to penetrate in the positive x_R direction. Eq. (42) shows that as $n = \sqrt{V_0/E - 1}$ increases, the resistance encountered by the particle becomes greater, resulting in a smaller penetration depth as given by Eq. (32).

From the perspective of probability, the higher the step barrier V_0 , the lower the probability of tunneling. The transition from the probability interpretation to the trajectory interpretation of the tunneling phenomenon requires the assistance of the quantum potential $Q(x)$. Figure 10 shows the complex trajectories of the particle moving over the total potential surface $V_T = Q(x) + V_0$ in the two cases of $n = 1$ and $n = 5$, indicating that the turning point P_4 is the place with the highest total potential in the overall tunneling trajectory, while the entry point P_0 and the exit point P_8 are the places with the lowest total potential. We also observe from the figure that when n is larger, the total potential surface becomes steeper, and the depth that the particle can penetrate is shallower.

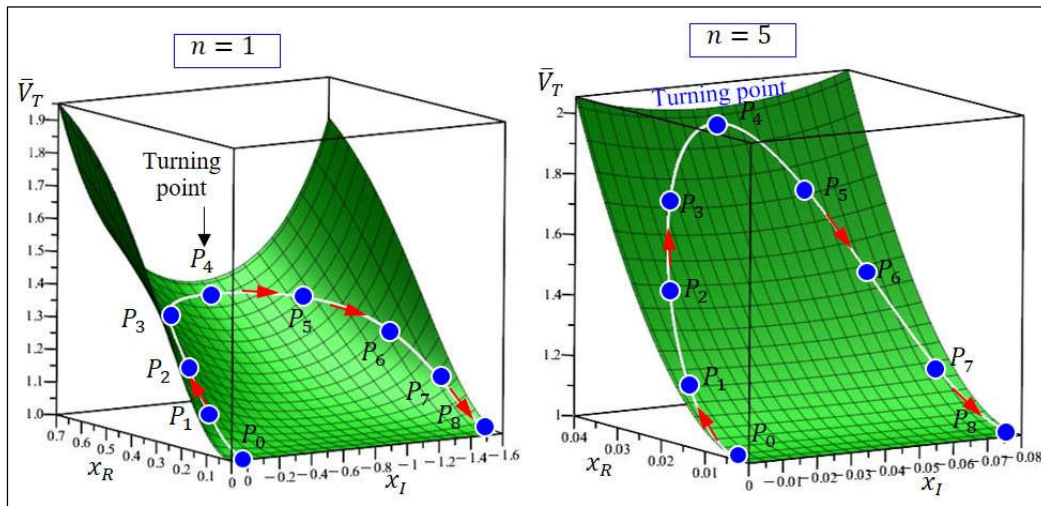


Figure 10. The complex trajectories of a tunneling particle on the total potential surface. When a particle undergoes quantum tunneling towards the real axis x_R , the height of the quantum potential increases with the tunneling depth. When the particle reaches the turning point (P_4), the quantum potential reaches its maximum. The figure shows that the higher the step barrier n , the steeper the tunnel trajectory and the shallower the tunneling depth. When the step barrier height n approaches infinity, the tunneling depth of the particle is zero, that is, the particle is immediately reflected when it encounters the surface of the step barrier.

7. Growth of Quantum YYD with Tunneling Intensity

The quantum YYD based on tunneling velocity $(p_A, p_B) = (\dot{x}_R^2, \dot{x}_I^2)$ is not always an ideal YYD and its shape depends on the intensity of the quantum tunneling effect. We find that the stronger the quantum tunneling effect, the clearer and more complete the quantum YYD. Once the intensity of the tunneling effect approaches zero, the quantum YYD disappears. The intensity of the tunneling effect is related to the relative magnitude between the incident particle's energy E and the step barrier height V_0 . This relative magnitude is measured by the dimensionless parameter $n = \sqrt{V_0/E - 1}$. The larger the value of n , the smaller the particle's energy is compared to the barrier height, and a larger tunneling effect is required for the particle to penetrate into the barrier.

What makes us curious is how the intensity of the tunneling effect affects the quantum YYD. Figure 11 shows the quantum YYDs corresponding to different values of n . We can see that the quantum YYD is born from $n = 0$, initially like a melon seed, and gradually grows into a perfect appearance as n increases. As $n \gg 1$, the quantum YYD no longer changes and the corresponding polar equation (10) converges to

$$r = \begin{cases} \cos^2(n^2 t), & 0 \leq t < T/2 \\ \sin^2(n^2 t), & T/2 \leq t < T \end{cases} = \begin{cases} (1 + \cos \theta)/2, & 0 \leq \theta \leq \pi \\ (1 - \cos \theta)/2, & \pi \leq \theta \leq 2\pi \end{cases} \quad (43)$$

where $T = \pi/n^2$ is the dimensionless tunneling time. The satisfaction of the complementarity condition $p_A + p_B = \cos^2(n^2 t) + \sin^2(n^2 t) = 1$ and the continuity condition $p_A(\pi) = p_B(2\pi) = 0$ shows that the quantum YYD described by Eq. (43) is an ideal YYD, whose inner curve is composed of two sections of cardioid.

Comparing Eq. (43) with Eq. (5), we are surprised to find that the ideal YYD generated by the tunneling motion resembles the Bohr's YYD designed in his coat of arms. If Bohr drew YYD based on Eq. (43) instead of Eq. (5), then the coat of arms he designed would completely reflect his principle of complementarity not only in image, but also in theory.

Figure 11 shows that when the parameter $n = \sqrt{V_0/E - 1}$ is much greater than 1, the tunneling-based YYD converges to a steady-state pattern, known as the ideal YYD. Since the ideal YYD is drawn based on the particle's tunneling trajectory, it is expected that the tunneling trajectory also converges to a steady-state pattern as $n \gg 1$. We will verify that this steady-state trajectory is a semicircle on the complex plane.

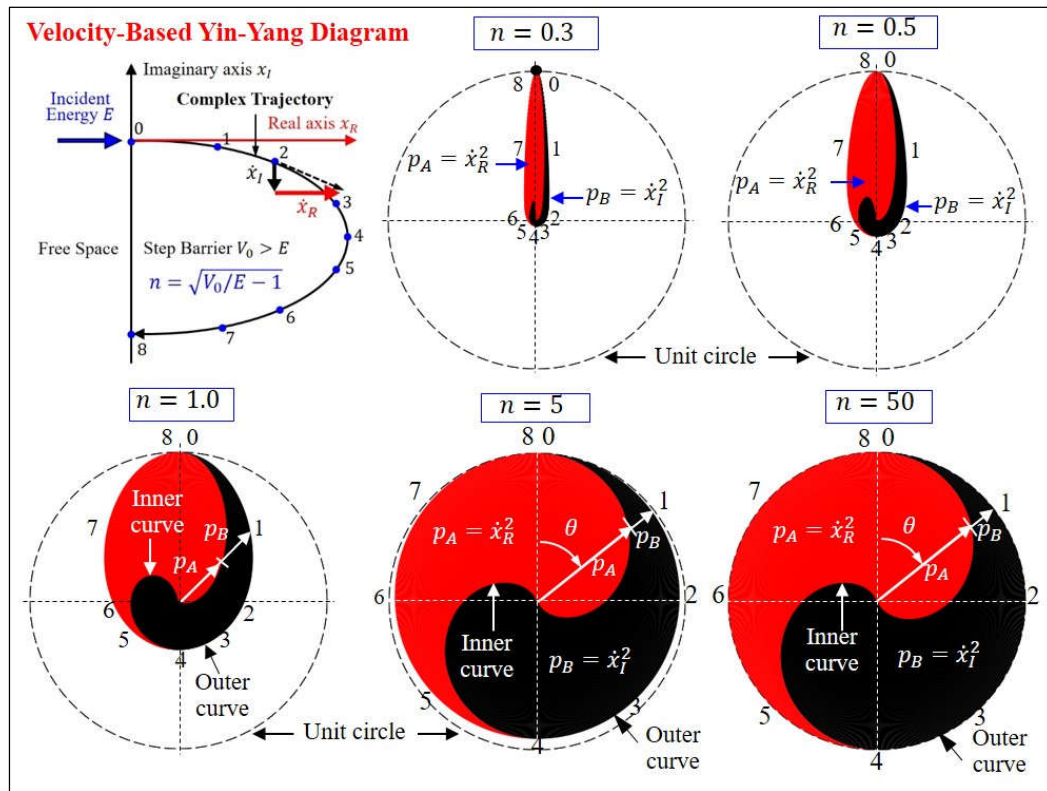


Figure 11. The evolution of the velocity-based YYD with the intensity of the tunneling effect $n = \sqrt{V_0/E - 1}$. When n is small, the real velocity component $\dot{x}_R$ (particlelike behavior) is dominant in the YYD, leading to a deformed YYD with a long strip shape concentrating near $\theta = 0$. When n increases, the imaginary velocity component $\dot{x}_I$ (the wavelike behavior) becomes comparable to $\dot{x}_R$, and the resulting YYD is fuller. As $n \gg 1$, the velocity-based YYD approaches the ideal YYD whose inner curve is composed of two sections of cardioid and whose outer curve is a unit circle.

Figure 12 shows the tunneling trajectories of the particle on the complex plane solved from Eq. (25) under several different values of n . The main feature of this series of trajectories indicates that as the value of n increases, the covering range of the particle's position x becomes narrower, as expressed by the relation $|nx| \approx 1/n$. Therefore, when n is large, we get the result $|nx| \ll 1$. With

this observation we can approximate the exponential function e^{nx} by its first-order expansion $1 + nx$, which is then used in Eq. (25) to yield the particle's velocity for $n \gg 1$ as

$$\frac{dx}{dt} = \frac{n \alpha_2 e^{nx} - e^{-nx}}{i \alpha_2 e^{nx} + e^{-nx}} \approx \frac{1 - in^2 x}{1 + ix} \approx 1 - in^2 x, \quad n \gg 1, \quad (44)$$

where we note $|nx| \ll 1$ and $1 + ix \approx 1$. With the initial condition $x(0) = 0$, the solution of Eq. (44) can be found readily as

$$x(t) = \frac{1}{n^2} \sin(n^2 t) - \frac{i}{n^2} (1 - \cos(n^2 t)) = x_R(t) + ix_I(t), \quad n \gg 1, \quad (45)$$

From Eq. (45), we can easily obtain the position, velocity and time when the particle reaches the turning point and the exit point. These results are identical to those summarized in Figure 7, when n is large. For example, the overall tunneling time T is the time that the real coordinate $x_R(t)$ returns to zero, and from Eq. (45) we have $T = \pi/n^2$, which is identical to Eq. (38). In addition, the penetration depth δ_p is the maximum value of the real part of $x(t)$, which is equal to $1/n^2$ from Eq. (45), and this result is identical to Eq. (32) as n is large.

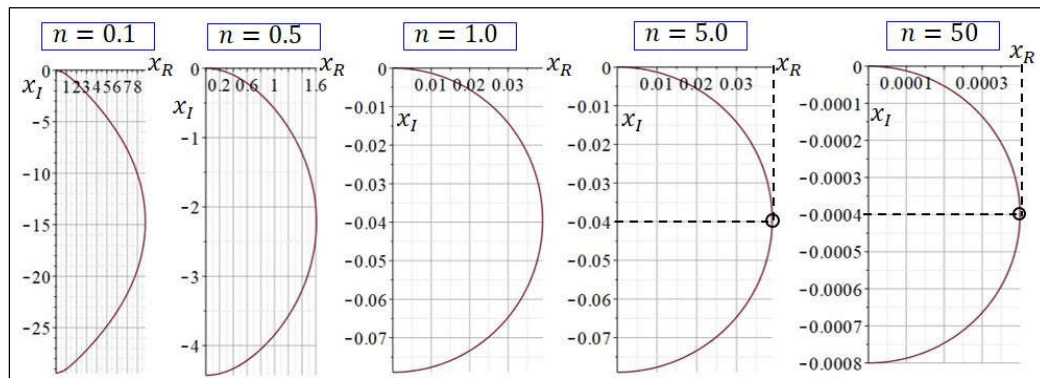


Figure 12. The change of the particle's tunneling trajectory on the complex plane with the parameter n . For large n , the particle's position x within the barrier is more restricted, as expressed by the relation $|x| \ll n^{-1}$. As n increases, the tunneling trajectory of the particle eventually approaches a perfect semicircle.

The tunneling trajectory expressed by Eq. (45) is a semicircle on the complex plane with radius $1/n^2$ and center $(0, -1/n^2)$:

$$(x_R)^2 + (x_I + 1/n^2)^2 = 1/n^4, \quad x_R \geq 0, \quad n \gg 1. \quad (46)$$

It is based on this ideal tunneling trajectory that we can construct the ideal quantum YYD. Differentiating Eq. (46) with respect to time, we obtain the real and imaginary velocities of the particle as

$$\dot{x}_R(t) = \cos(n^2 t), \quad \dot{x}_I(t) = -\sin(n^2 t), \quad n \gg 1. \quad (47)$$

Using Eq. (47) in Eq. (10), we obtain the ideal YYD as given by Eq. (43).

The YYD derived previously is based on the velocity distribution of a tunneling particle, so a natural question is whether the quantum YYD can also be constructed by the position distribution of a tunneling particle. The answer is yes, and surprisingly, when the value of n is large, the YYDs obtained by these two different methods are exactly the same. The construction procedure of the position-based YYD is the same as that of the velocity-based YYD, except that the quantitative measures p_A and p_B are given by the real and imaginary positions as:

$$p_A = (n^2 x_I + 1)^2, \quad p_B = (n^2 x_R)^2. \quad (48)$$

According to Eq. (46), the outer curve of the position-based YYD is $r = p_A + p_B = 1$, which is a unit circle, and the corresponding inner curve defined by Eq. (1) becomes

$$r = \begin{cases} (n^2 x_I + 1)^2, & 0 \leq t < T/2 \\ (n^2 x_R)^2, & T/2 \leq t < T \end{cases} = \begin{cases} \cos^2(n^2 t), & 0 \leq t < T/2 \\ \sin^2(n^2 t), & T/2 \leq t < T \end{cases} \quad (49)$$

where $x_R(t)$ and $x_I(t)$ are given by Eq. (45). It can be seen that Eq. (43) and Eq. (49) are equivalent, which implies that the velocity-based YYD and the position-based YYD becomes identical as $n \gg 1$.

For a general value of n not satisfying $n \gg 1$, the symmetric center of the tunneling trajectory is not at $(x_R, x_I) = (0, -1/n^2)$, but at $(0, -(1/n)\tan^{-1}(1/n))$ as can be seen from Figure 7. Hence, for a general position-based YYD, the quantitative measures p_A and p_B in Eq. (48) are replaced by

$$p_A = (n^2 x_I + n \tan^{-1}(1/n))^2, \quad p_B = (n^2 x_R)^2, \quad (50)$$

where the time history of the complex position $x(t) = x_R(t) + ix_I(t)$ is solved from Eq. (25) for a general value of n .

Figure 13 shows how the position-based YYD evolves with the tunneling intensity n . It can be seen that the YYD emerges from the origin with $n = 0$, initially resembling a walnut in appearance, and grows into an ideal YYD as n increases. Regardless of whether we use the velocity or position data to construct the YYD, we find that when the value of n is large, the two methods result in the same ideal YYDs, whose outer curve approaches a perfect circle, and whose inner curve approaches a cardioid.

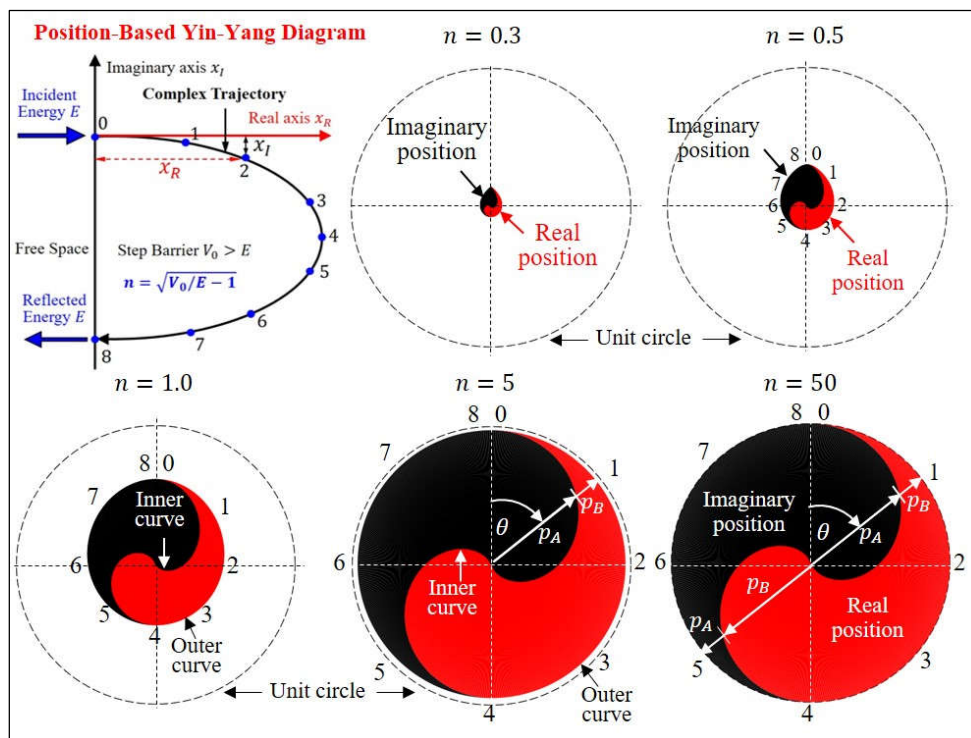


Figure 13. The growth of the position-based YYD with the tunneling intensity n . On plotting the position-based YYD, the quantitative measures are given by $p_A = (n^2 x_I + n \tan^{-1}(1/n))^2$ and $p_B = (n^2 x_R)^2$. As n increases from 0, the YYD grows from the origin, initially like the shape of a walnut. As $n \gg 1$, the position-based YYD becomes identical to the velocity-based YYD.

8. Conclusions

In this article we discovered Bohr's YYD in quantum tunneling dynamics and discussed four special forms of YYD relating to BPC. Bohr's YYD in his coat of arms is a popular YYD consisting of a unit circle and two inscribed circles, as shown in Figure 14a. Except for being similar to the

tunneling-based YYD in appearance, this popular YYD does not have any physical meaning. The second type of YYD is the traditional Taoist YYD, which is an ancient realization of BPC as shown in Figure 14b. The third type of YYD in Figure 14c visualizes a particle's tunneling motion by displaying the time evolution of the particle's position and velocity. The fourth type of YYD in Figure 14d is drawn based on the data of the two-path interference experiments for testing the wave-particle duality. Taoist YYD has been reproduced in this article by BPC, which in turn justifies Bohr's use of Taoist YYD as the icon of his principle. The only pity is that the YYD designed by Bohr in his cost of arms is not the Taoist YYD drawn based on his principle. However, Bohr did not need to borrow Taoist YYD to symbolize the wave-particle duality, because the wave-particle duality has its own representative YYDs, i.e., the tunneling-based YYD and the interference-based YYD, as revealed in this article. Our investigation shows that Bohr's YYD and its quantum version, the tunneling-based YYD, belong to the same category, while the Taoist YYD and its quantum version, the interference-based YYD, belong to another category.

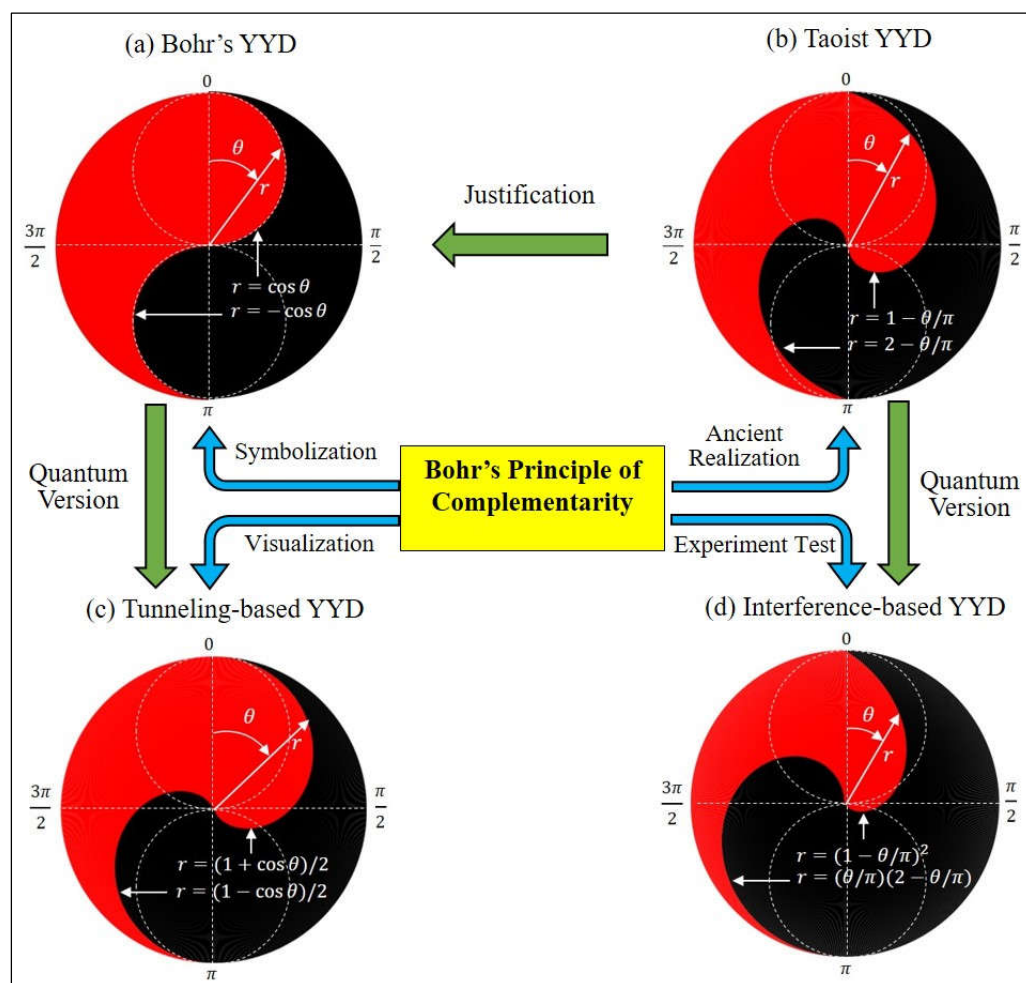


Figure 14. The interconnection of four YYDs and their relationship with BPC. (a) Bohr's YYD is based on a popular construction to symbolize the principle of complementarity. (b) Taoist YYD is proved to be an ancient realization of BPC, which justifies the role of YYD in Bohr's principle. (c) The tunneling-based YYD visualizes a particle's tunneling motion via a step barrier by displaying the time evolution of the particle's velocity. (d) The interference-based YYD is drawn by the data from two-path interferometers to test BPC. According to the inner curves of the four YYDs, Bohr's YYD and its quantum version, the tunneling-based YYD, belong to the same category, while Taoist YYD and its quantum version, the interference-based YYD, belong to another category.

The types of YYD are not limited to those shown in Figure 14. In fact, every system in nature has its accompanying YYD that can be constructed according to Eq. (1). Any complementary pair A and

B in a system, whether classical or quantum, can be employed to construct the system's YYD, if their quantitative measures p_A and p_B are available within a period of time. When the system further satisfies the conditions of continuity and complementarity, the obtained YYD turns out to be an ideal one, whose outer curve is a perfect circle with continuous inner curve. We can then judge the degree to which the system satisfies the condition of complementarity based on the deviation of the system's YYD from the ideal YYD.

Funding: This research was funded by Taiwan National Science and Technology Council under Award No. MOST 111-2221-E-006-215-MY3.

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