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Hani Khashan * and Eman Hussein

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S-Ideals: A Unified Framework for Ideal Structures via Multiplicatively Closed Subsets †

Hani A. Khashan [1,*] and Eman Hussein [2]

1 Department of Mathematics, Al al-Bayt University, Mafraq, Jordan
2 Department of Mathematics, Amman Arab University, Amman, Jordan
* Correspondence: hakhashan@aabu.edu.jo
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Abstract

In this paper, we study ideals defined with respect to arbitrary multiplicatively closed subsets $S \subseteq R$ of a commutative ring R. An ideal $I \subseteq R$ is called an S-ideal if for all $a, b \in R$, the condition $ab \in I$ and $a \in S$ implies $b \in I$. This is equivalent to the identity $I = S^{-1}I \cap R$, where $S^{-1}I$ is the extension of I in the ring of fractions $S^{-1}R$. The concept of S-ideals provides a unified framework encompassing several classical ideal types. For instance, r-ideals arise when $S = \mathrm{reg}(R)$, the set of regular elements. If $S = R \setminus P$ for a prime ideal P, then the S-ideals coincide with P-primary ideals. Ideals that admit primary decomposition correspond to S-ideals for which S is the complement of a finite union of prime ideals. Moreover, z_0-ideals are S-ideals when S is the complement of a union of minimal prime ideals of R. We generalize several results known for r-ideals to this broader setting and investigate structural and closure properties of S-ideals in various contexts. As an application, we give a characterization of the von Neumann regularity of the localization $S^{-1}R$ in terms of S-ideals. We also study the behavior of S-ideals in polynomial rings, idealizations, and amalgamated constructions with respect to different choices of S.

Keywords: S-ideals; r-ideals; multiplicatively closed subst

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1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. For such a ring R, we denote by $U(R)$ the set of units, by $\mathrm{reg}(R)$ the set of regular (i.e., non zero-divisor) elements, by $\mathrm{zd}(R)$ the set of zero divisors, by $N(R)$ the set of nilpotent elements, and by $J(R)$ the Jacobson radical of R. If A is a subset of R and I is an ideal of R, we define the ideal quotient

$$(I : A) = \{r \in R \mid rA \subseteq I\}.$$

In particular, the annihilator of A is defined by $\mathrm{Ann}(A) := (0 : A)$.

Multiplicatively closed subsets of a commutative ring have long played a fundamental role in commutative algebra, particularly in the localization of rings and modules. Moreover, they play a central role in the structure theory of ideals, offering a flexible framework for defining and analyzing diverse classes of ideals. For instance, the m.c.s. of regular elements of a ring R is used to define r-ideals, where a proper ideal I of R satisfies the condition: if $ab \in I$ and $a \in \mathrm{reg}(R)$, then $b \in I$.

We aim in this paper to provide a general framework for defining ideals using arbitrary multiplicatively closed subsets of a ring. Specifically, we introduce the notion of S-ideals. For an arbitrary multiplicatively closed subset S, a proper ideal I of a ring R is called an S-ideal if for all $a, b \in R$, the condition $ab \in I$ and $a \in S$ implies $b \in I$. Equivalently, I is an S-ideal if and only if $S^{-1}I \cap R = I$.

In this work, we explore the theory of S-ideals in depth, examining their foundational properties, characterizations, relationships to other types of ideals, and their behavior under various ring constructions and extensions. Beyond r-ideals, which are defined with respect to the multiplicatively closed set $S = \text{reg}(R)$, we show in this paper that several well-known classes of ideals naturally arise as S-ideals for suitable choices of multiplicatively closed subsets $S \subseteq R$. For instance, every P-primary ideal of R is an S-ideal when $S = R \setminus P$. More generally, any ideal that admits a primary decomposition can be viewed as an S-ideal where S is the complement of the finite union of the associated prime ideals.

Moreover, z_0-ideals, which are central in the theory of rings with zero-divisors, fit naturally within the S-ideal framework as well. Recall that an ideal $I \subseteq R$ is a z_0-ideal if, for each element $a \in I$, the intersection P_a of all minimal prime ideals containing a is also contained in I. We show that these ideals can be considered as S-ideals with respect to the multiplicatively closed subset $S = R \setminus \bigcup\{\mathfrak{p} \mid \mathfrak{p} \text{ minimal prime of } R\}$. These examples demonstrate how varying the choice of multiplicatively closed subsets enables a unified and generalized approach to studying ideal structures across various classes.

In addition to connecting familiar ideal classes to suitable multiplicatively closed subsets, we explore how certain specific forms of S lead to meaningful characterizations of S-ideals. For example, when $S = 1 + J$ for an ideal J of R, or when S is the set of powers of a fixed non-unit element, we provide explicit descriptions of the resulting S-ideals. Similarly, when S consists of primitive polynomials in a polynomial ring over R, or other carefully structured multiplicatively closed subsets, we are able to determine necessary and sufficient conditions for an ideal of R to be an S-ideal. These cases illustrate the flexibility of the S-ideal framework and highlight its potential for unifying diverse ideal-theoretic behaviors under a common generalization.

Section 2 of the paper is devoted to presenting several fundamental properties and characterizations of S-ideals. In addition to foundational results highlighted earlier, Proposition 1 provides a concise summary of key properties and includes a wide range of illustrative examples. One of the notable results we establish is that every maximal S-ideal of a ring R is prime (Proposition 4). Furthermore, we generalize the classical Prime Avoidance Lemma to the broader setting of multiplicatively closed subsets, offering a new version tailored to S-ideals (Corollary 1). When $S \subseteq \text{reg}(R)$, we explore connections between S-ideals and von Neumann regularity in the ring of fractions $S^{-1}R$. In particular, we provide two distinct characterizations of this regularity: one in terms of semiprime S-ideals and the other via minimal prime S-ideals (Theorems 2 and 3). We also identify rings where $\{0\}$ is the only S-ideal and provide a precise characterization in Proposition 6. For an arbitrary multiplicatively closed set S, Theorem 5 describes the structure of nonzero S-ideals within GCD-domains.

In the final section, we examine the behavior of S-ideals under various ring-theoretic constructions and extensions. These include localizations (Theorem 9), homomorphic images or quotient rings (Proposition 9), and direct products of rings (Proposition 10). We also study S-ideals in more advanced constructions such as idealization rings (Proposition 13) and the amalgamation of rings along an ideal, with key results captured in Theorems 10 and 11.

2. S-Ideals and Their Foundational Properties

In this section, we develop the foundational theory of S-ideals in commutative rings. We provide key definitions, explore characterizations, and relate S-ideals to known classes such as prime, semiprime and r-ideals. Several properties are examined with respect to different types of multiplicatively closed subsets. We also characterize when the localization $S^{-1}R$ is von Neumann regular in terms of S-ideals. These results unify and extend existing notions like r-ideals under a broader framework.

Definition 1. *Let I be a proper ideal of a ring R, and let S be a multiplicatively closed subset of R. We say that I is an S-ideal if for all $a, b \in \mathbb{R}$, the conditions $ab \in I$ and $a \in S$ imply that $b \in I$.*

If I is an S-ideal of a ring R, then $I \cap S = \emptyset$. Indeed if there is $s \in I \cap S$, then $s \cdot 1 \in I$ and $s \in S$ imply $1 \in I$, a contradiction. However, the converse is not true in general. For example, choose

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$S = \{2^k : k \geq 0\} \subseteq \mathbb{Z}$ and $I = 6\mathbb{Z}$. Then $I \cap S = \emptyset$ but I is not an S-ideal of $\mathbb{Z}$ since $2 \cdot 3 = 6 \in I$ and $2 \in S$ but $3 \notin I$. It is clear that if $S \subseteq T$ are multiplicatively closed subsets of a ring R, then any T-ideal of R is an S-ideal.

We begin with the following elementary result, whose proof is straightforward and is left to the reader.

Proposition 1. *Let S be a multiplicatively closed subset of a ring R.*

1. The class of $reg(R)$-ideals is the same as the class of r-ideals.
2. Every proper ideal in a ring R is an S-ideal if and only if $S = U(R)$.
3. The ideal $\{0\}$ of R is an S-ideal if and only if $S \subseteq reg(R)$.
4. If I is an S-ideal of R, then so is $\sqrt{I}$.
5. The intersection of any family of S-ideals is an S-ideal.
6. If I is an S-ideal and $A \subseteq R \backslash I$, then $(I : A)$ is an S-ideal. In particular, if $S \subseteq reg(R)$, then $Ann(A)$ is always an S-ideal.
7. If $S \subseteq reg(R)$, then every von Neumann regular ideal is an S-ideal.
8. If I is a proper ideal of R, then I is an S-ideal if and only if for ideals K and J of R, $KJ \subseteq I$ and $K \cap S \neq \emptyset$ imply $J \subseteq I$.
9. If K is an ideal of R with $K \cap S \neq \emptyset$ and I, J are S-ideals such that $IK = JK$, then $I = J$.
10. If I and J are ideals of R such that $J \cap S \neq \emptyset$ and IJ is an S-ideal, then $I = IJ$.
11. A prime (resp. primary) ideal P of R is an S-ideal if and only if $P \cap S = \emptyset$.

By using Zorn's Lemma, one can prove that if I is an ideal of a ring R such that $I \cap S = \varnothing$, then there exists a prime ideal P of R such that $I \subseteq P$ and $P \cap S = \varnothing$. Thus, every ideal disjoint from S is contained in a prime S-ideal.

Remark 1. *Any ideal I of a ring R is an S-ideal with respect to some multiplicatively closed subset of R. Indeed, the set $S := \{a \in R \mid ab \in I \Longrightarrow b \in I \ \ \forall b \in R\}$ is the largest multiplicatively closed set for which I is an S-ideal.*

Let R be a commutative ring and S a multiplicatively closed subset of R. For any ideal I of R, the S-compoment of I is defined as:

$$I_S := \{r \in R \mid \exists s \in S \text{ such that } sr \in I\}.$$

It is clear that I_S is an ideal of R containing I and it is well-known that $I_S = S^{-1}I \cap R$, [8]

Proposition 2. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. Then I_S is the smallest S-ideal of R containing I.*

Proof. Let $a, b \in R$ such that $ab \in I_S$ and $a \in S$. Then there is $s \in S$ such that $(sa)b = s(ab) \in I$. Hence, $b \in I_S$ and I_S is an S-ideal of R. Now, let J be an S-ideal such that $I \subseteq J$. Take $r \in I_S$ so that there is $s \in S$ such that $sr \in I \subseteq J$. Since J is an S-ideal and $s \in S$, then $r \in J$. Hence $I_S \subseteq J$, and I_S is the smallest S-ideal containing I. □

Let I be an ideal of a ring R and S be a multiplicatively closed subset of R. Following [11], the S-radical of I is defined by

$$\sqrt[S]{I} = \{a \in R : sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\},$$

Which is an ideal of R containing $\sqrt{I}$. We can easily conclude that $\sqrt[S]{I} = (\sqrt{I})_S$ and so $\sqrt[S]{I}$ is an S-ideal of R.

For a multiplicatively closed subset S of a ring R, the following theorem provides a useful characterization of S-ideals in R.

Theorem 1. *Let R be a ring, I a proper ideal of R, and S a multiplicatively closed subset of R. The following are equivalent:*

1. I is an S-ideal of R.
2. $(I : s) = I$ for every $s \in S$.
3. $I_S = S^{-1}I \cap R = I$.

Proof. (1) $\Longrightarrow$ (2) Assume I is an S-ideal and let $s \in S$. If $b \in (I : s)$. Then $sb \in I$ and so $b \in I$ as I is an S-ideal. Thus, $(I : s) \subseteq I$ and so $(I : s) = I$ as the reverse containment is obvious.

(2) $\Longrightarrow$ (3) Assume that for every $s \in S$, $(I : s) = I$ and let $b \in I_S$ so that there exists $s \in S$ such that $sb \in I$. Then $b \in (I : s) = I$. Therefore, $I_S \subseteq I$ and since clearly $I \subseteq I_S$, we get $I_S = I$.

(3) $\Longrightarrow$ (1) Suppose $I_S = I$ and let $a, b \in R$ such that $ab \in I$ and $a \in S$. Then $b \in I_S = I$ and I is an S-ideal of R. □

Recall that for an ideal I of a ring R, $Min(I)$ denotes the set of minimal prime ideals of R that contain I. The following proposition shows that minimal prime ideals over an S-ideal are themselves S-ideals.

Proposition 3. *Let S be a multiplicatively closed subset of a ring R and $P \in Min(I)$, where I is an S-ideal of R. Then P is an S-ideal.*

Proof. Let $a, b \in R$ such that $ab \in P$ and $a \in S$. by [7] [Theorem 2.1], there exist $x \notin P$ and $n \in \mathbb{N}$ such that $x(ab)^n = xa^n b^n \in I$. Since $a^n \in S$ and I is an S-ideal, we infer that $xb^n \in I \subseteq P$. It follows that $b^n \in P$ as $x \notin P$. Thus $b \in P$ and P is is an S-ideal of R. □

By Proposition 1 (3), we conclude that whenever $S \subseteq reg(R)$, then any minimal prime ideal of R is an S-ideal.

It is important to note that the conclusion of the preceding proposition may fail if the prime ideal P is not a minimal prime over I, as demonstrated in the following example.

Example 1. *Consider the non-prime ideal $I = \langle x^2 \rangle$ in $\mathbb{Z}[x]$. Then the prime ideal $P = \langle 2, x \rangle$ is not minimal over I. Consider the multiplicatively closed subset $S = \{f \in \mathbb{Z}[x] : f(0) \neq 0\}$ of $\mathbb{Z}[x]$. Then P is not an S-ideal since clearly, $P \cap S \neq \emptyset$. We prove that I is an S-ideal. Let $f = \sum_{j=0}^{m} a_j x^j \in S$ and $g = \sum_{i=0}^{n} b_i x^i \in \mathbb{Z}[x]$ such that $fg \in I = \langle x^2 \rangle$. Then $a_0 b_0 = 0$, $a_0 b_1 + a_1 b_0 = 0$ and $a_0 \neq 0$ imply $b_0 = b_1 = 0$. Therefore, $g \in I$ and I is an S-ideal of $\mathbb{Z}[x]$.*

Next, we prove that maximal S-ideals of a ring R are prime.

Proposition 4. *Let S be a multiplicatively closed subset of a ring R. Then any maximal S-ideal of R is prime.*

Proof. Suppose I is a maximal S-ideal of R. Let $a, b \in R$ such that $ab \in I$ and $a \notin I$. Then the ideal $(I : a)$ is proper in R and is an S-ideal by Proposition 1 (6). Since $I \subseteq (I : a)$ and I is a maximal S-ideal, then $b \in (I : a) = I$ and I is a prime ideal of R. □

Let $B \subseteq \bigcup_{i \in I} A_i$, where B and each A_i are subsets of a ring R. Recall that this inclusion is said to be *irreducible* if no A_i can be removed from the union; that is, for every $i \in I$, we have $B \nsubseteq \bigcup_{j \in I \setminus \{i\}} A_j$.

In the following, we generalize [13] [Theorem 3.8] concerning irreducible coverings of ideals by replacing the set of regular elements with an arbitrary multiplicatively closed set.

Proposition 5. *Let $I \subseteq \bigcup_{i=1}^{n} J_i$ be an irreducible inclusion of ideals in a ring R and let $S \subseteq R$ be a multiplicatively closed set. Suppose that J_1 is an S-ideal and that $J_i \cap S \neq \emptyset$ for all $i = 2, \ldots, n$. Then $I \subseteq J_1$.*

Proof. Since the inclusion is irreducible, we have $I \nsubseteq \bigcup_{i=2}^{n} J_i$, so there exists an element $a \in I \setminus \bigcup_{i=2}^{n} J_i$. In particular, $a \in J_1$. Now take any $x \in I \cap \bigcap_{i=2}^{n} J_i$; then $x + a \in I$, but $x + a \notin \bigcup_{i=2}^{n} J_i$ because otherwise we would have $a \in \bigcup_{i=2}^{n} J_i$, contradicting the choice of a. Therefore, $x + a \in J_1$, and since $a \in J_1$, it follows that $x \in J_1$. Hence, $I \cap \bigcap_{i=2}^{n} J_i \subseteq J_1$, and so $I \cdot \prod_{i=2}^{n} J_i \subseteq J_1$. Now, since each $J_i \cap S \neq \emptyset$, we conclude that $\prod_{i=2}^{n} J_i \cap S \neq \emptyset$. Since J_1 is an S-ideal, then by Proposition 1 (8), we conclude that $I \subseteq J_1$, as required. □

The following corollary is an interesting variant of the Prime Avoidance Lemma, generalized to an arbitrary multiplicatively closed set.

Corollary 1. *Let $Q \subseteq \bigcup_{i=1}^{n} P_i$, where Q and each P_i are ideals of a ring R, and assume the inclusion is irreducible. Let S be a multiplicatively closed subset of R. Suppose that P_1 is an S-ideal and that $P_i \cap S \neq \emptyset$ for all $i \geq 2$. Then $Q \subseteq P_1$.*

In particular, if in the above irreducible inclusion, $S \subseteq reg(R)$ and $P_1 \in Min(R)$, then $Q = P_1$ and $Q \in Min(R)$.

In the following theorem, we provide a characterization of the von Neumann regularity of the localization $S^{-1}R$ for an arbitrary multiplicatively closed subset $S \subseteq \text{reg}(R)$. This result extends the corresponding characterization for the case $S = \text{reg}(R)$, which appears in [13] [Proposition 3.6] in the literature concerning r-ideals.

Theorem 2. *Let R be a commutative ring and $S \subseteq \text{reg}(R)$ a multiplicatively closed subset. Then $S^{-1}R$ is a von Neumann regular ring if and only if every proper S-ideal of R is semiprime.*

Proof. $\Rightarrow$) Suppose $S^{-1}R$ is a von Neumann regular ring. Let I be a proper S-ideal of R, and let $x \in R$ such that $x^2 \in I$. Then $\frac{x^2}{1} \in S^{-1}I$ and since $S^{-1}R$ is von Neumann regular, there exists $\frac{y}{s} \in S^{-1}R$ such that $\frac{x}{1} = \left(\frac{x}{1}\right)^2 \cdot \frac{y}{s}$. Thus, $\frac{x}{1} \in S^{-1}I$, which implies there exists $s \in S$ such that $sx \in I$. Since I is an S-ideal and $s \in S$, it follows that $x \in I$. Hence I is semiprime.

$\Leftarrow$) Conversely, suppose every proper S-ideal of R is semiprime. Let $\frac{x}{s} \in S^{-1}R$. Let $I = (x^2)$ and consider the ideal I_S. If $I \cap S \neq \emptyset$, then $rx^2 \in S$ for some $r \in R$. Thus, $\frac{x}{s} = \frac{rx^3}{srx^2}$ is a unit in $S^{-1}R$ and so is a von Neumann regular element in $S^{-1}R$. Suppose $I \cap S = \emptyset$. Then I_S is an S-ideal of R with $x^2 \in I \subseteq I_S$. By assumption, I_S is semiprime in R and so $x \in I_S$. Hence, there exists $s \in S$ such that $sx \in I$, i.e., $sx = x^2 r$ for some $r \in R$. Then in $S^{-1}R$, $\frac{x}{s} = \frac{x^2 r}{s^2} = \left(\frac{x}{s}\right)^2 \cdot \frac{r}{1}$ and so again $\frac{x}{s}$ is a von Neumann regular element in $S^{-1}R$. Hence $S^{-1}R$ is a von Neumann regular ring. □

It is well known that in a von Neumann regular ring, for any prime ideal P containing an element x, the annihilator of x is contained in P. We recall the result established in [13] [Proposition 3.5], where the author characterizes the von Neumann regularity of the total ring of quotients $Q(R)$ in terms of r-ideals. In the following theorem, we extend this result to a more general setting by replacing $\text{reg}(R)$ with an arbitrary multiplicatively closed subset $S \subseteq \text{reg}(R)$.

Theorem 3. *Let R be a reduced ring, and let $S \subseteq \text{reg}(R)$ be a multiplicatively closed subset. Then the $S^{-}R$ is a von Neumann regular ring if and only if every prime S-ideal P of R is a minimal prime ideal of R.*

Proof. $\Rightarrow$) Suppose that $S^{-}R$ is a von Neumann regular ring. Let P be a prime S-ideal of R. Then $P \cap S = \emptyset$ and so $S^{-1}P$ is a prime ideal of $S^{-}R$. Assume, for contradiction, that $P \notin \text{Min}(R)$. Then there exists an element $a \in P$ such that $\text{Ann}_R(a) \nsubseteq P$. So there exists $r \in \text{Ann}_R(a)$ with $r \notin P$. Therefore, $\frac{r}{1} \notin S^{-1}P$, but $\frac{r}{1} \cdot \frac{a}{1} = \frac{0}{1}$ in $S^{-}R$, i.e., $\frac{r}{1} \in \text{Ann}_{R_{S^{-}R}}\left(\frac{r}{1}\right)$. Hence, the annihilator of $\frac{a}{1}$ is not contained in $S^{-1}P$, which contradicts the assumption that $S^{-}R$ is von Neumann regular. Thus, $P \in \text{Min}(R)$.

$\Leftarrow$) Conversely, assume that every prime S-ideal P of R is a minimal prime ideal. Let M be a maximal ideal of $S^{-}R$. Then $M = S^{-1}P$ for some prime ideal P of R disjoint from S. This prime ideal P is an S-ideal and so by assumption, $P \in \text{Min}(R)$. Hence, $M = S^{-1}P \in \text{Min}(S^{-}R)$. Therefore, every maximal ideal of $S^{-}R$ is minimal, and so $S^{-}R$ is von Neumann regular. □

Let S be a multiplicatively closed subset of a ring R. The saturation of S is the set $S^* = \{r \in R : \frac{r}{1}$ is a unit in $S^{-1}R\}$. It is clear that S^* is a multiplicatively closed subset of R and that $S \subseteq S^*$. Moreover, it is well known that $S^* = \{x \in R : xy \in S$ for some $y \in R\}$, see [8]. The set S is called saturated if $S^* = S$.

In the following proposition, we characterize rings in which $\{0\}$ is the only S-ideal where S is saturated.

Proposition 6. *Let S be a saturated multiplicatively closed subset of a ring R. Then the following are equivalent:*

1. The zero ideal is the only S-ideal of R.
2. R is a domain and $S = R\backslash\{0\}$.

Proof. $\Longrightarrow$) Suppose $\{0\}$ is the only ideal of R. It is well known that there is a prime ideal P of R with $P \cap S = \emptyset$. Thus, P is an S-ideal of R by Proposition 1 (11). By assumption, $\{0\} = P$ is a prime ideal and so R is a domain. Now, Suppose there exists $0 \neq x \notin R\backslash S$. Since S is saturated, then $\langle x \rangle \cap S = \emptyset$ and so by Proposition 2, $\langle x \rangle_S$ is an S-ideal of R. Therefore, $\langle x \rangle_S = \{0\}$ which is a contradiction as $0 \neq x \in \langle x \rangle_S$. Therefore, $S = R\backslash\{0\}$.

$\Longleftarrow$) We prove this implication without the assumption that S is saturated. Suppose R is a domain and $S = R\backslash\{0\}$. Then $S \subseteq reg(R)$ and so $\{0\}$ is an S-ideal by Proposition 1 (3). If I is any non-zero ideal of R, then $I \cap S \neq \emptyset$ and so I is not an S-ideal of R. Thus, $\{0\}$ is the only S-ideal of R. $\square$

If S is not saturated in Proposition 1, then we may find a ring R such that $\{0\}$ is the only S-ideal of R but $S \neq R - \{0\}$. For example, consider the multiplicatively closed subset $S = D - \{0, -1\}$ of any domain D. If I is an S-ideal of D, then $I \cap S = \emptyset$ and so $I \subseteq \{0, -1\}$. Therefore, $I = \{0\}$ and $\{0\}$ is the only S-ideal of R. Note that S is not saturated as $1 = (-1)(-1) \in S$ but $-1 \notin S$.

Note also that if R is a non domain and $S = R\backslash\{0\}$, then R has no S-ideals. Indeed, $\{0\}$ is not an S-ideal by Proposition 1 (3) and any nonzero ideal I of R is also not an S-ideal since $I \cap S \neq \emptyset$.

Next, we characterize non-zero principal S-ideals of any *GCD-domain.*

Theorem 4. *Let R be a GCD-domain, S a multiplicatively closed subset of R and $I = nR$ be a proper non-zero ideal of R. Then I is an S-ideal of R if and only if $\gcd(n, s) = 1$ for all $s \in S$.*

Proof. $\Longrightarrow$) Suppose $I = nR$ is a non-zero S-ideal of R. Assume, for contradiction, that there is a non unit element $d \in R$ such that $\gcd(n, s_0) = d$ for some $s_0 \in S$. Then there exists an irreducible element p of R such that $p \mid n$ and $p \mid s_0$. Write $s_0 = tp$ for some $t \in R$. Then $s_0 \cdot (\frac{n}{p}) = tp \cdot (\frac{n}{p}) = tn \in I$ and $s_0 \in S$. But, $n/p \notin I = n\mathbb{Z}$ (since $p \mid n$ and p is not a unit), contradicting the assumption that I is an S-ideal of R. Therefore, $\gcd(n, s) = 1$ for all $s \in S$.

$\Longleftarrow$) Suppose $\gcd(n, s) = 1$ for all $s \in S$ and let $a, b \in R$ such that $ab \in I$ and $a \in S$. Then $ab = tn$ for some $t \in R$ and $\gcd(a, n) = 1$. Since R is a *GCD-domain*, then $n \mid b$ and so $b \in I$. Thus, I is an S-ideal of R. $\square$

Moreover, we consider in the following theorem the non-principal case of Theorem 4. We start by the following lemma.

Lemma 1. *Let R be a GCD-domain, S a multiplicatively closed subset of R and I be a non-zero proper ideal of R. Then for every $s \in S$, there exists an element $r \in I$ such that $\gcd(r, s) = 1$.*

Proof. Assume, for contradiction, that there exists $s_0 \in S$ such that every element $r \in I$ satisfies $\gcd(r, s_0) \neq 1$, say, $d_r = \gcd(r, s_0)$ is a nonunit for all $r \in I$. Let $A = \{d_r : r \in I\}$ equipped with the divisibility partially ordered: $d_{r_1} \preceq d_{r_2}$ if and only if $d_{r_1} \mid d_{r_2}$. Then A is non-empty and every chain $\{d_{r_i} : i \in \Lambda\}$in A has a lower bound $d = \gcd\{r_i : i \in \Lambda\}$. By Zorn's Lemma, A has a minimal element $d = \gcd(r, s_0)$ for some $r \in I$. Let $r' = \frac{r}{d}$. Then $\gcd(r', s_0) = \gcd(\frac{r}{d}, s_0) = \gcd(\frac{r}{d}, \frac{ds_0}{d})$. Since $\gcd(\frac{r}{d}, \frac{s_0}{d}) = 1$, we have $\gcd(r', s_0) = \gcd(\frac{r}{d}, d) = d'$. If $d' \neq 1$, then $d' \mid d$ and since d is minimal in A, then d' must be a unit and $\gcd(r', s_0) = 1$. Now, $s_0 r' = \frac{s_0}{d} r \in I$ and $s_0 \in S$ imply $r' \in I$ since I is an S-ideal of R. This contradicts the assumption that $\gcd(r, s_0) \neq 1$ for every $0 \neq r \in I$. Therefore, for every $s \in S$, there is $r \in I$ such that $\gcd(r, s) = 1$ as needed. $\square$

Theorem 5. *Let R be a ring, S a multiplicatively closed subset of R and I a proper ideal of R. If $Rs + I = R$ for all $s \in S$, then I is an S-ideal of R. The converse is true if I is non-zero and R is a GCD-domain.*

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Proof. Let $a, b \in R$ such that $ab \in I$ and $a \in S$. Since by assumption, $Ra + I = R$, there exist $r \in R$ and $i \in I$ such that $1 = ra + i$. Multiplying both sides by b, we get $b = bra + bi \in I$. Therefore, I is an S-ideal of R as required. Now, assume R is a GCD-domain and I is a non-zero S-ideal of R. If $s \in S$, then by Lemma 1, there exists an element $r \in I$ such that $\gcd(r, s) = 1$. Thus, there exist $x, y \in R$ such that $1 = xs + yr \in Rs + I$. Therefore, $Rs + I = R$ for every $s \in S$ as needed. □

Corollary 2. *Let R be a ring, $0 \neq m \in R$ a non-unit, $S = \{m^n \mid n \geq 0\}$ and I a proper ideal of R. If $Rm + I = R$, then I is an S-ideal of R. The converse is true if I is non-zero and R is a GCD-domain.*

Corollary 3. *Let $S = \{m^k : k \geq 0\} \subseteq \mathbb{Z}$ where $m \in \mathbb{Z} \setminus \{0, \pm 1\}$. Then a proper ideal $I = n\mathbb{Z}$ of $\mathbb{Z}$ is an S-ideal if and only if $\gcd(m, n) = 1$.*

If we consider the case where m is a prime integer in Corollary 3, then we achieve an example of a ring in which every ideal disjoint with S is an S-ideal.

Corollary 4. *Let $S = \{p^k : k \geq 0\} \subseteq \mathbb{Z}$ where p is a prime integer and let $I = n\mathbb{Z}$ be a proper ideal of $\mathbb{Z}$. The following are equivalent:*

1. I is an S-ideal.
2. $I \cap S = \emptyset$.
3. $\gcd(p, n) = 1$.

Proof. The proof is straightforward and is left to the reader. □

The converse of Theorem 5 does not hold in general, as demonstrated by the following example.

Example 2. *Let k be a field and consider the ring $R = K[x, y]$. Define the multiplicatively closed subset $S = \{x^k : k \geq 0\} \subseteq R$ where $m = x \in R$ is a non-zero non-unit element. Let $I = (y)$ be the ideal of R generated by y. First, we verify that I is an S-ideal. Let $a \in S$ and $b \in R$ such that $ab \in I$. Since $a = x^k$ for some $k \geq 0$, we have*

$$x^k b \in I = (y) \Rightarrow x^k b = yf \quad \text{for some } f \in R.$$

Since x^k and y are coprime in the UFD R, then the factor y must divide b and so $b \in (y) = I$. Therefore, I is an S-ideal of R. On the other hand, $Rm + I = (x, y) \neq R$. Note that R is not a GCD-domain since there is no greatest common divisor of x and y in R.

Problem 1. *Can the converse of Theorem 1 be extended to a broader class of commutative rings? Specifically, can one identify ring-theoretic conditions weaker than being a GCD-domain under which the converse implication also holds?*

Proposition 7. *Let R be a ring, P a prime ideal of R, and let $S = R \setminus P$. Then an ideal I of R is an S-ideal if and only if I is P-primary.*

Proof. Suppose I is an S-ideal and $a, b \in R$ such that $ab \in I$ and $a \notin P$. Then $a \in S$ and by the S-ideal property, it follows that $b \in I$. Therefore, I is P-primary. Conversely, suppose I is P-primary. Let $a, b \in R$ such that $ab \in I$ and $a \in S$. Then $a \notin P = \sqrt{I}$ and hence $b \in I$ as I is P-primary. □

The following theorem is a generalization of the preceding proposition, where the multiplicatively closed set S is defined by excluding the union of finitely many prime ideals rather than a single prime ideal.

Theorem 6. *Let R be a commutative ring with identity, and let $P_1, \ldots, P_n$ be prime ideals of R. Define the multiplicatively closed set $S := R \setminus \bigcup_{i=1}^n P_i$. Then any S-ideal I of R has a primary decomposition $I = \bigcap_{i=1}^n Q_i$ where*

$$Q_i = I_{P_i} := \{r \in R : \exists s_i \notin P_i \text{ such that } s_i r \in I\}.$$

Proof. Assume I is an S-ideal, where $S = R \setminus \bigcup_{i=1}^{n} P_i$ and let $Q_i = I_{P_i} = \{r \in R : \exists s_i \notin P_i$ such that $s_i r \in I\}$. Then Q_i is an $(R \backslash P_i)$-ideal for each i by Proposition 2. Therefore, Q_i is a P_i-primary ideal of R by Proposition 7. If $r \in I$, then $1 \cdot r = r \in I$ with $1 \notin P_i$. Thus, $r \in I_{P_i}$ for all i and so $I \subseteq \bigcap_{i=1}^{n} Q_i$. Conversely, Let $r \in \bigcap_{i=1}^{n} Q_i$. Then for each i, there exists $s_i \notin P_i$ such that $s_i r \in I$. Let $J = \langle s_1, s_2, \ldots, s_n \rangle = \{t_1 s_1 + \cdots + t_n s_n \mid t_i \in R\}$ be the ideal generated by all of the s_i. We show that $J \cap S \neq \emptyset$. Suppose, to the contrary, that $J \subseteq \bigcup_{i=1}^{n} P_i$. Then by Prime Avoidance Lemma, $J \subseteq P_k$ for some $k \in \{1, 2, \cdots, n\}$. But $s_k \in J$ and $s_k \notin P_k$, a contradiction. Hence, there exists $s \in J \cap S$. Since $s \in J$, we can write $s = t_1 s_1 + \cdots + t_n s_n$ so that

$$sr = t_1(s_1 r) + \cdots + t_n(s_n r) \in I.$$

Since $s \in S$, it follows that $r \in I_S = I$. Therefore, $\bigcap_{i=1}^{n} Q_i \subseteq I$ and the equality $I = \bigcap_{i=1}^{n} Q_i$ holds. □

In the following theorem we describe the S-ideals of a ring R with respect to the multiplicatively closed subset $S = 1 + J$ for some proper ideal J of R.

Theorem 7. *Let R be a ring, J a proper ideal of R and set $S = 1 + J$. Let I be an ideal of R. If $J \subseteq I$, then I is an S-ideal of R. Moreover, The converse holds if $J \subseteq J(R)$ and I is maximal among proper S-ideals of R.*

Proof. Suppose $J \subseteq I$. Take any $s = 1 + j \in S$ and $r \in R$ such that $sr \in I$. Then $sr = r + jr \in I$, and since $jr \in I$ (as $j \in J \subseteq I$), it follows that $r = sr - jr \in I$. Thus, I is an S-ideal of R. Conversely, suppose $J \subseteq J(R)$ and I is maximal among proper S-ideals of R. Then $1 + J \subseteq U(R)$, and thus every proper ideal of R is an S-ideal. Suppose there is $j \in J \backslash I$. Then the ideal $I + Rj$ is strictly larger than I and still a proper ideal. Indeed, Suppose for contradiction $I + Rj = R$. Then $i + rj = 1$for some $r \in R$ and $i \in I$ and so $i = 1 - rj \in U(R)$, a contradiction. Thus, $I + Rj$ is an S-ideal which contradicts the maximality of I. Therefore, every $j \in J$ lies in I and $J \subseteq I$. □

However, the converse of Theorem 7 need not be true in general as we can see in the following example:

Example 3. *Let $R = \mathbb{Z}$ and let $J = 2\mathbb{Z}$, so that $S = 1 + J = \{1 + 2k \mid k \in \mathbb{Z}\} = \mathbb{Z} \backslash 2\mathbb{Z}$. Then $I = 4\mathbb{Z}$ is a $2\mathbb{Z}$-primary ideal of $\mathbb{Z}$ and so it is an S-ideal by Proposition 7. However, $J = 2\mathbb{Z} \nsubseteq 4\mathbb{Z} = I$.*

Recall that an ideal I of a ring R is called a z_0-ideal if for every $a \in I$, the intersection P_a of all minimal prime ideals containing a is contained in I, i.e., $P_a \subseteq I$. In [13] [Theorem 2.19], it is proved that every z_0-ideal of a ring R is an r-ideal. More generally, we next determine a multiplicatively closed subset S such that every z_0-ideal is an S-ideal.

Proposition 8. *Let R be a commutative ring and define the multiplicatively closed subset*

$$S = R \setminus \bigcup \{P \mid P \in \mathrm{Min}(R)\}.$$

Then every z_0-ideal of R is an S-ideal.

Proof. Suppose I is a z_0-ideal and let $ab \in I$ with $a \in S$. Since $ab \in I$, by the definition of z_0-ideals, we have $P_{ab} \subseteq I$. Also, since $a \in S$, it follows that $a \notin P$ for any $P \in \mathrm{Min}(R)$, hence $a \notin P_{ab}$. Therefore, clearly $b \in P_{ab} \subseteq I$, and so I is an S-ideal. □

As the following example indicates, the converse of Proposition 8 does not necessarily hold.

Example 4. *Let $R = \mathbb{Z}[x]/(x^2)$, and let $I = (0)$ be the zero ideal of R. We know that the unique minimal prime ideal of R is $\mathrm{Min}(R) = \{(x)\}$. Then*

$$S = R \setminus \bigcup \{P \mid P \in \mathrm{Min}(R)\} = R \setminus (x) = \{a + bx \in R \mid a, b \in \mathbb{Z},\ a \neq 0\}.$$

Since clearly $S \subseteq \mathrm{reg}(R)$, then $I = (0)$ is an S-ideal by Proposition 1 (3). However, if we take $a = 0 \in I$, then $P_a = (x)$ is not contained in I. Therefore, I is not a z_0-ideal of R.

In [13] [Definition 3.11], the concept of r-multiplicatively closed sets (briefly, r-m.c.s) was introduced using the original m.c.s $S = \mathrm{reg}(R)$. To allow broader applications, we now introduce S-multiplicatively closed sets (briefly, S-m.c.s), where S is any multiplicatively closed set of R.

Definition 2. *Let S be a subset of a ring R. We say that a subset T of R is an S-multiplicatively closed subset (briefly, S-m.c.s) if T contains at least one element $s \neq 1$ from S and for all $s \in S \cap T$ and $t \in T$, we have $st \in T$. We say that T is S-saturated if it is an S-m.c.s and for all $a, b \in R$, $ab \in T$ implies $a \in T$ and $b \in T$.*

Remark 2. *Similar to the case of r-m.c.s T where we may assume for practical purposes that $\mathrm{reg}(R) \subseteq T$, [13], we may similarly assume that any S-m.c.s T of R contains S. This is because the set $T' = T \cup S \cup \{st : s \in S, t \in T\}$ is again an S-m.c.s larger than T containing S and clearly for any ideal I of R, $I \cap T = \emptyset$ if and only if $I \cap T' = \emptyset$.*

Lemma 2. *Let $\{I_i : i \in \Delta\}$ be a family of S-ideals of R. Then the set $T = R \setminus \bigcup_{i \in \Delta} I_i$ is an S-saturated m.c.s.*

Proof. Since each I_i is an S-ideal, $I_i \cap S = \emptyset$. Hence, $S \subseteq T$. Now, suppose $s \in S \cap T$ and $t \in T$. Then for all i, $s, t \notin I_i$ and since I_i is an S-ideal, then $st \notin I_i$. Thus, $st \in T$, proving that T is an S-mcs. To show that$T$ is S-saturated, assume $ab \in T$. Then $ab \notin I_i$ for all $i \in \Delta$ and so $a, b \notin I_i$ for all $i \in \Delta$. Thus, $a, b \in T$. □

We now present an analogue of [13] [Proposition 3.14] by replacing $\mathrm{reg}(R)$ with an arbitrary S.

Theorem 8. *Let S be a subset of a ring R. Then a subset T of R is an S-saturated m.c.s of R such that $S \subseteq T$ if and only if $T = R \setminus \bigcup_{I \in \mathcal{A}} I$ for some family $\mathcal{A}$ of S-ideals of R.*

Proof. $\Rightarrow$) Assume T is an S-saturated m.c.s. of R and $S \subseteq T$. Let

$$\mathcal{A} = \{I \subseteq R : I \text{ is an } S\text{-ideal and } I \cap T = \emptyset\}.$$

By the definition of $\mathcal{A}$, we have clearly, $T \subseteq R \setminus \bigcup_{I \in \mathcal{A}} I$. To prove the reverse inclusion, let $r \in R \setminus \bigcup_{I \in \mathcal{A}} I$ and suppose that $r \notin T$. If $\langle r \rangle \cap T \neq \emptyset$, say $ar \in T$ for some $a \in R$, then T being S-saturated implies $r \in T$, a contradiction. Therefore, $\langle r \rangle \cap T = \emptyset$ and so $\langle r \rangle \cap S = \emptyset$ as $S \subseteq T$. By Proposition 2, the ideal $\langle r \rangle_S$ is an S-ideal and $\langle r \rangle_S \cap T \neq \emptyset$. Thus, $r \in \langle r \rangle_S \subseteq \bigcup_{I \in \mathcal{A}} I$, a contradiction. Therefore, $r \in T$ and the equality$T = R \setminus \bigcup_{I \in \mathcal{A}} I$ holds.

$\Leftarrow$) Lemma 2. □

3. S-Ideals in Ring Extensions and Constructions

In this section, we investigate the behavior of S-ideals under various ring-theoretic constructions and extensions. We examine how the concept of S-ideals is preserved or transformed when passing to localized rings (rings of fractions), quotient rings, and finite Cartesian products. In addition, we study the extension of S-ideals to more structured rings, including polynomial rings, idealization rings, and amalgamated algebras. This analysis provides insight into the structural robustness of S-ideals and illustrates their applicability across a wide range of ring-theoretic settings.

If S and T are multiplicatively closed subsets of a ring R, then

$$T^{-1}S := \left\{ \frac{s}{t} : s \in S,\ t \in T \right\}$$

is multiplicatively closed subsets of $T^{-1}R$. In particular, $\{1\}^{-1}S = \{\frac{s}{1} : s \in S\}$ is multiplicatively closed in $T^{-1}R$.

Theorem 9. *Let I a proper ideal of a ring R and let S, T be multiplicatively closed subsets of R. If I is an S-ideal of R and $I \cap T = \emptyset$, then $T^{-1}I$ is an $T^{-1}S$-ideal of $T^{-1}R$.*

Proof. Suppose there is $\frac{s}{t} \in T^{-1}S \cap T^{-1}I$. Then there exists $t' \in T$ such that $t's \in I$ and I being an S-ideal implies $t' \in I \cap T$, a contradiction. Thus, $T^{-1}S \cap T^{-1}I = \emptyset$. Let $\frac{a}{t_1} \in T^{-1}S$ and $\frac{b}{t_2} \in T^{-1}R$, and suppose $\left(\frac{a}{t_1}\right)\left(\frac{b}{t_2}\right) = \frac{ab}{t_1t_2} \in T^{-1}I$. Then there exists $t_3 \in T$ such that $t_3ab \in I$. Since I is an S-ideal and $a \in S$, then $bt_3 \in I$ and so $\frac{b}{t_2} = \frac{bt_3}{t_2t_3} \in T^{-1}I$. Hence, $T^{-1}I$ is an $T^{-1}S$-ideal of $T^{-1}R$. □

In particular, if $S = T$, then all elements of $T^{-1}S$ are units in $T^{-1}R$. Thus, Proposition 1 (2) implies the following corollary:

Corollary 5. *Let S be a multiplicatively closed subset of a ring R. Then any proper ideal of $S^{-1}R$ is an $S^{-1}S$-ideal.*

However, the converse of Theorem 9 is not true in general.

Example 5. *Let $R = \mathbb{Z}$, $S = \{3^n : n \geq 0\}$, $T = \mathbb{Z} \setminus \{0\}$ and $I = 6\mathbb{Z}$. Then $T^{-1}R = \mathbb{Q}$, $T^{-1}I = 6\mathbb{Q}$ and $T^{-1}S = \left\{\frac{3^n}{m} : n \geq 0,\ m \in \mathbb{Z} \setminus \{0\}\right\}$. We show that $T^{-1}I$ is a $T^{-1}S$-ideal in $\mathbb{Q}$: Let $x \in \mathbb{Q}$ and suppose there exists $s \in T^{-1}S$ such that $sx \in 6\mathbb{Q}$, that is, $sx = 6q$ for some $q \in \mathbb{Q}$. Since $s \in T^{-1}S$, write $s = \frac{3^n}{m}$ for some $n \geq 0$, $m \in \mathbb{Z} \setminus \{0\}$. Then $\frac{3^n}{m}x = 6q$ and so $x = \frac{6mq}{3^n} \in 6\mathbb{Q} = T^{-1}I$. Therefore, $T^{-1}I$ is a $T^{-1}S$-ideal in $\mathbb{Q}$. On the other hand, by Corollary 3, $I = 6\mathbb{Z}$ is not an S-ideal in $\mathbb{Z}$ since $\gcd(6,2) \neq 1$.*

Proposition 9. *Let $f : R_1 \to R_2$ be a ring homomorphism and S be a multiplicatively closed subset of R_1. Then the following statements hold.*

1. If J is an $f(S)$-ideal of R_2, then $f^{-1}(J)$ is an S-ideal of R_1.
2. If f is an epimorphism and I is an S-ideal of R_1 containing $Ker(f)$, then $f(I)$ is an $f(S)$-ideal of R_2.

Proof. (1) Let $a, b \in R_1$ with $a \in S$ and $ab \in f^{-1}(J)$. Then $f(ab) = f(a)f(b) \in J$ with $f(a) \in f(S)$ and since J is an $f(S)$-ideal of R_2, $f(b) \in J$. Thus, $b \in f^{-1}(J)$.

(2) Let $a, b \in R_2$ such that $a \in f(S)$ and $ab \in f(I)$. Write $a = f(s)$ for some $s \in S$ and since f is onto, there is $y \in R_1$ such that $b = f(y)$. Since $f(s)f(y) \in f(I)$ and $Ker(f) \subseteq I$, we have $sy \in I$ and so $y \in I$ as I is an S-ideal of R_1. Therefore, $b = f(y) \in f(I)$ and $f(I)$ is an $f(S)$-ideal of R_2. □

Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S. If we denote $r + I \in R/I$ by $\bar{r}$, then $\bar{S} = \{\bar{s} : s \in S\}$ is a multiplicatively closed subset of R/I. In view of Proposition 9, we conclude the following result for $\overline{S}$-ideals of R/I.

Corollary 6. *Let S be a multiplicatively closed subset of a ring R and I, J are two ideals of R with $I \subseteq J$. Then J is an S-ideal of R if and only if J/I is an $\overline{S}$-ideal of R/I.*

Proof. We apply the canonical epimorphism $\pi : R \to R/I$ in Proposition 9. □

Corollary 7. *Let n be a positive integer and let $\bar{S}$ be a multiplicatively closed subset of the ring $\mathbb{Z}_n$. Then a proper ideal $I = \langle \bar{m} \rangle$ of $\mathbb{Z}_n$ is an $\bar{S}$-ideal if and only $\gcd(m, s) = 1$ for every $s \in S$.*

Proof. Consider the canonical epimorphism $\pi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ in Proposition 4 and use Theorem 5. □

Let $\{R_i\}_{i\in\Lambda}$ be a family of rings and for each $i \in \Lambda$, let S_i be a multiplicatively closed subset of R_i. The following proposition investigates $(\prod_{i\in\Lambda} S_i)$-ideals in the Cartesian product ring $\prod_{i\in\Lambda} R_i$.

Proposition 10. *Let $\{R_i\}_{i\in\Lambda}$ be any family of rings and for each $i \in \Lambda$, let $S_i \subseteq R_i$ be a multiplicatively closed subset and $I_i \subseteq R_i$ be an ideal. Define:*

$$R = \prod_{i\in\Lambda} R_i, \quad S = \prod_{i\in\Lambda} S_i, \quad I = \prod_{i\in\Lambda} I_i.$$

Then I is an S-ideal of R if and only if I_i is an S_i-ideal of R_i for each $i \in \Lambda$.

Proof. $\Rightarrow$) Assume $I = \prod_{i\in\Lambda} I_i$ is an S-ideal in $R = \prod_{i\in\Lambda} R_i$. Fix $j \in \Lambda$ and suppose $s_j \in S_j$, $r_j \in R_j$ with $s_j r_j \in I_j$. Define elements in R as follows:

$$s = (s_i) \in S \text{ where } s_i = \begin{cases} s_j & \text{if } i = j, \\ 1 & \text{otherwise,} \end{cases} \qquad r = (r_i) \in R \text{ where } r_i = \begin{cases} r_j & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $sr \in I$ because $s_j r_j \in I_j$ and $s_i r_i = 0 \in I_i$ for $i \neq j$. Since I is an S-ideal of R, it follows that $r \in I$, hence $r_j \in I_j$. Therefore, I_j is an S_j-ideal of R_j for all $j \in \Lambda$.

$\Leftarrow$) Assume each I_i is an S_i-ideal in R_i. Let $s = (s_i) \in S$, $r = (r_i) \in R$ such that $sr = (s_i r_i) \in I = \prod_{i\in\Lambda} I_i$. Then $s_i r_i \in I_i$ for all $i \in \Lambda$. Since each I_i is an S_i-ideal and $s_i \in S_i$, it follows that $r_i \in I_i$ for all $i \in \Lambda$. Thus, $r \in I$ and hence, I is an S-ideal in R. □

Let R be a commutative ring with identity and $S \subseteq R$ a multiplicatively closed subset. If I is an S-ideal of R, then it is easy to verify that $I[x]$, the ideal generated by I in $R[x]$, is also an S-ideal in $R[x]$. That is, the S-ideal property is preserved under extension to the polynomial ring.

However, in general, the set $S[x] \subseteq R[x]$, consisting of polynomials with coefficients in S, is not a multiplicatively closed subset of $R[x]$. For example, if $S = \{1, 3\}$ in $\mathbb{Z}_6$, then $f(x) = 3x + 1$ and $g(x) = x + 3$ are in $S[x]$ but $f(x)g(x) = 3x^2 + 4x + 3 \notin S[x]$.

To address this issue, we consider instead the set of monomials of the form sx^n with $s \in S$ and $n \geq 0$. Define:

$$S' = \{sx^n : s \in S,\ n \geq 0\} \subseteq R[x].$$

It is straightforward to check that S' is a multiplicatively closed subset of $R[x]$.

In the following proposition, we characterize the S'-ideals of $R[x]$ in terms of S-ideals of R.

Proposition 11. *Let R be a commutative ring with identity, S a multiplicatively closed subset of R, and let*

$$S' = \{sx^n : s \in S,\ n \geq 0\} \subseteq R[x].$$

Then for any ideal I of R, we have:

$$I \text{ is an } S\text{-ideal in } R \iff I[x] \text{ is an } S'\text{-ideal in } R[x].$$

Proof. $\Rightarrow$) Suppose I is an S-ideal in R. Let $f(x) \in R[x]$ and $sx^n \in S'$ with $s \in S$, $n \geq 0$ such that $(sx^n)f(x) \in I[x]$. Write $f(x) = \sum_{j=0}^{d} b_j x^j$. Then: $(sx^n)f(x) = \sum_{j=0}^{d} sb_j x^{n+j} \in I[x]$. So for all j, $sb_j \in I$. Since I is an S-ideal and $s \in S$, we get $b_j \in I$ for all j, so $f(x) \in I[x]$. Hence, $I[x]$ is an S'-ideal.

$\Leftarrow$) Suppose $I[x]$ is an S'-ideal. Let $r \in R$ and $s \in S$ with $sr \in I$. Then $sx^0 \cdot r = (sr) \in I \Rightarrow (sx^0)(r) \in I[x]$. Since $sx^0 \in S'$ and $r \in R[x]$, the assumption implies $r \in I[x]$, i.e., $r \in I$. Thus, I is an S-ideal. □

We now study S-ideals of $R[x]$ with respect to other types of multiplicatively closed subsets, beyond those inherited directly from R. This broader approach allows us to explore more refined structures within the polynomial ring.

For a polynomial $f \in R[x]$, the content of f, denoted $C(f)$, is the ideal of R generated by its coefficients. A polynomial $f \in R[x]$ is said to be primitive if its content $C(f) = R$. It is proved in [12] [Proposition 33.1] that $T = \{f \in R[x] : C(f) = R\}$ is a complement of a union of prime ideals of $R[x]$ and so it is a (regular) multiplicatively closed subset of $R[x]$. The ring of fractions of $R[x]$ at T is denoted by $R(x) = T^{-1}R[x]$, and it is called the **Nagata ring** of R. By [12] [Proposition 33.1], for any ideal I of R, $T^{-1}(I[x]) \cap R[x] = I[x]$ and so $I[x]$ is an T-ideal of $R[x]$.

In particular, if we consider the multiplicatively subset $N = \{f \in R[x] : f \text{ is monic}\} \subseteq T$, then also $I[x]$ is an N-ideal of $R[x]$ for every ideal I of R. The ring of fractions of $R[x]$ at N is denoted by $R\langle x\rangle = N^{-1}R[x]$ and it is well-known in the literature, see [12].

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While $I[x]$ is an S-ideal of $R[x]$ for every ideal $I \subseteq R$ when $S = T$ or $S = N$ as defined above, this property does not necessarily hold for arbitrary multiplicatively closed subsets $S \subseteq R[x]$ that contain T or N.

Proposition 12. *Let S be a multiplicatively closed subset of a ring R. Define*

$$T := \{f \in R[x] : C(f) \cap S \neq \emptyset\}$$

Then J is an S-ideal of R if and only if $J[x]$ is an T-ideal of $R[x]$.

Proof. $\Longrightarrow$) Suppose J is an S-ideal of R, and let $g(x), f(x) \in R[x]$ such that $g(x) \in T$ and $g(x)f(x) \in J[x]$. Since $g(x) \in T$, there exists $s \in S$ such that $s \in C(g)$. Write $f = \sum_{j=0}^{m} b_j x^j$ and $g = \sum_{i=0}^{n} a_i x^i$. Then $gf = \sum_{k=0}^{m+n} c_k x^k$, with $c_k = \sum_{i+j=k} a_i b_j$. Since $gf \in J[x]$, we have $c_k \in J$ for all k. In particular, for each j, $a_i b_j \in J \quad$ for all i. Let $s \in C(g) = \langle a_0, \ldots, a_n \rangle$, so there exist $r_0, \ldots, r_n \in R$ such that $s = \sum_{i=0}^{n} r_i a_i$. Then for each j, we compute

$$sb_j = \left(\sum_{i=0}^{n} r_i a_i\right) b_j = \sum_{i=0}^{n} r_i a_i b_j \in J,$$

since each $a_i b_j \in J$. Thus, $sb_j \in J$ for all j, and since $s \in S$ and J is an S-ideal, it follows that $b_j \in J$. Hence, all coefficients of f lie in J and $f \in J[x]$. Therefore, $J[x]$ is a T-ideal.

$\Longleftarrow$) Suppose $J[x]$ is a T-ideal and let $a, b \in R$ with $ab \in J$ and $a \notin S$. Define $f(x) = b$, $g(x) = a$, so $g(x)f(x) = ab \in J[x]$ and $g(x) \in T$. Then by assumption, $b = f(x) \in J[x]$ and so $b \in J$. Therefore, J is an S-ideal of R. $\square$

The following corollaries are special cases of the preceding proposition.

Corollary 8. *Let P be a prime ideal of a ring R, and define*

$$T := \{f \in R[x] : \text{the constant term of } f \notin P\}.$$

Then I is an S-ideal of R with $S := R \setminus P$ if and only if $I[x]$ is an T-ideal of $R[x]$.

Corollary 9. *Let I be an ideal of ring R. Then I is an $(R \backslash \{0\})$-ideal of R if and only if $I[x]$ is an $(R[x] \backslash \{0\})$-ideal of $R[x]$.*

Corollary 10. *Let $S \subseteq$ be a multiplicatively closed subset of a commutative ring R, and define*

$$T = \{f \in R[x] : \text{the leading coefficient of } f \text{ lies in } S\}.$$

Then I is an S-ideal of R with if and only if $I[x]$ is an T-ideal of $R[x]$.

Recall that the idealization of an R-module M denoted by $R(+)M$ is the commutative ring $R \times M$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. If S is a multiplicatively closed subset of R, then clearly the set $S(+)K = \{(s, m) : s \in S, m \in K\}$ is a multiplicatively closed subsets of the ring $R(+)M$ for any submodule K of M.

Definition 3. *Let R be a ring and let N be a proper submodule of an R-module M. Let S be a multiplicatively closed subset of R. Then N is called an S-submodule of M if whenever $s \in S$ and $m \in M$, $sm \in N$ implies $m \in N$.*

Equivalently, N is an S-submodule of M if and only if $S^{-1}N \cap M = N$.

Next, for a submodule K of an R-module M, we determine the relation between S-ideals of R and $S(+)K$-ideals of $R(+)M$.

Proposition 13. *Let N, K be submodules of an R-module M, S be a multiplicatively closed subset of R and I be an ideal of R where $IM \subseteq N$. Then $I(+)N$ is an $S(+)K$-ideal of $R(+)M$ if and only if I is an S-ideal of R and N is an S-submodule of M.*

Proof. $\Longrightarrow$) Suppose $I(+)N$ is an $S(+)K$-ideal of $R(+)M$. Let $a, b \in R$ such that $ab \in I$ and $a \in S$. Then $(a,0)(b,0) \in I(+)N$ with $(a,0) \in S(+)K$ and by assumption, $(b,0) \in I(+)N$. Thus, $b \in I$ and I is an S-ideal of R. Now, let $s \in S$ and $m \in M$ such that $sm \in N$. Then $(s,0)(0,m) = (0,sm) \in I(+)N$ with $(s,0) \in S(+)K$. Again by assumption, $(0,m) \in I(+)N$ and so $m \in N$. Thus, N is an S-submodule of M.

$\Longleftarrow$) Suppose I is an S-ideal of R and N is an S-submodule of M. Let $(r_1, m_1), (r_2, m_2) \in R(+)M$ such that $(r_1, m_1)(r_2, m_2) \in I(+)N$ and $(r_1, m_1) \in S(+)K$. Then $r_1 r_2 \in I$ and $r_1 \in S$ and I being an S-ideal implies $r_2 \in I$. Since $r_1 m_2 + r_2 m_1 \in N$ and $r_2 m_1 \in IM \subseteq N$, then $r_1 m_2 \in N$. Since N is an S-submodule of M and $r_1 \in S$, then $m_2 \in N$. Therefore, $(r_2, m_2) \in I(+)N$ and $I(+)N$ is an $S(+)K$-ideal of $R(+)M$. $\square$

The converse of the proposition 13 is not true in general. That is, even if I is an S-ideal of R, the ideal $I(+)N$ of $R(+)M$ need not be an $S(+)K$-ideal if N is not an S-submodule of M. We demonstrate this with the following example:

Example 6. *Consider the ring $R = \mathbb{Z}$ and the R-module $M = \mathbb{Z}_6$ and let $S = \mathbb{Z} \setminus 2\mathbb{Z}$. Define the ideal $I = \{0\}$ of $\mathbb{Z}$, and the submodule $N = \{\bar{0}, \bar{3}\}$ of $\mathbb{Z}_6$. Then $IM = 0 \subseteq N$ and I is an S-ideal of $\mathbb{Z}$. However, N is not an S-submodule of M since if we take $s = 3 \in S$ and $m = \bar{2} \in \mathbb{Z}_6$, then $sm \in N$, but $m = \bar{2} \notin N$. Consequently, although I is an S-ideal, the ideal $I(+)N$ fails to be an $S(+)M$-ideal of $R(+)M$ since $(3, \bar{1}) \in S(+)M$ and $(3, \bar{1})(0, \bar{2}) = (0, \bar{0}) \in I(+)N$ but $(0, \bar{2}) \notin I(+)N$. This confirms that the converse of the theorem is not valid when N is not an S-submodule.*

Let R and R' be two rings, J be an ideal of R' and $f : R \to R'$ be a ring homomorphism. The set $R \bowtie^f J = \{(r, f(r) + j) : r \in R, j \in J\}$ is a subring of $R \times R'$ called the amalgamation of R and R' along J with respect to f. In particular, if $Id_R : R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{Id_R} J = \{(r, r + j) : r \in R, j \in J\}$ is the amalgamated duplication of a ring along an ideal J. Many properties of this ring have been investigated and analyzed over the last two decades, see for example [5], [6].

Let I be an ideal of R and K be an ideal of $f(R) + J$. Then $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in K\}$ are ideals of $R \bowtie^f J$, [6]. For a multiplicatively closed subset S of R, one can easily verify that $S \bowtie^f J = \{(s, f(s) + j) : s \in S, j \in J\}$ and $W = \{(s, f(s)) : s \in S\})$ are multiplicatively closed subsets of $R \bowtie^f J$. Let T be a multiplicatively closed subset of R'. Then clearly, the set $\bar{T}^f = \{(s, f(s) + j) : s \in R, j \in J, f(s) + j \in T\}$ is also a multiplicatively closed subset of $R \bowtie^f J$.

Next, for a multiplicatively closed subset S of a ring R, we determine when the ideal $I \bowtie^f J$ is an $(S \bowtie^f J)$-ideal in $R \bowtie^f J$.

Theorem 10. *Consider the amalgamation of rings R and R' along the ideals J of R' with respect to a homomorphism f. Let S be a multiplicatively closed subset of R and I be an ideal of R. The following statements are equivalent.*

1. $I \bowtie^f J$ is an $(S \bowtie^f J)$-ideal of $R \bowtie^f J$.
2. $I \bowtie^f J$ is an W-ideal of $R \bowtie^f J$.
3. I is a S-ideal of R.

Proof. Note that clearly, I is proper in R of and only if $I \bowtie^f J$ is proper in $R \bowtie^f J$.

(1) $\Longrightarrow$ (2) Clear as $W \subseteq S \bowtie^f J$.

(2) $\Longrightarrow$ (3) Suppose $I \bowtie^f J$ is an $(S \bowtie^f J)$-ideal of $R \bowtie^f J$. Let $a, b \in R$ such that $ab \in I$ and $s \in S$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$ with $(a, f(a)) \in W$. By assumption, $(b, f(b)) \in I \bowtie^f J$ and so $b \in I$. Therefore, I is an S-ideal of R.

(3) $\Longrightarrow$ (1) Suppose I is an S-ideal of R. Let $(a, f(a) + j_1)(b, f(b) + j_2) = (ab, (f(a) + j_1)(f(b) + j_2)) \in I \bowtie^f J$ for $(a, f(a) + j_1), (b, f(b) + j_1) \in R \bowtie^f J$ where $(a, f(a) + j_1) \in S \bowtie^f J$. Then $ab \in I$ and

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$a \in S$ and as I is an S-ideal, then $b \in I$. Thus, $(b, f(b) + j_2) \in I \bowtie^f J$ and $I \bowtie^f J$ is an $(S \bowtie^f J)$-ideal of $R \bowtie^f J$. □

Corollary 11. *Consider the amalgamation of rings R and R' along the ideal J of R' with respect to a homomorphism f. Let S be a multiplicatively closed subset of R. The $(S \bowtie^f J)$-ideals of $R \bowtie^f J$ containing $\{0\} \times J$ are of the form $I \bowtie^f J$ where I is an S-ideal of R.*

Proof. From Theorem 10, $I \bowtie^f J$ is an $(S \bowtie^f J)$-ideal of $R \bowtie^f J$ for any S-ideal I of R. Let K be an $(S \bowtie^f J)$-ideal of $R \bowtie^f J$ containing $\{0\} \times J$. Consider the surjective homomorphism $\varphi : R \bowtie^f J \to R$ defined by $\varphi(a, f(a) + j) = a$ for all $(a, f(a) + j) \in R \bowtie^f J$. Then $Ker(\varphi) = \{0\} \times J \subseteq K$ and so $I := \varphi(K)$ is an S-ideal of R by Proposition 9. Since $\{0\} \times J \subseteq K$, we conclude that $K = I \bowtie^f J$ as needed. □

In the following theorem, we clarify the relation between T-ideals of R' and $\bar{T}^f$-ideal of $R \bowtie^f J$.

Theorem 11. *Consider the amalgamation of rings R and R' along the ideals J of R' with respect to an epimorphism f. Let K be an ideal of R' and T be a multiplicatively closed subset of R'. Then $\bar{K}^f$ is an $\bar{T}^f$-ideal of $R \bowtie^f J$ if and only if K is an T-ideal of R'.*

Proof. Note that clearly, K is proper in R' if and only if $\bar{K}^f$ is proper in $R \bowtie^f J$.

Suppose $\bar{K}^f$ is an $\bar{T}^f$-ideal of $R \bowtie^f J$. Let a',$b' \in R'$ such that $a'b' \in K$ and $a' \in T$. Choose $a, b \in R$ such that $f(a) = a'$ and $f(b) = b'$. Then $(a, a'), (b, b') \in R \bowtie^f J$ with $(a, a')(b, b') = (ab, a'b') \in \bar{K}^f$ and $(a, a') \in \bar{T}^f$. By assumption, we have $(b, b') \in \bar{K}^f$ and so $b' \in K$. Therefore, K is an T-ideal of R'. conversely, suppose K is an T-ideal of R'. Let $(a, f(a) + j_1)(b, f(b) + j_2) = (ab, (f(a) + j_1)(f(b) + j_2)) \in \bar{K}^f$ for $(a, f(a) + j_1), (b, f(b) + j_1) \in R \bowtie^f J$ where $(a, f(a) + j_1) \in \bar{T}^f$. Then $(f(a) + j_1)(f(b) + j_2) \in K$ with $(f(a) + j_1) \in T$ and so $(f(b) + j_2) \in K$ as K is an T-ideal of R'. It follows that $(b, f(b) + j_2) \in \bar{K}^f$ and $\bar{K}^f$ is an $\bar{T}^f$-ideal of $R \bowtie^f J$. □

Corollary 12. *Let R,I, J, K, S and T be as in Theorems 10 and 11. Then*

1. $I \bowtie J$ is an $(S \bowtie J)$-ideal of $R \bowtie J$ if and only if I is an S-ideal of R.
2. $\bar{K}$ is a $\bar{T}$-ideal of $R \bowtie J$ if and only if K is a T-ideal of R.

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