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Proof of the Binary Goldbach Conjecture

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Abstract

In this article the proof of the binary Goldbach conjecture via Chen's weak conjecture are established (any integer greater than three is the sum and the difference of two positive primes). To this end, a "localised" algorithm is developed for the construction of two recurrent sequences of extreme Goldbach decomponents (U_{2n}) and (V_{2n}), ((U_{2n}) dependent of (V_{2n})) verifying: for any integer $n \geq 2$ (U_{2n}) and (V_{2n}) are positive primes and $U_{2n} + V_{2n} = 2n$. To form them, a third sequence of primes (W_{2n}) is defined for any integer $n \geq 3$ by W_{2n} = Sup ($p \in \mathcal{P}$: $p \leq 2n - 3$), $\mathcal{P}$ denoting the set of positive primes. The Goldbach conjecture has been proved for all even integers $2n$ between 4 and $4.10^{18.}$ and in the neighbourhood of $10^{100}, 10^{200}$ and 10^{300} for intervals of amplitude 10^9. The table of extreme Goldbach decomponents, compiled using the programs in Appendix 15 and written with the Maxima and Maple scientific computing software, as well as files from ResearchGate, Internet Archive, and the OEIS, reaches values of the order of $2n = 10^{5000}$. Algorithms for locating Goldbach's decomponentss for very large values of $2n$ are also proposed. In addition, a global proof by strong recurrence "finite ascent and descent method" on all the Goldbach decomponents is provided by using sequences of primes (Wq_{2n}) defined by: Wq_{2n} = Sup ($p \in \mathcal{P}$: $p \leq 2n - q$) for any odd positive prime q, and a further proof by Euclidean divisions of $2n$ by its two assumed extreme Goldbach decomponents is announced by identifying uniqueness, coincidence and consistency of the two operations. Next, a majorization of U_{2n} by $n^{0.525}$, $0.7 \ln^{2.2}(n)$ with probability one and $5 \ln^{1.3}(n)$ on average for any integer n large enough is justified. Finally, the Lagrange-Lemoine-Levy (3L) conjecture and its generalization called "Bachet-Bézout-Goldbach"(BBG) conjecture are proven by the same type of method. In Aditional notes, we provide heuristic estimates for Goldbach's comet and presented a graphical synthesis using a reversible Goldbach tree (parallel algorithm).

Keywords: prime number theorem; binary Goldbach conjecture; Chen's weak conjecture; Lagrange-Lemoine-Levy conjecture; Bachet-Bézout-Goldbach conjecture; Goldbach decomponents; computational number theory; gaps between consecutive primes; Goldbach comet; Goldbach tree (parallel algorithm)

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1. Overview

Number theory, "the queen of mathematics" studies the structures and properties defined on integers and primes (Euclid [15], Hadamard [18], Hardy, Wright [20], Landau [26], Tchebychev [44]). Many problems and conjectures have been formulated simply, but they remain very difficult to prove. These main components include:

- **Elementary arithmetic.**
  - Operations on integers, determination and properties of primes. (Basic operations, congruence, gcd, lcm, ………..).
  - Decomposition of integers into products or sums of primes.
    (Fundamental theorem of arithmetic, decomposition of large integers, cryptography and Goldbach's conjecture, Filhoa, Jaimea, de Oliveira Gouveaa, Keller-Füchter, [16]).
- **Analytical number theory.**
  - Distribution of primes: Prime Number Theorem, the Riemann hypothesis, (Hadamard [18], De la Vallée-Poussin [45], Littlewood [29] and Erdos [14], ,.....).

– Gaps between consecutive primes (Bombieri,Davenport [3], Cramer [9], Baker,Harmann,Iwaniec, Pintz [4,5,24], Granville [17], Maynard [31], Tao [43], Shanks [40], Tchebychev [44] and Zhang [50]).

• **Algebraic, probabilistic, combinatorial and algorithmic number theories.**

– Modular arithmetic.
  – Diophantine approximations and equations.
  – Arithmetic and algebraic functions.
  – Diophantine and number geometry.
  – Computational number theory.

2. Definitions Notations and Background

The integers h, m, M, n, N, k, K ,p, q, Q, r,…..used in this article are always positive. (2.1)

The symbol " / " means: such as or knowing that.. (2.2)

Let $\mathcal{P}$ be the infinite set of positive primes p_k (called simply primes) (2.3)

($p_1 = 2$; $p_2 = 3$; $p_3 = 5$; $p_4 = 7$; $p_5 = 11$; $p_6 = 13$;. ....)

For any non-zero integer K $\mathcal{P}_K = \{ p \in \mathcal{P}: p \leq 2K \}$ (2.4)

Writing the large numbers calculated in Appendix 15 is simplified by defining the following constants:

$M = 10^9$; $R = 4.10^8$; $G = 10^{100}$; $S = 10^{500}$; $T = 10^{1000}$ (2.5)

$p_k\# = \prod_1^k p_j$ is the primoriale of p_k. (2.6)

$\ln(x)$ denotes the neperian logarithm of the strictly positive real number x. (2.7)

$\exp(x)$ denotes the exponential of the real number x. (2.8)

Lambert's function is defined as the solution to the complex-valued functional equation in " w(z)":

$z = w \exp(w)$ (2.9)

where z is a given complex number and w is the unknown complex-valued function.

Since functional solutions of (2.9) are not injective, the Lambert's function is multivalued (multibranch).

Main branch: LambertW(0, x) is the inverse function of f defined on [-1; +∞ [by:

$f(x) = x.\exp(x)$ (2.10)

Secondary branch: LambertW(-1, x). This branch is defined for values of x less than $\frac{-1}{e}$ It corresponds to values of x that are generally negative and is used where the main branch does not apply.

Remark. Analytical extensions are defined by entire series.

Let (W_{2n}) be the sequence of primes defined by

$\forall\, n \in \mathbb{N} + 3\ W_{2n} = \text{Sup}\ (p \in \mathcal{P}: p \leq 2n - 3)$ (2.11)

For any odd prime q, let (Wq_{2n}) be the sequence of primes defined by

$\forall\, n \in \mathbb{N}\ n \geq \frac{(q+3)}{2}\ Wq_{2n} = \text{Sup}\ (p \in \mathcal{P}: p \leq 2n - q)$ (2.12)

Any sequence denoted by (G_{2n}) = (U_{2n} ; V_{2n}) verifying (2.11) is called a *Goldbach sequence*.

$\forall\, n \in \mathbb{N} + 2\ \ U_{2n}, V_{2n} \in \mathcal{P}$ and $U_{2n} + V_{2n} = 2n$ (2.13)

U_{2n} and V_{2n} are also known as "*Goldbach partitions, pairs or decomponents*".

Iwaniec,Pintz [24] have shown that for a sufficiently large integer n there is always a prime between $n - n^{23/42}$ and n. Baker and Harman [4,5] concluded that there is a prime in the interval [n; $n + o\ (n^{0.525})$]. Thus this results provides an increase of the gap between two consecutive primes p_k and p_{k+1} of the form

$\forall\ \varepsilon > 0\ \exists\ k_\varepsilon \in \mathbb{N}^*\ \mid\ \forall\, k \in \mathbb{N}\ k \geq k_\varepsilon\ p_{k+1} - p_k < \varepsilon.p_k^{0.525}$ (2.14)

The results obtained on the Cramer-Granville-Maier-Nicely conjecture [1,3,9,17,30,32] imply the following majorization.

For any real $c > 2$ and for any integer $k \geq 500$

$p_{k+1} - p_k \leq 0.7\ \ln^c(p_k)$ (*with probability one*) (2.15) and

p_{k+1} - p_k ≤ 20.ln(p_k) *(on average)* (2.16)

The following abbreviations have been adopted:

● Lagrange-Lemoine-Levy conjecture (3L) conjecture (2.17)

● Bachet-Bézout-Goldbach conjecture (BBG) conjecture (2.18)

● (Extreme) Goldbach decomponents (E).G.D. (2.19)

3. Introduction

Chen [7], Hardy, Littlewood [21], Hegfollt, Platt [22], Ramaré, Saouter [35], Tao [43], Tchebychev [44] and Vinogradov [47] have taken important steps and obtained promising results on the Goldbach conjecture (any integer $n \geq 2$ is the mean arithmetic of two primes). Indeed, Helfgott, Platt [22] proved the ternary Goldbach conjecture in 2013.

Silva, Herzog, Pardi [41] held the record for calculating the terms of Goldbach sequences after determining pairs of primes (U_{2n}; V_{2n}) verifying

$\forall\, n \in \mathbb{N}$ | $4 \leq 2n \leq 4.10^{18}$ $U_{2n} + V_{2n} = 2n$ (3.1)

Goldbach's conjecture has also been verified for all even integers $2n$ satisfying

$$10^{5k} \leq 2n \leq 10^{5k} + 10^{8} : k = 3, 4, 5, 6, \ldots\ldots\ldots, 20$$

and

$$10^{10k} \leq 2n \leq 10^{10k} + 10^{9} : k = 20, 21, 22, 23, 24, \ldots\ldots\ldots, 30$$

by Deshouillers, te Riele, Saouter [11].

In previous research work there is no explicit construction of recurrent Goldbach sequences.

In this article, for any integer n greater than two the E.G.D. U_{2n} and V_{2n} are computed iteratively using a simple and efficient "localised" algorithm.

Using Maxima and Maple scientific computing software on a personal computer Silva's record is broken and many E.G.D. are calculated up to the neighbourhood of

$2n = 10^{500}$, $10^{1000,}$ 10^{5000} and G.D. around 10^{10000} (see Sainty [37]

"In Researchgate, Internet Archive, and OEIS, E.G.D. files are supplied: E.G.D. File S around $2n = 10^{S}$ for $S = 1, 2, 3, \ldots\ldots\ldots\ldots, 10000$").

The binary Goldbach conjecture can be proved globally by strong recurrence on all G.D. using (Wq_{2n}) sequences of primes in the same way via Goldbach(-) conjecture (any even integer greater than one is the difference of two primes) demonstrated in Teorem 4.

Remark.

1. **Chen conjecture:** *For any integer* $K \geq 1$ *there are infinitely many pairs of primes with a difference equal to* $2K$.

2. **De Polignac conjecture:** *Same as Chen, but with consecutive pairs of primes.*

3. **What we know:** April 2013, Yitang Zhang [50] demonstrates that the smallest even integer $2K$ verifying the conjecture is greater than 70 million.

In 2014 James Maynard [31] then Terence Tao [43] lowered this limit to 246.

We validate Chen's weak conjecture by verifying directly in the primes tables that all even gaps from 2 to 246 are possible (see Appendix 16).

In addition, the (3L) conjectures [10],[23,25,28,48] and its generalization called (BBG) conjecture are validated.

Using case disjunction reasoning we construct two recurrent E.G.D. sequences of primes (V_{2n}) and (U_{2n}) according to the sequence (W_{2n}) by the following process

Firstly,

$U_4 = 2$ and $V_4 = 2$ (3.2)

For any integer n greater than two

● *Either*

($2n$ - W_{2n}) is a prime

then V_{2n} and U_{2n} are defined directly in terms of W_{2n}.

● *Either*

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$(2n - W_{2n})$ is a composite number

then V_{2n} and U_{2n} are determined from the previous terms of the sequence (G_{2n}).

(This process can be reversed by first determining the increasing sequence of primes less than Inf $(2n - W_{2k} \in \mathcal{P}: k \in \mathbb{N})$*, which saves a lot of computing time when programming).*

4. Theorem (Chen's Weak or Goldbach(-) Conjecture)

$\forall K \in \mathbb{N}^* \ \exists \ p, q \in \mathcal{P} \ \mid \ p - q = 2K$ (4.1)

If $K \geq 2$ $3 \leq q \leq 2K$ *and* $3 + 2K \leq p \leq 4K$

Practical method on some examples:

First of all (5 - 3 = 2), then we begin the process at (7 - 3 = 4); we will select the smallest primes for which the difference is precisely 6 (11 - 5 = 6), then 8 (11 - 3 = 8), then 10

(13 - 3 = 10),........., then $2K$ (demonstration established by strong recurrence, by the asurd and feedback). All pairs of Goldbach(-) partitions obtained by this method for K between 2 and 123 are listed in Appendix 16 to validate it using Tao results.

Proof. An other proof can also be established by strong recurrence on the integer $K \geq 2$. Let $\mathcal{P}_{Chen}(K)$ be the following property

" $\forall K \in \mathbb{N}^* \ \exists \ p, q \in \mathcal{P} \ \mid \ p - q = 2K$ $3 \leq q \leq 2K$ *and* $2K + 3 \leq p \leq 4K$ " (4.2)

► $\mathcal{P}_{Chen}(2)$ is true: 7 - 3 = 4 $q = 3 \leq 4$ and $p = 7 \leq 4 \times 2 = 8$

► Let's show

$\forall M \in \mathbb{N} \ \mid \ 2 \leq M \leq K$ then $\mathcal{P}_{Chen}(M) \implies \mathcal{P}_{Chen}(K+1)$

We reason through the absurd

Let $p, q \in \mathcal{P}_K \ \mid \ p \geq q$

$\forall P, Q \in \mathcal{P} \ \mid \ P \geq Q \ \exists \ h, m \in \mathbb{N} \ \mid$

$$P = p + 2h \text{ and } Q = q + 2m$$

we assume that

$$P - Q = p + 2h - q - 2m \neq 2(K+1) \quad (4.3)$$

Therefore

$$p - q \neq 2(K + 1 - h + m) \quad (4.4)$$

You can always choose $h \geq m$ and $h - m \leq K + 1$.

The set $\{2(K + 1 - h + m) > 0;$ $2h$ and $2m$ are any gaps between primes$\}$ contains all even integers between 2 and $2K$ (according to the recurrence hypothesis on $\mathcal{P}_{Chen}(K)$).

However the strong recurrence hypothesis asserts that

$\forall M \in \mathbb{N} \ / \ M \leq K \ \exists \, p, q \in \mathcal{P} \ \mid \ p - q = 2M$ (4.5)

By choosing: $M = K + 1 - h + m$

this contradicts (4.4).

So

$\exists \, h, m \in \mathbb{N} \ \mid \ P - Q = p + 2h - q - 2m = 2(K+1)$ (4.6)

knowing

$p, p + 2h, q, q + 2m \in \mathcal{P}$ $h \geq m$ and $h - m \leq K + 1$

Thus validating the heredity of property $\mathcal{P}_{Chen}(K)$.

The property $\mathcal{P}_{Chen}(K)$ is therefore true. As a result Goldbach(-) conjecture is validated.

5. Corollary

Let (R_{2K}) *and* (Q_{2K}) *be two sequences of primes determined by*

R_{2K} = Inf $(p \in \mathcal{P}: p - 2K \in \mathcal{P})$ *and* $\underline{Q_{2K}}$ = Inf $(p \in \mathcal{P}: 2K + p \in \mathcal{P}) = R_{2K} - 2K$ (5.1)

They are defined for any integer $K \in \mathbb{N}^*$ (5.2)

and satisfy

$\lim R_{2K} = +\infty$ (5.3)

$\forall K \in \mathbb{N}^*$ $R_{2K}, \underline{Q_{2K}} \in \mathcal{P}$ *and* $R_{2K} - Q_{2K} = 2K$ (5.4)

$\forall K \in \mathbb{N}^* \mid 2 \leq K \leq 16$ $3 \leq \underline{Q_{2K}} \leq 2K$ and $2K + 3 \leq R_{2K} \leq 4K$ (5.5)

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For any integer K large enough

$$3 \leq \underline{Q_{2K}} \leq (2K)^{0.525} \text{ and } 2K + 3 \leq R_{2K} \leq 2K + (2K)^{0.525} \quad (5.6)$$

Proof.

(5.1); (5.2): According to the previous Theorem 4, the sequences (R_{2K}) and ($\underline{Q_{2K}}$) are defined by strong recurrence (finite descent).

(5.3): $R_{2K} \geq 2K \Longrightarrow \lim R_{2K} = +\infty$

(5.4): By construction, these sequences thus verify: $R_{2K} - Q_{2K} = 2K$

(5.5): The property can be verified directly term-to-term by examining the sequence proposed above.

(5.6): This property is verified up to $2K = 246$ by calculations on the previous list.

We prove this result by recurrence

First of all, we order the Goldbach(-) decomponents at a fixed prime q, so as to obtain the estimate (5.6) more easily.

Let q_r be the $(r+1)$th prime:

We examine the sequences of primes $(T_r(K))_{K\in\mathbb{N}}$ satisfying:

$T_1(K) = 2K + 3$

$(T_1(K); 2K) \rightarrow$ (5;2); (7 4); (11;8); (13;10); (17;14); (19;16); (23;20); (29;26); (29;28);..

$T_2(K) = 2K + 5$

$(T_2(K); 2K) \rightarrow$ (7;2); (11;6); (13;8); (17;12); (19;14); (23;18); (29;24); (31;26); (37;32).

$T_3(K) = 2K + 7$

$(T_3(K); 2K) \rightarrow$ (11;4); (13;6); (17;10); (19;12); (23;16); (29;22); (31;24); (37;30)..........

$T_4(K) = 2K + 11$

$(T11(K); 2K) \rightarrow$ (13;2); (17;6); (19;8); (23;12); (29;18); (31;20); (37;26); (41;30); (43;34).

$(T13(K); 2K) \rightarrow$ (17;4); (19;6); (23;10) ;(29;16); (31;18); (37;24); (41;28); (43;30; (47;34)..

. ...

$T_r(K) = 2K + q_r$ ($K \in \mathbb{N}^*$: $T_r(K)$ and q_r are primes) (see Appendix 16)

For any integer K satisfying $(2K)^{0.525} > q_r$ the property holds for $T_r(K)$.

Therefore it is generally validated for all $K > K_0$, since we obtain all possible cases of Chen's weak conjecture starting with $T_1(K)$, then $T_2(K)$, then $T_3(K)$. ... for $(2K)^{0.525} \leq q_r$.

(can be proved by strong recurrence using the same method as in Theorem 4 by "finite descent").

Let $a = \frac{40}{21}$ and $P_a(r)$ be the following property

"For any integer M | $2M < (q_r)^a$ there exists at least a prime $q < q_r$ | $2M + q \in \mathcal{P}$ "

▶ $P_a(K_0)$ is true (see Appendix 16).

▶ Let's show: $P_a(r) \Longrightarrow P_a(r+1)$

$$q_{r+1} \leq q_r + q_r^{0.525} \quad (5.6)$$

It is assumed that M |

$T_{r+1}(K) - q_{r+1} \neq 2M$ knowing $2M < (q_{r+1})^{c_p}$

$\forall$ $T_m(R)$, $q_m \in \mathcal{P}$ $\exists$ $h, s \in \mathbb{N}$ | $T_{r+1}(K) = T_m(R) + 2h$ and $q_{r+1} = q_m + 2s$ (5.7)

then

$$T_m(R) - q_m \neq 2(M + s - h) \quad (5.8)$$

which is impossible according to the hypothesis of strong recurrence since

$2(M + s - h)$ is less than Sup $(q_m)^a$ and that all primes $T_m(R)$ and q_m satisfy the recurrence hypothesis.

We deduce that: $Pc_p(r) \Longrightarrow Pc_p(r+1)$

Thus the property (5.6) is true.

6. Lemma (Goldbach's Fundamental Lemma)

Let q be an odd prime; then

there exists integers n_0 *and* n_q |

For any integer $n \geq n_q$ *there exists an integer s* |

$$2n - Wq_{2s} \in \mathcal{P} \ (6.1)$$

Let (Zq_{2n}) *be the sequence of primes defined by*

$\forall\, n \in \mathbb{N}\ n \geq n_q\ Zq_{2n} = \text{Inf}\,(2n - Wq_{2k} \in \mathcal{P}: k \in \mathbb{N})$ (6.2)

All G.D. are contains in the set $\{(2n - Zq_{2n};\ Zq_{2n}): n \in \mathbb{N} + 3\}$

For any integer $n \geq n_0\ Zq_{2n} \leq (2n)^{0.525}$ (6.3)

$$Zq_{2n} \leq o\ (n^{0.525})\ (6.4)$$

Proof. The proofs of propositions (6.1), (6.2) and (6.3) are established following the same principle of strong recurrence as in Theorem 4 and Corollary 5 by "return, absurd and finite descent"

(6.1): For any integer $n > 3$ and for any odd primes $r, q \mid 3 \leq r < q$,

there exists an integer M_r |

$$2n - Wq_{2k} = 2n - 2M_r - Wr_{2k} = 2(n - M_r) - Wr_{2k}$$

or

$$2(n+1) - Wq_{2k} = 2(n + 1 - M_r) - Wr_{2k}$$

then by recurrence and the absurd the property is validated.

If there were no integer k such that $2(n+1- M_r) - Wr_{2k} \in \mathcal{P}$, then there would be no integer k such that $2(n+1- M_r) - Wr_{2k} \in \mathcal{P}$, contradicting the recurrence hypothesis.

(6.2): By strong recurrence:

If

$2(n+1) - Wq_{2(n+1)} \in \mathcal{P}$ then the proof is directly validated

(see Baker, Harman [4])

else

$$2(n+1) - Wq_{2k} = 2(n + 1 - M_r) - Wr_{2k} = Zp_{2(n+1-M_r)}$$

Then, the property is validated following the recurrence hypothesis hence

$$2(n + 1 - M_r) \leq 2n$$

and then

$$Zp_{2(n+1-M_r)} \leq (2n)^{0.525}$$

(*Proof to develop*).

Remark. A better estimate of the following form can be obtained by the same method with probability one or on average using the results of Bombieri [3], Cramer [9], Granville [17],

Nicely [32] and Maier [30]:

$\exists\, m_0 \in \mathbb{N} \mid \forall\ n \in \mathbb{N}: n \geq m_0;$

For any real $c > 2\ \ U_{2n} < 1.7\ \ln(n)^c$ (*with probability one*) (6.5)

and

$\exists\, K' \geq 3.5 \mid U_{2n} < K'.\ln^{1.3}(n)$ (*on average*) (6.6)

7. Principle of Proof

To determine the E.G.D. three sequences of primes $(W_{2n}), (V_{2n}), (U_{2n})$ are defined and they verify the following properties

$\lim\ V_{2n} = +\infty.$ (7.1)

$\forall\ n \in \mathbb{N} + 2\ \ V_{2n}$ is defined as a function of $W_{2n} = \text{Sup}\,(p \in \mathcal{P}: p \leq 2n - 3)$ (7.2)

(W_{2n}) is an increasing sequence of primes that contains all of them except $p_1 = 2$ (7.3)

$$\lim\ W_{2n} = +\infty\ (7.4)$$

(U_{2n}) is a complementary sequence to (W_{2n}) of negligible primes with respect to $2n$ (7.5) For any integer $n \geq 3$

- *If* $(2n - W_{2n})$ *is a prime*

then V_{2n} and U_{2n} are defined by

$V_{2n} = W_{2n}$ and $U_{2n} = 2n - W_{2n}$ (7.6)

- *Otherwise, if* $(2n - W_{2n})$ *is a composite number*

we search for two previous terms of the sequence (G_{2n}), $U_{2(n-k)}$ and $V_{2(n-k)}$ satisfying the following conditions

$U_{2(n-k)}, V_{2(n-k)}, [U_{2(n-k)} + 2k] \in \mathcal{P}$ (7.7) $U_{2(n-k)} + V_{2(n-k)} = 2(n - k)$

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which is always possible (see Theorem 4 and "*Goldbach's fundamental Lemma* 6")

So by setting

$V_{2n} = V_{2(n-k)}$ and $U_{2n} = U_{2(n-k)} + 2k$ (7.8)

two new primes V_{2n} and U_{2n} satisfying (4.10) are generated |

$U_{2n} + V_{2n} = 2n$ (7.9)

This process is then repeated incrementing n by one unit ($n \leftarrow n+1$).

• *Remark.* Using the same method as in Theorem 4, we can the following equivalent property by strong recurrence: For any integer n greater than 48

$\mathcal{P}_{ret}(n)$: " There exists an integer K such that $2K + U_{2(n-k)} \in \mathcal{P}$ " (7.10)

To this end,.

▶ $\mathcal{P}_{ret}(49)$ is true.

▶ The heredity of the property $\mathcal{P}_{ret}(n)$: $\mathcal{P}_{ret}(n) \Longrightarrow \mathcal{P}_{ret}(n+1)$

can be proved by the absurd and returning to the previous terms by noting that

For any integer r: $r \leq n$, there is at least one integer M_r |

$$U_{2(n+1-k)} = 2M_r + U_{2(r+1-k)}$$

then

$$2K + U_{2(n+1-k)} = 2(K + M_r) + U_{2(r+1-k)}$$
$$= 2P + U_{2(r+1+M_r-P)} \quad (7.11)$$

By posing: $P = K + M_r$ and $r + 1 + M_r \leq n$

Now, according to the recurrence hypothesis on $\mathcal{P}_{ret}(n)$ there exists an integer P |

$$2P + U_{2(r+1+M_r-P)} \in \mathcal{P} \quad (7.12)$$

then there exists an integer K |

$$2K + U_{2(n+1-k)} \in \mathcal{P} \quad (7.13)$$

In summary, the property $\mathcal{P}_{ret}(n)$ is hereditary and, as a result, verifiable.

We apply the same type of reasoning using Theorem 4 to the general case with the sequence (Wq_{2n}), showing:

For any integer $n > 2$ there exists an integer K |

$$2K + q_{2n} \in \mathcal{P}$$

8. Theorem (Goldbach Conjecture)

(i) *There exists at least a recurrent sequence* $(G_{2n}) = (U_{2n}; V_{2n})$ *of primes satisfying the following conditions.*

For any integer $n \geq 2$

$U_{2n}, V_{2n} \in \mathcal{P}$ and $U_{2n} + V_{2n} = 2n$ (8.1)

(*Any integer* $n \geq 2$ *is the mean arithmetic of two primes*)

(ii) **A***n algorithm can be used to explicitly compute any E.G.D.* U_{2n} and V_{2n} (8.2)

Proof.

■ **GLOBAL STRONG RECURRENCE:**

The proof can be made using the following strong recurrence principle.

Let $P_G(n)$ be the property defined for any integer $n \geq 2$ by

$P_G(n)$: " For any integer p satisfying $2 \leq p \leq n$ there exists two primes U_{2p} and V_{2p} such their sum is equal to $2p$ ".

$(\forall p \in \mathbb{N} \mid 2 \leq p \leq n \; U_{2p}, V_{2p} \in \mathcal{P}$ and $U_{2p} + V_{2p} = 2p)$

Let's show by strong recurrence that $P_G(n)$ is true for any integer $n \geq 2$

▶ $P_G(2)$ is true: it suffices to choose $U_4 = V_4 = 2$.

▶ Let's show that the property $P_G(n)$ is hereditary: $P_G(n) \Longrightarrow P_G(n+1)$

Assume property $P_G(n)$ is true.

• *If* $(2(n+1) - W_{2(n+1)})$ *is a prime*

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

$V_{2(n+1)} = W_{2(n+1)}$ and $U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$ (8.3)

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• *Otherwise, if* $(2(n+1) - W_{2(n+1)})$ *is a composite number*

then there exists an integer k to obtain two terms $U_{2(n+1-k)}$ and $V_{2(n+1-k)}$ satisfying the following conditions

$U_{2(n+1-k)}, V_{2(n+1-k)}$ and $U_{2(n+1-k)} + 2k \in \mathcal{P}$ (8.4) $U_{2(n+1-k)} + V_{2(n+1-k)} = 2(n+1-k)$

we use the previous terms of the sequence (G_{2n}).

For any integer $q \mid 1 \leq q \leq n - 3$ we have

$$3 \leq U_{2(n-q)} \leq n.$$

Then there exists an integer k $1 \leq k \leq n - 3$ |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \text{ (8.5)}$$

following Theorem 4 since all primes smaller than $(2n)^{0.525}$ are in the set $\{U_{2k}: k \leq n\}$

(If there were no such primes, we would have a contradiction with the *Goldbach(-) Conjecture* (Theorem 4) or with *Goldbach's fundamental Lemma 6*). In fact, in an equivalent way (see the previous remark) we can copy the proof of Teorem 4 by performing a similar strong recurrence "finite descent feedback and absurd" directly on the set

$\{U_{2k}: k \leq n\}$ |

$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P}$ (8.6)

The smallest integer k | $R_{2n} \in \mathcal{P}$ is denoted by k_n.

So by setting

$U_{2n} = U_{2(n-k_n)} + 2k_n$ and $V_{2n} = V_{2(n-k_n)} \in \mathcal{P}$ (8.7)

(*These two terms are primes*)

In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n - k_n)$ were determined.

$U_{2(n-k_n)} + V_{2(n-k_n)} = 2(n - k_n)$ (8.8)

By adding the term $2k_n$ to each member of the equality (8.6) it follows

$U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)} = 2(n - k_n) + 2k_n$ (8.9)

$\Leftrightarrow [\,U_{2(n-k_n)} + 2k_n\,] + V_{2(n-k_n)} = 2n$ (8.10)

$\Leftrightarrow U_{2n} + V_{2n} = 2n$ (8.11)

Two new primes $V_{2(n+1)}$ and $U_{2(n+1)}$ satisfying $(U_{2(n+1)} + V_{2(n+1)} = 2(n+1))$ are generated.

It follows that $P_G(n+1)$ is true. Then the property $P_G(n)$ is hereditary:

$$P_G(n) \implies P_G(n+1).$$

Therefore for any integer $n \geq 2$ the property $P_G(n)$ is true.

It follows

$\forall\ n \in \mathbb{N} + 2$ there are two primes U_{2n} and V_{2n} and such their sum is $2n$: $U_{2n} + V_{2n} = 2n$

■ **ALGORITHM:**

For any integer $n \geq 3$

• *If* $(2n - W_{2n})$ *is a prime*

then V_{2n} and U_{2n} are defined by

$V_{2n} = W_{2n}$ and $U_{2n} = 2n - W_{2n}$ (8.12)

• *Otherwise, if* $(2n - W_{2n})$ *is a composite number*

we use the previous terms of the sequence (G_{2n}).

For any integer $q \mid 1 \leq q \leq n - 3$ we have

$$3 \leq U_{2(n-q)} \leq n.$$

Then there exists an integer k $1 \leq k \leq n - 3$ |

$$R_{2n} = U_{2(n-k)} + 2k \in \mathcal{P} \text{ (8.13)}$$

following Theorem 4 since all primes smaller than $(2n)^{0.525}$ are in the set $\{U_{2k}: k \leq n\}$

(If there were no such primes, we would have a contradiction with the Theorem 4 or with *Goldbach's fundamental Lemma 6*), see previous **GLOBACH STRONG RECURRENCE**

Finally, for any integer $n \geq 3$ this algorithm determines two sequences of primes (U_{2n}) and (V_{2n}) verifying Goldbach's conjecture.

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9. Lemma

The sequence (U_{2n}) *verifies the following majorization*

For any integer $n \geq 65$

$U_{2n} \leq (2n)^{0.525}$ (9.1)

and

$U_{2n} = o\,(n^{0.525})$ (9.2)

Proof. According to the programm 12.2 and Appendix 14 the majorization (9.1) is verified for any integer n | $65 \leq n \leq 2000$.

For any integer $n > 2000$ the proof is established by recurrence. For this purpose let $P_{bhip}(n)$ be the following property

$P_{bhip}(n)$: " $U_{2n} \leq (2n)^{0.525}$ ". (9.3)

▶ $P_{bhip}(2000)$ is true according to program 13.2 and the table in appendix 14.

▶ For any integer $n \geq 2000$ let's show that $P_{bhip}(n)$ is hereditary:

$$P_{bhip}(n) \Longrightarrow . P_{bhip}(n+1)$$

Assume that $P_{bhip}(n)$ is true: then

● *If* $(2(n+1) - W_{2(n+1)})$ *is a prime*

then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by

$V_{2(n+1)} = W_{2(n+1)}$ and $U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$ (9.4)

According to the results in [4,5,24] (see Lemma 9) there is a constant $K > 0$ such that

$2(n+1) - K.\ [2(n+1)]^{0.525} < W_{2(n+1)} < 2(n+1)$

$\Longrightarrow U_{2(n+1)} = 2(n+1) - W_{2(n+1)} < K\ .[2(n+1)]^{0.525}$

$\Longrightarrow U_{2(n+1)} \leq K\ .[2(n+1)]^{0.525}$

● *Otherwise, if* $(2(n+1) - W_{2(n+1)})$ *is a composite number*

$\exists\ p \in \mathbb{N}^*$ | $U_{2(n+1)} = U_{2(n+1-p)} + 2p$ (9.5)

According to [4,5,24]

$U_{2(n+1)} = 2p + U_{2(n+1-p)} = 2p + 2(n+1-p) - W_{2(n+1-p)} = 2(n+1) - W_{2(n+1-p)}$ (9.6)

Via " *Goldbach's fundamental Lemma 6* " it follows that

$U_{2(n+1)} < K.[2(n+1)]^{0.525}$ (9.7)

$P_{bhip}(n+1)$ is true then $P_{bhip}(n)$ is hereditary.

So for any integer $n \geq 2000$ the property $P_{bhip}(n)$ is true.

Finally $U_{2(n+1)} \leq [2(n+1)]^{0.525}$

● *Remark.* A more precise estimate can be obtained using the Cipolla or Axler frames [2,8].

10. Propositions

A) Link between Goldbach conjecture and the fundamental theorem of arithmetic.

A log-exp correspondence is established by linking the sum and product of primes via Goldbach's conjecture and the fundamental theorem of arithmetic, since if G.D. of $2n$ *are* p' *and* q', *and if* $2n$ *decomposes into factors* P'' *and* Q'' |

$(p', q' \in \mathcal{P}$ | $p' \gg q'$ *and* $P'' \gg Q''$); *then,*

$2n = P''.Q'' = p' + q'$ *and* $p' - q' = 2K$

$ln(P''.Q'') = ln(P'') + ln(Q'')$

$= ln(p' + q') = ln(p'(1 + \frac{q'}{p'}))$

$\approx ln(p') + \frac{q'}{p'}$

By choosing p' = *next or prevprime*(P'') $(P'' = p' +/- a)$ *we obtain a* q' *localization of the form*

$q' \approx [\ p'.ln(Q'')] \approx [\ p'.ln(\frac{2n}{P''})] \approx [\ p'.ln(\frac{2n}{p'})]$

.then

$2n \approx p'(1 + ln(\frac{2n}{p'}))$

$2n \approx p'(1 + ln(\frac{2n}{p''})) \approx p'(1 + ln(\frac{2n}{p' +/- a}))$

$2n \approx p'(1 + ln(2n / (p'(1 +/- \frac{a}{p'}))))$

$2n \approx p'(1 + ln((\frac{2n}{p'}).(1 / (1 +/- \frac{a}{p'}))))$

$2n \approx p'(1 + ln(\frac{2n}{p'}) + ln(1/(1 +/- \frac{a}{p'})))$

$2n \approx p'(1 + ln(\frac{2n}{p'}))$ -/+ a

You can solve equations like these using the scientific software Maple via the command,

$$solve(2n +/- a = x.(1 + ln(\frac{2n}{x})), x)$$

to locate p′ and proceed by successive next or prevprime to determine two G.D. of 2n, (programming possible in Algorithm 14). This procedure appears to generalise Pocklington's theorem, and we observe that the G.D. and their number G(E) are related to the number of prime factors in the decomposition of 2n.

Examples:

- *evalf(solve([90 = x*(1 + ln(96/x)), x < 96], x)); {x = 64.12418697}; p′ = 67 q′ = 29*
- *evalf(solve([1000 = x*(1 + ln(1100/x)), x < 1100], x)); { x = 665.6361412}*

prevprime(665); 661

isprime(1100 - 661); true; p′ = 661 q′ = 439

- *evalf(solve([9700 = x*(1 + ln(10000/x)), x < 10000], x)); { x = 7652.697929}*

prevprime(7652); 7649

isprime(10000 - 7649); true; p′ = 7649 q′ = 4351

- *evalf(solve([99950 = x*(1 + ln(100000/x)), x < 100000], x)); { x = 96854.43333}*

a: = prevprime(96799); a: = 96797 # obtained after 3 or 4 iterations of the command prevprime()

isprime(100000 - a); true; p′ = 96799 q′ = 3201

Solutions are:

x_0 = Re(- (2n +/- a) / LambertW(-1,- (2n +/- a) / (2n.e)))

and

x_1= Re(- (2n +/- a) / LambertW(- (2n +/- a) / (2n.e)))

Remarks. For any composite number n greater than three,

- *gcd(n, p′)= gcd(n ,2n - p′) = gcd(n ,q′) = gcd(n ,K)= gcd(n,p′.q′) = gcd(n,n² - K²) = 1*
- *gcd(K,p′) = gcd(K,q′) = gcd(n.K, p′) = gcd(n.K, q′) = 1*
- *The smallest E.G.D. of 2n is less than the square of its greatest prime factor.*
- *For any non-zero integer R, the smallest of G.D.'s of R.p_k# is greater than p_k.*

B) Method of locating G.D. products (Difference in squares: N² - K² or decentered dichotomy by geometric mean (see code RSA, [37]).

Locally (around 2n), there exists a sub-sequence $(p'_s, q'_{s'})$ of G.D. of 2s such that the product sequence $M_s = p'_s.q'_s = s^2 - k^2$ is almost increasing (the variations of the geometric mean almost follow those of the arithmetic mean; indeed, if

$p'_{m+1} \geq p'_m, p'_{m+1} \approx p'_m \gg q'_m, q'_{m+1}$ *and* $q'_{m+1} \geq q'_m$

then

$$p'_{m+1}.q'_{m+1} - p'_m.q'_m = (p'_{m+1} - p'_m).q'_{m+1} + p'_m.(q'_{m+1} - q'_m) \geq 0).$$

If we choose: $q'_m = q'_{m+1}$, *we minimize and better controls the deviation* $M_{s+1} - M_s$

Thus, it is possible to determine Goldbach decomponents of 2n by the following algorithm, choosing a neighborhood of 2n of amplitude c.ln²(n) in agreement with the estimates made on the G(E) distribution function associated with the Goldbach comet.

Another possible method.

By off-center dichotomy using geometric means, similar to that used to crack RSA codes (see Sainty [37]).

```
>
n2:= 1000;
```

 # To determine two G.D.s of 2n = 1000, we choose two decomponents of a lower integer, m2 and two decomponents of a higher integer, r2 to 2n; we easily calculate

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m2 < 2n = n2 < r2 and their differences km2 and kr2; then we examine their products which are assumed to preserve order, (if the initial decomponents are well chosen:
$p'_1 \leq p'_2, p'_1 \approx p'_2 \gg q'_1, q'_2$ *and* $q'_2 \geq q'_1$ *,*$(p'.\ q' = n^2 - k^2)$*; we then define admissible bounds for k from a =* $p'_1.q'_1$ *and b =.* $p'_2.q'_2$

min2 = trunc(evalf(sqrt(n² - b),Digits)) and max2 = trunc(evalf(sqrt(n² - a),Digits)); decomponents of 2n are deduced by iterating the nextprime() command from n + min2, (choose a gap of the order of c.ln²(n) between m2 and r2.

```
pinf: = prevprime(735); pinf: = 733
qinf: = nextprime(17); qinf: = 19
psup: = nextprime(1050); psup: = 1051
qsup: = nextprime(29); qsup: = 31
m2: = pinf + qinf; m2: = 752
r2: = psup + qsup; r2: = 1082
km2: = pinf - qinf; km2: = 714
kr2: = psup - qsup; kr2: = 1020
a: = m2*m2 - km2*km2; a: = 55708 # a: = pinf.qinf
b: = r2*r2 - kr2*kr2; b: = 130324 # b: = psup.qsup
min2: = trunc(evalf(sqrt(0.25*n2*n2 - b),digits); min2: = 466
max2: = trunc(evalf(sqrt(0.25*n2*n2 - a),digits); max2: = 485
n:= trunc(0.5*n2);
em: = nextprime(n + min2 - 1); em: = 967
nextprime(em); 971
em2: = 0.5*n2 + max2; em2: = 985.0
q: = n2 - 971; q: = 29
isprime(q); true
```

C) Euclidean divisions of 2*n* by its presumed Goldbach decomponents

To determine two Goldbach decomponents of 2n, the following parameters can be used:

If $p' + q' = 2n$, $(p', q' \in \mathcal{P} \mid p' \gg q')$ *then we perform the Euclidean division of p′ by q′ under the following conditions:*

$$p' = m.q' + r \quad 0 < r < q' \quad r \wedge q' = 1 \quad r \wedge m = 1$$

We deduce that $q' = \frac{(2n - r)}{(m+1)}$ *or* $2n = (m + 1).q' + r$ *(dual view point).*

which leads to the algorithm.

(To develop)

Implementation:

We perform the Euclidean division of 2n by odd primes in ascending order.

3,5,7,11,.....

20 = 3 x 6 + 2 = (3 x 5 + 2) + 3 = 17 + 3
22 = 3 x 7 + 1 = (3 x 6 + 1) + 3 = 19 + 3
24 = 3 x 8 = 5 x 4 + 4 = (5 x 3 + 4) + 5 = 19 + 5
26 = 3 x 8 + 2 = (3 x 7 + 2) + 3 = 23 + 3
28 = 3 x 9 + 1 = (3 x 8 + 1) + 3 = 5 x 5 + 3 = (5 x 4 + 3) + 5 = 23 + 5
30 = 3 x 10 = 5 x 6 = 7 x 4 + 2 = (7 x 3 + 2) + 7 = 23 + 7
32 = 3 x 10 + 2 = (3 x 9 + 2) + 3 = 29 + 3
34 = 3 x 10 + 4 = (3 x 9 + 4) + 3 = 31 + 3
36 = 3 x 12 = 5 x 7 + 1 = (5 x 6 + 1) + 5 = 31 + 5
38 = 3 x 12 + 2 = (3 x 11 + 2) + 3 = 35 + 3 = 5 x 7 + 3 = (5 x 6 + 3) + 5 = 33 + 5
= 7 x 5 + 3 = (7 x 4 + 3) + 7 = 31 + 7

. ..

500 = 3 x 166 + 2 = (3 x 165 + 2) + 3 = 497 + 3 = 5 x 100 = 7 x 71 + 3 = (7 x 70 + 3) + 7 = 493 + 7 = 11 x 45 + 5 = (11 x 44 + 5) + 11 = 489 + 11 = 13 x 38 + 6 = (13 x 37 + 6) + 13 = 487 + 13

For large integers, we will begin Euclidean division with a prime divisor of the order of nextprime(trunc(c.ln(n)).

Remark. This point of view allows us to give another proof of the Binary Goldbach Conjecture equivalent but more explicit by identifying uniqueness, coincidence and consistency using euclidean division of $2n$ *by* $p_k \in \mathcal{P}$ $p_k > n$: $2n = p_k + R_k$ *and division of* $2n$ *by* $q_r \in \mathcal{P}$ $p_k \gg q_r$ *which gives*

$$2n = m.q_r + t_r$$

hence $2n = ((m-1).q_r + t_r) + q_r = D_r + q_r$;

p_k, q_r *are increasing sequences,* R_k *and* $D_r = (m-1).q_r + t_r$ *are decreasing sequences. By uniqueness of Euclidean division and since* $D_r \gg q_r$ *and* $p_k \gg q_r, R_k$,

$$[\frac{2n}{q}] = m = 1 + [\frac{p}{q}]$$

(D_r *is the result of the euclidean division of* $2n$ *by* q_r*)*

We deduce that there exists integers i_n *and* j_n *such that:*

$$p_{i_n} = D_{j_n} \text{ and } R_{i_n} = q_{j_n}.$$

11. Theorem

For any integer $n \geq 3$ *it is easy to check*

(W_{2n}) *is a positive increasing sequence of primes* (11.1)

$\{ W_{2n}: n \in \mathrm{IN}+3 \} \cup \{2\} = \mathcal{P}$ (11.2)

$\lim W_{2n} = +\infty$ (11.3)

(U_{2n}) and (V_{2n}) *are sequences of primes and the set* $\{ U_{2k}: k \leq n \}$ (11.4) *contains all primes less than* $\ln(n)$

$n \leq V_{2n} \leq W_{2n}$ (11.5)

$3 \leq 2n - W_{2n} \leq U_{2n} \leq n$ (11.6)

$\lim V_{2n} = +oo$ (11.7)

Proof.

(11.1): For any integer $n \geq 2$ $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Therefore, $W_{2n} \leq W_{2(n+1)}$. So the sequence (W_{2n}) is increasing.

(11.2): Any prime except $p_1 = 2$ is odd, hence the result.

(11.3): $\lim W_{2n} = \lim p_k = +oo$

(11.4): By definition $V_{2n} = W_{2n}$ or there exits an integer $k \leq n - 2$ | $V_{2n} = V_{2(n-k)}$.

So the terms of the sequence $(V_{2n)})$ are primes.

(11.5): According to Lemma 9, for any integer $n \geq 65$

$U_{2n} < (2n)^{0.525}$

therefore

$U_{2n} < (2n)^{0.55} < n$

and

$V_{2n} = 2n - U_{2n} > 2n - n > n$

For any integer n | $3 \leq n \leq 65$ verification is carried out according to the computer program in paragraph 14.3 and the table in appendix 15.

We can also see that by construction $V_{2n} \geq U_{2n}$ because if we assume the opposite then V_{2n} is not the largest prime number verifying

$\frac{1}{2}(U_{2n} + V_{2n}) = n$.

So

$V_{2n} \geq n$

According to (11.5) $n \leq V_{2n} \implies U_{2n} = 2n - V_{2n} \leq 2n - n \leq n$ (11.6)

$V_{2n} \leq W_{2n} \implies 2n - W_{2n} \leq 2n - V_{2n} = U_{2n}$ (11.7)

By (11.5) for any integer $n \geq 2$: $n \leq V_{2n}$

$\lim V_{2n} = +oo$.

12. Lemma

We dissociate the following cases mod 6 for any even integer 2n: $n \geq 3$ | $p + q = 2n$ $p, q \in \mathcal{P}$

1. *If* $2n = 6m$ *then* $(p; q) = (6r + 5; 6(m - r - 1) + 1)$ *or* $(6r+1; 6(m - 1 - r) + 5)$
2. *If* $2n = 6m + 2$ *then* $(p; q) = (6r + 1; 6(m - r) + 1)$
3. *If* $2n = 6m + 4$ *then* $(p; q) = (6r + 5; 6(m- 1 - r) + 5)$

Table 1. Sum of integers 1, 5 mod 6 (in $\mathbb{Z}/6\mathbb{Z}$).

$p + q$ mod 6	1	5
1	2	0
5	0	4

(*To adapt with* $2n = 30m + k$).

Table 2. Sum of integers 1, 7, 11, 13, 17, 19, 23, 29 mod 30 (in $\mathbb{Z}/30\mathbb{Z}$).

+ mod 30	1	7	11	13	17	19	23	29
1	2	8	12	14	18	20	24	0
7	8	14	18	20	24	26	0	6
11	12	18	22	24	28	0	4	10
13	14	20	24	26	0	2	6	12
17	18	24	28	0	4	6	10	16
19	20	26	0	2	6	8	12	18
23	24	0	4	6	10	12	16	22
29	0	6	10	12	16	18	22	28

Proof.

13. Properties

For any integer $k \geq 2$ *there are infinitely many integers* n | $U_{2n} = p_k$ (13.1) $V_{2n} \sim 2n$ $(n \to +\infty)$ (13.2)

For any integer $n \geq 5000$

$U_{2n} \ll V_{2n}$ *and* $\lim (\frac{U_{2n}}{V_{2n}}) = 0$ (13.3)

(*Cesaro sums of E.G.D.* U_{2n} *and* V_{2n} *have interesting properties*)

The smallest integer n | $U_{2n} \neq 2n - W_{2n}$ *is obtained for* $n = 49$ *and* $G_{98} = (79; 19)$ (13.4)

(*This type of terms increases in the Goldbach sequence* (G_{2n}) *as n increases in the sense* of the *Schnirelmann density and there are an infinite number of them; their proportion per interval can be computed using the results given in* [39]).

The sequence (G_{2n}) *is "extremal " in the sense that for any integer* $n \geq 2$ (13.5)

V_{2n} and U_{2n} *are the largest and smallest possible primes* | $U_{2n} + V_{2n} = 2n$.

The Cramer-Granville-Maier-Nicely conjecture [9],[17],[30,32] *is verified with probability one. It leads to the following majorization*

For any integer $p \geq 500$

$U_{2p} \leq 0.7\,[\ln(2p)]^{(2.2 - \frac{1}{p})}$ (*with probability one*) (13.6)

The proof is similar to that of Lemma 9 *and is validated by the studying functions of the type*

$f: x \to a.\, g(x) + b[\ln(g(x))]^c$ $(a,b > 0;\ c > 2)$ *with*

$g: x \to 0.7\,[\ln(x)]^{(c - \frac{1}{x})}$ *and* $h: x \to 0.7\,[\ln(x)]^{(2.2 - \frac{1}{x})}$ *by using Maple software.*

A better estimate can be obtained via [29–31].

According to Bombieri [3] *and using the same method as in the proof of Lemma* 9,

we obtain the following estimate of U_{2n}

$\forall\ \varepsilon > 0$ $U_{2n} = O\,(\ln^{1.3+\varepsilon}(n))$ (*on average*) (13.7)

14. Algorithm

14.1. Algorithm Written in Natural Language

Inputs:
Input four integer variables: k, N, n, P
Input: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$,. ...., p_N the first N primes.
: $n \leftarrow 3$
: $P = M, R, G, S$ or T as indicated in paragraph **2**
Algorithm body:
A) Compute: $W_{2n} = \text{Sup}(p \in \mathcal{P}: p \leq 2n - 3)$

If $T_{2n} = (2n - W_{2n})$ is a prime
$U_{2n} \leftarrow T_{2n}$ and $V_{2n} \leftarrow W_{2n}$ (14.1.1)
otherwise
B) If T_{2n} is a composite number
Let: $k = 1$
B.1) While $U_{2(n-k)} + 2k$ is a composite number
assign to k the value $k + 1$ ($k \longleftarrow k + 1$).
return to **B1)**
End while
Assign to k the value k_n ($k_n \longleftarrow k$)
Let:

$$U_{2n} = U_{2(n-k_n)} + 2k_n \text{ and } V_{2n} = V_{2(n-k_n)} \quad (14.1.2)$$

Assign to n the value $n + 1$ ($n \longleftarrow n + 1$ and return to **A)**
End:
Outputs for integers less than 10^4:
Print ($2n$ = ●; $2n$ - 3 = ●; W_{2n} = ●; T_{2n} = ●; V_{2n} = ●; U_{2n} = ●)
Outputs for large integers:
Print ($2n$ - P = ●; $2n$ - 3 - P = ●; W_{2n} - P = ●; T_{2n} = ●; $V_{2n} - P$ = ● ; U_{2n} = ●)

14.2. Program Written with Maxima Software for 2n around $\mathbf{10^{1000}}$_

```
c: 10**1000; for n: c + 40000 step 2 thru c + 40100 do
(b:2,test: 0, b: next_prime(b) ,e: n - b ,
if primep(e)
then print(n - c, b, e - c)
else while test = 0 do (e: n -b, if primep(e)
   then test:1, print(n - c, b, e - c)
else test: 0 ,b:next-prime(b)) ;
```

14.3. Program Written with Maplesoft Maple for 2n Around $\mathbf{10^{1000}}$

```
G: = 10^1000:
V: = [1, 11, 13, 17, 19, 23, 29]:
A: = G + 500000:
B: = A + 59:
b:=2:
st: = time():
for q from A by 6 to B do # Program modulo 30. using the results of Lemma 11
 Possibility of inverting the two loops or defining three similar structures with s: = 0, 1, 2.
 for s from 0 to 2 do
```

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n: = *q* + *s* + *s*:

b: = **trunc**(0.59*b* - 20); # *Improving computation time: the idea is to recognise that for any integer n large enough there exists a Goldbach decomponent* p'_n *and a successor* p'_{n+1} *such that* *(E)* : $|\ p'_{n+1} - p'_n\ | < k.\ln^2(n)$*; this reduces the number of " nextprime(●) " operations which take up the most computing time.*

(*If G* = 10^{500}: *Computingtime is around* 10 *sec for thirty terms);The algorithm can be refined by exploiting frame* (E). *Cesàro averages can also be used to determine the initial condition for* b.

t:= 0:

R: = [[1, 5], [1,5]]: *Q*: = [[1, 7, 11, 13, 17, 19, 23, 29], [1, 13, 19], [11, 17, 23], [7, 13, 17, 19, 23, 29], [1, 7, 19], [11, 17, 23, 29], [1, 11, 13, 19, 23, 29], [1, 7, 13], [17, 23, 29], [1, 7, 11, 17, 19, 29], [1, 19, 7, 13], [11, 23, 29], [1, 23, 7, 17, 11, 13], [7, 19, 13], [11, 17, 29]]:

while *t* = 0 **do**.

b: = *nextprime*(*b* + 100); # *Additional test possible by improving Lemma 11. (with V mod 30).*

 # Possibility of replacing nextprime with a faster procedure (see Sainty [37]).(the computation time is greatly reduced by replacing with b:=nextprime(b + k(b,G)), k(b,G) constant around 150 for G=10^{1000}*, k(b,G) chosen randomly with the rand procedure or very slowly increasing as a function of b and G), but in general we don't obtain the E.G.D. but any Goldbach decomponents.*

e: = *n* - *b*;

K: = *e* **mod** 6;

if *K* ***in*** *R*[*s*+1] **then**

if *isprime*(e)

Then *t*: = 1;

print(*n* - *G*, *b*, *e* - *G*);

end if;

end if;

end do:

end do:

end do:

Computingtime:= *time*() - *st*;

Comments: *Possible test with igcd(n, b) = 1 and igcd(n, 2n - b) = 1 (or igcd(n,b.(n-b)) = 1) then isprime(b) and isprime(2n - b) may be faster than the nextprime() command, if we can improve the igcd algorithm.*

RESULTS:

G = 10^{1000}

b: ***b:= nextprime(b+rand(100..150)) b:= nextprime(b+100) b:= nextprime(b+150)***

n - G b n - G - b		
500000, 54133, 445867	500000, 139387, 360613	500000, 361069, 138931
500002, 40693, 459309	500002, 40693, 459309	500002, 40693, 459309
500004, 422393, 77611	500004, 731447, -231443	500004, 535637, -35633
500006, 49157, 450849	500006, 54139, 445867	500006, 277789, 222217
500008, 222991, 277017	500008, 205651, 294357	500008, 205651, 294357
500010, 259451, 240559	500010, 100109, 399901	500010, 138959, 361051
500012, 521981, -21969	500012, 40693, 459319	500012, 40693, 459319
500014, 622561, -22547	500014, 261823, 238191	500014, 145501, 354513
500016, 342929, 157087	500016, 82913, 417103	500016, 198659, 301357
500018, 25097, 474921	500018, 300889, 199129	500018, 26309, 473709
500020, 95083, 404937	500020, 12583, 487437	500020, 77347, 422673
500022, 201821, 298201	500022, 233591, 266431	500022, 160709, 339313

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500024, 226337, 273687	500024, 159871, 340153	500024, 162553, 337471
500026, 255859, 244167	500026, 106087, 393939	500026, 106087, 393939
500028, 8147, 491881	500028, 608459, -108431	500028, 263009, 237019
500030, 83833, 416197	500030, 30347, 469683	500030, 151813, 348217
500032, 43261, 456771	500032, 43261, 456771	500032, 24049, 475983
500034, 162251, 337783	500034, 201833, 298201	500034, 400031, 100003
500036, 179203, 320833	500036, 186859, 313177	500036, 145037, 354999
500038, 12601, 487437	500038, 95101, 404937	500038, 854257, -354219
500040, 608471,-108431	500040, 121763, 378277	500040, 121763, 378277
500042, 157103, 342939	500042, 9029, 491013	500042, 8161, 491881
500044, 145531, 354513	500044, 148663, 351381	500044, 145987, 354057
500046, 440303, 59743	500046, 304847, 195199	500046, 304847, 195199
500048, 162577, 337471	500048, 157109, 342939	500048, 12611, 487437
500050, 258637, 241413	500050, 40459, 459591	500050, 163729, 336321
500052, 111791, 388261	500052, 8171, 491881	500052, 100151, 399901
500054, 139661, 360393	500054, 223037, 277017	500054, 155291, 344763
500056, 126397, 373659	500056, 49207, 450849	500056, 126397, 373659
500058, 40739, 459319	500058, 301349, 198709	500058, 208277, 291781
500060, 106121, 393939	***Computtime***:= **188.250 sec**	500060, 67547, 432513
om ***Computtime***:= **179.343 sec**		***Computime***:= **163.828 sec**

b:= *nextprime(b+rand(150..175))* ***b***:= *nextprime(b+rand(140..160))*

n-G b n-b-G	***n-G b n-b-G***	
500000, 139387, 360613	500000, 112429,-387571	Record: 116 sec; see in researchgate files PDFGOLDBACHTEST4,10 (*For n from* G+5000000 *to* 5000058 *by* 2), [37].
500002, 90481, 409521	500002, 40693, 459309	
500004, 422393, 77611	500004, 277787, 222217	
500006, 145007, 354999	500006, 82903, 417103	
500008, 604339, -104331	500008, 148627, 351381	
500010, 138959, 361051	500010, 139397, 360613	
500012, 221021, 278991	500012, 40693, 459319	
500014, 334843, 165171	500014, 145501, 354513	
500016, 297779, 202237	500016, 388313, 111703	
500018, 167267, 332751	500018, 258329, 241689	
500020, 54577, 445443	500020, 77347, 422673	
500022, 139409, 360613	500022, 453683, 46339	
500024, 336491, 163533	500024, 67511, 432513	
500026, 12589, 487437	500026, 221197, 278829	
500028, 263009, 237019	500028, 263009, 237019	
500030, 145517, 354513	500030, 112459, 387571	
500032, 334861, 165171	500032, 178681, 321351	
500034, 163697, 336337	500034, 208253, 291781	
500036, 318979, 181057	500036, 274019, 226017	
500038, 221047, 278991	500038, 14071, 485967	

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500040, 761591, -261551	500040, 162257, 337783	
500042, 178691, 321351	500042, 361111, 138931	
500044, 54601, 445443	500044, 52903, 447141	
500046, 174989, 325057	500046, 582299, -82253	
500048, 84229, 415819	500048, 8167, 491881	
500050, 163729, 336321	500050, 67537, 432513	
500052, 159899, 340153	500052, 111791, 388261	
500054, 155291, 344763	500054, 126641, 373413	
500056, 166183, 333873	500056, 126397, 373659	
500058, 151841, 348217	500058, 40739, 459319	
Computtime*:= 174.438 *sec	***Computtime*:= 138.578 *sec***	

500000, 9473, 490527
500002, 24019, 475983
500004, 8123, 491881
500006, 9479, 490527
500008, 25087, 474921
500010, 57917, 442093
500012, 8999, 491013
500014, 9001, 491013
500016, 40697, 459319
500018, 9491, 490527
500020, 9007, 491013
500022, 139409, 360613
500024, 9011, 491013
500026, 9013, 491013
500028, 8147, 491881
500030, 26321, 473709
500032, 24049, 475983
500034, 54167, 445867
500036, 57943, 442093
500038, 9511, 490527
500040, 57947, 442093
500042, 8161, 491881
500044, 24061, 475983
500046, 162263, 337783
500048, 8167, 491881
500050, 12613, 487437
500052, 8171, 491881
500054, 9041, 491013
500056, 9043, 491013
500058, 40739, 459319
Computingtime: 343.453 sec

$G = 10^{2000}$

n* - *G n* - *b* - *G b n* - *G b n* - *b* - *G
40000, 39957, 43 40050, 86117, -46067
40002, 39091, 911 40052, 503, 39549
40004, 39957, 47 40054, 97, 39957
40006, 39549, 457 40056, 89393, -49337

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40008, 25369, 14639 40058, 101, 39957
40010, 39957, 53 40060, 103, 39957
40012, 39549, 463 40062, 971, 39091
40014, 17737, 22277 40064, 107, 39957
40016, 39957, 59 40066, 109, 39957
40018, 39957, 61 40068, 977, 39091
40020, 39091, 929 40070, 113, 39957
40022, 39141, 881 40072, 523, 39549
40024, 39957, 67 40074, 983, 39091
40026, 35443, 4583 40076, 16937, 23139
40028, 39957, 71 40078, 937, 39141
40030, 39957, 73 40080, 4637, 35443
40032, 39091, 941 40082, 941, 39141
40034, 35443, 4591 40084, 127, 39957
40036, 39957, 79 40086, 4643, 35443
40038, 39091, 947 40088, 131, 39957
40040, 39957, 83 40090, 541, 39549
40042, 23139, 16903 40092, 4649, 35443
40044, 39091, 953 40094, 137, 39957
40046, 39957, 89 40096, 139, 39957
40048, 39549, 499 40098, 31991, 8107
40100, 1009, 39091
$G = 10^{3000}$
$n - G$ b $n - b - G$
100000, 36529, 63471
100002, 77069, 22933
100004, 22717, 77287
100006, 181873, -81867
100008, 12239, 87769
100010, 4547, 95463
100012, 4549, 95463
100014, 22727, 77287
100016, 59497, 40519
100018, 24847, 75171
100020, 12251, 87769
100022, 12253, 87769
100024, 4561, 95463
100026, 22739, 77287
100028, 22741, 77287
100030, 4567, 95463
100032, 12263, 87769
100034, 36563, 63471
100036, 42649, 57387
100038, 12269, 87769
100040, 23143, 76897
100042, 36571, 63471
100044, 43973, 56071
100046, 4583, 95463
100048, 24877, 75171
100050, 12281, 87769
$G = 10^{5000}$

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n - G b n - b - G n - G b n - b - G n - G b n - b - G
100000, 31147, 68853 100050, 12611, 87439 100100, 31247, 68853
100002, 309371, -209369 100052, 12613, 87439 100102, 31249, 68853
100104, 105071, -4967
100106, 13649, 86457
100004, 31151, 68853 100054, 13597, 86457 100108, 640669, -540561
100006, 31153, 68853 100056, 105023, -4967 100110, 12671, 87439
100008, 12569, 87439 100058, 12619, 87439 100112, 31259, 68853
100114, 87991, 12123
100116, 122033, -21917
100118, 18379, 81739
100010, 13553, 86457 100060, 54151, 45909
100012, 31159, 68853 100062, 108971, -8909
100014, 108923, -8909 100064, 103091, -3027
100016, 12577, 87439 100066, 87943, 12123
100018, 592237, -492219 100068, 18329, 81739
100020, 104987, -4967 100070, 13613, 86457
100022, 12583, 87439 100072, 31219, 68853
100024, 13567, 86457 100074, 264881, -
100026, 18287, 81739 100076, 12637, 87439
100028, 12589, 87439 100078, 107971, -7893
100030, 31177, 68853 100080, 12641, 87439
100032, 61871, 38161 100082, 76913, 23169
100034, 13577, 86457 100084, 13627, 86457
100036, 31183, 68853 100086, 12647, 87439 100038, 108947, -8909 10038, 108947, -8909 100088, 61927, 38161
100040, 12601, 87439 100090, 13633, 86457
100042, 31189, 68853 100092, 12653, 87439
100044, 457091, -357047 100094, 61933, 38161
100046, 18307, 81739 100096, 87973, 12123
100048, 13591, 86457 100098, 12659, 87439
100120, 31267, 68853
100122, 61961, 38161
100124, 31271, 68853
100126, 13669, 86457
100128, 12689, 87439
100130, 31277, 68853
100132, 76963, 23169
100134, 122051, -21917
100136, 12697, 87439
100138, 13681, 86457
100140, 18401, 81739
100142, 12703, 87439
100144, 13687, 86457
100146, 152993, -52847
100148, 13691, 86457
100150, 13693, 86457
1000000, 35509, 964491
1000002, 113, 999889
1000004, 69193, 930811
1000006, 95233, 904773

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1000008, 69197, 930811
1000010, 31873, 968137
1000012, 35521, 964491
1000014, 69203, 930811
1000016, 127, 999889
1000018, 35527, 964491
1000020, 131, 999889

Maple program corrected and improved, (see Sainty [37]).

15. Appendix

Application of Algorithm 14: Table of extreme Goldbach partitions U_{2n} and V_{2n} computed from program 14.2 ($2 \leq 2n \leq 10^{1000} + 4020$).

The ** sign in the table below indicates the results given by the algorithm 14 in case **B)** of return to the previous terms of the sequence (G_{2n}).

WATCH OUT !

To simplify the display of large numbers n ($2n > 10^9$) the results are entered as follows:

$$2n - P, (2n - 3) - P, W_{2n} - P, T_{2n}, V_{2n} - P \text{ and } U_{2n}$$

with

$$P = M, R, G, S, \text{ or } T \text{ constants defined in (2.3)}$$

$2n$ $2n - 3$	W_{2n}	$T_{2n}=2n - W_{2n}$	V_{2n}	U_{2n}
4 1	X	X	**2**	**2**
6 3	3	3	**3**	**3**
8 5	5	3	**5**	**3**
1 10 7	7	3	**7**	**3**
112 9	7	5	**7**	**5**
14 11	11	3	**11**	**3**
16 13	13	3	**13**	**3**
18 15	13	5	**13**	**5**
20 17	17	3	**17**	**3**
22 19	19	3	**19**	**3**
24 21	19	5	**19**	**5**
26 23	23	3	**23**	**3**
28 25	23	5	**23**	**5**
30 27	23	7	**23**	**7**
32 29	29	3	**29**	**3**
34 31	31	3	**31**	**3**
36 33	31	5	**31**	**5**
38 35	31	7	**31**	**7**
40 37	37	3	**37**	**3**
80 77	73	7	**73**	**7**

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82 79	79	3	**79**	**3**
84 81	79	5	**79**	**5**
86 83	83	3	**83**	**3**
88 85	83	5	**83**	**5**
90 87	83	7	**83**	**7**
92 89	89	3	**89**	**3**
94 91	89	5	**89**	**5**
96 93	89	7	**89**	**7**
****98 95**	**89**	**9**	**79**	**19**
100 97	97	3	**97**	**3**
120 117	113	7	**113**	**7**
****122** 119	113	9	**109**	**13**
124 121	113	11	**113**	**11**
126 123	113	13	**113**	**13**
****128** 125	113	15	**109**	**19**
130 127	127	3	**127**	**3**
132 129	127	5	**127**	**5**
134 131	131	**3**	**131**	**3**
136 133	131	5	**131**	**5**
138 135	131	7	**131**	**7**
140 137	137	3	**137**	**3**
****500 497**	**491**	**9**	**487**	**13**
502 499	499	3	**499**	**3**
504 501	499	5	**499**	**5**
506 503	503	3	**503**	**3**
508 505	503	5	**503**	**5**
510 507	503	7	**503**	**7**
1000 997	997	3	**997**	**3**
1002 999	997	5	**997**	**5**
1004 1001	997	7	**997**	**7**
****1006 1003**	**997**	**9**	**983**	**23**
1008 1005	997	11	**997**	**11**
1010 1007	997	13	**997**	**13**

1012 1009	1009	3	1009	3
1014 1011	1009	5	1009	5
1016 1013	1013	3	1013	3
1018 1015	1013	5	1013	5
10002 9999	9973	29	9973	29
10004 10001	9973	31	9973	31
**10006 10003	9973	33	9923	83
**10008 10005	9973	35	9967	41
10010 10007	10007	3	10007	3
10012 10009	10009	3	10009	3
10014 10011	10009	5	10009	5
10016 10013	10009	7	10009	7
**10018 10015	10009	9	10007	11
10020 10017	10009	11	10009	11
$2n$ - M ($2n$ - 3) - M	W_{2n} - M	$T_{2n} = 2n - W_{2n}$	V_{2n} - M	U_{2n}
+1000 +997	+993	7	+993	7
**+1002 +999	+993	9	+931	71
+1004 +1001	+993	11	+993	11
+1006 +1003	+993	13	+993	13
**+1008 +1005	+993	15	+919	89
+1010 +1007	+993	17	+993	17
+1012 +1009	+993	19	+993	19
+1014 +1011	+1011	3	+1011	3
+1016 +1013	+1011	5	+1011	5
+1018 +1015	+1011	7	+1011	7
**+1020 +1017	+1011	9	+931	89
$2n$ - R ($2n$ - 3) - R	W_{2n} - R	$T_{2n} = 2n - W_{2n}$	V_{2n} - R	U_{2n}
**+1000 +997	+979	21	+903	97
+1002 +999	+979	23	+979	23
**+1004 +1001	+979	25	+951	53
**+1006 +1003	+979	27	+903	103
+1008 +1005	+979	29	+979	29

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+1010 +1007	+979	31	+979	31
**+1012 +1009	+979	33	+951	61
**+1014 +1011	+979	35	+ 781	233
+1016 +1013	+979	37	+979	37
**+1018 +1015	+979	39	+951	67
+1020 +1017	+1017	3	+1017	3
2n - G (2n - 3) - G	W_{2n}- G	T_{2n} = 2n - W_{2n}	V_{2n}- G	U_{2n}
**+10000 +9997	+9631	369	+7443	2557
**+10002 +9999	+9631	371	+9259	743
+10004 +10001	+9631	373	+9631	373
**+10006 +10003	+9631	375	+8583	1423
**+10008 + 10005	+9631	377	+6637	3371
+10010 +10007	+9631	379	+9631	379
**+10012 +10009	+9631	381	+8583	1429
+10014 +10011	+9631	383	+9631	383
**+10016 +10013	+9631	385	+9259	757
**+10018 +10015	+9631	387	+4491	5527
+10020 +10017	+9631	389	+9631	389
2n-S (2n-3)-S	W_{2n}- S	T_{2n} = 2n - W_{2n}	V_{2n}- S	U_{2n}
**+20000 +19997	+18031	1969	+17409	2591
**+20002 +19999	+18031	1971	+ 17409	2593
+20004 +20001	+18031	1973	+18031	1973
**+20006 +20003	+18031	1975	+16663	3343
**+20008 +20005	+18031	1977	+16941	3067
+20010 +20007	+18031	1979	+18031	1979
**+20012 +20009	+18031	1981	+5671	14341
**+20014 +20011	+18031	1983	+4101	15913
**+20016 +20013	+18031	1985	+3229	16787
+20018 +20015	+18031	1987	+18031	1987
**+20020 +20017	+18031	1989	+16941	3079
2n-T (2n-3)-T	W_{2n}-T	T_{2n} = 2n - W_{2n}	$V_{2n} - T$	U_{2n}
**+40000 +39997	+29737	10263	+ 21567	18433

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**+40002 +39999	+29737	10265	+ 22273	17729	
+40004 +40001	+29737	10267	+29737	10267	
**+40006 +40003	+29737	10269	+21567	18439	
+40008 +40005	+29737	10271	+29737	10271	
+40010 + 40007	+29737	10273	+29737	10273	
**+40012 +40009	+29737	10275	+10401	29611	
**+40014 +40011	+29737	10277	-56003	96017	
**+40016 +40013	+29737	10279	+27057	12959	
**+40018 +40015	+29737	10281	+25947	14071	
**+40020 +40017	+29737	10283	+24493	15527	

16. Appendix

7-3=4	11-5=6	11-3=8	13-3=10	17-5=12	17-3=14	19-3=16	23-5=18
23-3=20	29-7=22	29-5=24	29-3=26	31-3=28	37-7=30	37-5=32	37-3=34
41-5=36	41-3=38	43-3=40	47-5=42	47-3=44	53-7=46	53-5=48	53-3=50
59-7=52	59-5=54	59-3=56	61-3=58	67-7=60	67-5=62	67-3=64	71-5=66
71-3=68	73-3=70	79-7=72	79-5=74	79-3=76	83-5=78	83-3=80	89-7=82
89-5=84	89-3=86	101-13=88	97-7=90	97-5=92	97-3=94	101-5=96	101-3=98
103-3=100	107-5=102	107-3=104	109-3=106	113-5=108	113-3=110	131-19=112	127-13=114
127-11=116	131-13=118	127-7=120	127-5=122	127-3=124	131-5=126	131-3=128	137-7=130
137-5=132	137-3=134	139-3=136	149-11=138	151-11=140	149-7=142	149-5=144	149-3=146
151-3=148	157-7=150	157-5=152	157-3=154	163-7=156	163-5=158	163-3=160	167-5=162
167-3=164	173-7=166	173-5=168	173-3=170	179-7=172	179-5=174	179-3=176	181-3=178
191-11=180	193-11=182	191-7=184	191-5=186	191-3=188	193-3=190	197-5=192	197-3=194
199-3=196	211-13=198	211-11=200	233-31=202	211-7=204	211-5=206	211-3=208	223-13=210
229-17=212	227-13=214	223-7=216	223-5=218	223-3=220	227-5=222	227-3=224	229-3=226
233-5=228	233-3=230	239-7=232	239-5=234	239-3=236	241-3=238	251-11=240	271-29=242
251-7=244	251-5=246						

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17. Appendix

$T_r(K)$

	q_1 = 3	q_2 = 5	q_3 = 7	q_4 = 1	q_5 = 1	q_6 = 1	q_7 = 1	q_8 = 2	q_9 = 2	q_{10} = 3	q_{11}=37
2K = 2	5	7		13		19			31		
2K = 4	7		11		17		23				41
2K = 6		11	13	17	19	23		29		37	43
2K = 8	11	13		19				31	37		
2K = 10	13				23		29			41	47
2K = 12		17	19	23		29	31		41	43	
2K =14	17	19				31		37	43		
2K = 16	19		23		29					47	59
2K = 18		23		29	31		37	41	47		61
2K =20	23			31		37		43			67
2K=22			29				41			53	
2K=24		29	31		37	41	43	47	53		71
2K=26	29	31		37		43					73
2K=28	31				41		47			59	
2K=30			37	41	43	47		53	59	61	
2K=32		37		43					61		79
2K=34	37		41		47		53				
2K=36		41	43	47		53		59		67	83
2K=38	41	43						61	67		
2K=40	43		47		53		59			71	
2K=42		47		53		59	61		71	73	89
2K=44	47					61		67	73		
2K=46			53		59						
2K=48		53		59	61		67	71		79	
2K=50	53			61		67		73	79		97
2K=52			59				71			83	
2K=54		59	61		67	71	73		83		
2K=56	59	61		67		73		79			
2K=58	61				71					89	
2K=60			67	71	73		79	83	89		

18. Perspectives and Generalizations

18.1.

Other Goldbach sequences (G'_{2n}) independent of (G_{2n}) may be studied using the increasing sequences of primes (W'_{2n}) defined by

For any integer $n \geq 3$

$W'_{2n} = \text{Sup}\,(p \in \mathcal{P} : p \leq f(n))$ (18.1.1)

f is a function defined on the interval J = [3; +∞[and satisfying the following conditions

• f is strictly increasing on the interval J

• $f(3) = 3$ and $\lim_{x \to +\infty} f(x) = +\infty$

• $\forall x \in I \; f(x) \leq 2x - 3$

For example, one of the following functions defined on J can be selected.

■ $f: x \to a\,x + 3 - 3a$ ($a \in \mathbb{R}$: $0 < a \leq 2$)

■ $g: x \to [\,4\sqrt{3x} - 9\,]$ ([x] is the integer part of the real x)

■ $h: x \to 6 \ln\left(\frac{x}{3}\right) + 3$

18.2.

Using this method it would be interesting to study the Schnirelmann density [39] of primes 3, 5, 7, 11 ,......... .. in the sequence (U_{2n}) on variable intervals and the Caesaro sums of U_{2n} E.D.G.'s with a view to more efficient programming for their calculation.

18.3.

It is possible to exceed the values shown in the table of 2n = 10^{1000} (many E.G.D have been calculated for values of 2n in the order of 10^{2000}, 10^{5000} and G.D. in the order of 10^{10000} Sainty [37]), by perfecting this algorithm, exploiting the fact that one of Goldbach's decomponents can be chosen equal to 4p + 3, (G.D. are primes of the form

$6m + 1$ or $6m + 5$ and can be expressed more precisely using primes of the form $30m + r$:

$r \in$ [1,7,11,13,17,19,23,29], see Table mod 30, Lemma 11), by using De Pocklington Theorem [6,34,36], Primality tests [37], Cipolla-Axler-Dusart type functions and improvment of primes frames [2],[8,12,13,37] via a new Prime number Theorem to better identify the terms of (G_{2n}), supercomputers and more efficients software as C++, or Assembleur compilation.

18.4 G.D.'s of order $2n = 10^{10000}$ can be determined more quickly by replacing the instruction b:= 2 by b:= trunc($c.b + d$) and b: = $nextprime(b)$ with

b: = $nextprime(b + k(b, G))$, where $k(b, G)$ is a constant of around 150 for $G = 10^{1000}$ and is chosen randomly using the rand procedure or increases very slowly as a function of b and G. An increasing sequence of primes, b_k, can also be determined in stages by replacing the initial value b:= 2 by b:= trunc($k_0.b$ - $k_1.\ln^s(n)$ - k_2) and by setting c: = trunc($a.\ln^d(b)$),

$1 \leq d$, $s \leq 2$ and b: = $b + c$ for each stage, followed by b: = $nextprime(b)$ until the next stage, (see Sainty [37]); Note that for any even integer $2n$ large enough there exists G.D.

$p'_n, p'_{n+1}, q'_n, q'_{n+1} \mid p'_n + q'_n = 2n$ and $p'_{n+1} + q'_{n+1} = 2(n+1)$ with

$p'_{n+1} - p'_n$ and $q'_{n+1} - q'_n < k.\ln^2(n)$. It is therefore advisable to develop adaptive algorithms based on this model using A.I., as a function of the program's G parameter.

18.5 Diophantine equations and conjectures of the same nature ((3L) conjecture [9,21,23,26,27,44]) can be processed using similar reasoning and algorithms.

■ To validate the (3L) conjecture we study the following sequences of primes (Wl_{2n}), (Vl_{2n}) and (Ul_{2n}) defined by

For any integer $n \geq 3$ Wl_{2n}= Sup ($p \in \mathcal{P}: p \leq n - 1$) (18.5.1)

• If $Tl_{2n} = (2n + 1 - 2\, Wl_{2n})$ is a **prime**

then let

$$Vl_{2n} = Wl_{2n} \text{ and } Ul_{2n} = Tl_{2n} \quad (18.5.2)$$

• If Tl_{2n} is a **composite number**

then there exists an integer k $1 \leq k \leq n - 3$ |

$$Ul_{2(n-k)} + 2k \in \mathcal{P} \quad (18.5.3)$$

then let

$Vl_{2n} = Vl_{2(n-k)}$ and $Ul_{2n} = Ul_{2(n-k)} + 2k$ (18.5.4)

■ Using the same type of reasoning a generalization, the (BBG) conjecture of the following form can be validated

• Let K and Q be two odd integers prime to each other:

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For any integer n | $2n \geq 3(K+Q)$ there exist two primes Ub_{2n} and Vb_{2n} verifying

$$K.\ Ub_{2n} + Q.\ Vb_{2n} = 2n \ (18.5.5)$$

- Let K and Q be two integers of different parity prime to each other:

For any integer n | $2n \geq 3(K+Q)$ there are two primes Ub_{2n} and Vb_{2n} verifying $K\ .Ub_{2n} + Q.\ Vb_{2n} = 2n+1$ (18.5.6)

18.6. Remark

- **GOLDBACH (-):**

R_{2K} = Inf ($p \in \mathcal{P}: p - 2K \in \mathcal{P}$) and $\underline{Q_{2K}}$ = Inf ($p \in \mathcal{P}: 2K + p \in \mathcal{P}$) = R_{2K} - $2K$

- **GOLDBACH (+):**

V_{2K} = Sup ($p \in \mathcal{P}: 2K - p \in \mathcal{P}$) and $\underline{U_{2K}}$ = Inf ($p \in \mathcal{P}: 2K - p \in \mathcal{P}$) = $2K$ - V_{2K}

(It is possible to envisage symmetries in the Goldbach triangle).

- For any integer n greater than one there exists two integers K_{Min} and K_{Max} such that the G.D. of $2n$ *are* $n - K$ *and* $n + K$ | $K_{Min} \leq K \leq K_{Max}$.

18.7.

The sequences ($W\mathrm{q}_{2n}$) generate all the G.D. and may enable us to better estimate the values of distribution function G of the Goldbach's Comet, probably of type:

$$0.57\ \frac{E}{\ln^2(E)} < G(E) < 3.62\ \frac{E}{\ln^2(E)},$$ (Vella-Chemla [46], Woon [49]).

Average value of $G(E) \approx 1.62\ \frac{E}{\ln^2(E)}$

19. Conclusions

19.1.

A recurrent and explicit Goldbach sequence $(G_{2n}) = (U_{2n}\,;\ V_{2n})$ verifying

$\forall\ n \in \mathbb{N} + 2\ U_{2n}, V_{2n} \in \mathcal{P}$ and $U_{2n} + V_{2n} = 2n$

has been developed using an simple and efficient "localised" algorithm. The Goldbach conjecture has been proved by strong recurrence (absurd and finite descent), and a reversible Goldbach tree uniquely associated with each even integer $2n$: $2n \geq 8$ allows a better understanding of this conjecture. A relation (Proposition 10) is established between the fundamental theorem of arithmetic and the Goldbach conjecture (sum and product of primes), allowing fast computation of G.D. of very large even integers via a "localisation" of G.D.'s using a generalized Pocklington-type algorithm and further proof of Goldbach's binary conjecture via Euclidean divisions of $2n$ by primes and consistent increasing and decreasing sequences.

19.2.

The records of Silva [41] and Deshouillers, te Riele, Saouter [11] are beaten on a personal computer. Hundreds E.G.D. U_{2n} and V_{2n} are obtained for values around

$2n = 10^{1000}$, twenty-six around $2n = 10^{2000}$, seventy-five around $2n = 10^{5000}$ and ten G.D. around $2n = 10^{10000}$ for a computation time of less than three hours (see Sainty [37]).

19.3.

For a given integer $n \geq 49$ the evaluation of the terms U_{2n} and V_{2n} does not require the computing of all previous terms U_{2k} and V_{2k} | $1 \leq k < n-1$; we will only consider those that verify:

$U_{2k} \leq 5.\ln^{1.3}(2n)$ and $2n - 5.\ln^{1.3}(2n) \leq V_{2k} \leq 2n$ (*on average*) **(**19.3.1)

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This property allows any E.G.D U_{2n} and V_{2n} to be calculated quite quickly, the upper limit being defined by the scientific software and the computer's ability to determine the largest prime preceding $2n$ - 2 (*next or prevprime*($2n$ - 2) function).

19.4.

Therefore the (BBG), the (3L) and the binary Goldbach(- /+) conjectures "Any even integer greater than three is the sum and difference of two primes" are true.

In fact these two conjectures are intertwined.

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