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Article

Energy and Momentum of Strong Gravitational Waves In Free Space

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Abstract

We extend the Rosen and Virbhadra [9] results to the case of two interacting gravitational waves.

Keywords: gravitational waves; theory of general relativity; energy; momentum densities

1. Introduction

Many phenomena in the theory of general relativity have had a more plausible explanation thanks to the interaction of two gravitational waves described by the metric of Jordan and Ehlers [1]. The different interactions of these gravitational waves, allowed to understand the notion of time shift and Faraday rotation [2–4] which are of capital importance in the field of gravitation. In the investigations of these different phenomena, the methods of solving Einstein's field equations have always occupied an important place in obtaining the solutions as well as their interpretations in relativity theory [5]. Recently, it has been proved that soliton generating methods have a better capacity than other methods in solving and explaining phenomena emanating from field equations [6]. Among these soliton generating methods, we have inverse scattering method (ISM) of Pomeransky [7] which has allowed to highlight the different phenomena contained in the works [2–4]. We note, that in these different works the choice of Cartesian coordinates instead of cylindrical coordinates allow to have values and physical quantities finite as the suggested refs [8,9]. The objective of this paper is to extend this principle in the case of the interaction of two gravitational waves. The organization of this work is hierarchical according to the established order, in Sect. 2 we present the metric and the different tensors $g_{\mu\nu}$ and $g^{\mu\nu}$. In Sect. 3, we calculate the different quantities of energy and momentum densities resulting from the interaction of the two gravitational waves. Sect. 4 conclusions and perspectives.

2. Strong Gravitational Waves

We introduce the Jordan and Ehlers metric [1,10] as well as the Einstein field equations, which is presented as follows:

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi} d\phi^2 - e^{2\psi} (dz + \omega d\phi)^2, \quad (1)$$

$$\psi_{,tt} - \frac{\psi_{,\rho\rho}}{\rho} - \psi_{,\rho\rho} = \frac{e^{4\psi}}{2\rho^2} (\omega_{,t}^2 - \omega_{,\rho}^2), \quad (2)$$

$$\omega_{,tt} + \frac{\omega_{,\rho\rho}}{\rho} - \omega_{,\rho\rho} = 4(\omega_{,\rho} \psi_{,\rho} - \omega_{,t} \psi_{,t}), \quad (3)$$

$$\gamma_{,\rho} = \rho(\psi_{,t}^2 + \psi_{,\rho}^2) + \frac{e^{4\psi}}{4\rho} (\omega_{,t}^2 + \omega_{,\rho}^2), \quad (4)$$

$$\gamma_{,t} = 2\rho\psi_{,t} \psi_{,\rho} + \frac{e^{4\psi}}{2\rho} \omega_{,t} \omega_{,\rho}. \quad (5)$$

We take into account that (ρ, z, ϕ) represents the cylindrical coordinates and t the time. We specify that the different functions $\psi = \psi(t, \rho)$, $\omega = \omega(t, \rho)$ and $\gamma = \gamma(t, \rho)$ depend on ρ and t . In this metric including the Einstein field equations, ψ and ω represents two dynamic degree of freedom of the gravitational field and γ plays the role of the gravitational energy of the system. We introduce the following Cartesian coordinates [9]:

$$x = \rho \cos \phi, \quad (6)$$

and

$$y = \rho \sin \phi. \quad (7)$$

We calculate the different quantities that have the expression :

$$d\rho = \frac{x dy - y dx}{\rho}, \quad (8)$$

and

$$d\phi = \frac{x dy - y dx}{\rho^2}. \quad (9)$$

By replacing these different quantities previously calculated, we obtain the following metric:

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - (\frac{x dx + y dy}{\rho})^2) - e^{-2\psi}(\frac{x dy - y dx}{\rho})^2 - e^{2\psi}(dz + \omega(\frac{x dy - y dx}{\rho^2}))^2. \quad (10)$$

We use the coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, this gives for the covariant metric tensors $g_{\mu\nu}$ the components:

$$g_{00} = e^{2(\gamma-\psi)}, \quad (11)$$

$$g_{11} = \frac{-e^{-2\psi}}{\rho^2}(e^{2\gamma}x^2 + y^2(1 + (\frac{\omega}{\rho})^2e^{4\psi})), \quad (12)$$

$$g_{12} = \frac{-e^{-2\psi}}{\rho^2}(e^{2\gamma} - 1 - (\frac{\omega}{\rho})^2e^{4\psi})xy, \quad (13)$$

$$g_{22} = \frac{-e^{-2\psi}}{\rho^2}(e^{2\gamma}y^2 + x^2(1 + (\frac{\omega}{\rho})^2e^{4\psi})), \quad (14)$$

$$g_{13} = \frac{e^{2\psi}}{\rho^2}\omega y, \quad (15)$$

$$g_{23} = -\frac{e^{2\psi}}{\rho^2}\omega x, \quad (16)$$

$$g_{33} = -e^{2\psi}. \quad (17)$$

The determinant of the metric is:

$$g = -e^{4(\gamma-\psi)}. \quad (18)$$

We determine the different components of the contravariant tensor $g^{\mu\nu}$ which has the following form:

$$g^{00} = e^{2(\psi-\gamma)}, \quad (19)$$

$$g^{11} = \frac{-\rho^2 e^{2\psi}}{(e^{2\gamma}x^2 + y^2(1 + (\frac{\omega}{\rho})^2e^{4\psi}))}, \quad (20)$$

$$g^{12} = \frac{-\rho^2 e^{2\psi}}{(e^{2\gamma} - 1 - (\frac{\omega}{\rho})^2e^{4\psi})xy}, \quad (21)$$

$$g^{22} = \frac{-\rho^2 e^{2\psi}}{(e^{2\gamma}y^2 + x^2(1 + (\frac{\omega}{\rho})^2e^{4\psi}))}, \quad (22)$$

$$g^{13} = \frac{\rho^2 e^{-2\psi}}{\omega y}, \quad (23)$$

$$g^{23} = -\frac{\rho^2 e^{-2\psi}}{\omega x}, \quad (24)$$

$$g^{33} = -e^{-2\psi}. \quad (25)$$

These different tensors allow us to evaluate the different quantities, namely the energy and the momentum densities.

3. Energy and Momentum Densities

We measure the different physical quantities mentioned in this text, for that we first evaluate the energy and then the momentum.

3.1. Energy

We measure the amount of reasonable energy radiated during the interaction of these two gravitational waves, in this procedure we present the general formula which is written in the general form [9,11]:

$$H_{\mu}^{\nu\sigma} = (-g)^{-\frac{1}{2}} g_{\mu\lambda} \left[-g(g^{\nu\lambda} g^{\sigma\tau} - g^{\sigma\lambda} g^{\nu\tau}) \right]_{,\tau}, \quad (26)$$

and

$$H_{\mu}^{\nu\sigma} = -H_{\mu}^{\sigma\nu}. \quad (27)$$

Using this assumption, we obtain the following different quantities:

$$H_0^{01} = \frac{(e^{2\gamma} - 1 - \omega^2 e^{4\psi})}{\rho^2} x, \quad (28)$$

$$H_0^{02} = \frac{(e^{2\gamma} - 1 - \omega^2 e^{4\psi})}{\rho^2} y, \quad (29)$$

$$H_1^{01} = \frac{-2y^2(\gamma_{,t} - \frac{e^{4\psi}}{\rho^2} \omega(\omega_{,t} - \gamma_{,t} \omega + 2\psi_{,t} \omega))}{\rho^2}, \quad (30)$$

$$H_1^{02} = \frac{2xy(\gamma_{,t} - \frac{e^{4\psi}}{\rho^2} \omega(\omega_{,t} - \gamma_{,t} \omega + 2\psi_{,t} \omega))}{\rho^2}, \quad (31)$$

$$H_1^{03} = \frac{(2\gamma_{,t} \omega - \omega_{,t} - 4\psi_{,t} \omega)}{\rho^2} y, \quad (32)$$

$$H_2^{01} = \frac{2xy(\gamma_{,t} - \frac{e^{4\psi}}{\rho^2} \omega(\omega_{,t} - \gamma_{,t} \omega + 2\psi_{,t} \omega))}{\rho^2}, \quad (33)$$

$$H_2^{02} = \frac{-2x^2(\gamma_{,t} - \frac{e^{4\psi}}{\rho^2} \omega(\omega_{,t} - \gamma_{,t} \omega + 2\psi_{,t} \omega))}{\rho^2}, \quad (34)$$

$$H_2^{03} = \frac{(\omega_{,t} - 2\gamma_{,t} \omega + 4\psi_{,t} \omega)}{\rho^2} x, \quad (35)$$

$$H_3^{01} = \frac{\rho^2(2\gamma_{,t} \omega - \omega_{,t} - 4\psi_{,t} \omega)}{\omega^2 y}, \quad (36)$$

$$H_3^{02} = \frac{\rho^2(\omega_{,t} - 2\gamma_{,t} \omega + 4\psi_{,t} \omega)}{\omega^2 x}, \quad (37)$$

$$H_3^{03} = 4\psi_{,t} - 2\gamma_{,t}, \quad (38)$$

$$H_0^{12} = H_0^{23} = H_0^{13} = H_0^{03} = 0. \quad (39)$$

These different quantities allow to evaluate the quantities momentum densities produced during the interaction of these two gravitational waves.

3.2. Momentum Densities

To evaluate the different quantities momentum caused by the wave during its propagation, we use the following general expression [9]:

$$\theta_{\mu}^{\nu} = \frac{1}{16\pi} H_{\mu,\sigma}^{\nu\sigma}. \quad (40)$$

Using the previous results, we obtain the following expressions:

$$\theta_0^0 = \frac{1}{8\pi\rho} (\gamma_{,\rho} e^{2\gamma} - \omega e^{4\psi} (\omega_{,\rho} + 2\psi_{,\rho})), \quad (41)$$

$$\theta_1^0 \approx \frac{1}{8\pi\rho^2} \gamma_{,t} x, \quad (42)$$

$$\theta_2^0 \approx \frac{1}{8\pi\rho^2} \gamma_{,t} y, \quad (43)$$

$$\theta_0^1 = -\frac{1}{8\pi\rho^2} (\gamma_{,t} e^{2\gamma} - \omega e^{4\psi} (\omega_{,t} + 4\psi_{,t} \omega)) x, \quad (44)$$

$$\theta_0^2 = -\frac{1}{8\pi\rho^2} (\gamma_{,t} e^{2\gamma} - \omega e^{4\psi} (\omega_{,t} + 4\psi_{,t} \omega)) y, \quad (45)$$

$$\theta_0^3 = \theta_3^0 = 0. \quad (46)$$

3.3. Conclusions and Perspectives.

In summary, in this paper we have extended the results of Rosen and Virbhadra [9] to the case of the interaction of two gravitational waves. In this new study, we have obtained additional information indicating that the energy and momentum during the interaction of these gravitational waves admit reasonable values. These different results obtained in association with the soliton generation methods could, under certain conditions, bring additional information on the gravitational waves observed by the LIGO-Virgo scientific team [12] in particular on the amount of energy and momentum when using an exact solution of the Einstein field equations [13–15].

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