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Article

On Regularity and Stability Properties of G-Stochastic Differential Equations with Jumps

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Abstract

This paper deals with a system of G-stochastic differential equations with jumps, driven by G-Brownian motion and G-Lévy process. By using the Burkholder-Davis-Gundy inequalities, we prove a moment estimate and the temporal Hölder regularity of the solution, under the Linear growth and the global Lipschitz conditions of the coefficients with respect to the state variable uniformly in the time variable. Moreover, different stability properties are proved. Some examples like Black-Scholes market driven by G-Brownian motion are employed in order to support our theoretical results.

Keywords: sublinear expectation; G-Brownian motion; G-Lévy process; jump process; G-stochastic differential equation; moment estimate; temporal regularity; stability

1. Introduction

During the last eight decades, the theory of stochastic (partial) differential equations has played a major role in applied mathematics, due to its importance in modeling many complex phenomena with nondeterministic nature in applied sciences. We can find in the literature a huge number of papers concerned with such kind of equations, see for example [1–3] and the references therein.

In the recent years, the need to model the uncertainty or the ambiguity, due to an incomplete information, or vague concepts and principles gives rise to the so-called the theory of G-stochastic differential equations (briefly G-SDEs), see for instance [6,17,19–21]. Such theory developed quickly due to the great interest from many Mathematicians, which led to publication of many papers about the G-SDEs and their applications, see e.g. [4,11,24], besides a few works concerned with their qualitative properties, see for instance [12,13].

Modeling of some random phenomena of volatility uncertainty, which are discontinuous, specially the financial ones fails with the only use of the G-Brownian motion, due to the fact that this process has the property of the continuity of the paths. Therefore, the process with jumps, like G-Lévy process is more suitable in modeling such phenomena than the G-Brownian motion. The theory of G-Lévy process has been first introduced by Hu and Peng [10]. Later on, such theory has motivated some authors to deal with the SDEs driven by G-Lévy process. Namely, Faizullah et al. [7] have proved the well-posedness of stochastic functional differential equations driven by G-Lévy process, besides the stability of their solutions under linear growth and Lipschitz conditions. Similar results have been obtained under non-Lipschitz condition by Wang et al. in [28]. More recently, Ma et al. [14] have proved the stability of solutions for a class of stochastic delay non-linear system driven by G-Lévy process, under one-sided polynomial growth and local Lipschitz conditions. In addition, the only result about the existence of relaxed optimal control for G-SDEs with controlled jumps, has been

established by Redjil et al. in [23], and more recently, Saci et al. [25] have proved the well-posedness of a system of fractional SDEs driven by G-Brownian motion with delay.

To the best of our knowledge, there is no work deals with the temporal regularity and stability of the solutions to G-SDEs with jumps. Motivating by this and by the above facts, we aim in this work to study the temporal regularity, besides the stability with respect to the initial condition and to the coefficients of the solution for the following system of G-stochastic differential equations with jumps driven by G-Brownian motion $B := (B_t)_{t \in [0, T]}$, on some space of sublinear expectation $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathbb{F}^{\mathbb{P}})$, where $\mathbb{F}^{\mathbb{P}}$ is the universal filtration.

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t + \gamma(t, X_t)d\langle B \rangle_t + \int_{\mathbb{R}_0^d} f(t, X_{t-}, \theta) \tilde{N}(dt, d\theta), \\ X_0 &= x, \end{cases} \quad (1.1)$$

for any $t \in (0, T]$, where $x \in \mathbb{R}^n$ be a given initial condition, $\langle B \rangle := (\langle B \rangle_t)_{t \in [0, T]}$ be the quadratic variation process of B and

$$b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

$$\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d},$$

$$\gamma : [0, T] \times \mathbb{R}^n \rightarrow (\mathbb{R}^{d^2})^{\times n}$$

and

$$f : [0, T] \times \mathbb{R}^n \times \mathbb{R}_0^d \rightarrow \mathbb{R}^n$$

are given deterministic functions, for $n, d \in \mathbb{N}_0$, with \mathbb{N}_0 and \mathbb{R}_0^d are $\mathbb{N} - \{0\}$ and $\mathbb{R}^d - \{0\}$ respectively. The jumps are confined to the set \mathbb{R}_0^d , and

$$\tilde{N}(dt, d\theta) = N(dt, d\theta) - V(d\theta)dt,$$

where N is an independent Poisson measure with compensator is $V(d\theta)dt$.

This paper has the following plan: in Section 2 we give some preliminaries about G-expectation, G-Brownian motion and G-stochastic integrals, besides background about G-Lévy processes. Section 3 is devoted to introduce the result of the existence and the uniqueness of solution. Further, we state and prove a result that is concerned with a moment estimate of such solution. In the main Section 4 we deal with the temporal regularity.

2. Preliminaries

In this section we recall the notions and main results from the framework of G-stochastic calculus, mainly based on the references [5,6,18–20,26,27].

2.1. G-Expectation and G-Brownian Motion

Let $\Omega := \{\omega \in C(\mathbb{R}_+, \mathbb{R}^d) : \omega(0) = 0\}$, equipped with the topology of uniform convergence on compact intervals, $\mathcal{B}(\Omega)$ the associated Borel σ -algebra, $\Omega_t := \{\omega_{\cdot \wedge t} : \omega \in \Omega\}$, B the canonical process and \mathbb{P}_0 be the Wiener measure on Ω . Let $\mathbb{F} := \mathbb{F}^B = \{\mathcal{F}_t\}_{t \geq 0}$ be the raw filtration generated by B , which is only left-continuous. Further, consider the right-limit filtration $\mathbb{F}^+ := \{\mathcal{F}_t^+, t \geq 0\}$, where $\mathcal{F}_t^+ := \mathcal{F}_{t+} := \bigcap_{s > t} \mathcal{F}_s$.

Given a probability measure \mathbb{P} on $(\Omega, \mathcal{B}(\Omega))$, we consider the right-continuous \mathbb{P} -completed filtrations $\mathcal{F}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_t^+)$ and $\hat{\mathcal{F}}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$, where the \mathbb{P} -negligible set $\mathcal{N}^{\mathbb{P}}(\mathcal{G})$ on a σ -algebra \mathcal{G} is defined as

$$\mathcal{N}^{\mathbb{P}}(\mathcal{G}) := \{D \subset \Omega : \text{there exists } \tilde{D} \in \mathcal{G} \text{ such that } D \subset \tilde{D} \text{ and } \mathbb{P}[\tilde{D}] = 0\}.$$

Lemma 2.1 ([26], Lemma 2.1). Let \mathbb{P} be an arbitrary probability measure on $(\Omega, \mathcal{F}_\infty)$. For every $\hat{\mathcal{F}}_t^\mathbb{P}$ -measurable random variable $\hat{\xi}$, there exists a \mathbb{P} -a.s unique \mathcal{F}_t -measurable random variable ξ such that $\xi = \hat{\xi}$, \mathbb{P} -a.s.

For every $\hat{\mathbb{P}}^\mathbb{P}$ -progressively measurable process \hat{X} , there exists a unique \mathbb{F} -progressively measurable process X such that $X = \hat{X}$, $dt \times \mathbb{P}$ -a.s. Moreover, if \hat{X} is \mathbb{P} -almost surely continuous, then X can be chosen to \mathbb{P} -almost surely continuous.

The G -expectation is defined by Peng in [17,19,20] through the nonlinear heat equation in the following sense. A d -dimensional random vector X is said to be G -normally distributed under the G -expectation $\hat{\mathbb{E}}[\cdot]$, if for each bounded and Lipschitz continuous function φ on \mathbb{R}^d , $\varphi \in \text{Lip}(\mathbb{R}^d)$, the function u defined by

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)], \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

is the unique, bounded Lipschitz continuous viscosity solution of the following parabolic equation

$$\frac{\partial u}{\partial t} - G(D^2u) = 0, \quad \text{on } (t, x) \in (0, +\infty) \times \mathbb{R}^d \quad \text{and} \quad u(0, x) = \varphi(x),$$

where $D^2u = (\partial_{x_i x_j}^2 u)_{1 \leq i, j \leq d}$ is the Hessian matrix of u and the nonlinear operator G is defined by

$$G(A) := \frac{1}{2} \sup_{\gamma \in \Gamma} \{tr(\gamma \gamma^* A)\}, \quad \gamma \in \mathbb{R}^{d \times d}, \quad (2.1)$$

where A is a $d \times d$ symmetric matrix and Γ is a given non empty, bounded and closed subset of $\mathbb{R}^{d \times d}$. Here, v^* denotes the transpose of the vector v . This G -normal distribution is denoted by $N(0, \Sigma)$, where $\Sigma := \{\gamma \gamma^*, \gamma \in \Gamma\}$.

Peng [19,20] has showed that the G -expectation $\hat{\mathbb{E}} : \mathcal{H} := \text{Lip}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is a consistent sublinear expectation on the lattice \mathcal{H} of real functions i.e. it satisfies

- (i) **Sub-additivity:** for all $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$,
- (ii) **Monotonicity:** for all $X, Y \in \mathcal{H}$, $X \geq Y \Rightarrow \hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$,
- (iii) **Constant preserving:** for all $c \in \mathbb{R}$, $\hat{\mathbb{E}}[c] = c$,
- (v) **Positive homogeneity:** for all $\lambda \geq 0$, $X \in \mathcal{H}$, $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$.

Now, let $\text{Lip}(\Omega)$ be the set of random variables of the form $\zeta := \varphi(B_{t_1}, B_{t_2}, \dots, B_{t_n})$ for some bounded Lipschitz continuous function φ on $\mathbb{R}^{d \times n}$ and $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T$. The coordinate process $(B_t, t \geq 0)$ is called G -Brownian motion whenever B_1 is G -normally distributed under $\hat{\mathbb{E}}[\cdot]$ and for each $s, t \geq 0$ and $t_1, t_2, \dots, t_n \in [0, t]$ we have

$$\hat{\mathbb{E}}[\varphi(B_{t_1}, \dots, B_{t_n}, B_{t+s} - B_t)] = \hat{\mathbb{E}}[\psi(B_{t_1}, \dots, B_{t_n})],$$

where $\psi(x_1, \dots, x_n) = \text{hat}\mathbb{E}[\varphi(x_1, \dots, x_n, \sqrt{s}B_1)]$. This property implies that the increments of the G -Brownian motion are independent and that $B_{t+s} - B_t$ and B_s are identically $N(0, s\Sigma)$ -distributed.

A remarkable result of Peng [19,20] is that if \mathcal{H} is a lattice of real functions on Ω such that $\text{Lip}(\Omega) \subset \mathcal{H}$, then the G -expectation $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ is a consistent sublinear expectation.

For $p \in [0, +\infty)$, we denote by $\mathbb{L}_G^p(\Omega)$ the closure of $\text{Lip}(\Omega)$ under the Banach norm

$$\|X\|_{\mathbb{L}_G^p(\Omega)}^p := \hat{\mathbb{E}}[|X|^p].$$

For each $t \geq 0$, let $L^0(\Omega_t)$ be the set of \mathcal{F}_t -measurable functions. We set

$$\text{Lip}(\Omega_t) := \text{Lip}(\Omega) \cap L^0(\Omega_t), \quad \mathbb{L}_G^p(\Omega_t) := \mathbb{L}_G^p(\Omega) \cap L^0(\Omega_t).$$

2.2. G-Stochastic Integrals

For $p \in [0, +\infty)$, we let $M_G^{0,p}(0, T)$ be the space of \mathbb{F} -progressively measurable, \mathbb{R}^d -valued elementary processes of the form

$$\eta(t) = \sum_{i=0}^{n-1} \eta_i 1_{[t_i, t_{i+1})}(s),$$

where $0 \leq 0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$, $n \geq 1$ and $\eta_i \in \text{Lip}(\Omega_{t_i})$. Let $M_G^p(0, T)$ be the closure of $M_G^{0,p}(0, T)$ under the norm

$$\|\eta\|_{M_G^p(0, T)}^p := \hat{\mathbb{E}}\left[\int_0^T |\eta(t)|^p ds\right].$$

For each $\eta \in M_G^{0,2}(0, T)$, the G-stochastic integral is defined pointwisely by

$$I_t(\eta) = \int_0^t \eta_s d_G B_s := \sum_{j=0}^{n-1} \eta_j (B_{t \wedge t_{j+1}} - B_{t \wedge t_j}),$$

with $I(\eta) := I_T(\eta)$, the mapping $I : M_G^{0,2}(0, T) \rightarrow \mathbb{L}_G^2(\Omega_t)$ is continuous and thus can be continuously extended to $M_G^2(0, T)$.

The quadratic variation process of G-Brownian motion can be formulated in $\mathbb{L}_G^2(\Omega_t)$ by the continuous $d \times d$ -symmetric-matrix-valued process defined by

$$\langle B \rangle_t^G := B_t \otimes B_t - 2 \int_0^t B_s \otimes d_G B_s, \quad (2.2)$$

whose diagonal is constituted of nondecreasing processes. Here, for $a, b \in \mathbb{R}^d$, the $d \times d$ -symmetric matrix $a \otimes b$ is defined by $(a \otimes b)x = (a \cdot x)b$ for $x \in \mathbb{R}^d$, where " \cdot " denotes the scalar product in \mathbb{R}^d .

Define the mapping $\mathcal{J} : M_G^{0,1}(0, T) \mapsto \mathbb{L}_G^1(\Omega_T)$:

$$\mathcal{J} = \int_0^T \eta_t d \langle B \rangle_t^G := \sum_{j=0}^{n-1} \eta_j (\langle B \rangle_{t_{j+1}}^G - \langle B \rangle_{t_j}^G).$$

Then \mathcal{J} can be uniquely extended to $\mathcal{Q} : M_G^1(0, T) \rightarrow \mathbb{L}_G^1(\Omega_T)$, where

$$\mathcal{Q} := \int_0^T \eta_t d \langle B \rangle_t^G, \quad \eta \in M_G^1(0, T).$$

Now, we have the following 'isometry' (formulated for the case $d = 1$, for simplicity).

Lemma 2.2 ([20]).

Assume $d = 1$ and let $\eta \in M_G^2(0, T)$. We have

$$\hat{\mathbb{E}} \left[\left(\int_0^T \eta(s) d_G B_s \right)^2 \right] = \hat{\mathbb{E}} \left[\int_0^T \eta^2(s) d \langle B \rangle_s^G \right].$$

2.3. A Dual Representation of G-Expectation

Denis et. al [5] and Denis and Martini [6] have proved the following dual representation of the G-expectation in terms of a weakly compact (tight) family \mathcal{P} of possibly mutually singular probability measures on $(\Omega, \mathcal{B}(\Omega))$. This duality expresses the G-expectation as a robust expectation with respect to \mathcal{P} . We refer to the above papers for explicit constructions of \mathcal{P} . Soner et al. [26,27] have performed an in-depth analysis of such a construction and its consequences on the G-stochastic analysis and in particular the question of aggregation of processes.

Proposition 2.3 ([5,6]). *There exists a family of weakly compact probability measures \mathcal{P} on $(\Omega, \mathcal{B}(\Omega))$ such that for each $\xi \in \mathbb{L}_G^1(\Omega)$*

$$\hat{\mathbb{E}}[\xi] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi]. \quad (2.3)$$

Moreover, the set function

$$c(A) := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(A), \quad A \in \mathcal{B}(\Omega),$$

defines a regular Choquet capacity.

This leads to the following ([6,26]).

Definition 2.4. *A set $A \in \mathcal{B}(\Omega)$ is called polar if $c(A) = 0$ or equivalently if $\mathbb{P}(A) = 0$ for all $\mathbb{P} \in \mathcal{P}$. We say that a property holds \mathcal{P} -quasi-surely (q.s.) if it holds \mathbb{P} -almost-surely for all $\mathbb{P} \in \mathcal{P}$ i.e. outside a polar set. A probability measure \mathbb{P} is called absolutely continuous with respect to \mathcal{P} if $\mathbb{P}(A) = 0$ for all $A \in \mathcal{N}_{\mathcal{P}}$.*

Denote by $\mathcal{N}_{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_{\infty})$ the \mathcal{P} -polar sets. We shall use the following universal filtration $\mathbb{F}^{\mathcal{P}}$ for the possibly mutually singular probability measures $\{\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$ (cf. [27]).

$$\mathbb{F}^{\mathcal{P}} := \{\hat{\mathcal{F}}_t^{\mathcal{P}}\}_{t \geq 0}, \quad \text{where } \hat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} (\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}), \quad \text{for } t \geq 0. \quad (2.4)$$

The dual formulation of the G-expectation suggests the following aspect of aggregation.

Lemma 2.5 ([26], Proposition 3.3). *Let $\eta \in M_G^2(0, T)$. Then, η is Itô-integrable for every $\mathbb{P} \in \mathcal{P}$. Moreover, for every $t \in [0, T]$,*

$$\int_0^t \eta(s) d_G B_s = \int_0^t \eta(s) dB_s, \quad \mathbb{P} - \text{a.s. for every } \mathbb{P} \in \mathcal{P}, \quad (2.5)$$

where the right hand side is the usual Itô integral. Consequently, the quadratic variation process $\langle B \rangle^G$ defined in (2.2) agrees with the usual quadratic variation process quasi-surely.

In the sequel, we will drop the notation G from both the G-stochastic integral and the G-quadratic variation.

Lemma 2.6 ([5], Lemma 29). *If $(\mathbb{P}_n)_{n=1}^{\infty} \subset \mathcal{P}$ converges weakly to $\mathbb{P} \in \mathcal{P}$. Then for each $\xi \in \mathbb{L}_G^1(\Omega_T)$, $\mathbb{E}^{\mathbb{P}_n}[\xi] \rightarrow \mathbb{E}^{\mathbb{P}}[\xi]$.*

Considering the properties of the quadratic variation process $\langle B \rangle$ in the G-framework and the dual formulation of the G-expectation, we have the following Burkholder-Davis-Gundy-type estimates.

Lemma 2.7 ([8]). *For each $p \geq 2$ and $\eta \in M_G^p(0, T)$, there exists some constant C_p depending only on p and T such that*

$$\hat{\mathbb{E}} \left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r dB_r \right|^p \right] \leq C_p \hat{\mathbb{E}} \left[\left(\int_s^t |\eta_r|^2 dr \right)^{p/2} \right] \leq C_p |t - s|^{\frac{p}{2} - 1} \int_s^t \hat{\mathbb{E}}[|\eta_r|^p] dr. \quad (2.6)$$

If $\bar{\sigma}$ is a positive constant such that $\frac{d\langle B \rangle_t}{dt} \leq \bar{\sigma}$ quasi-surely, then, for each $p \geq 1$ and $\eta \in M_G^p(0, T)$,

$$\hat{\mathbb{E}} \left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r d\langle B \rangle_r \right|^p \right] \leq \bar{\sigma}^p |t - s|^{p-1} \int_s^t \hat{\mathbb{E}}[|\eta_r|^p] dr. \quad (2.7)$$

2.4. Representation of G-Lévy Processes

Now we present the definition of Lévy processes

Definition 2.8. Let $(X_t)_{t \geq 0}$ be a d -dimensional càdlàg process on sublinear expectation space $(\Omega, H, \widehat{\mathbb{E}})$. We say that X is a Lévy process if:

- 1– $X_0 = x$
 - 2– $\forall t, s \geq 0$, the increment $X_{t+s} - X_t$ is independent of $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ for every $n \in \mathbb{N}$ and every partition $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq t$.
 - 3– The distribution of the increment $X_{t+s} - X_t$, $t, s \geq 0$ is stationary (does not depend on time).
- Moreover, we say that a Lévy process X is a G -Lévy process, if satisfies additionally following conditions:
- 4– There exists a d -dimensional Lévy process $(X_t^c, X_t^d)_{t \geq 0}$ such that for each $t \geq 0$, $X_t = X_t^c + X_t^d$.
 - 5– The processes X^c and X^d satisfy the following growth conditions:

$$\lim_{t \rightarrow 0} \widehat{\mathbb{E}}[|X_t^c|^3] t^{-1} = 0; \widehat{\mathbb{E}}[|X_t^d|] \leq C_t \text{ for all } t \geq 0.$$

Remark 1. Condition 5 implies that X^c is a d -dimensional generalized G -Brownian motion (in particular it has continuous paths) and the jump part X^d is of finite variation.

Peng and Hu [10] noticed that each G -Lévy process X can be characterized by a non local operator G_X .

Theorem 2.9 ([16], Theorem 7). Let U the subset defined as in the Lévy-Khintchine representation Theorem [10]. Consider the canonical space $\Omega = D_0([0, T], \mathbb{R}^n)$ of all càdlàg functions taking values in \mathbb{R}^n with the Skorokhod topology, then there exists a sublinear expectation $\widehat{\mathbb{E}}$ on $D_0([0, T], \mathbb{R}^n)$ such that the canonical process $(X_t)_{t \geq 0}$ is a G -Lévy process satisfying Lévy Khintchine representation with the same set U .

In the rest of this paper, we assume that the Lévy process $(X_t)_{t \geq 0}$ has finite activity, i.e.

$$\alpha := \sup_{v \in \vartheta} v(\mathbb{R}_0^d) < \infty. \quad (2.8)$$

3. Formulation and Well-Posedness of Prb. (1.1)

Before recalling the well-posedness result of Prb. (1.1), let us first introduce its form, mainly based on [16]. To do this, let $(\Omega, H, \widehat{\mathbb{E}}, \mathbb{F}^{\mathcal{P}})$ be the space of sublinear expectation, we associate the universal filtration $\mathbb{F}^{\mathcal{P}}$:

$$\mathbb{F}^{\mathcal{P}} := \{\widehat{\mathcal{F}}_t^{\mathcal{P}}\}_{t \geq 0}.$$

where:

$$\widehat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} (\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}) \quad \text{for } t \geq 0.$$

By considering an independent Poisson measure N whose compensator $V(d\theta)dt$, where the jumps are confined to the set \mathbb{R}_0^d , we have

$$\widetilde{N}(dt, d\theta) = N(dt, d\theta) - V(d\theta)dt.$$

Let X be an \mathbb{R}^n -dimensional process. The space $\widehat{M}_G^p(0, T)$ is the completion of the space of n -dimensional elementary processes under the following norm

$$\|X\|_{\widehat{M}_G^p(0, T)}^p := \int_0^T \widehat{\mathbb{E}}[|X_t|^p] dt, \quad p \geq 1.$$

Furthermore, to adjust the space of integrands for the jump measure, Paczka [16] has defined the space $\widehat{\mathcal{H}}_G^2([0, T] \times \mathbb{R}_0^d)$ as a completion of $\mathcal{H}_G^S([0, T] \times \mathbb{R}_0^d)$ with respect to the following norm:

$$\|f\|_{\widehat{\mathcal{H}}_G^2([0, T] \times \mathbb{R}_0^d)}^2 := \int_0^T \widehat{\mathbb{E}} \left[\sup_{v \in \vartheta} \int_{\mathbb{R}_0^d} f^2(s, \theta) v(d\theta) \right] ds,$$

where $\mathcal{H}_G^S([0, T] \times \mathbb{R}_0^d)$ is a collection of all processes defined on $[0, T] \times \mathbb{R}_0^d \times \Omega$ of the form:

$$f(s, \theta)(w) = \sum_{k=1}^{n-1} \sum_{l=1}^m F_{k,l}(w) 1_{]t_k, t_{k+1}[}(s) \Psi_l(\theta), \quad n, m \in \mathbb{N}_0,$$

where $(t_i)_{i=1}^n$ is a partition of $[0, T]$, $(\Psi_l)_{l=1}^m \subset C_{b, \text{lip}}(\mathbb{R}^d)$ are functions with disjoint supports such that:

$$\begin{aligned} \Psi_l(0) &= 0 \text{ and } F_{k,l} = \phi_{k,l}(X_{t_1}, \dots, X_{t_{k-1}} - X_{t_k}), \\ \phi_{k,l} &\in C_{b, \text{lip}}(\mathbb{R}^{d \times k}), \end{aligned}$$

with the following well defined two norms

$$\|f\|_{\mathcal{H}_G^p([0, T] \times \mathbb{R}_0^d)} := \hat{\mathbb{E}} \left[\int_0^T \sup_{v \in \vartheta} \int_{\mathbb{R}_0^d} |f(s, \theta)|^p v(d\theta) ds \right]^{\frac{1}{p}}, \quad p = 1, 2.$$

Now, we give the Burkholder-Davis-Gundy type inequality for G-stochastic integral with respect to G-Lévy process.

Lemma 3.1 ([28], Lemma 2.3).

Let $f \in \mathcal{H}_G^2([0, T] \times \mathbb{R}_0^d)$ and let $K_t := \int_0^t \int_{\mathbb{R}_0^d} f(s, \theta) \tilde{N}(ds, d\theta)$. Then, there exists a càdlàg modification \tilde{K}_t of K_t , $\forall t \in [0, T]$ such that

$$\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |\tilde{K}_t|^2 \right] \leq C_T \hat{\mathbb{E}} \left[\int_0^T \sup_{v \in \vartheta} \int_{\mathbb{R}_0^d} |f(s, \theta)|^2 v(d\theta) ds \right], \quad (3.1)$$

for some positive constant C_T , where $\mathcal{H}_G^2([0, T] \times \mathbb{R}_0^d)$ is the topological completion of $\mathcal{H}_G^S([0, T] \times \mathbb{R}_0^d)$ with respect to the following norm:

$$\|f\|_{\mathcal{H}_G^2([0, T] \times \mathbb{R}_0^d)}^2 := \hat{\mathbb{E}} \left[\int_0^T \sup_{v \in \vartheta} \int_{\mathbb{R}_0^d} |f(s, \theta)|^2 v(d\theta) ds \right].$$

In order to fulfill the well-posedness of Prb.(1.1), Paczka [16] has imposed the following assumption:

Assumption A.

- For any $x \in \mathbb{R}^n$, the functions $b(\cdot, x)$, $\gamma(\cdot, x)$ and $\sigma(\cdot, x)$ take values in $\hat{M}_G^2(0, T)$ and $f(\cdot, x, \cdot)$ takes value in $\hat{\mathcal{H}}_G^2([0, T] \times \mathbb{R}_0^d)$. Moreover, $\gamma(\cdot, x)$ is a symmetric $d \times d$ matrix with each element.
- The functions b, γ, σ and f are Lipschitz continuous with respect to the space variable x uniformly in time.

Definition 3.2 ([16], Definition 33). Let the stochastic process $X := (X(t))_{t \in [0, T]} \in \hat{M}_G^2(0, T)$. We say that X is a solution of Prb.(1.1), if it satisfies

$$\begin{aligned} X(t) &= x + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dB_s \\ &+ \int_0^t \gamma(s, X(s)) d\langle B \rangle_s + \int_0^t \int_{\mathbb{R}_0^d} f(s, X(s^-), \theta) \tilde{N}(ds, d\theta), \end{aligned} \quad (3.2)$$

for every $t \in [0, T]$.

Theorem 3.3 (Well-posedness, [16], Theorem 34). Assume that **Assumption A** is fulfilled. Then, Prb.(1.1) admits a unique solution $X \in \hat{M}_G^2(0, T)$ in the sense of Definition 3.2.

To prove a moment estimate of the solution, we need to assume that the functions b, σ, γ and f satisfy the linear growth condition with respect to the space variable x uniformly in t , i.e.

Assumption \mathcal{H} . $\exists c > 0$ such that for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$, it holds

$$\max\{|b(t, x)|, |\sigma(t, x)|, |\gamma(t, x)|, |f(t, x)|\} \leq c(1 + |x|), \quad \forall t \in [0, T]. \quad (3.3)$$

Proposition 3.4. (Moment estimate). Let $X := (X(t))_{t \in [0, T]}$ be the solution of the Prb.(1.1). Under the assumption of Theorem 3.3, the solution X satisfies for every $t \in [0, T]$ the following

$$\sup_{s \in [0, t]} \hat{\mathbb{E}}[|X(s)|^2] \leq C < \infty, \quad (3.4)$$

for some positive constant C , which depends on x, t and the boundedness constant.

To prove Proposition 3.4, we need also the following two lemmas.

Lemma 3.5 (Grönwall lemma, [9]). Let g be a $[0, +\infty)$ -valued continuous function defined on $[0, T]$ (for $T > 0$ be fixed). If there exists $q > 0$ s.t. the following inequality holds for some two positive constants C_1 and C_2 ,

$$g(t) \leq C_1 + C_2 \int_0^t (t-s)^{q-1} g(s) ds, \quad \forall t \in (0, T].$$

Then, $\exists C > 0$, s.t.

$$g(t) \leq C_1 C.$$

Lemma 3.6. (Basic inequality). Let $n \in \mathbb{N}_0$, $p \geq 2$ and let $a_i \geq 0$, $i \in \{1, \dots, n\}$. It is true that

$$\left(\sum_{i=1}^n a_i \right)^p \leq n^{p-1} \sum_{i=1}^n a_i^p. \quad (3.5)$$

Proof of Proposition 3.4

Let $T > 0$ be fixed and let X be the solution of the problem. The use of the basic inequality (3.5) and the sub-additivity of the G-expectation, gives us

$$\begin{aligned} \hat{\mathbb{E}}(|X(t)|^2) &\leq 5|x|^2 + 5\hat{\mathbb{E}}\left(\left|\int_0^t b(r, X(r))dr\right|^2\right) + 5\hat{\mathbb{E}}\left(\left|\int_0^t \sigma(r, X(r))dB_r\right|^2\right) \\ &+ 5\hat{\mathbb{E}}\left(\left|\int_0^t \gamma(r, X(r))d\langle B \rangle_r\right|^2\right) + 5\hat{\mathbb{E}}\left(\left|\int_0^t \int_{\mathbb{R}_0^d} f(r, X(r^-), \theta) \tilde{N}(dr, d\theta)\right|^2\right) \\ &=: 5|x|^2 + 5(I_1 + I_2 + I_3 + I_4), \end{aligned} \quad (3.6)$$

for any $t \in [0, T]$.

We use Hölder inequality, the **Assumption \mathcal{H}** and the inequality (3.5), to estimate I_1 as follows

$$\begin{aligned} I_1 &:= \hat{\mathbb{E}}\left(\left|\int_0^t b(r, X(r))dr\right|^2\right) \leq t \hat{\mathbb{E}}\left(\int_0^t |b(r, X(r))|^2 dr\right) \\ &\leq 2t c^2 \int_0^t (1 + \hat{\mathbb{E}}|X(r)|^2) dr \leq 2T^2 c^2 + 2Tc^2 \int_0^t \sup_{r \in [0, t]} \hat{\mathbb{E}}[|X(r)|^2] dr. \end{aligned} \quad (3.7)$$

To estimate I_2 , we use Burkholder-Davis-Gundy inequality (2.6) (with $p = 2$), **Assumption \mathcal{H}** and the inequality (3.5) as follows

$$\begin{aligned} I_2 &:= \hat{\mathbb{E}}\left(\left|\int_s^t \sigma(r, X(r))dB_r\right|^2\right) \leq C_2 \int_0^t \hat{\mathbb{E}}|\sigma(r, X(r))|^2 dr \\ &\leq 2 C_2 c^2 \int_0^t (1 + \hat{\mathbb{E}}|X(r)|^2)dr \leq 2 C_2 c^2 T + 2 C_2 c^2 \int_0^t \sup_{r \in [0, t]} \hat{\mathbb{E}}[|X(r)|^2]dr. \end{aligned} \quad (3.8)$$

We argue as above to estimate I_3 as follows

$$I_3 := \hat{\mathbb{E}}\left(\left|\int_0^t \gamma(r, X(r))d\langle B \rangle_r\right|^2\right) \leq 2 \bar{\sigma}^2 T^2 c^2 + 2 \bar{\sigma}^2 T c^2 \int_0^t \sup_{r \in [0, t]} \hat{\mathbb{E}}|X(r)|^2 dr. \quad (3.9)$$

By virtue of Burkholder-Davis-Gundy inequality in Lemma 3.1, the **Assumption \mathcal{H}** , the inequality (3.5), condition (2.8) and the fact that the set $\{r : X_{r^-} \neq X_r\}$ is countable, we estimate I_4 as

$$\begin{aligned} I_4 &:= \hat{\mathbb{E}}\left(\left|\int_0^t \int_{\mathbb{R}_0^d} f(r, X(r^-), \theta) \tilde{N}(dr, d\theta)\right|^2\right) \\ &\leq \hat{\mathbb{E}}\left(\sup_{\tau \in [0, t]} \left|\int_0^\tau \int_{\mathbb{R}_0^d} f(r, X(r^-), \theta) \tilde{N}(dr, d\theta)\right|^2\right) \\ &\leq C_t \hat{\mathbb{E}}\left(\int_0^t \sup_{v \in \emptyset} \int_{\mathbb{R}_0^d} |f(r, X(r^-), \theta)|^2 v(d\theta) dr\right) \\ &\leq 2 C_t c^2 \hat{\mathbb{E}}\left(\int_0^t \sup_{v \in \emptyset} \int_{\mathbb{R}_0^d} (1 + |X(r^-)|^2) v(d\theta) dr\right) \\ &\leq 2 C_t c^2 \alpha \hat{\mathbb{E}}\left(\int_0^t (1 + |X(r^-)|^2) dr\right) \leq 2 C_t c^2 \alpha \int_0^t (1 + \hat{\mathbb{E}}|X(r^-)|^2) dr \\ &\leq 2 C_t c^2 \alpha \int_0^t (1 + \hat{\mathbb{E}}|X(r)|^2) dr \leq 2 C_T c^2 \alpha T + 2 C_T c^2 \alpha \int_0^t \sup_{r \in [0, t]} \hat{\mathbb{E}}|X(r)|^2 dr. \end{aligned}$$

By replacing the estimates (3.7), (3.8), (3.9) and (3.10) in (3.6) we obtain

$$\hat{\mathbb{E}}(|X(t)|^2) \leq C_1 + C_2 \int_0^t \sup_{r \in [0, t]} \hat{\mathbb{E}}|X(r)|^2 dr,$$

for any $t \in [0, T]$, where

$$C_1 := 5|x|^2 + 2c^2 T(T + C_2 + \bar{\sigma}^2 T + C_T \alpha),$$

and

$$C_2 := 10c^2(T + C_2 + \bar{\sigma}^2 T + C_T \alpha).$$

Consequently,

$$\sup_{t \in [0, T]} \hat{\mathbb{E}}(|X(t)|^2) \leq C_1 + C_2 \int_0^t \sup_{r \in [0, t]} \hat{\mathbb{E}}|X(r)|^2 dr.$$

Finally, an application of Grönwall Lemma 3.5 with $\rho = 1$ completes the proof.

4. Temporal Regularity of the Solution

This section is devoted to study the temporal regularity of the solution in the following sense.

Definition 4.1. Let $T > 0$ be fixed. The solution $X := (X(t))_{t \in [0, T]}$ of Prb.(3.2), is said to be time Hölder continuous with exponent $\theta \in (0, 1)$, if there exists a positive constant C such that

$$\sup_{t, s \in [0, T], t \neq s} \frac{\hat{\mathbb{E}}(|X(t) - X(s)|^2)}{|t - s|^{2\theta}} \leq C.$$

The main result of this section is the following.

Theorem 4.2. Let $T > 0$ be fixed. According to the Assumptions \mathcal{A} and \mathcal{H} , the solution X of Prb.(1.1) is time Hölder continuous with exponent $\frac{1}{2}$.

Proof. Let $T > 0$, $t, s \in [0, T]$ and X be the solution of the problem. Without loss in generality, we assume that $s < t$. From Equation (3.2), the basic inequality (3.5) and the sub-additivity of the G-expectation, we have

$$\begin{aligned} \hat{\mathbb{E}}(|X(t) - X(s)|^2) &\leq 4\hat{\mathbb{E}}\left(\left|\int_s^t b(r, X(r))dr\right|^2\right) \\ &+ 4\hat{\mathbb{E}}\left(\left|\int_s^t \sigma(r, X(r))dB_r\right|^2\right) \\ &+ 4\hat{\mathbb{E}}\left(\left|\int_s^t \gamma(r, X(r))d\langle B \rangle_r\right|^2\right) \\ &+ 4\hat{\mathbb{E}}\left(\left|\int_s^t \int_{\mathbb{R}_0^d} f(r, X_{r^-}, \theta)\tilde{N}(dr, d\theta)\right|^2\right) \\ &=: 4(J_1 + J_2 + J_3 + J_4). \end{aligned} \quad (4.1)$$

In order to estimate J_1, J_2, J_3 and J_4 , we follow the same steps used above for the estimates I_1, I_2, I_3 and I_4 respectively, besides the use of the Proposition 3.4 as follows:

$$\begin{aligned} J_1 &:= \hat{\mathbb{E}}\left(\left|\int_s^t b(r, X(r))dr\right|^2\right) \leq 2c^2(t-s) \int_s^t (1 + \sup_{r \in [s, t]} \hat{\mathbb{E}}[|X(r)|^2])dr \\ &\leq 2c^2(1+C)(t-s)^2, \end{aligned} \quad (4.2)$$

$$\begin{aligned} J_2 &:= \hat{\mathbb{E}}\left(\left|\int_s^t \sigma(r, X(r))dB_r\right|^2\right) \leq 2C_2c^2 \int_s^t (1 + \sup_{r \in [s, t]} \hat{\mathbb{E}}|X(r)|^2)dr \\ &\leq 2C_2c^2(1+C)(t-s), \end{aligned} \quad (4.3)$$

$$\begin{aligned} J_3 &:= \hat{\mathbb{E}}\left(\left|\int_s^t \gamma(r, X(r))d\langle B \rangle_r\right|^2\right) \leq 2\bar{\sigma}^2(t-s)c^2 \int_s^t (1 + \sup_{r \in [s, t]} \hat{\mathbb{E}}|X(r)|^2)dr \\ &\leq 2\bar{\sigma}^2c^2(1+C)(t-s)^2, \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} J_4 &:= \hat{\mathbb{E}}\left(\left|\int_s^t \int_{\mathbb{R}_0^d} f(r, X(r^-), \theta)\tilde{N}(dr, d\theta)\right|^2\right) \\ &\leq 2C_T c^2 \alpha \int_0^t (1 + \sup_{r \in [s, t]} \hat{\mathbb{E}}|X(r)|^2)dr \leq 2C_T c^2 \alpha (1+C)(t-s). \end{aligned} \quad (4.5)$$

Coming back to estimate (4.1), by replacing the estimates (4.2), (4.3), (4.4) and (4.5) in it we obtain

$$\hat{\mathbb{E}}\left(|X(t) - X(s)|^2\right) \leq C(t - s), \quad (4.6)$$

for any $t, s \in [0, T]$ such that $s < t$, where $C = 8c^2(1 + C)(T + C_2 + \bar{\sigma}^2T + C_T\alpha)$. By this, the desired result is obtained. \square

5. Some Stability Properties of the Solution

5.1. Stability with Respect to the Initial Condition

To deal with the stability with respect to the initial condition, let us first rewrite the Prb.(1.1) as follows,

$$\begin{cases} dX_t^x &= b(t, X_t^x)dt + \sigma(t, X_t^x)dB_t + \gamma(t, X_t^x)d\langle B \rangle_t + \int_{\mathbb{R}^d} f(t, X_t^x, \theta)\tilde{N}(dt, d\theta), \forall t \in (0, T], \\ X_0^x &= x, \end{cases}$$

where $(X_t^x)_{t \in [0, T]}$ be its unique solution.

Theorem 5.1. *Suppose that b, σ, γ and f satisfy the Assumption \mathcal{A} , then the application:*

$$\Phi : \mathbb{R}^d \rightarrow \hat{M}_G^2(0, T)$$

defined by $(\Phi(x))_t = (X_t^x)$ is continuous.

Proof. Let (x_n) a sequence of \mathbb{R}^d converges to x , such that $X_0^x = x$ and $X_0^n = x_n$, we prove that

$$\lim_{n \rightarrow +\infty} \hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t^x|^2 \right] = 0.$$

We have

$$\begin{aligned} |X_t^n - X_t^x|^2 &\leq C|x_n - x|^2 + C \left(\int_0^t |b(s, X_s^n) - b(s, X_s^x)| ds \right)^2 \\ &\quad + C \left(\int_0^t |\sigma(s, X_s^n) - \sigma(s, X_s^x)| dB_s \right)^2 + C \left(\int_0^t |\gamma(s, X_s^n) - \gamma(s, X_s^x)| d\langle B \rangle_s \right)^2 \\ &\quad + C \left(\int_0^t \int_{\mathbb{R}^d} |f(s, X_s^n, \theta) - f(s, X_s^x, \theta)| \tilde{N}(ds, d\theta) \right)^2. \end{aligned}$$

By the inequalities of Schawrtz and G-Burkholder Davis Gundy (3.1 and 2.7), we obtain:

$$\begin{aligned} \hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t^x|^2 \right] &\leq C|x_n - x|^2 + T \hat{\mathbb{E}} \left[\int_0^t |b(s, X_s^n) - b(s, X_s^x)|^2 ds \right] \\ &\quad + C \hat{\mathbb{E}} \left[\int_0^t |\sigma(s, X_s^n) - \sigma(s, X_s^x)|^2 ds \right]. \\ &\quad + 2 \bar{\sigma}^2 T C^2 \int_0^T \sup_{r \in [0, T]} \hat{\mathbb{E}} |\gamma(r, X_r^n) - \gamma(r, X_r^x)|^2 dr. \\ &\quad + C \hat{\mathbb{E}} \left[\int_0^T \sup_{v \in \vartheta} \int_{\mathbb{R}^d} |f(s, X_s^n, \theta) - f(s, X_s^x, \theta)|^2 v(d\theta) ds \right]. \end{aligned}$$

The Lipschitz condition in Assumption \mathcal{A} implies that

$$\hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t^x|^2 \right] \leq C|x_n - x|^2 + K \int_0^t \hat{\mathbb{E}} \sup_{t \leq T} |X_s^n - X_s^x|^2 ds.$$

Applying Gronwall's lemma, we conclude that

$$\hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t^x|^2 \right] \leq C|x_n - x|^2 \exp[Kt].$$

Moreover,

$$\lim_{n \rightarrow +\infty} x_n = x,$$

implies that

$$\lim_{n \rightarrow +\infty} \hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t^x|^2 \right] = 0.$$

□

5.2. Stability with Respect to Coefficients

In this part, we establish the stability with respect to coefficients b, σ, γ and f . To do this, we consider sequences of functions $(b_n), (\sigma_n), (\gamma_n)$ and (f_n) , and the equation

$$\begin{cases} dX_t^n = b_n(t, X_t^n)dt + \sigma_n(t, X_t^n)dB_t + \gamma_n(t, X_t^n)d\langle B \rangle_t + \int_{\mathbb{R}_0^d} f_n(t, X_t^n, \theta) \tilde{N}(dt, d\theta) \\ X_0^n = x. \end{cases} \quad (5.1)$$

Theorem 5.2. *Suppose that the functions $b, b_n, \sigma, \sigma_n, \gamma, \gamma_n, f$ and f_n satisfy the **Assumption A**. Suppose furthermore that for each $T > 0$ and each compact set K , there exists $C > 0$ such that*

- i) $\sup_{t \leq T} (|b_n(t, x)| + |\sigma_n(t, x)| + |\gamma_n(t, x)| + |f_n(t, x)|) \leq C(1 + |x|)$,
- ii) $\lim_{n \rightarrow \infty} \sup_{t \leq T} \sup_{x \in K} \|h_n(t, x) - h(t, x)\|_2 = 0$, where (h, h_n) can be $(b, b_n), (\sigma, \sigma_n), (\gamma, \gamma_n)$ and (f, f_n) .

Then

$$\lim_{n \rightarrow +\infty} \hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t|^2 \right] = 0,$$

where (X_t^n) et (X_t) are respectively the solutions of (5.1) and (1.1).

Proof. for every $n \in \mathbb{N}$, let (X_t^n) the solution of (5.1), we have

$$\begin{aligned} |X_t^n - X_t|^2 &\leq \left(\int_0^t |b_n(s, X_s^n) - b_n(s, X_s)| ds \right)^2 + \left(\int_0^t |b_n(s, X_s) - b(s, X_s)| ds \right)^2 \\ &\quad + \left(\int_0^t |\sigma_n(s, X_s^n) - \sigma_n(s, X_s)| dB_s \right)^2 + \left(\int_0^t |\sigma_n(s, X_s) - \sigma(s, X_s)| dB_s \right)^2 \\ &\quad + \left(\int_0^t |\gamma_n(s, X_s^n) - \gamma_n(s, X_s)| d\langle B \rangle_s \right)^2 + \left(\int_0^t |\gamma_n(s, X_s) - \gamma(s, X_s)| d\langle B \rangle_s \right)^2 \\ &\quad + \left(\int_0^t \int_{\mathbb{R}_0^d} |f_n(s, X_s^n, \theta) - f_n(s, X_s, \theta)| \tilde{N}(ds, d\theta) \right)^2 \\ &\quad + \left(\int_0^t \int_{\mathbb{R}_0^d} |f_n(s, X_s, \theta) - f(s, X_s, \theta)| \tilde{N}(ds, d\theta) \right)^2. \end{aligned}$$

Using the G-Burkholder Davis Gundy inequality, we obtain

$$\begin{aligned} \hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t|^2 \right] &\leq (T + C_1) \int_0^t \hat{\mathbb{E}}[|X_s^n - X_s|^2] ds \\ &\quad + C_2 \int_0^t |b_n(s, X_s^n) - b(s, X_s)|^2 ds \\ &\quad + C_3 \hat{\mathbb{E}} \left[\int_0^t |\sigma_n(s, X_s^n) - \sigma(s, X_s)|^2 ds \right] \\ &\quad + C_4 \int_0^T \sup_{r \in [0, T]} \hat{\mathbb{E}} |\gamma(r, X_r^n) - \gamma(r, X_r^x)|^2 dr \\ &\quad + C_5 \hat{\mathbb{E}} \left[\int_0^T \sup_{v \in \vartheta} \int_{\mathbb{R}_0^d} |f(s, X_s^n, \theta) - f(s, X_s^x, \theta)|^2 v(d\theta) ds \right] \\ &\leq (T + C_1) \int_0^T \hat{\mathbb{E}}[|X_s^n - X_s|^2] ds + K_n \\ &\leq (T + C_1) \int_0^T \hat{\mathbb{E}}[\sup_{s \leq t} |X_s^n - X_s|^2] ds + K_n, \end{aligned}$$

such that

$$\begin{aligned} K_n &= C_2 \int_0^t |b_n(s, X_s^n) - b(s, X_s)|^2 ds + C_3 \hat{\mathbb{E}} \left[\int_0^t |\sigma_n(s, X_s^n) - \sigma(s, X_s)|^2 ds \right] \\ &\quad + C_4 \int_0^T \sup_{r \in [0, T]} \hat{\mathbb{E}} |\gamma(r, X_r^n) - \gamma(r, X_r^x)|^2 dr \\ &\quad + C_5 \hat{\mathbb{E}} \left[\int_0^T \sup_{v \in \vartheta} \int_{\mathbb{R}_0^d} |f(s, X_s^n, \theta) - f(s, X_s^x, \theta)|^2 v(d\theta) ds \right]. \end{aligned}$$

By Gronwall's lemma, we conclude that

$$\hat{\mathbb{E}} \left[\sup_{t \leq T} |X_t^n - X_t|^2 \right] \leq K_n \exp(T + C_1)T.$$

Then, by *i)* and *ii)* it is easy to see that $K_n \rightarrow 0$ when $n \rightarrow +\infty$, which completes the proof. \square

6. Examples

Financial markets are typical fields where information is subject to uncertainty, to illustrate our work, in this section we present two important examples.

Example 1

In optimal portfolio choice problems, the volatility and the risk premium processes are unknown and hard to accurately estimate from reliable data, we need to consider a family of different models or scenarios instead of one fixed asset process based on a given prior or estimate. To cope with any skeptical attitude towards a given model and quantify ambiguity aversion, the decision maker needs to perform a robust portfolio optimization that survives all given scenarios. For this reason, we need to consider \mathcal{P} , a family of probability measures.

Example 2

Lévy processes are very important in financial mathematics, in particular in the modelling of stock prices. The term of brownian motion is aimed to give a plausible description of the uncertainty of the stock price. On the other hand, the volatility parameter is used to control the size of random fluctuations of the stock price.

On some sublinear expectation space $(\Omega, H, \hat{\mathbb{E}})$ equipped by the universal filtration $F^{\mathcal{P}}$, we consider the following Black-Scholes market driven by G -Brownian motion $(B_t)_t$:

(Bond price)

$$dS_0(t) = rS_0(t)dt + \phi(t, S_0(t))d\langle B \rangle_t, S_0(0) = 1.$$

(Stock price)

$$dS_1(t) = S_1(t)[\mu dt + \gamma d\eta(t)] + \phi(t, S_1(t))d\langle B \rangle_t, S_1(0) = x > 0.$$

where $r, \mu, \gamma \neq 0$ are constants and

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z \tilde{N}(dt, d\theta),$$

is a pure jump Lévy martingale with respect to each probability measure $P \in \mathcal{P}$. To ensure that $S_1(t) \geq 0$ for all $t \geq 0$, we assume as before that $\gamma z \geq -1$ a.s. v . Assume in addition that

$$\int_{\mathbb{R}} v(dz) > \frac{\mu - r}{\gamma}, \quad (6.1)$$

which means, the total mass of the jump measure v exceeds $\frac{\mu - r}{\gamma}$.

Let us consider $\phi \equiv 0$, the normalized stock price $\bar{S}_1(t) \geq 0$ is given by

$$\bar{S}_1(t) = \frac{1}{S_0(t)} S_1(t) = \exp(-\rho t) S_1(t).$$

Note that

$$d\bar{S}_1(t) = \bar{S}_1(t^-)[(\mu - r)dt + \gamma d\eta(t)], \bar{S}_1(0) = x.$$

We seek an equivalent local martingale measure Q of the process $\bar{S}_1(t)$. To this end, we apply Girsanov theorem for jump processes (see [15], Theorem 1.34), besides we try to find a solution $\theta(z) \leq 1$ of the equation given by:

$$\int_{\mathbb{R}} \gamma(t, z) \theta(t, z) v(dz) = \alpha(t), \text{ for a. a. } (t, w), \quad (6.2)$$

with

$$dX(t) = \alpha(t, w)dt + \int_{\mathbb{R}} \gamma(t, z) \tilde{N}(dt, d\theta),$$

which in this case gets the form

$$\int_{\mathbb{R}} \theta(z) v(dz) = \frac{\mu - r}{\gamma}, \quad (6.3)$$

where the process X is a one-dimensional Lévy process. Assume that there exists a process $\theta(s, z) \leq 1$ such that Equation (6.2) holds.

By inequality (6.1) we see that if $A \subset \mathbb{R}$ with $\frac{\mu - r}{\gamma} < v(A) < \infty$ then

$$\theta(z) = \frac{\mu - r}{\gamma v(A)} \psi_A(z),$$

is a possible solution. If v is concentrated on one point z_0 , i.e. if

$$v(\mathbb{R}) = v(\{z_0\}),$$

which means that $\eta(t)$ is a Poisson process multiplied by z_0 , then this is the only solution. On the other hand, if there exist two sets $A, B \subset (-1, \infty)$ such that $A \cap B = \emptyset$ and

$$v(A) > 0, v(B) > 0.$$

Subsequently, we see that there are infinitely many solutions $\theta(z)$ of Equation (6.3) such that $\theta(z) < 1$.

Fix a solution $\theta(z) < 1$ of Equation (6.3) and define

$$Z(t) = Z^\theta(t) = \exp\left\{\int_0^t \int_{\mathbb{R}} \ln(1 - \theta(z)) N(ds, dz) + \frac{\mu - r}{\gamma} t\right\}; 0 \leq t \leq T,$$

and

$$dQ = Q^\theta(t) = Z^\theta(T) dP \text{ on } \hat{\mathcal{F}}_T.$$

Then by [15, Lemma 1.33], $\bar{S}_1(t)$ is a local martingale with respect to Q .

7. Conclusion

In reality, G-SDEs with jumps are more suitable to model many random phenomena with discontinuity in finance. This work has dealt with a system of SDEs with jumps driven by G-Brownian motion and G-Lévy process. Precisely, under the linear growth and the Lipschitz conditions we have proved a moment estimate of the solution and the temporal Hölder regularity, besides the stability with respect to the initial condition and to the coefficients. In order to support and show the effectiveness of our theoretical results we have employed an important examples in Finance, like Black-Scholes market driven by G-Brownian motion.

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