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How Terminal Potentials Determine the Energy Flow in Electric Circuits

Minsu Oh
Department of Electrical and Computer Engineering, Tufts University, MA, USA

Abstract: In electric circuits, energy is delivered by electromagnetic fields through the space, as suggested by Kraus and Carver. They provide insights by showing electric/magnetic fields and Poynting vector directions around a DC circuit. In this work, the energy flow mechanism by Kraus and Carver is further studied theoretically for arbitrary terminal potentials of a DC power source.

Keywords: *energy flow in circuits, electromagnetic fields, electromagnetic theories, Kraus and Carver's method, Poynting theorem, Poynting vectors, power flow.*

1 Introduction

The waterfall analogy of electricity is commonly used to help conceptualize voltage and current in most educational or academic environments [1–3]. While it does a good job in that purpose, it may not provide a full description of how energy in electric circuits is transported. It is pointed out that electromagnetic fields in space deliver energy in circuits [4,5]. Kraus and Carver especially provided great insights into the subject by visualizing electric/magnetic fields and Poynting vector distribution around a DC circuit in 1973 (Ref. [4], figure 10-19 on page 417). They show that the energy enters the load through the space from the battery's terminals, as shown in Fig. 1. They also state that the dissipated power in the load has the same magnitude as the total power entering the load from the space, which is the integral of the Poynting vector over a surface enclosing the load (Ref. [4], page 418). If a power source with a higher electromotive force is used, then the power dissipated in the load increases as well, according to Ohm's law. Thus, it can be inferred that the energy flux around the circuit would change as a function of electromotive force. The electromotive force of a power source, such as battery, is determined by the potential difference between its terminals. It is the purpose of this work to discuss the subject suggested by Kraus and Carver for arbitrary terminal potentials of the power source. In Section 2 below, relationships between source terminal potentials and energy flow magnitude/direction are derived based on a shielded DC circuit model. Simulation results are also provided which support the findings.

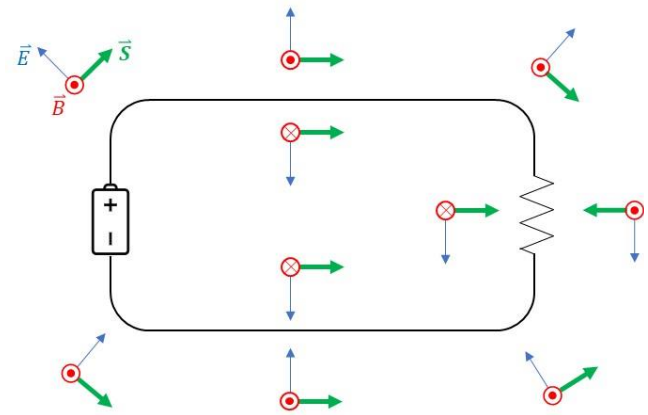

Fig. 1. Energy flow around a DC circuit. $\vec{E}$ and $\vec{B}$ represent electric and magnetic fields, respectively. $\vec{S}$ represents Poynting vector. The concept is explained by Kraus and Carver [4].

2 Derivation

Prior to the derivation, let us make a note about the power source (or a battery). The power source's terminals are not constrained to be at potentials with opposite signs, + and -, although this is what is commonly seen in commercially available batteries. In fact, a power source should be understood as anything that can create electromotive force or make charges "move". For example, suppose two sets of identical capacitor plates: one set of plates with the charge density +Q and -Q, and the other set with +2Q and -2Q. If you connect the plates charged with +Q and +2Q (or -Q and -2Q), there will be a movement of charges until both plates reach the equal potential at +1.5Q. Consider three cases in Fig. 2 where the battery's terminal potentials are different while the electromotive force stays the same value. The three circuits in Fig. 2 are all identical in that the voltage and current across the resister are the same. Assuming no resistance of the wire and battery, the dissipated power in the resistor is equal to the product of the battery's electromotive force and the current ($P_R = V_s I$). The + and – sings around the wire in Fig. 2 represent surface charges [6–8].

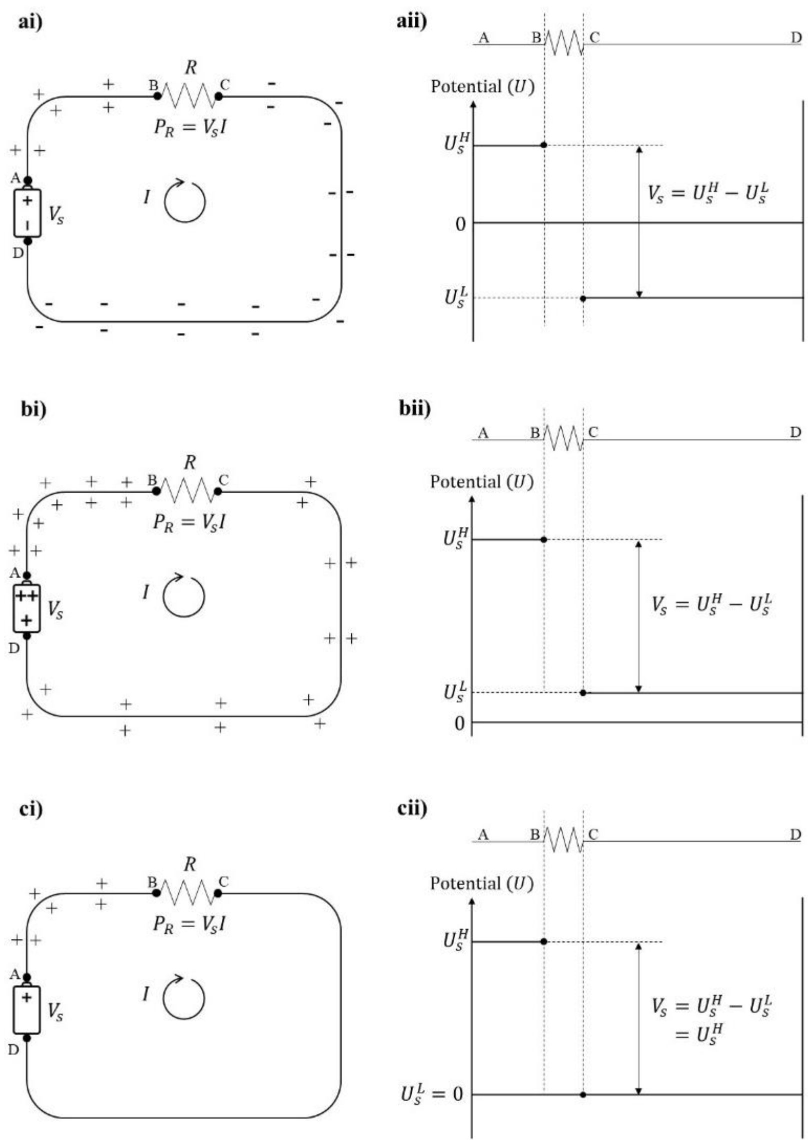

Fig. 2. Three identical DC circuits in steady state. Surface charges in the wire are represented with + and – signs. The electric potential along the circuit components is illustrated. U_s^H and U_s^L are potentials of the source terminals relative to the ground (*U = 0*). P_R is the power consumed in the resistor. *R* is the resistance of the resistor. *I* is current. V_s is the electromotive force of the power source. An infinite conductivity of the wire and no internal resistance of the battery are assumed.

Case 1: $U_s^H > 0 > U_s^L$

Suppose that the circuit in Fig. 2a is shielded with a separate, grounded conductor along the wire as shown in Fig. 3. The wire and shield are not connected, and the current flows only in the wire. Since the shield is grounded, the potential and electric fields outside the shield are zero everywhere. This, in steady state, indicates that there is no electromagnetic energy flow outside the shield due to the Poynting vector being zero. For a cylindrical, infinitely long wire, the electric and magnetic fields are visualized with the corresponding Poynting vector in Figs. 3a and 3b. Since the wire has an infinite conductivity, the potentials in the wires are constant as U_s^H and U_s^L in Regions 1 and 3, respectively. This indicates that the electric field and Poynting vector within the wire are zero. Thus, energy in the circuit only transports through the space between the wire and shield. Because the energy must be conserved, the sum of energy flows from both terminals is expected to be equal to V_sI. To derive this, isotropic materials are assumed in this work such that $\vec{D} = \varepsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$ or $\vec{D} \parallel \vec{E}$ and $\vec{B} \parallel \vec{H}$ [9].

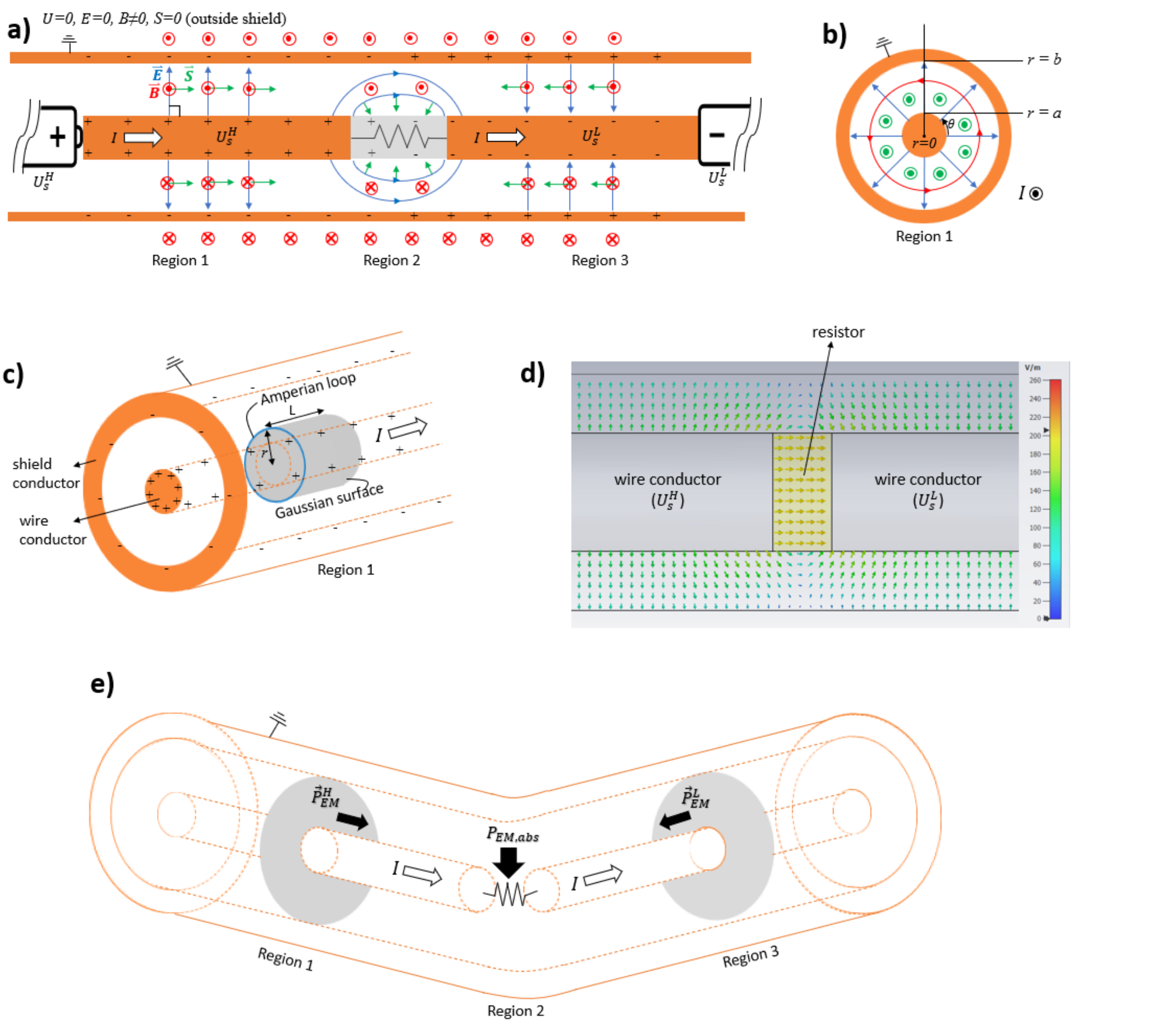

Fig. 3. Electric and magnetic fields around the circuit for Case 1. a) and b): cross-section of the wire along and perpendicular to the wire's longitudinal axis, respectively. The resistor is visualized in grey with the lumped element symbol. The electric potential (U), electric field ($\vec{E}$), magnetic field ($\vec{B}$), and Poynting vector ($\vec{S}$) are denoted. r and θ are radius and angle. c): oblique view with a Gaussian surface and an Amperian loop. d): cross-section view of simulated electric fields, where U_s^H = 1V and U_s^L = -1V are used. The electric field strength, in V/m, is represented by color. e): electromagnetic power flow ($\vec{P}_{EM}$) in Regions 1 and 3 and power absorbed ($P_{EM,abs}$) in the resistor. The thickness of the power flow arrows represents relative power magnitude.

a) <u>Electric Fields</u>

According to Gauss's law,

$$\oint \vec{D}\cdot d\vec{a} = Q_{f,enc} \qquad (1a)$$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{f,enc}}{\varepsilon} \qquad (1b)$$

, where $\vec{E}$ and $\vec{D}$ are electric and electric displacement fields, respectively. $d\vec{a}$ is the surface normal vector with the magnitude of an infinitesimal area on the Gaussian surface (surface in grey in Fig. 3c, for example). $Q_{f,enc}$ is the total free charge enclosed in the volume that the Gaussian surface makes. ε is permittivity of the material where the fields are present. From Eq. (1b), the electric field around the wire is obtained as

$$\vec{E} = E\hat{r}$$
$$= \left(\frac{Q_{f,enc}}{2\pi L\varepsilon}\frac{1}{r}\right)\hat{r}, \qquad a \le r \le b \qquad (2)$$

, where r and L are radius and height of the Gaussian surface in Fig. 3c. $\hat{r}$ is a unit vector in the radial direction. Since the shield is grounded (U = 0), the potential differences between the shield and wire are U_s^H and U_s^L for Regions 1 and 3, respectively. These potential differences can also be calculated by integrating the electric field along the path. That is,

For Region 1:

$$\begin{aligned} U_s^H &= \int_a^b \vec{E}_{Region\ 1}\, d\vec{r} \\ &= \frac{Q_{f,enc,Region\ 1}}{2\pi L\varepsilon}\int_a^b \frac{1}{r}dr \\ &= \frac{Q_{f,enc,Regoin\ 1}}{2\pi L\varepsilon}\, ln\frac{b}{a} \\ &= rE_{Region\ 1}\, ln\frac{b}{a} \end{aligned} \tag{3a}$$

For Region 3:

$$\begin{aligned} U_s^L &= \int_a^b \vec{E}_{Region\ 3}\, d\vec{r} \\ &= rE_{Region\ 3}\, ln\frac{b}{a} \end{aligned} \tag{3b}$$

From the left-hand and right-hand sides of Eq. (3), the electric fields around the wire can also be expressed by

$$E_{Region\ 1} = \frac{U_s^H}{ln\frac{b}{a}}\frac{1}{r}\,, \qquad a \le r \le b \tag{4a}$$

$$E_{Region\ 3} = \frac{U_s^L}{ln\frac{b}{a}}\frac{1}{r}\,, \qquad a \le r \le b \tag{4b}$$

b) Magnetic Fields

According to Ampere's law,

$$\oint \vec{H}\cdot d\vec{l} = I_{f,enc} \tag{5a}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu I_{f,enc} \tag{5b}$$

, where $\vec{H}$ and $\vec{B}$ are magnetic "H" and "B" fields, respectively (it seems that there is no universal agreement as to which field should be called "magnetic field" [10]). $d\vec{l}$ is the path vector with the magnitude of an infinitesimal length along the Amperian loop (loop in blue in Fig. 3c, for example). $I_{f,enc}$ is the total free current passing through the Amperian loop. μ is permeability of the material where the fields are present. From Eq. (5b), the magnetic field around the wire is obtained as

For both Regions 1 and 3:

$$\begin{aligned} \vec{B} &= B\hat{\theta} \\ &= \frac{\mu I}{2\pi r}\hat{\theta} \\ &= \frac{\mu}{2\pi r}\frac{V_s}{R}\hat{\theta}, \qquad r \ge a \end{aligned} \tag{6}$$

$\hat{\theta}$ is a unit vector in the angular direction.

c) Poynting Vector and Power Flow

According to Poynting's theorem, the energy flow in electromagnetic fields is given by

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} \tag{7}$$

By inserting Eqs. (4) and (6) into Eq. (7), the magnitude of the Poynting vector ($\vec{S}$) writes

$$\begin{aligned} |S_{Region\ 1}| &= \frac{1}{\mu}|E_{Region\ 1}||B| \\ &= \frac{1}{\mu}\left(\frac{U_s^H}{ln\frac{b}{a}}\frac{1}{r}\right)\left(\frac{\mu}{2\pi r}\frac{V_s}{R}\right) \\ &= \frac{U_s^H V_s}{2\pi R ln\frac{b}{a}}\frac{1}{r^2} \end{aligned} \tag{8a}$$

$$\begin{aligned} |S_{Region\ 3}| &= \frac{1}{\mu}|E_{Region\ 3}||B| \\ &= \frac{1}{\mu}\left(-\frac{U_s^L}{ln\frac{b}{a}}\frac{1}{r}\right)\left(\frac{\mu}{2\pi r}\frac{V_s}{R}\right) \\ &= -\frac{U_s^L V_s}{2\pi R ln\frac{b}{a}}\frac{1}{r^2} \end{aligned} \tag{8b}$$

Since the unit of Poynting vector is $\left[{}^{J}/_{s\cdot m^2}\right]$, the amount of power passing through an area can be obtained by integrating the Poynting vector over that area. Therefore, from Eq. (8), the power flow around the wire is

$$|P_{EM}^{H}| = \int \vec{S}_{Region\ 1} \cdot d\vec{a} = \int_0^{2\pi}\int_a^b \left(\frac{U_s^H V_s}{2\pi R ln\frac{b}{a}} \frac{1}{r^2}\right) r\,dr\,d\theta = \frac{U_s^H V_s}{2\pi R ln\frac{b}{a}}\ 2\pi\ ln\frac{b}{a} = \frac{U_s^H V_s}{R} \tag{9a}$$

Likewise,

$$|P_{EM}^{L}| = \int \vec{S}_{Region\ 3} \cdot d\vec{a} = -\frac{U_s^L V_s}{R} \tag{9b}$$

In calculation of Eqs. (9a) and (9b), note that the orientation of the surface normal vector ($d\vec{a}$) is chosen to be the same as that of the Poynting vector in each Region. Since there is no energy input/output from outside the shield, the energy flow in the circuit is confined in the space between the shield and wire. Thus, the total amount of electromagnetic power absorbed in the resistor is

$$|P_{EM,abs}| = |P_{EM}^{H}| + |P_{EM}^{L}| = (U_s^H - U_s^L)\frac{V_s}{R} = \frac{{V_s}^2}{R} = V_s I \tag{9c}$$

From the classical circuit theory, the power consumed in the resistor is

$$P_R = V_s I \tag{10}$$

From Eqs. (9c) and (10),

$$|P_{EM,abs}| = P_R \tag{11}$$

Moreover, from Eqs. (9a) and (9b), the power flow ratio from the battery's terminals is obtained by

$$|P_{EM}^{H}| : |P_{EM}^{L}| = |U_s^H| : |U_s^L| \tag{12}$$

Case 2: $U_s^H > U_s^L > 0$

Now, let us consider the case of Fig. 2b. The fields in this case are illustrated in Fig. 4. In analogy to Case 1 above, the electric and magnetic fields around the wire are given by Eqs. (4) and (6). Putting these equations into Eq. (7), the magnitude of the Poynting vector is obtained by

$$|S_{Region\ 1}| = \frac{1}{\mu}|E_{Region\ 1}||B| = \frac{1}{\mu}\left(\frac{U_s^H}{ln\frac{b}{a}}\frac{1}{r}\right)\left(\frac{\mu}{2\pi r}\frac{V_s}{R}\right) = \frac{U_s^H V_s}{2\pi R ln\frac{b}{a}}\frac{1}{r^2} \tag{13a}$$

$$|S_{Region\ 3}| = \frac{1}{\mu}|E_{Region\ 3}||B| = \frac{1}{\mu}\left(\frac{U_s^L}{ln\frac{b}{a}}\frac{1}{r}\right)\left(\frac{\mu}{2\pi r}\frac{V_s}{R}\right) = \frac{U_s^L V_s}{2\pi R ln\frac{b}{a}}\frac{1}{r^2} \tag{13b}$$

Therefore,

$$|P_{EM}^{H}| = \int \vec{S}_{Region\ 1}\, d\vec{a} = \frac{U_s^H V_s}{R} \tag{14a}$$

$$|P_{EM}^{L}| = \int \vec{S}_{Region\ 3} \cdot d\vec{a} = \frac{U_s^L V_s}{R} \tag{14b}$$

Since the energy in the circuit flows in the same direction in both Regions 1 and 3, as shown in Fig. 4, and $U_s^H > U_s^L > 0$, the total amount of electromagnetic power absorbed in the resistor is

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$$
\begin{aligned}
|P_{EM,abs}| &= |P^H_{EM}| - |P^L_{EM}| \\
&= (U^H_s - U^L_s)\frac{V_s}{R} \qquad (14c) \\
&= P_R
\end{aligned}
$$

From Eqs. (14a) and (14b), the power flow ratio from/into the two terminals is

$$|P^H_{EM}|:|P^L_{EM}| = |U^H_s|:|U^L_s| \qquad (15)$$

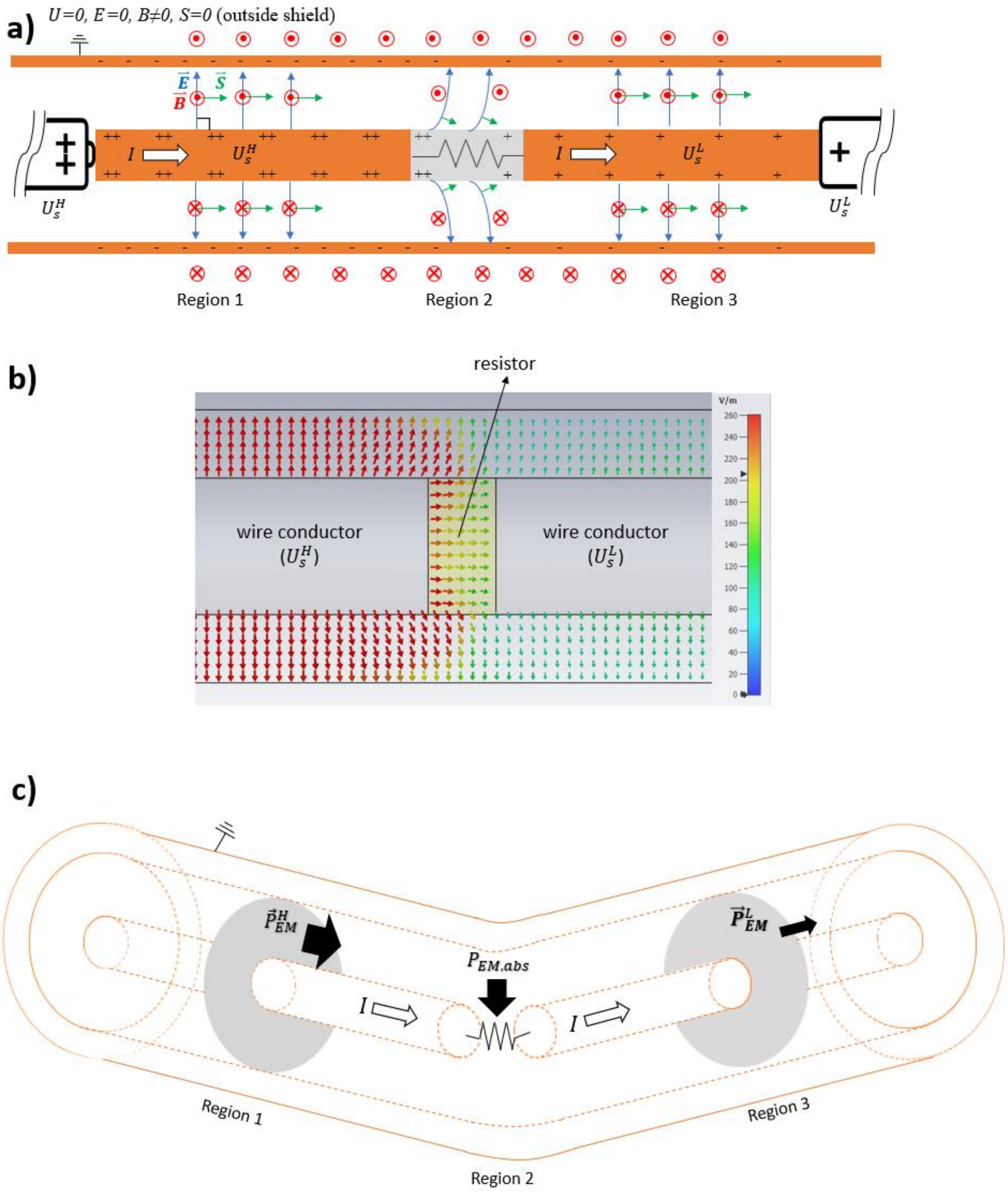

Fig. 4. Electric and magnetic fields around the circuit for Case 2. a): cross-section of the wire along and perpendicular to the wire's longitudinal axis, respectively. The resistor is visualized in grey with the lumped element symbol. The electric potential (U), electric field ($\vec{E}$), magnetic field ($\vec{B}$), and Poynting vector ($\vec{S}$) are denoted. r and θ are radius and angle. b): cross-section view of simulated electric fields, where U_s^H = 3V and U_s^L = 1V are used. The electric field strength, in V/m, is represented by color. Structure dimensions are same as the one in Fig. 3d. c): electromagnetic power flow ($\vec{P}_{EM}$) in Regions 1 and 3 and power absorbed ($P_{EM,abs}$) in the resistor. The thickness of the power flow arrows represents relative power magnitude.

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Case 3: $U_s^H > U_s^L = 0$

Lastly, let us consider the case of Fig. 2c. The fields in this case are illustrated in Fig. 5. Likewise in Cases 1 and 2 above, the following is obtained

$$|P_{EM}^H| = \frac{U_s^H V_s}{R} = \frac{V_s^2}{R} \tag{16a}$$

$$|P_{EM}^L| = \frac{U_s^L V_s}{R} = 0 \tag{16b}$$

$$|P_{EM,abs}| = |P_{EM}^H| = \frac{V_s^2}{R} = P_R \tag{16c}$$

Therefore,

$$|P_{EM}^H| : |P_{EM}^L| = |U_s^H| : |U_s^L| \tag{17}$$

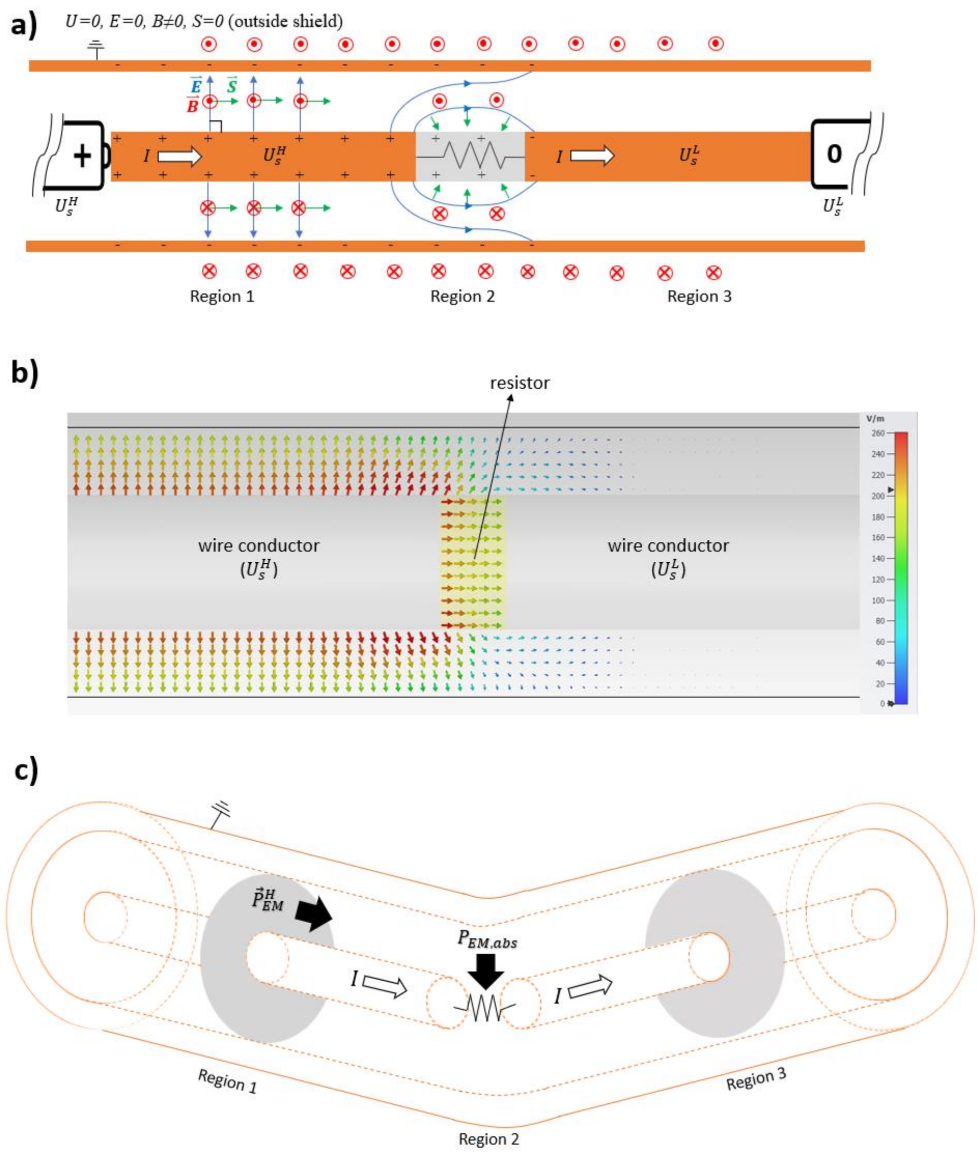

Fig. 5. Electric and magnetic fields around the circuit for Case 3. a): cross-section of the wire along and perpendicular to the wire's longitudinal axis, respectively. The resistor is visualized in grey with the lumped element symbol. The electric potential (U), electric field ($\vec{E}$), magnetic field ($\vec{B}$), and Poynting vector ($\vec{S}$) are denoted. r and θ are radius and angle. b): cross-section view of simulated electric fields, where U_s^H = 2V and U_s^L = 0V are used. The electric field strength, in V/m, is represented by color. Structure dimensions are same as the one in Fig. 3d. c): electromagnetic power flow ($\vec{P}_{EM}$) in Regions 1 and 3 and power absorbed ($P_{EM,abs}$) in the resistor. The thickness of the power flow arrows represents relative power magnitude.

3 Summary and Conclusion

The energy flow mechanism in electric circuits is described by Kraus and Carver by using electric/magnetic fields and Poynting vectors [4]. Based on the insights provided by Kraus and Carver, this work discusses the subject for arbitrary terminal potentials of the power source. In conclusion, the magnitude and direction of energy flow in circuits are determined by the terminal potentials in a way that the power absorbed in the load is only dependent on the terminals' potential difference, as shown in Eqs. 12, 15, and 17.

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