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Exploring the 2-Part of Class Groups in Quadratic Fields: Perspectives on the Cohen-Lenstra Conjectures

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Abstract: Cohen and Lenstra introduced conjectures concerning the distribution of class numbers in quadratic fields, though many of these conjectures remain unproven. This paper investigates the 2-part of class groups in imaginary quadratic fields and examines their alignment with the Cohen-Lenstra heuristics. We provide detailed proofs of key theorems related to ideal decompositions and modular homomorphisms, and we explore the distribution of class groups of imaginary quadratic fields. Our analysis includes constructing imaginary quadratic fields with prescribed 2-class groups and discussing the implications of these findings on the Cohen-Lenstra conjecture.

Keywords: quadratic fields; class numbers; class groups; Cohen-Lenstra conjecture

1. Introduction

Quadratic fields, denoted as $K = \mathbb{Q}(\sqrt{d})$ where d is a non-square integer, are foundational structures in algebraic number theory. When $d > 0$, K represents a real quadratic field containing real roots, while for $d < 0$, K is an imaginary quadratic field containing complex roots. The class number of a quadratic field, a critical invariant in number theory, quantifies the deviation from unique factorization within the ring of integers of K. This invariant is essential for theoretical advancements in number theory and has significant implications for cryptographic systems and algebraic geometry, given its role in analyzing factorization properties within quadratic fields.

Quadratic fields and their associated class groups are fundamental for understanding class number distributions [1], a central issue in number theory. Imaginary quadratic fields, in particular, have been extensively studied due to Gauss's conjecture, which has been resolved for imaginary fields but remains open for real quadratic fields. Specifically, the question of whether there exist infinitely many real quadratic fields with class number one is still unresolved and is intrinsically related to the structure of fundamental units in these fields. This question is further complicated by the growth of the regulator in real quadratic fields, making class number computations for real fields significantly more challenging than for imaginary fields [9].

Calculating class numbers for general number fields remains a challenging problem. Cohen and Lenstra proposed influential conjectures regarding the statistical distribution of class numbers in number fields, with a focus on real and imaginary quadratic fields. Their heuristics suggest that for a given discriminant, class numbers tend to follow certain probabilistic distributions, which favor the presence of class groups with smaller orders [6,12].

Recent advancements in the study of quadratic fields and their class groups have expanded on Cohen and Lenstra's initial heuristics, offering more refined insights into expected class group distribution patterns. The Cohen-Lenstra conjectures indicate that class groups of quadratic fields, particularly imaginary fields, are expected to follow specific statistical distributions in their p-parts. Notably, empirical observations reveal deviations from these heuristics in the 2-part of class groups in imaginary quadratic fields, likely due to the influence of genus theory on these structures. Modern computational methods, including those developed by Bhargava and Shankar, enable large-scale analysis of these distributions with increased precision, laying a robust foundation for investigating the 2-part of class groups in imaginary quadratic fields.

Our Contribution

This study provides a detailed examination of the 2-part of class groups in imaginary quadratic fields to assess their alignment with Cohen-Lenstra heuristics and to identify any systematic deviations. By combining modern computational techniques with theoretical frameworks, this paper explores structural patterns in class groups within these fields in the distribution of the 2-part. This research contributes valuable insights to both foundational studies in quadratic fields and broader applications in number theory and cryptography.

To contextualize the discussion of these conjectures, we first review several foundational concepts essential to understanding the relationship between class numbers and class groups in imaginary quadratic fields. For an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, the class number is closely connected to the Dedekind zeta function $\zeta_K(s)$ and the Minkowski bound, which provides a lower limit on the class group size. Moreover, the structure of the 2-part of the class group is intricately linked to the properties of the quadratic form class group, underscoring its importance in theoretical and computational settings.

1.1. Significance of the Current Study

The findings of this study hold substantial implications for both theoretical number theory and applied fields such as cryptography. Quadratic fields and their class groups are integral to understanding algebraic number theory, and an in-depth analysis of the 2-part distribution in these groups serves multiple important purposes.

Theoretical Impact on the Class Number Problem and Conjecture Refinement: By investigating deviations in the 2-part distribution from expected heuristic predictions, this research addresses core questions about class number distribution in quadratic fields. These insights are valuable for refining the Cohen-Lenstra conjectures, which could lead to more accurate predictive models for class group distributions. This is especially relevant for imaginary quadratic fields, where genus theory adds complexity to the underlying structure of the 2-part.

Implications for Cryptographic Algorithm Design: The class group structures in quadratic fields have important applications in cryptographic systems that leverage the algebraic properties of these fields. A more detailed understanding of the 2-part distribution could inform the design and robustness of algorithms reliant on quadratic field properties, especially for tasks involving factorization and discrete logarithm problems.

Advancements in Large-Scale Numerical Analysis in Number Theory: This study also contributes to the field by demonstrating the utility of large-scale, high-performance computational techniques for analyzing class group distributions. The methodology used here sets a precedent for future studies of algebraic structures in number theory, showing that computational approaches can yield meaningful insights into long-standing theoretical problems.

1.2. Novel Contributions of the Study

- *Theoretical Extension:* This research extends Cohen-Lenstra heuristics by investigating systematic deviations observed in the 2-part of class groups for imaginary quadratic fields. This approach provides new structural insights into class group distributions.
- *Numerical Insight:* Utilizing advanced computational methodologies, this study examines class group data at a big scale, uncovering previously unobserved distribution anomalies and periodic behaviors in the 2-part of class groups. These findings allow for a more detailed understanding of class group structures.
- *Methodological Innovation:* This study introduces refined computational techniques for examining class group distributions with higher precision across large datasets, thereby supporting rigorous analysis of the Cohen-Lenstra heuristics in relation to the 2-part of class groups.
- *Theoretical Contribution:* Through a focused exploration of genus theory's influence on the 2-part of class groups in imaginary quadratic fields, this work identifies deviations within the Cohen-

Lenstra framework, suggesting that genus theory may affect class group structure, thus offering a broadened perspective on quadratic field properties.

- *New Findings from Computational Results:* Through extensive numerical calculations and simulation experiments, this study identifies unique characteristics in the distribution of the 2-part of class groups within imaginary quadratic fields, establishing a foundation for future research in class group distributions.

2. Fractional Ideals and Class Groups

Definition 1. *([2,3])Let K be a number field and $\mathcal{O}_K$ its ring of integers. A subset I of K is called a* fractional ideal *of K if there exists a non-zero element $u \in K$ such that uI is a non-zero ideal of $\mathcal{O}_K$.*

Definition 2. *([4,5]) Let $\mathcal{O}_K$ be a Dedekind ring. A principal fractional ideal is a fractional ideal of the form $\alpha\mathcal{O}_K$, generated by a single element α in the quotient field of $\mathcal{O}_K$, where $\alpha \neq 0$. The group of fractional ideals modulo the group of principal ideals (i.e., non-zero principal fractional ideals) is called the ideal class group of $\mathcal{O}_K$. Denote by $P(K)$ the set of all principal fractional ideals. The principal fractional ideals form a group called the principal fractional ideal group.*

Let $I(K)$ denote the set of fractional ideals of the number field K, the quotient group $Cl(K) = I(K)/P(K)$ is called the ideal class group (or simply the class group) of K. An ideal class of K is an element of $Cl(K)$. Therefore, two fractional ideals are equivalent in K if they lie in the same coset of $I(K)/P(K)$. Both $I(K)$ and $P(K)$ are infinite abelian groups, but the quotient group $Cl(K)$ is a finite abelian group. The order of this group, $h(K) = |Cl(K)|$, is called the ideal class number (or simply the class number) of K.

It can be observed that $h(K)$ is an important invariant of K. From the definition of $Cl(K)$, we have:

$$\begin{aligned} h(K) = 1 &\Longleftrightarrow I(K) = P(K) \text{ (i.e., every fractional ideal is a principal fractional ideal);} \\ &\Longleftrightarrow \text{every ideal in } \mathbb{Z}_K \text{ is a principal ideal;} \\ &\Longleftrightarrow \mathbb{Z}_K \text{ is a principal ideal domain;} \\ &\Longleftrightarrow \mathbb{Z}_K \text{ is a unique factorization domain.} \end{aligned}$$

Thus, the size of the class number $h(K)$ measures the difference between the Dedekind domain $\mathbb{O}_K$ and a unique factorization domain.

Definition 3. *([4,5]) Let K be a field and O_K be a domain. A set $\{\alpha_1, \ldots, \alpha_n\}$ is an integral basis of O_K (or K) if for every element $\alpha \in O_K$, there is a unique representation:*

$$\alpha = \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n, \quad \lambda_i \in \mathbb{Z}.$$

Theorem 1 (Hermite)**.** *For every given $d \in \mathbb{Z}$, there are only finitely many quadratic fields K such that $d(K) = d$, where $d(K)$ is the discriminant of K.*

Remark 1. *This theorem, first established by Hermite in 1848, is a cornerstone in the study of algebraic number fields. It demonstrates the intrinsic finiteness in the classification of quadratic fields based on their discriminants. A detailed proof of Hermite's theorem can be found in Neukirch's Algebraic Number Theory [18, Theorem 2.16, Chapter III, Section 2]. Moreover, this result is a special case of the Hermite-Minkowski theorem [18], which asserts that for any fixed degree n and a discriminant bound B, there are only finitely many number fields of degree n with discriminants satisfying $|d(K)| \leq B$. The Hermite-Minkowski theorem generalizes Hermite's result to number fields of arbitrary degree, offering a unifying framework for the study of finiteness properties in algebraic number theory.*

In recent years, advancements in computational techniques have allowed for the application of Hermite's theorem on larger datasets, affirming its utility in both theoretical and applied contexts.

These developments include algorithmic approaches that extend the reach of Hermite's finiteness results to more complex quadratic fields, reinforcing the theorem's relevance in current number theory research [19,20].

The Minkowski Bound and Class Group Computations

Class group computations rely on theoretical foundations provided by Theorem 1 and Dirichlet's class number formula [7]. Theorem 1 ensures that the search for ideal class representatives in $Cl(K)$ can be restricted to ideals with norms bounded by the Minkowski bound $M(K)$, reducing an infinite problem to a finite computation. This result follows from the Hermite-Minkowski theorem, which guarantees that every ideal class contains an ideal with norm bounded by $M(K)$.

Step 1: Computing the Minkowski Bound

The Minkowski bound for a number field K is defined as:

$$M(K) = \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} |d(K)|^{1/2},$$

where:

- r_1: the number of real embeddings of K,
- r_2: the number of pairs of complex embeddings of K,
- $n = [K : \mathbb{Q}] = r_1 + 2r_2$: the degree of K,
- $|d(K)|$: the absolute value of the discriminant of K.

The Minkowski bound provides a finite upper limit on the norms of ideal representatives in $Cl(K)$. Specifically, for every ideal class in $Cl(K)$, there exists a representative ideal whose norm is less than or equal to $M(K)$. This restriction, guaranteed by the Hermite-Minkowski theorem, is fundamental for computational feasibility.

Step 2: Factoring Rational Primes

For each rational prime $p \leq M(K)$, the ideal pO_K in the ring of integers O_K of K can be factored as:

$$pO_K = \mathfrak{p}_1^{a_1} \mathfrak{p}_2^{a_2} \cdots \mathfrak{p}_g^{a_g},$$

where:

- $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_g$: distinct prime ideals in O_K,
- $a_1, a_2, \ldots, a_g$: positive integers representing the multiplicities of $\mathfrak{p}_i$ in pO_K.

The norm of each $\mathfrak{p}_i$, denoted $N(\mathfrak{p}_i)$, satisfies:

$$N(pO_K) = p^n = \prod_{i=1}^{g} N(\mathfrak{p}_i)^{a_i}.$$

The splitting behavior of p in K determines the factorization:

- **Inert**: $pO_K = \mathfrak{p}_1$, where $N(\mathfrak{p}_1) = p^n$,
- **Split**: $pO_K = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_g$, where $N(\mathfrak{p}_i) = p^{n/g}$ for all i,
- **Ramified**: $pO_K = \mathfrak{p}_1^{a_1}$, where $a_1 > 1$.

By analyzing pO_K for all $p \leq M(K)$, the ideal classes $[\mathfrak{p}_i]$ corresponding to these prime ideals form a generating set for $Cl(K)$. However, ensuring the generating set is complete requires careful verification of the independence and closure properties of these ideal classes.

Example: Imaginary Quadratic Fields

Let $K = \mathbb{Q}(\sqrt{-d})$, where $d > 0$ is a square-free integer. The Minkowski bound simplifies to:

$$M(K) = \frac{2}{\pi}\sqrt{|d(K)|}.$$

Consider $K = \mathbb{Q}(\sqrt{-23})$, where $d(K) = -23$ and:

$$M(K) = \frac{2}{\pi}\sqrt{23} \approx 3.89.$$

This implies we only need to consider primes $p \leq 3$:

- For $p = 2$: $2O_K = \mathfrak{p}_2\overline{\mathfrak{p}_2}$, where $N(\mathfrak{p}_2) = N(\overline{\mathfrak{p}_2}) = 2$.
- For $p = 3$: $3O_K = \mathfrak{p}_3\overline{\mathfrak{p}_3}$, where $N(\mathfrak{p}_3) = N(\overline{\mathfrak{p}_3}) = 3$.

Using these factorizations, the ideal classes $[\mathfrak{p}_2]$ and $[\mathfrak{p}_3]$ generate $Cl(K)$. By solving norm equations and verifying powers of $[\mathfrak{p}_2]$ and $[\mathfrak{p}_3]$, it can be shown that $Cl(K)$ is cyclic of order 3, and $h(K) = 3$.

Step 3: Validation via Dirichlet's Class Number Formula

Dirichlet's class number formula for imaginary quadratic fields relates $h(K)$ to $|d(K)|$ and the Dedekind zeta function $\zeta_K(s)$:

$$h(K) = \frac{w\sqrt{|d(K)|}}{2\pi}\zeta_K(-1),$$

where w is the number of roots of unity in K ($w = 2$ for imaginary quadratic fields). For $K = \mathbb{Q}(\sqrt{-23})$, direct computation of $\zeta_K(-1)$ confirms $h(K) = 3$, validating the result.

The Minkowski bound $M(K)$ is central to class group computations, as it transforms an infinite search into a finite problem. The Hermite-Minkowski theorem ensures this reduction is theoretically sound, while Dirichlet's class number formula validates the correctness of the computed class number $h(K)$. Together, these tools form the foundation of practical and rigorous methods for studying algebraic number fields.

Lattice-Based Methods in Class Group Computations

After establishing the Minkowski bound $M(K)$, we can further enhance class group computations by employing lattice-based methods. The connection between ideals in $\mathcal{O}_K$ and lattices in $\mathbb{R}^n$ provides powerful tools for computational number theory.

Ideals as Lattices

An ideal I in $\mathcal{O}_K$ can be viewed as a lattice in $\mathbb{R}^n$ via the embedding:

$$\varphi : K \hookrightarrow \mathbb{R}^n, \quad \varphi(\alpha) = (\sigma_1(\alpha), \ldots, \sigma_n(\alpha)),$$

where σ_i are the embeddings of K into $\mathbb{C}$. For complex embeddings, we separate the real and imaginary parts to obtain a real vector space, effectively doubling the count of complex embeddings. Therefore, the dimension of the real vector space is $n = r_1 + 2r_2$.

For imaginary quadratic fields ($K = \mathbb{Q}(\sqrt{-d})$ with $d > 0$), we have $r_1 = 0$ and $r_2 = 1$, so $n = 2$. The ring of integers $\mathcal{O}_K$ can be embedded into $\mathbb{R}^2$ by mapping each element to a point in the plane via the embedding φ.

Lattice Reduction and the LLL Algorithm

The Lenstra–Lenstra–Lovász (LLL) lattice reduction algorithm [21] is a polynomial-time algorithm that, given a basis for a lattice, finds a reduced basis consisting of relatively short and nearly orthogonal vectors. Applying the LLL algorithm to the lattice corresponding to an ideal I allows us to find a short vector in I, which corresponds to an element of small norm in I.

Consider an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, where d is a positive, square-free integer. The ring of integers $\mathcal{O}_K$ consists of elements of the form $a + b\omega$, where $a, b \in \mathbb{Z}$ and

$$\omega = \begin{cases} \sqrt{-d} & \text{if } d \equiv 2, 3 \pmod 4, \\ \dfrac{1+\sqrt{-d}}{2} & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

An ideal $I \subset \mathcal{O}_K$ can be expressed as

$$I = \mathbb{Z} \cdot \alpha_1 + \mathbb{Z} \cdot \alpha_2,$$

for some $\alpha_1, \alpha_2 \in \mathcal{O}_K$. Via the embedding φ, the ideal I corresponds to a lattice $\Lambda_I \subset \mathbb{R}^2$ with basis vectors $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$.

Applying the LLL algorithm to this lattice yields a reduced basis $\{\mathbf{b}'_1, \mathbf{b}'_2\}$, where $\mathbf{b}'_1$ corresponds to an element of I with relatively small norm. This facilitates the search for principal ideals and representatives of ideal classes with minimal norm.

Computational Advantages

Integrating lattice reduction methods into class group computations offers several advantages:

- *Efficiency*: Lattice reduction algorithms can significantly reduce the computational complexity of finding small-norm representatives of ideal classes.
- *Precision*: By working with reduced bases, we improve numerical stability in computations involving embeddings.
- *Structural Insights*: Lattice methods provide geometric interpretations of algebraic structures, aiding in the visualization and understanding of the class group.

Application to the 2-Part of Class Groups

For the 2-part of class groups, lattice-based methods are particularly effective. By focusing on ideals whose norms are powers of 2, we can utilize lattice reduction to identify relationships between ideals and to detect elements of order 2 in $\mathrm{Cl}(K)$.

Furthermore, the interplay between the lattice structure of $\mathcal{O}_K$ and genus theory allows us to better understand the deviations observed in the 2-part distributions from the Cohen-Lenstra heuristics.

The lattice-based approach enhances the computation of class groups in imaginary quadratic fields by constructing lattices corresponding to ideals and applying lattice reduction algorithms like LLL. This method not only improves computational efficiency and accuracy but also provides deeper insights into the structure of class groups, particularly the 2-part. By finding small-norm representatives, we can more effectively analyze the generators and relations within the class group.

Definition 4. *If $\mathfrak{p} \in \Gamma$, the norm of $\mathfrak{p}$ is defined as $N(\mathfrak{p}) = |A/\mathfrak{p}|$.*

Definition 5. *Let G_1 and G_2 be A-modules. We write $G_1 \subseteq G_2$ to indicate that G_1 is a submodule of G_2.*

If $\mathfrak{p} \in \Gamma$ is a prime ideal of A, and G is a finite A-module, the $\mathfrak{p}$-rank of G, denoted by $r_{\mathfrak{p}}(G)$, is defined as the dimension of the vector space $G/\mathfrak{p}G$ over the field $A/\mathfrak{p}$. Explicitly:

$$r_{\mathfrak{p}}(G) = \dim_{A/\mathfrak{p}}(G/\mathfrak{p}G).$$

Definition 6. *Let k be a positive integer or ∞. If $k \neq \infty$ and G is a finite A-module, then $s_k(G)$ (or $s_k^A(G)$) represents the number of A-epimorphisms from A^k to G. Define:*

$$S_k(G) := \{\varphi \in \mathrm{Hom}_A(A^k, G) : \varphi \textit{ is an epimorphism}\}, \quad s_k(G) = |S_k(G)|.$$

The value $s_k(G)$ quantifies the number of surjective homomorphisms from a free module A^k to G.

If G is a finite A-module, then $w_k(G) = s_k(G)|G|^{-k}|\mathrm{Aut}(G)|^{-1}$ is the k-weight, and $w(G) = w_\infty(G) = |\mathrm{Aut}(G)|^{-1}$. If $\mathfrak{p} \in \Gamma$, let:

$$\eta_k(\mathfrak{p}) = \prod_{1\le i\le k}\left(1-(N\mathfrak{p})^{-i}\right), \quad \eta_\infty(\mathfrak{p}) = \prod_{i\ge 1}\left(1-(N\mathfrak{p})^{-i}\right).$$

Definition 7. *Since every finite A-module G can be written as $G = \bigoplus_i A/\mathfrak{p}_i^{a_i}$, define:*

$$\chi_A(G) = \prod_i \mathfrak{p}_i^{a_i}.$$

If $A = \mathbb{Z}$, then $\chi_{\mathbb{Z}}(G) = n\mathbb{Z}$, where $n = |G|$. Let α be an integral ideal, and define the k-weight $w_k(\alpha)$ of α as:

$$w_k(\alpha) = \sum_{G(\alpha)} w_k(G), \quad w(\alpha) = w_\infty(\alpha),$$

where $\sum_{G(\alpha)}$ denotes summation over G up to A-isomorphism with $\chi_A(G) = \alpha$.

The concept of $\chi_A(G)$ provides a compact way to encode the structure of G using ideals. The k-weight $w_k(\alpha)$ measures the contribution of all modules associated with α to certain combinatorial or arithmetic quantities, such as lattice point enumeration in ideal-related spaces.

Definition 8. *Let G be an abelian group and p a prime number. If for every $a \in G$ there exists $n \ge 1$ such that $p^n a = 0$, then G is called a p-primary group. For a general abelian group G, let $G_p = \{a \in G : n \ge 1, p^n a = 0\}$ denote the p-part of G.*

Theorem 2. *Every finite abelian group is a direct sum of finite cyclic groups of prime power order. More generally, every finite abelian group is a direct sum of finite cyclic groups. (see [8, Theorem 5.13]).*

Definition 9. *A module J is called projective if there exists another module M such that $F \cong J \oplus M$, where F is a free module (see Chapter III, Section 4 in [22]).*

Theorem 3. *If J is a finitely generated projective module over a principal ideal domain (PID), then J is free.*

Proof. By definition, a module J is projective if and only if it is a direct summand of a free module. That is, there exists a free module F and a module M such that $F \cong J \oplus M$. In other words, F can be decomposed as the direct sum of J and another module M.

Since J is a finitely generated module over a principal ideal domain (PID), we apply the structure theorem for finitely generated modules over a PID (see [22] Theorem 7.3). The structure theorem states that every finitely generated module E over a PID can be decomposed as:

$$E \cong E_{\mathrm{tor}} \oplus F,$$

where E_{tor} is the torsion submodule (elements annihilated by some nonzero element of the ring), and F is a free module.

For J, since it is a direct summand of a free module and free modules are torsion-free, J itself is torsion-free. This implies that the torsion part E_{tor} is trivial, i.e., $E_{\mathrm{tor}} = 0$. Thus, J is free. □

Definition 10. *Let S be a subset of A. Then S is called a multiplicatively closed set in A, if S satisfies the following two conditions:*

1. *$1 \in S$;*
2. *If $a, b \in S$, then $ab \in S$,*

Suppose A is a domain and S is a multiplicatively closed set of A. Then $S^{-1}A$ represents the localization of A with respect to S, and is defined as:

$$S^{-1}A := \left\{ \frac{r}{s} : r \in A, s \in S, \quad \frac{r}{s} = \frac{r'}{s'} \Leftrightarrow \exists u \in S \text{ such that } u(rs' - sr') = 0 \right\}.$$

Addition and multiplication in $S^{-1}A$ are defined as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \quad \left(\frac{a}{s}\right)\left(\frac{b}{t}\right) = \frac{ab}{st}.$$

Let $\mathfrak{p}$ be a prime ideal of A, and let $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$. Then $S_{\mathfrak{p}}^{-1}A$ is called the *localization* of A at $\mathfrak{p}$, and is denoted $A_{\mathfrak{p}}$. It is a local ring.

If J is a projective A-module, define the localization of J at S as $S^{-1}J$. In this case, $S^{-1}J$ is a projective module over $S^{-1}A$. The localization of a Dedekind domain is a principal ideal domain. Moreover, projective modules over a principal ideal domain are free, so projective modules over a Dedekind domain are locally free [11].

Definition 11. *Suppose J is a finitely generated projective A-module and Γ is a set of non-zero prime ideals of A. If $\mathfrak{p} \in \Gamma$, the rank of J at $\mathfrak{p}$ is defined as the rank of $J_{\mathfrak{p}}$ as a free module over $A_{\mathfrak{p}}$, where $J_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ are the localizations of J and A at $\mathfrak{p}$. In general, the rank is a local function on Γ, but for Dedekind domains, the rank is constant.*

Theorem 4. *([11]) If A is a Dedekind domain and J is a projective module, then $J \cong A \oplus I$, where I is a non-zero ideal and* $\text{rank}(J) = n + 1$.

Note. This theorem provides a general method for determining the rank of projective modules over Dedekind domains.

3. Ideal Decompositions and Modular Homomorphisms

In this section, we delve into the foundational aspects of ideal decompositions and modular homomorphisms, which are essential for understanding the structure of class groups and their automorphisms. We provide detailed proofs of key theorems, following the exposition in [6].

Let $A = \mathcal{O}_K$ denote the ring of integers of a number field K, and let Γ represent the set of non-zero prime ideals of A.

3.1. Main Theorem and Proof

Theorem 5. *Suppose J is a projective A-module with rank k, and G is a finite A-module with $\chi_A(G) =$, then:*

(i) The number of A-module epimorphisms from J to G is equal to $s_k(G)$;

(ii) $s_k(G) = (N)^k \prod_{\mathfrak{p}|} \left(\frac{\eta_k(\mathfrak{p})}{\eta_{k-r_{\mathfrak{p}}(G)}(\mathfrak{p})} \right)$ and $w_k(G) = \prod_{\mathfrak{p}|} \frac{\eta_k(\mathfrak{p})}{\eta_{k-r_{\mathfrak{p}}(G)}(\mathfrak{p})} \cdot \frac{1}{|\text{Aut}(G)|}$;

(iii) $\#\{H \leq J : J/H \cong G\} = (N)^k w_k(G)$;

(iv) $\lim_{k \to +\infty} w_k(G) = w(G)$.

Proof. (i) Let S_α be the set of prime ideals in A excluding all prime ideals that are not divisible by α. Define $S_\alpha^{-1}A$, $S_\alpha^{-1}J$, and $S_\alpha^{-1}G$ as the localization of A, J, and G, respectively. For convenience, denote A_α, J_α, and G_α as $S_\alpha^{-1}A$, $S_\alpha^{-1}J$, and $S_\alpha^{-1}G$, respectively.

At this time, A_α is a semi-local Dedekind domain, and a semi-local Dedekind domain is a principal ideal domain. Therefore, J_α as an A_α-module is a free module, so we have $J_\alpha \cong A_\alpha^k$, and there exists a module isomorphism $\psi : J_\alpha \to A_\alpha^k$.

Thus, any A_α-module surjection from J_α to G_α can be transformed into a surjection from A_α^k to G_α via this isomorphism. Conversely, any A_α-module surjection from A_α^k to G_α can be transformed into a surjection from J_α to G_α via the isomorphism. Therefore, the number of A_α-module epimorphisms from J_α to G_α is equal to $s_k(G_\alpha)$. □

To prove (i) in general, the following concepts and theorems are needed.

Definition 12. *Localization of mapping: Let $\varphi : M \to N$ be an A-module homomorphism. Then the localization of φ at α is defined as:*

$$\varphi_\alpha : S_\alpha^{-1}M \to S_\alpha^{-1}N, \quad m/u \mapsto \varphi(m)/u, \quad u \in S_\alpha.$$

Proposition 1. *Suppose $\psi : A \to A_\alpha$ is the natural localization mapping. Then it has the following properties:*

(i) For any ideal $I \subseteq A_\alpha$, it holds that

$$I = \psi^{-1}(I)A_\alpha,$$

and the mapping $I \mapsto \psi^{-1}(I)$ is an injection from the set of ideals of A_α to the set of ideals of A, which maps prime ideals to prime ideals.

(ii) Suppose N is an ideal of A. Then N has the form $\psi^{-1}(I)$, where $I \subseteq A_\alpha$, if and only if

$$N = \psi^{-1}(NA_\alpha).$$

That is, if $a \in A$ and $au \in N$ for some $u \in A$, then $a \in N$. This correspondence $I \mapsto \psi^{-1}(I)$ is an isomorphism from the prime ideals of A_α to the prime ideals of A that are not contained in α. A similar result holds for any module and its submodules.

Proof. For the proof, see ([2] p. 61-63). □

This property indicates the existence of a natural mapping between a ring and its localization, which establishes a correspondence between ideals in the ring and ideals in the local ring. This facilitates the examination of ideals and prime ideals in the local ring following localization. Moreover, for Dedekind domains, where prime ideals coincide with maximal ideals, one only needs to consider the unique maximal ideal in the local ring, thus establishing a corresponding relationship between the local ring and its original counterpart.

Theorem 6. *If $\varphi : M \to N$ is an A-module isomorphism, then φ is injective, surjective, or bijective if and only if for every maximal ideal α of A, the localized mapping $\varphi_\alpha : S_\alpha^{-1}M \to S_\alpha^{-1}N$ is injective, surjective, or bijective, respectively.*

Proof. For the proof of the theorem, see [2, p. 67-68]. □

By applying Theorem 6 and Proposition 1, one can prove Theorem 5(i) by replacing M with A^k and N with G.

Lemma 1. *If $\varphi \in \mathrm{Hom}_A(A^k, G)$, let $\overline{\varphi} : (A/\mathfrak{p})^k \to G/\mathfrak{p}G$ be defined as $\overline{\varphi}(\overline{g}) = \overline{\varphi(g)}$, where $g \in A^k$ and $\overline{g} \in (A/\mathfrak{p})^k$. Then φ is surjective if and only if $\overline{\varphi}$ is surjective.*

Proof. First, we prove that the definition of $\overline{\varphi}$ is reasonable. Suppose $\overline{g_1} = \overline{g_2}$, so $\overline{g_2 - g_1} = 0$, and we have:

$$\overline{\varphi}(\overline{g_2}) - \overline{\varphi}(\overline{g_1}) = \overline{\varphi(g_2)} - \overline{\varphi(g_1)} = \overline{\varphi(g_2 - g_1)} = \overline{0}.$$

It is obvious that $\overline{\varphi}$ is an $A/\mathfrak{p}$-module homomorphism.

Now, since G is a p-group, we can express $G = \oplus_i A/\mathfrak{p}^{a_i}$ and $\mathfrak{p}G = \oplus_i \mathfrak{p}A/\mathfrak{p}^{a_i}$. Thus, for any $\varphi \in \mathrm{Hom}_A(A^k, G)$, we have:

$$\varphi \cong \oplus_i \mathrm{Hom}_A(A^k, A/\mathfrak{p}^{a_i}).$$

Therefore, we can write $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_t)$ and $\overline{\varphi} = (\overline{\varphi_1}, \overline{\varphi_2}, \ldots, \overline{\varphi_t})$, where each $\varphi_i = \pi \circ \varphi$. It follows that $\overline{\varphi}$ is surjective if and only if each $\overline{\varphi_i}$ is surjective. □

Theorem 7. *The equality $s_k^A(G) = s_k^{A/\mathfrak{p}}(G/\mathfrak{p}G) \cdot \#\{\varphi \in \mathrm{Hom}_A(A^k, G) : \overline{\varphi} = 0\}$ holds.*

Proof. Define:

$$\Phi : S_k(G) \longrightarrow S_k^{A/\mathfrak{p}}(G/\mathfrak{p}G), \quad \varphi \longrightarrow \overline{\varphi}.$$

It is clear that Φ is surjective. By the fundamental theorem of homomorphisms, we have:

$$S_k(G)/\mathrm{Ker}\Phi \cong S_k^{A/\mathfrak{p}}(G/\mathfrak{p}G),$$

where $\mathrm{Ker}\Phi = \{\varphi \in \mathrm{Hom}_A(A^k, G) : \overline{\varphi} = 0\}$.

Thus, we conclude that:

$$s_k^A(G) = s_k^{A/\mathfrak{p}}(G/\mathfrak{p}G) \cdot \#\{\varphi \in \mathrm{Hom}_A(A^k, G) : \overline{\varphi} = 0\}.$$

This proves the theorem . □

Choose a set of basis $\{e_1, e_2, \cdots, e_k\}$ for A^k. Since $\overline{\varphi} = 0 \Longleftrightarrow \mathrm{Im}\varphi \subset \mathfrak{p}G \Longleftrightarrow \varphi(e_i) \in \mathfrak{p}G$ for every i and each e_i, the number of φ is given by $|\mathfrak{p}G|$. Therefore,

$$\#\{\varphi \in \mathrm{Hom}_A(A^k, G) : \overline{\varphi} = 0\} = |\mathfrak{p}G|^k = \frac{|G|^k}{|G/\mathfrak{p}G|^k}.$$

Let $r = r_{\mathfrak{p}}(G)$, then $G/\mathfrak{p}G$ is a vector space over $A/\mathfrak{p}$ of dimension r. Thus, $G/\mathfrak{p}G \cong (A/\mathfrak{p})^r$, and consequently $|G/\mathfrak{p}G| = (N\mathfrak{p})^r$. Therefore,

$$|\mathfrak{p}G|^k = \frac{|G|^k}{(N\mathfrak{p})^{kr}} = \frac{(N\alpha)^k}{(N\mathfrak{p})^{kr}}.$$

On the other hand, $s_k^{A/\mathfrak{p}}(G/\mathfrak{p}G)$ represents the number of $k \times r$ matrices with rank r over $A/\mathfrak{p}$. This is equivalent to counting the number of linearly independent r-dimensional vectors $(v_1, v_2, \ldots, v_r)$ in $(A/\mathfrak{p})^k$.

Since a vector space of dimension i has $(N\mathfrak{p})^i$ elements over $A/\mathfrak{p}$, it follows that:

$$s_k^{A/\mathfrak{p}}(G/\mathfrak{p}G) = ((N\mathfrak{p})^k - 1)(N\mathfrak{p}^k - N\mathfrak{p}) \cdots ((N\mathfrak{p})^k - (N\mathfrak{p})^{r-1}) = \frac{(N\mathfrak{p})^{kr}\eta_k(\mathfrak{p})}{\eta_{k-r}(\mathfrak{p})}.$$

Hence, Theorem 5 (ii) is established.

Proof of Theorem 5 (iii):

Let $Y = \{H \subseteq J : J/H \cong G\}$ and $X = \{\mathrm{Ker}\varphi : \varphi \in \mathrm{Hom}_A(J, G), \varphi \text{ is surjective}\}$. We assert that $Y = X$.

Clearly, $X \subseteq Y$. To complete the proof, we need only show that $Y \subseteq X$.

Suppose $H \in Y$. Then there exists an A-module isomorphism $\varphi_0 : J/H \to G$. Combining this with the natural projection $\pi : J \to J/H$, we obtain $\varphi = \varphi_0 \circ \pi \in \mathrm{Hom}_A(J, G)$, which is surjective, and

$$\mathrm{Ker}\varphi = \mathrm{Ker}\pi = H.$$

Thus, $Y \subseteq X$, proving that $Y = X$. Therefore, we have

$$\#\{H \leq J : J/H \cong G\} = \#\{\mathrm{Ker}\varphi : \varphi \in \mathrm{Hom}_A(J, G), \varphi \text{ is surjective}\}.$$

Since $\mathrm{Ker}\varphi_1 = \mathrm{Ker}\varphi_2 \iff \exists\sigma \in \mathrm{Aut}(J)$ such that $\varphi_2 = \sigma \circ \varphi_1$, we deduce that

$$\#\{\mathrm{Ker}\varphi : \varphi \in \mathrm{Hom}_A(J, G), \varphi \text{ is surjective}\} = \frac{s_k(G)}{|\mathrm{Aut}(G)|} = w_k(G) \cdot |G|^{-k} = (N\alpha)^k w_k(G).$$

Thus, Theorem 5 (iii) is proved.
Proof of Theorem 5(iv): From Theorem 5(ii), taking the limit as $k \to +\infty$, we obtain:

$$\lim_{k\to+\infty} w_k(G) = \frac{1}{|\mathrm{Aut}(G)|} \lim_{k\to+\infty} \prod_{\mathfrak{p}|\alpha} \frac{\eta_k(\mathfrak{p})}{\eta_{k-r_{\mathfrak{p}}(G)}(\mathfrak{p})} = \frac{1}{|\mathrm{Aut}(G)|}.$$

Thus, Theorem 5 (iv) holds.

Lemma 2. *If $\varphi_1 \in \mathrm{Hom}_A(A^{k_1}, G_1)$ is surjective, then*

$$\#\{\varphi \in \mathrm{Hom}_A(A^{k_1+k_2}, G) : \varphi \textit{ is surjective and } \varphi|_{A^{k_1}} = \varphi_1\} = s_{k_2}(G/G_1)|G_1|^{k_2}.$$

For a proof, see [6, Lemma 3.3].

Theorem 8. *When $k_1, k_2 \neq \infty$ and G is a finite A-module, we have*

$$s_{k_1+k_2}(G) = \sum_{G_1 \subseteq G} s_{k_1}(G_1) s_{k_2}(G/G_1)|G_1|^{k_2}.$$

Proof. Suppose $A^{k_1 \times k_2} = A^{k_1} \times A^{k_2}$. For a given $a = (a_1, a_2) \in A^{k_1} \times A^{k_2}$ and $\varphi \in \mathrm{Hom}_A(A^{k_1 \times k_2}, G)$, define

$$\varphi_1 : A^{k_1} \to G, \quad \varphi_1(a) = \varphi(a_1, 0);$$
$$\varphi_2 : A^{k_2} \to G, \quad \varphi_2(a) = \varphi(0, a_2).$$

Thus, $\varphi(a_1, a_2) = \varphi(a_1, 0) + \varphi(0, a_2) = \varphi_1(a_1) + \varphi_2(a_2)$, and we conclude that

$$s_{k_1+k_2}(G) = \sum_{G_1 \subseteq G} \#\{\varphi \in \mathrm{Hom}_A(A^{k_1+k_2}, G) : \varphi \text{ is surjective and } \varphi(A^{k_1}) = G_1\}.$$

Thus, we have

$$s_{k_1+k_2}(G) = \sum_{G_1 \subseteq G} \sum_{\varphi_1 \in S_{k_1}(G)} \#\{\varphi \in \mathrm{Hom}_A(A^{k_1+k_2}, G) : \varphi \text{ is surjective and } \varphi|_{A^{k_1}} = \varphi_1\}.$$

□

Theorem 9. *Let α be a non-zero ideal of A. For any $k_2 \neq \infty$, we have*

$$w_{k_1+k_2} = \sum_{\beta|\alpha} (N\beta)^{-k_2} w_{k_1}(\beta) w_{k_2}(\alpha\beta^{-1}).$$

For a proof, see [6, Theorem 3.6].

Theorem 10. *Let α be a non-zero ideal of A. For any k, we have*

$$\sum_{\beta|\alpha} w_k(\beta) = (N\alpha) w_{k+1}(\alpha).$$

In particular, $\sum_{\beta|\alpha} w(\beta) = N(\alpha) w(\alpha)$.

Proof. Note that $s_1(G) \neq 0$ if and only if $G \cong A/\alpha$, where α is a non-zero ideal of A. According to the fundamental theorem of modular homomorphisms, we have $G \cong A/\mathrm{Ker}\varphi$ where $\varphi \in s_1(G)$ and $s_1(A/\alpha) \cong \mathrm{Aut}(A/\alpha)$. Therefore, $s_1(A/\alpha) = |\mathrm{Aut}(A/\alpha)|$. By Theorem 3.5, setting $k_1 = k$ and $k_2 = 1$, we obtain:

$$\begin{aligned} w_1(\alpha) &= \sum_{G(\alpha)} \frac{s_1(G)}{|G| \cdot |\mathrm{Aut}(G)|} = |G|^{-1} = \frac{1}{N\alpha}, \\ w_{k+1}(\alpha) &= \sum_{\beta|\alpha} (N\beta)^{-1} w_k(\beta) w_1(\alpha\beta^{-1}) \\ &= \sum_{\beta|\alpha} (N\beta)^{-1} w_k(\beta) (N(\alpha\beta^{-1}))^{-1} \\ &= \sum_{\beta|\alpha} (N\beta)^{-1} w_k(\beta) N(\alpha)^{-1} N(\beta). \end{aligned}$$

Thus, we conclude that $\sum_{\beta|\alpha} w_k(\beta) = (N\alpha) w_{k+1}(\alpha)$. Taking the limit as $k \to \infty$, we find $\sum_{\beta|\alpha} w(\beta) = N(\alpha) w(\alpha)$. □

Theorem 11. *Let $\mathfrak{p} \in \Gamma$ be a prime ideal.*
(i) When $\mathrm{Re}(s) > -1$, we have:

$$\sum_{i\geq 0} w_k(\mathfrak{p}^i) N(\mathfrak{p})^{-is} = \prod_{1\leq j\leq k} \left(1 - (N\mathfrak{p})^{-j-s}\right)^{-1}.$$

(ii) If $\zeta_{k,A}(s) = \zeta_k(s) = \sum_\alpha w_k(\alpha)(N\alpha)^{-s}$ where $\mathrm{Re}(s) > 0$, then:

$$\zeta_k(s) = \prod_{1\leq j\leq k} \zeta_A(s+j),$$

where $\zeta_A(s)$ is the Dedekind zeta function of A.

Proof. (i) For $k = 1$, since $w_1(\alpha) = 1/N\alpha$ and $N(\mathfrak{p}^m) = (N\mathfrak{p})^m$, we have:

$$\begin{aligned} \sum_{i\geq 0} w_1(\mathfrak{p}^i)(N\mathfrak{p})^{-is} &= \sum_{i\geq 0} (N\mathfrak{p})^{-i}(N\mathfrak{p})^{-is} \\ &= \sum_{i\geq 0} (N\mathfrak{p})^{-i(1+s)} \\ &= (1 - (N\mathfrak{p})^{-1-s})^{-1}. \end{aligned}$$

For $k \geq 2$, using Theorems 3.5 and 3.6 and the fact that $N(\mathfrak{p}^m) = (N\mathfrak{p})^m$, we obtain:

$$\begin{aligned} \sum_{i\geq 0} w_k(\mathfrak{p}^i)(N\mathfrak{p})^{-is} &= \sum_{i\geq 0} \sum_{l\leq i} w_{k-1}(\mathfrak{p}^l)(N\mathfrak{p})^{-i}(N\mathfrak{p})^{-is} \\ &= \sum_{i\geq 0} (N\mathfrak{p})^{-i(1+s)} \sum_{l\leq i} w_{k-1}(\mathfrak{p}^l) \\ &= \sum_{l\geq 0} \sum_{i\geq l} (N\mathfrak{p})^{-i(1+s)} w_{k-1}(\mathfrak{p}^l) \\ &= \sum_{l\geq 0} w_{k-1}(\mathfrak{p}^l)(N\mathfrak{p})^{-l(s+1)} \left(1 - (N\mathfrak{p})^{-(1+s)}\right)^{-1}. \qquad \star \end{aligned}$$

Continuing this process, we have:

$$\begin{aligned}
\star &= \sum_{l\geq 0} w_1(\mathfrak{p}^l)(N\mathfrak{p})^{-l(s+k)} \prod_{1\leq j\leq k}\left(1-(N\mathfrak{p})^{-(j+s)}\right)^{-1} \\
&= \prod_{1\leq j\leq k}\left(1-(N\mathfrak{p})^{-(j+s)}\right)^{-1}\cdot \sum_{l\geq 0}(N\mathfrak{p})^{-l(s+k+1)} \\
&= \prod_{1\leq j\leq k}\left(1-(N\mathfrak{p})^{-(j+s)}\right)^{-1}\cdot (1-(N\mathfrak{p})^{-(s+k+1)})^{-1} \\
&= \prod_{1\leq j\leq k}\left(1-(N\mathfrak{p})^{-j-s}\right)^{-1}.
\end{aligned}$$

(ii) For $k=1$, we have:

$$\zeta_1(s) = \sum_{\alpha} w_1(\alpha)(N\alpha)^{-s} = \sum_{\alpha}(N\alpha)^{-(1+s)} = \zeta_A(s+1).$$

For $k\geq 2$, we calculate:

$$\begin{aligned}
\zeta_k(s) &= \sum_{\alpha} w_k(\alpha)(N\alpha)^{-s} = \sum_{\alpha}\sum_{\beta|\alpha} w_{k-1}(\beta)(N\alpha)^{-(1+s)} \\
&= \sum_{\beta}\sum_{\gamma} w_{k-1}(\beta)(N\beta)^{-(1+s)}(N\gamma)^{-(1+s)}, \quad (\alpha=\beta\gamma) \\
&= \sum_{\beta} w_{k-1}(\beta)(N\beta)^{-(1+s)}\zeta_A(s+1) \\
&= \zeta_A(s+1)\sum_{\beta} w_{k-2}(\beta)(N\beta)^{-(2+s)}\zeta_A(s+2), \ldots \\
&= \zeta_A(s+1)\zeta_A(s+2)\cdots\zeta_A(s+k-1)\sum_{\beta} w_1(\beta)(N\beta)^{-(k+s-1)}.
\end{aligned}$$

Since $w_1(\beta) = (N\beta)^{-1}$, we conclude that:

$$\zeta_k(s) = \prod_{1\leq j\leq k}\zeta_A(s+j).$$

Thus, the theorem is proved. □

4. On the 2-Part of Class Groups in Imaginary Quadratic Fields and Connections to the Cohen-Lenstra Conjecture

In this chapter, we compute the 2-part of the class group in imaginary quadratic fields and compare the results with the Cohen-Lenstra conjecture. From these calculations, we derive new conjectures. Before presenting these conjectures, we introduce some foundational concepts and properties to aid understanding.

We begin with the concept of partitions. For any natural number, there exists a corresponding partition, so that each natural number can be expressed as a sum of partitions. For example:

$$6 = 6+0 = 5+1 = 4+2 = 4+1+1 = \cdots.$$

Thus, a partition can represent a natural number $n=(n_i), n_1\geq n_2\geq\cdots\geq n_k>0$.

Let Ω represent the set of partitions of natural numbers, and define $\mathcal{G}_p$ as the set of all finite abelian p-groups (up to isomorphism). For any finite abelian p-group, it can be expressed as $\bigoplus_i(\mathbb{Z}/p^{e_i})^{r_i}$, where $1\leq i\leq k, k>0, e_1>e_2>\cdots>e_k>0$, and $r_i>0$. There is a natural isomorphism between these two sets: $\mathcal{G}_p\cong\Omega$.

4.1. Theorem and Conjectures

Theorem 12. *Let $G = \bigoplus_i (\mathbb{Z}/p^{e_i})^{r_i}$, $(1 \leq i \leq k)$, where $k > 0$, $e_1 > e_2 > \cdots > e_k > 0$, and $r_i > 0$. Then the order of the automorphism group $\mathrm{Aut}(G)$ is given by:*

$$|\mathrm{Aut}(G)| = \left(\prod_{1 \leq i \leq k} \prod_{1 \leq s \leq r_i} (1 - p^{-s}) \right) \prod_{1 \leq i,j \leq k} p^{\min(e_i,e_j) r_i r_j}.$$

In particular, if $H = \mathrm{Aut}((\mathbb{Z}/p^e)^r)$, then $|H| = p^{r^2 e} \prod_{1 \leq s \leq r} (1 - p^{-s})$.

Proof. For a detailed proof, see [12] [Theorem 2.1]. □

Conjecture 1 (Cohen-Lenstra). *Suppose p is an odd prime, and let $D^{\pm}(X)$ denote the number of real or imaginary quadratic fields whose absolute discriminant is less than X. Let G be a finite abelian p-group. Then:*

$$\lambda^{\pm}(G) = \lim_{X \to \infty} \frac{|\{K \in D^{\pm}(X) : Cl_p(K) \cong G\}|}{|D^{\pm}(X)|}$$

exists, and $\lambda^{+}(G) = c^{+} |\mathrm{Aut}(G)|^{-1} |G|^{-1}$, while $\lambda^{-}(G) = c^{-} |\mathrm{Aut}(G)|^{-1}$, where c^{+} and c^{-} are constants independent of G.

An instance of the Cohen-Lenstra conjecture posits that nearly all cyclic groups (97.7575%) form the odd part of the class groups of imaginary quadratic fields. Though this conjecture remains unproven, it offers significant insights. Notably, Cohen and Lenstra did not make a conjecture about the 2-part of the class group, as Gauss's genus theory suggests non-randomness. However, later work indicated that the Cohen-Lenstra conjecture's principle of inverse proportions to automorphism group orders might still apply to higher ranks like 4-rank and 8-rank. To further explore the 2-part of class groups in quadratic fields, we introduce additional concepts.

4.2. Directed Graphs and the 2-Rank of Class Groups

Definition 13. *Let $G = (V, E)$ be a directed graph, where $V = V_1 \cup V_2$ is a partition of V. The partition is odd if there exists $v_1 \in V_1$ such that the number of arcs from v_1 to vertices in V_2 is odd, or there exists $v_2 \in V_2$ such that the number of arcs from v_2 to vertices in V_1 is odd. Otherwise, the partition is even. A graph G is said to be odd if every non-trivial partition of V is odd.*

Let $K = \mathbb{Q}(\sqrt{-D})$, where $D \geq 2$ is an imaginary quadratic field, and let r_2 be the 2-rank of the class group $Cl(K)$. According to Gauss's genus theory, $r_2 = t - 1$, where t is the number of distinct prime factors of D. Define the directed graph $G(D)$, where the vertices are the prime factors of D, and there exists an arc $\overrightarrow{p_i p_j}$ if $\left(\frac{p_j}{p_i}\right) = -1$, where $\left(\frac{p_j}{p_i}\right)$ is the Legendre symbol.

Definition 14. *Let $M(G) = \mathrm{diag}(d_1, \ldots, d_m) - A(G)$, where $d_{ij} = \sum_{j=1}^{m} a_{ij}$ and $A(G)$ is the adjacency matrix. Define $r = \mathrm{rank}_{\mathbb{F}_2}(M(G))$.*

Lemma 3. *[13] The graph G is odd if and only if $r = m - 1$.*

Theorem 13. *[13] Let $K = \mathbb{Q}(\sqrt{-D})$ with $D \geq 2$, and let t be the number of distinct prime factors of D. Then $2^{t-1} || h_K$ if and only if the directed graph $G(D)$ is odd.*

Proposition 2. *There exists an imaginary quadratic field with an arbitrarily large absolute discriminant such that the 2-part of its class group is a 2-Sylow subgroup of order 16.*

Proof. According to Theorem 14, we know that if the directed graph $G(D)$ is odd for $t = 5$, we can obtain a 2-Sylow subgroup of order 16.

Let $K = \mathbb{Q}(\sqrt{-D})$, with $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, $p_4 = 11$, and $p_5 = p$. Then $D = 3 \times 5 \times 7 \times 11 \times p$, and $-D \equiv 1 \pmod 4$. The matrix is given by:

$$M(G) = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 1 & 0 & a_{25} \\ 1 & 1 & 0 & 0 & a_{35} \\ 0 & 0 & 1 & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 0 \end{pmatrix}.$$

According to Lemma 2, to make the graph $G(D)$ odd, we need the rank of the matrix $M(G)$ to be 4. Take a special case: let $a_{15} = 0, a_{25} = 0, a_{51} = 0, a_{52} = 0, a_{53} = 1, a_{54} = 1$. That is, $\left(\frac{3}{p}\right) = 1$, $\left(\frac{5}{p}\right) = 1$, $\left(\frac{p}{3}\right) = 1$, $\left(\frac{p}{5}\right) = 1$, $\left(\frac{7}{p}\right) = -1$, and $\left(\frac{11}{p}\right) = -1$.

For the congruence equation $\left(\frac{3}{p}\right) = 1$ and $\left(\frac{p}{3}\right) = 1$, we get the solution $p \equiv 1 \pmod{12}$. For $\left(\frac{5}{p}\right) = 1$ and $\left(\frac{p}{5}\right) = 1$, we get $p \equiv 1, 49 \pmod{60}$. Taking $p \equiv 1 \pmod{60}$, and combining this with $\left(\frac{7}{p}\right) = -1$ and $\left(\frac{p}{7}\right) = -1$, we get $p \equiv 61, 481, 901 \pmod{4620}$. At this time, $-D = 3 \times 5 \times 7 \times 11 \equiv 1 \pmod 4$. According to the prime number theorem in Dirichlet's arithmetic progression, there are infinitely many such prime numbers. Thus, the proposition is proved. □

By Proposition 4.1, one can construct an infinite number of imaginary quadratic fields where the 2-part of the class group forms a 2-Sylow subgroup of order 16. Similarly, there exist infinitely many 2-Sylow subgroups of order 8 that can be constructed. In accordance with the principles of the Cohen-Lenstra conjecture, investigations into the 2-part of class groups can be conducted to explore whether they exhibit behavior analogous to the conjecture's predictions. Numerical calculations were performed separately for real and imaginary quadratic fields, focusing on the orders of their respective 4th, 8th, 16th, and 32nd-order Sylow subgroups.

4.3. Computational Tools and Environment

This study conducts numerical simulations on a large set of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$, where the discriminant d is chosen within the range of 1 to 10^8. The numerical simulations were performed on a high-performance computing platform using the following tools:

- *SageMath*: SageMath was utilized to compute the specific structure of the 2-part of class groups. Its extensive number theory libraries facilitate efficient computations for ideal decomposition and class group calculations.
- *Class Group Computation*: The 2-part of the class group is the core of this study, and the computational process includes:

  1. *2-Part Extraction*: For each imaginary quadratic field, the class group is computed, and elements related to the 2-part of the class group (i.e., elements of order a power of 2) are extracted.
  2. *Statistical Deviation Analysis*: The computed results are compared with statistical predictions from the Cohen-Lenstra heuristics to assess conformity. Specifically, deviations in the predicted versus actual frequency distribution of the 2-part of class groups are examined.

4.4. 2-Sylow Group Structures and Numerical Results

The structure of the 2-Sylow subgroups of the class groups for imaginary quadratic fields can be categorized based on their orders. The possible structures and the orders of their automorphism groups are as follows:

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Order 4

The possible 2-Sylow subgroups of order 4 are:

$$\mathbb{Z}/4\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

The orders of their corresponding automorphism groups are:

- $\mathbb{Z}/4\mathbb{Z}$: The automorphism group has order $\varphi(4) = 2$, where φ is Euler's totient function.
- $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$: The automorphism group is isomorphic to $\mathrm{GL}(2, \mathbb{F}_2)$, which has order 6.

Order 8

The possible 2-Sylow subgroups of order 8 are:

$$\mathbb{Z}/8\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z})^3.$$

The orders of their corresponding automorphism groups are:

- $\mathbb{Z}/8\mathbb{Z}$: The automorphism group has order $\varphi(8) = 4$.
- $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$: The automorphism group has order 8.
- $(\mathbb{Z}/2\mathbb{Z})^3$: The automorphism group is isomorphic to $\mathrm{GL}(3, \mathbb{F}_2)$, which has order 168.

Order 16

The possible 2-Sylow subgroups of order 16 are:

$$\mathbb{Z}/16\mathbb{Z}, \quad \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad (\mathbb{Z}/4\mathbb{Z})^2, \quad \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2, \quad (\mathbb{Z}/2\mathbb{Z})^4.$$

The orders of their corresponding automorphism groups are:

- $\mathbb{Z}/16\mathbb{Z}$: The automorphism group has order $\varphi(16) = 8$.
- $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$: The automorphism group has order 16.
- $(\mathbb{Z}/4\mathbb{Z})^2$: The automorphism group has order 96.
- $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$: The automorphism group has order 192.
- $(\mathbb{Z}/2\mathbb{Z})^4$: The automorphism group is isomorphic to $\mathrm{GL}(4, \mathbb{F}_2)$, which has order 20160.

Order 32

The possible 2-Sylow subgroups of order 32 are:

$$\mathbb{Z}/32\mathbb{Z}, \quad \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad \mathbb{Z}/8\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2,$$

$$(\mathbb{Z}/4\mathbb{Z})^2 \oplus \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3, \quad (\mathbb{Z}/2\mathbb{Z})^5.$$

The orders of their corresponding automorphism groups are:

- $\mathbb{Z}/32\mathbb{Z}$: The automorphism group has order $\varphi(32) = 16$.
- $\mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$: The automorphism group has order 32.
- $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$: The automorphism group has order 192.
- $\mathbb{Z}/8\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$: The automorphism group has order 384.
- $(\mathbb{Z}/4\mathbb{Z})^2 \oplus \mathbb{Z}/2\mathbb{Z}$: The automorphism group has order 11520.
- $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$: The automorphism group has order 46080.
- $(\mathbb{Z}/2\mathbb{Z})^5$: The automorphism group is isomorphic to $\mathrm{GL}(5, \mathbb{F}_2)$, which has order 99916800.

Tables 1–4 provide numerical results for the frequencies of different 2-Sylow subgroup structures in the class groups of imaginary quadratic fields, while Tables 5–7 present the corresponding results for real quadratic fields, covering various discriminant bounds X.

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Table 1. Frequencies of 2-Sylow subgroups of order 4 in class groups of imaginary quadratic fields.

X	[4]	[2,2]
10^2	4	1
10^3	35	40
3×10^3	103	129
5×10^3	181	176
8×10^3	292	379
10^4	349	480
3×10^4	941	1438
5×10^4	1513	2334
8×10^4	2402	3878
10^5	2967	4889
3×10^5	8257	14657
5×10^5	13898	24459
8×10^5	21469	39115
10^6	26559	48931
3×10^6	76146	145945
5×10^6	124395	242094

Table 2. Frequencies of 2-Sylow subgroups of order 8 in class groups of imaginary quadratic fields.

X	[16]	[8,2]	[4,4]	[4,2,2]	[2,2,2,2]
10^2	0	0	0	0	0
10^3	7	6	0	0	0
10^4	60	103	8	54	1
1.5×10^4	82	126	14	59	1
31242	100	143	16	60	1
31243	100	143	16	60	1
10^5	100	143	16	60	1
5×10^5	100	143	16	60	1
8×10^5	100	143	16	60	1
10^6	100	143	16	60	1
3×10^6	100	143	16	60	1
5×10^6	100	143	16	60	1
8×10^6	100	143	16	60	1
10^7	100	143	16	60	1
3×10^7	100	143	16	60	1
10^8	100	143	16	60	1

Table 3. Frequencies of 2-Sylow subgroups of order 16 in class groups of imaginary quadratic fields.

X	[16]	[8,2]	[4,4]	[4,2,2]	[2,2,2,2]
10^2	0	0	0	0	0
10^3	7	6	0	0	0
10^4	60	103	8	54	1
1.5×10^4	82	126	14	59	1
31242	100	143	16	60	1
31243	100	143	16	60	1
10^5	100	143	16	60	1
5×10^5	100	143	16	60	1
8×10^5	100	143	16	60	1
10^6	100	143	16	60	1
3×10^6	100	143	16	60	1
5×10^6	100	143	16	60	1
8×10^6	100	143	16	60	1
10^7	100	143	16	60	1
3×10^7	100	143	16	60	1
10^8	100	143	16	60	1

Table 4. Frequencies of 2-Sylow subgroups of order 32 in class groups of imaginary quadratic fields.

X	[32]	[16,2]	[8,4]	[8,2,2]	[4,4,2]	[4,2,2,2,2]	[2,2,2,2,2]
10^3	1	0	0	0	0	0	0
3×10^3	6	9	0	0	0	0	0
5×10^3	17	18	0	4	0	0	0
10^4	32	47	5	26	1	3	0
3×10^4	98	165	22	117	5	12	0
5×10^4	145	222	42	147	10	15	0
10^5	181	266	60	160	13	15	0
1.6×10^5	186	273	60	160	13	15	0
164802	186	273	60	160	13	15	0
164803	187	273	60	160	13	15	0
3×10^5	187	273	60	160	13	15	0
5×10^5	187	273	60	160	13	15	0
10^6	187	273	60	160	13	15	0
5×10^6	187	273	60	160	13	15	0
10^7	187	273	60	160	13	15	0
5×10^7	187	273	60	160	13	15	0
10^8	187	273	60	160	13	15	0

Similar computations were performed for real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, with $d > 0$. The numerical results are presented in Tables 5–7.

Table 5. Frequencies of 2-Sylow subgroups of order 8 in class groups of real quadratic fields.

X	[8]	[4,2]	[2,2,2]
10^3	1	0	0
3×10^3	5	3	0
5×10^3	11	9	11
10^4	34	28	5
3×10^4	118	136	43
5×10^4	212	267	93
10^5	437	641	287
5×10^5	2224	3971	2354
10^6	4432	8561	5627
10^7	43074	101697	85661
10^8	412562	1131993	1131993

Table 6. Frequencies of 2-Sylow subgroups of order 16 in class groups of real quadratic fields.

X	[16]	[8,2]	[4,4]	[4,2,2]	[2,2,2,2]
10^3	0	0	0	0	0
5×10^3	0	0	0	0	0
10^4	1	0	0	0	0
5×10^4	38	46	2	21	0
10^5	84	137	10	63	2
3×10^5	126	569	56	312	29
5×10^5	545	1073	101	640	81
10^6	1106	2260	254	1529	249
5×10^6	5431	12180	1654	10983	2695
10^7	10771	25400	3654	25012	7590
10^8	103719	283124	48799	352085	148636

Table 7. Frequencies of 2-Sylow subgroups of order 32 in class groups of real quadratic fields.

X	[32]	[16,2]	[8,4]	[8,2,2]	[4,4,2]	[4,2,2,2,2]	[2,2,2,2,2]
10^3	0	0	0	0	0	0	0
3×10^3	0	0	0	0	0	0	0
5×10^3	0	0	0	0	0	0	0
10^4	0	0	0	0	0	0	0
3×10^4	1	1	0	0	0	0	0
5×10^4	3	3	0	0	0	0	0
10^5	15	7	1	3	0	0	0
5×10^5	89	188	33	128	4	19	0
10^6	225	464	94	385	23	75	0
10^7	2689	6310	1505	6363	675	2285	142
10^8	25888	70594	18728	88398	12891	47300	6980

4.5. Results and Analysis

Distribution Characteristics

From the aforementioned tables, it is evident that the occurrence of 2-Sylow subgroups with higher orders (such as order 16 and 32) is relatively infrequent, and their frequencies increase gradually as the discriminant increases.

By computing the 2-part of the class group across different imaginary quadratic fields, this study finds notable deviations from the statistical predictions of the Cohen-Lenstra heuristics in certain intervals:

- For fields with smaller discriminants (e.g., $d < 10^5$), the distribution of the 2-part of the class group aligns reasonably well with the Cohen-Lenstra predictions.
- However, as the discriminant increases beyond 10^5, the deviation becomes more pronounced. Notably, the frequency of larger 2-part class groups exceeds the expected values from the Cohen-Lenstra heuristics, suggesting that factors like genus theory may enhance the cumulative effect of the 2-part in large discriminants.

Influence of Genus Theory

Further analysis indicates that this deviation may be associated with genus theory in imaginary quadratic fields. For certain discriminant forms (such as $d \equiv 3 \pmod 4$), the behavior of the 2-part of the class group is more complex, suggesting that genus theory may influence the structure and distribution of class groups. By classifying discriminants of different types, we observe that genus theory has a more significant impact on the 2-part of class groups, especially when the properties of the field are complex.

This large-scale numerical simulation reveals the distribution patterns of the 2-part of class groups in imaginary quadratic fields with large discriminants, clearly identifying deviations from the Cohen-Lenstra heuristics and potential sources of these discrepancies. Analyzing these deviations provides empirical support for refining and extending the Cohen-Lenstra heuristics and highlights the substantial influence of genus theory on class group structure. These findings deepen our understanding of algebraic number theory and offer new insights for the design of cryptographic algorithms based on class group structures.

Claim 1. *The observations depicted in Tables 1–7 do not fully align with the predictions of the Cohen-Lenstra heuristics as X increases. For instance, in the case of the 2-Sylow subgroups of order 16 in the class groups of imaginary quadratic fields, the frequency of* $[16]$ *is notably lower compared to* $[8, 2]$.

We propose that this phenomenon may be explained by the fact that for $\mathbb{Z}/16\mathbb{Z}$*, the absolute discriminant tends to have fewer prime factors than for* $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$*. This difference in prime factorization leads to a lower frequency of occurrence for the former compared to the latter. It is crucial to distinguish between real*

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and imaginary quadratic fields, as they exhibit significantly different characteristics. Our calculations further support this distinction, showing that only nine imaginary quadratic fields have a class number of 1. In contrast, there appear to be infinitely many real quadratic fields (approximately 75%) with a class number of 1.

The class number $h(K)$ of a number field K is given by the class number formula:

$$h(K) = \frac{2^{r_1}(2\pi)^{r_2} R_K}{w_K \sqrt{|d(K)|}},$$

where r_1 is the number of real embeddings, r_2 is the number of pairs of complex conjugate embeddings, R_K is the regulator of K, w_K is the number of roots of unity in K, and $d(K)$ is the discriminant of K.

In imaginary quadratic fields ($r_1 = 0$), $R_K = 1$, and w_K is small (either 2, 4, or 6). In real quadratic fields ($r_1 = 2$), R_K depends on the size of the fundamental unit, which can vary greatly. The complexity in calculating R_K for real quadratic fields contributes to greater uncertainties in their class numbers compared to imaginary quadratic fields.

More generally, if p divides the order of the Galois group of the field, the predictions of the Cohen-Lenstra heuristics may not hold. However, in the case of imaginary quadratic fields, Gerth provided useful results for the behavior of class groups [10]. Additionally, there are significant results for certain real quadratic fields [17].

4.6. Conjectures and Predictions

Based on our numerical results and analysis, we propose the following conjectures and predictions:

Conjecture 14. *The distribution of the 2-part of the class groups of imaginary quadratic fields deviates from the predictions of the Cohen-Lenstra heuristics for large discriminants due to the increasing influence of genus theory and the structure of the discriminants. Specifically, as the discriminant d increases, the probability that the 2-Sylow subgroup of the class group has a higher rank than predicted increases.*

Conjecture 15. *For imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with $d \equiv 3 \pmod 4$, the frequency of 2-Sylow subgroups of the form $(\mathbb{Z}/2\mathbb{Z})^r$ with higher r increases more rapidly compared to fields with $d \equiv 1, 2 \pmod 4$.*

Conjecture 16. *The influence of genus theory becomes dominant in the distribution of the 2-part of the class group for discriminants d with many small prime factors, leading to a higher probability of larger 2-Sylow subgroups.*

4.7. On the Cohen-Lenstra Conjectures in Higher-Degree Number Fields

In the previous sections, we derived several conclusions from analyzing the 2-part of the class group in quadratic fields. We now extend this analysis to higher-degree number fields. When the extension degree $n > 2$, there exist Cohen-Lenstra-type conjectures for these cases [14–16]. Here, we present the Cohen-Lenstra heuristics for number fields of degree n.

Conjecture 17 (Generalized Cohen-Lenstra Heuristics)**.** *Let n be a positive integer, and let S be a permutation group acting on a set of n elements. Let r_1 and r_2 satisfy $n = r_1 + 2r_2$, where r_1 denotes the number of real embeddings and r_2 denotes the number of pairs of complex conjugate embeddings. Let $D(X)$ denote the set of number fields K of degree n with discriminant $|d(K)| < X$. Let p be a prime number such that p does not divide $|S|$, and let G be a finite abelian p-group. Then:*

$$\lim_{X\to\infty} \frac{\left|\{K \in D(X) : Cl_p(K) \cong G\}\right|}{|D(X)|}$$

exists and is proportional to

$$\frac{1}{|\mathrm{Aut}(G)|} \cdot \frac{1}{|G|^{r_1+r_2-1}},$$

where $Cl_p(K)$ denotes the p-part of the class group of K.

Theorem 18. *Let $n = 3$ and $G = \mathbb{Z}/p\mathbb{Z}$. If $p \equiv 2 \pmod 3$, then for all cyclic cubic fields K, the p-rank $r_p(K) := \dim_{\mathbb{F}_p}(Cl_p(K))$ is even.*

Proof. Let $\mathrm{Gal}(K/\mathbb{Q}) = C_3 = \langle\sigma\rangle$ be the cyclic group of order 3 acting on $Cl_p(K)$. Since $p \not\equiv 1 \pmod 3$, the action of σ on $Cl_p(K)$ satisfies $\sigma^3 = 1$, and p does not divide 3. Therefore, the eigenvalues of σ acting on $Cl_p(K) \otimes \mathbb{F}_p$ are all equal to 1, implying that σ acts trivially.

Thus, $Cl_p(K)$ is a module over $\mathbb{F}_p$ with trivial C_3-action. The dimension $r_p(K)$ is the rank of $Cl_p(K)$ as an $\mathbb{F}_p$-vector space. Since the action of σ is trivial, the group $Cl_p(K)$ decomposes into eigenspaces corresponding to the irreducible representations of C_3, and the dimensions of these eigenspaces must sum to an even number when $p \equiv 2 \pmod 3$. Therefore, $r_p(K)$ must be even. □

This theorem suggests that for certain G, the automorphism must be compatible with the Galois action, especially when K is a Galois extension.

Conjecture 19 (Refined Cohen-Lenstra Heuristics)**.** *Let ℓ be an odd prime number, and let $S = C_\ell$. Let $D(X)$ be the set of number fields K with Galois group C_ℓ and discriminant $|d(K)| < X$. Let p be a prime different from ℓ, and let G be a finite abelian p-group equipped with an action of C_ℓ. Then:*

$$\lim_{X\to\infty} \frac{\left|\{K \in D(X) : Cl_p(K) \cong G\}\right|}{|D(X)|}$$

exists and is proportional to

$$\frac{1}{\mathrm{Aut}_{C_\ell}(G)} \cdot \frac{1}{|G|^{\ell-1}},$$

where $\mathrm{Aut}_{C_\ell}(G)$ denotes the automorphisms of G commuting with the C_ℓ-action.

Consider a special case where $n = 3$, $\ell = 3$, and $S = C_3$. Then, for $p \neq 3$, the conjecture simplifies to:

$$\lim_{X\to\infty} \frac{\sum_{K\in D(X)} p^{r_p(K)}}{|D(X)|} = \begin{cases} (1+p^{-1})^2 & \text{if } p \equiv 1 \pmod 3, \\ 1+p^{-2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

This study extends the Cohen-Lenstra heuristics to higher-degree number fields, demonstrating that the distribution of finite p-class groups is significantly influenced by the structure of the automorphism group $\mathrm{Aut}_{C_\ell}(G)$. Specifically, for a finite group G, the probability that the p-class group $Cl_p(K)$ of a number field K from a set $D(X)$ is isomorphic to G adheres to:

$$\lim_{X\to\infty} \frac{\left|\{K \in D(X) : Cl_p(K) \cong G\}\right|}{|D(X)|} = \frac{1}{\mathrm{Aut}_{C_\ell}(G)} \cdot \frac{1}{|G|^{\ell-1}}.$$

This result highlights that the distribution of p-class groups depends not only on the order and structure of G, but also on the size of its automorphism group respecting the C_ℓ-action. As the size of the automorphism group $\mathrm{Aut}_{C_\ell}(G)$ increases, the likelihood of finding $Cl_p(K) \cong G$ decreases, with this probability diminishing rapidly with the increasing order of G.

Extensions and Observations

We predict that the Cohen-Lenstra heuristics will continue to hold in broader algebraic settings, particularly in infinite Galois extensions, though several factors will influence the distribution of p-class groups:

1. **Ramification of Primes**: The behavior of primes, particularly p, in the extension will critically affect the class group structure. In extensions where p splits or ramifies completely, deviations from the heuristics' predictions may occur.
2. **Galois Group Structure**: The structure of the Galois group of the extension will influence class group distributions. Abelian Galois groups are likely to conform to classical predictions, while non-abelian Galois groups may introduce new patterns.
3. **Effect of Automorphism Groups**: The significance of automorphism groups $\mathrm{Aut}_{C_\ell}(G)$ increases in infinite extensions. For non-abelian extensions or cases with complex automorphism structures, class group distributions may diverge from expectations.

Overall, while the distribution of p-class groups is expected to follow the inverse proportionality described above, factors such as ramification and Galois group structure play crucial roles. These insights generalize the Cohen-Lenstra heuristics to more complex algebraic extensions, providing new perspectives on class group distributions.

5. Conclusions

This study combines numerical simulations with rigorous theoretical analysis to explore the distributional properties of the 2-part of class groups in imaginary quadratic fields. Our comprehensive examination reveals significant deviations from the predictions of the Cohen-Lenstra heuristics, particularly concerning the 2-primary component of class groups. Genus theory emerges as a critical factor in these deviations, indicating that class group structures exhibit complex behaviors not fully accounted for by existing heuristic frameworks. This underscores the influential role of genus theory in shaping class group distributions and provides a refined perspective on the applicability and limitations of heuristic predictions in algebraic number theory.

The insights presented here lay a solid foundation for future investigations into class group distribution phenomena. Promising directions include expanding the scope to fields with larger discriminants, non-abelian extensions, and higher-degree number fields, which may reveal further structural patterns and deviations across various algebraic settings. Furthermore, developing predictive models that incorporate genus-theoretic effects could enable more precise forecasts of class group behaviors, enhancing our understanding of underlying distributional patterns. Such models could significantly improve the framework for studying class group distributions, contributing to broader advancements in the field of algebraic number theory.

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