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A Consumer Preferences. An Influence of Society on the Individual

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Abstract: In this paper, we define the consumer preference relation from scratch. We study algebraic and topological properties for the preference magma that reflects the consumer. We also consider the influence of society on an individual's preference.

Keywords: factorization; magma; preference; product; society

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1. Introduction

A preference theory studies the key aspects of individual choices, focusing on their identification. It answers the questions of how to measure and determine how strong an individual's preferences are for various options. Additionally, it constructs appropriate functions for their representation, to facilitate decision-making and their subsequent analysis. Preference theory is crucial for economics and decision science. It allows us to model consumer behavior, determine optimal choice strategies, and predict the effects of economic policy, market changes, and other occurrences.

The revealed preference theory initiated by the American economist Paul Samuelson in [14], is a method of analyzing consumer choice, emphasizing the relationship between consumer behavior and preferences. It assumes that based on the decisions made by the consumer, their preferences can be determined, assuming that the consumer's behavior is rational. Revealed preferences allow for the creation of a utility function for the consumer, which the consumer reveals through their consumption decisions (assuming that preferences remain unchanged). More details can be found in the book [2], where the authors present the foundations and development of revealed preference theory, discussing its assumptions, axioms, theorems, and applications. Authors also demonstrate how this theory can be used to study various economic issues such as demand, welfare, equilibrium, auctions, games, and social decisions. Unfortunately, this theory is heavily criticized by many economists. Raising many questions. In particular, the following questions arise: Are the assumptions about rationality, full information, and consistency of consumer preferences realistic and adequate for describing real behavior? Is the utility function a good tool for measuring and comparing satisfaction from different baskets of goods? Are consumer preferences stable and independent of external factors such as social, cultural, and marketing environments? Is the Equivalent Budgetary Consumption Model sufficient to account for the impact of income on consumer preferences? Is the consumer able to compare all available choice options and choose the best one for themselves? Is the consumer aware of their needs and preferences, or are they shaped by the decision-making process? Is the consumer always selfish and maximizing their utility, or do they consider the utility of other entities? Is consumer consistent in their time preferences, or do they succumb to impulses and temptations? In the article [3], published in 2014, Dryzek criticizes the theory of revealed preferences, arguing that it is too narrow and limited to capture the complexity and diversity of human rationality. Author proposes an alternative concept of rationality based on pluralism of values, norms, and goals, rather than a uniform utility function. It is also worth noting the article [8], where authors analyze the relationship between the theory of revealed preferences and behavioral economics, which deals with the influence of psychological, social, and emotional factors on economic decisions. Authors point out the limitations and challenges that

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revealed preference theory poses to behavioral economics, as well as the opportunities and benefits that arise from combining these two approaches.

We join the group of researchers who disagree with classical revealed preference theory. We believe that the current theory may not capture their actual complexity, especially in the context of products with diverse properties and preference weights, and when preferences are nonlinear. Therefore, we propose an approach to their analysis using algebraic structures and topological properties, which may better capture the complexity and dynamics of consumer preferences and reflect human behavior in a real way. Our motivation is thus the need to develop mathematical tools that can more precisely model individuals' preferences in the context of products with various properties and weights of preference. We have taken our first tentative steps in [4]. In our work, we will focus on precisely defining a magma with preference as our consumer, on the impact of society, and beginning considerations on the change of preferences over time and their effects on consumer choices, which we will continue in the next papers.

The Section 2 focuses on exploring a magma with a preference relation (Definition 2.4), defining it as a set of elements in the form of pairs (p, w), where p represents the product, and w is the weight of the preference for this product. Let us recall that a set G together with an internal operation is called a magma. We defined the internal operation of adding preferences, which is in our view very close to consumer behavior. We also introduced a modified operation for not-loose products, taking into account the function δ. We also discuss the topological properties of the magma H, stating that with the topology of a linear order, it is a topological space (Proposition 2.12), regular (Corollary 2.17) and, under certain assumptions, also connected (Corollary 2.15).We also analyze properties regarding accumulation points (Proposition 2.18), compactness (Proposition 2.13) and local connectedness (Corollary 2.16), and it was also stated that H is an open domain by itself (Remark 2.21).

Recall that a square-free element is one that does not divide the square of a non-invertible element. A square-free factorization of an element means expressing that element as a finite product of square-free elements. The theory of square-free factorizations comes from research on the Jacobian conjecture in the algebraic context in the article [7]. The same authors explicitly began to explore the theory in [6] and later continued in [5]. Subsequently, Matysiak thoroughly explored the properties of square-free factorizations in [11,12]. The Section 3 explores the method of decomposing consumer preferences based on square-free factorizations, where each product attribute is analyzed as a square-free element. We also have attributes that arise as the product of two relatively prime such elements, and these are unique. The factorization of consumer preferences is described as a product of appropriate powers of product attributes, depending on the factorizations of types 1s, 2s, and 3s, that were described in the articles [11,12]. This section also presents practical examples **??**, illustrating how consumer preferences towards attributes such as price, quality, and brand can be interpreted in the context of square-free factorization. Finally, we discuss the meaning of square-free ideals and the semigroup of product features, which are useful for modeling the impact of individual attributes on consumer decisions, optimizing marketing strategies, and segmenting the market. Recall, that the set G with an internal and associative operation, is called a semigroup. Let us also recall, that an ideal I is called square-free if, for every x there is an implication, that if $x^2 \in I$, then $x \in I$. Square-free ideals have been defined and studied in the paper [13]. Also in the works [9,10], it turned out that the sums of square-free ideals form a topology called $Ł_2$ and many properties were investigated. And these properties can be used in the semigroup of product features.

In the last Section 4 we investigate the external influence on individual preferences by modeling it using a function: $E : H \times S \to H$, where S is a magma that is algebraically very similar to H, but applies to society. This function is defined as $E((p, w),(p, s)) = (p, \alpha w + (1 - \alpha)s)$, where α is a parameter determining the strength of the influence of society, and s is the average social preference for the product p (Definition 4.1). The investigation of the function E reveals its key properties, such as the fixed point of the function (Remark 4.4), indicating the state of balance of preferences, and its surjectivity (Proposition 4.6), which guarantees that every possible pair of preferences can be achieved.

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In Example 4.5 we have shown an example of its application. This function is not injective (see Example 4.7), but meets the Lipschitz condition with a constant equal to 1 (Proposition 4.8), which implies that changes in preferences are proportional to changes in the function's arguments. With the help of an appropriate homotopy function, we also show that changes in preferences can occur without sudden jumps. (Proposition 4.9). We also analyze the geometry of the function E (Proposition 4.10) and its gradient (Remark 4.11), which indicates the fastest and constant increase of preferences. Additionally, we pointed out the formulas for the area of the regular region of the function E (Proposition 4.12), for the change in its value over a time interval (Proposition 4.13) and the value of the function E in infinite time (Corollary 4.14). Example 4.15 pokazuje, demonstrates how these theoretical tools can be applied to analyze real changes preferences in different time contexts.

2. Individual Preferences as an Algebraic Structure

In this section, we define a consumer H as an algebraic structure consisting of products and their associated preference weights, and we study the algebraic and topological properties of this structure.

Let us begin with the definition of the preference relation we propose.

Definition 2.1. A relation $\preceq$ is called a preference relation if it is connected and reflexive.

Remark 2.2. Traditionally, the preference relation is additionally defined as a transitive relation. However, it happens that the condition $A \preceq B \preceq C$ does not entail $A \preceq C$, but it can entail $C \preceq A$.

Definition 2.3. A magma H is called a magma with a preference relation if in the magma H there is a preference relation for all elements of H.

Definition 2.4. Let us define a magma H with a preference. Let the elements be (p, w), where p is a product and w is a preference weight for product p. Let us define an inner action for any $n \in \mathbb{N}$:

$$\oplus : H^n \to H$$

as

$$(p_1, w_1) \oplus (p_2, w_2) \oplus \cdots \oplus (p_n, w_n) =$$
$$= \left(\frac{p_1 w_1 + p_2 w_2 + \cdots + p_n w_n}{w_1 + w_2 + \cdots + w_n}, \frac{w_1 + w_2 + \cdots + w_n}{n} \right).$$

Remark 2.5. In the above definition, the set of weights is any interval, but we can standardly use $[0, 1]$ (intervals of the form $[a, b]$ are isomorphic). It cannot be a set of type $\{0.0, 0.1, \ldots, 0.9, 1.0\}$ because the operation $\oplus$ would not be an inner operation.

Remark 2.6. The operation $\oplus$ is not associative. Therefore H cannot be a semigroup.

Remark 2.7. We define a preference relation in a magma H as $(p, w) \preceq (r, v) \Leftrightarrow w \leqslant v$.

Proposition 2.8. *In the magma H we have* $2(p, w) = (p, w)$.

Proof. $2(p, w) = (p, w) \oplus (p, w) = \left(\frac{pw + pw}{w + w}, \frac{w + w}{2} \right) = \left(\frac{2pw}{2w}, \frac{2w}{2} \right) = (p, w)$. □

Corollary 2.9. *In the magma H we cannot distinguish one neutral element. Each pair* (p, w) *is itself a neutral element.*

Remark 2.10. If $w \in [0, 1]$, then for each product p the element $(p, 0)$ is an irreducible and square-free (unique) element in H.

The result of $\oplus$ may sometimes suggest that considering two products that are not loose products indicates a preference for the part of the not-loose product. In reality, this is not the case. We can modify the operation of $\oplus$ for not-loose products as follows.

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Recall that any real number can be written in the form $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ denotes the integer part (floor) of x, and $\{x\}$ denotes the fractional part. Furthermore, the number $\lceil x \rceil$ is called the ceiling of x, i.e., the rounding of x up to an integer.

Let's define the δ function as follows:

$$\delta(x) = \begin{cases} \lfloor x \rfloor, & \text{where} \quad \{x\} \leqslant 0.5, \\ \lceil x \rceil, & \text{where} \quad \{x\} > 0.5 \end{cases}$$

Then, on H we can introduce a modified $\oplus'$ operation for non-loose products:

$$(p_1, w_1) \oplus' (p_2, w_2) = \left(\delta\left(\frac{w_1 p_1}{w_1 + w_2} \right) + \delta\left(\frac{w_2 p_2}{w_1 + w_2} \right), \delta\left(\frac{w_1 + w_2}{2} \right) \right),$$

with $\delta(w \cdot p) = p \cdot \delta(w)$, where p, p_1, p_2 are products, w, w_1, w_2 are preference weights. An analogous formula holds for any finite number of products.

Remark 2.11. The set H with the action $\oplus'$ is not a semigroup, but a magma. As a counterexample to verify it yourself, we propose to check that $((p_1, 0.3) \oplus' (p_2, 0.8)) \oplus' (p_3, 0.1) \neq (p_1, 0.3) \oplus' ((p_2, 0.8) \oplus' (p_3, 0.1))$.

The next results concern the study of the topological properties of the magma H.

Proposition 2.12. *The magma H with the linear order topology is a topological magma, that is, H with this topology is a topological space and the operation $\oplus$ is a continuous operation.*

Proof. The open sets in the linear order topology on the magma H are the sets of the form

$$T = \{(p, w) : \exists a < b : a < w < b\},$$

whereby we assume that the empty set is the set of products for which we have not decided in terms of preferences.

Now we show that the operation $\oplus$ is continuous. Let the sequences $\{(p_n, w_n)\}$ and $\{q_n, v_n\}$ belong to H such that the sequence $\{(p_n, w_n)\}$ converges to (p, w), and the sequence $\{(q_n, v_n)\}$ converges to (q, v). We will show that the sequence $\{(p_n, w_n) \oplus (q_n, v_n)\}$ converges to $(p, w) \oplus (q, v)$.

By assumption we have that for every $\epsilon > 0$ there exists N such that $n > N$ satisfying $|w_n - w| < \epsilon$ and $|v_n - v| < \epsilon$. Since the arithmetic average is a continuous operation, then

$$\left| \frac{w_n + v_n}{2} - \frac{w + v}{2} \right| = \left| \frac{w_n}{2} - \frac{w}{2} + \frac{v_n}{2} - \frac{v}{2} \right| \leqslant \left| \frac{w_n}{2} - \frac{w}{2} \right| + \left| \frac{v_n}{2} - \frac{v}{2} \right| < \epsilon$$

Similarly, weighted average is a continuous operation, so we have:

$$\left| \frac{p_n w_n + q_n v_n}{w_n + v_n} - \frac{pw + qv}{w + v} \right| < \epsilon.$$

Therefore the sequence $\{(p_n, w_n) \oplus (q_n, v_n)\}$ converges to $(p, w) \oplus (q, v)$. □

Proposition 2.13. *The magma H with its linear order topology is not a compact space.*

Example 2.14. For the sequence $(p_n, w_n) \in H$ there exists a subsequence that does not converge. The sequence (w_n) is a sequence of subintervals of the form (w_i, w_j) for $i < j$. For example, it has the form $(0.2, 0.4), (0.5, 0.6), \ldots$. For a convergent subsequence of (w_n) to exist, the limits (w_i) and (w_j) must be the same. This is not the case if $(w_i) \in (a, b)$ and $(w_j) \in (c, d)$ for $a < b < c < d$.

Corollary 2.15. *The magma H with the linear order topology is a connected space for $n \neq 2$, where $|P| = n$ (P is the set of products).*

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Proof. By the definition of an open set as a neighborhood of a point p, for $|P| > 2$ we cannot factorize it into two open sets, because each open set contains only one p and its selected weights. So in this case it is a connected space. The space H for $n = 2$ is not a connected space, because we can decompose it into the union of two disjoint open sets:

$$S_1 = \{(p_1; w_1) : w_1 \in (a, b)\},$$

$$S_2 = \{(p_2; w_2); w_2 \in (c; d)\},$$

where a, c and b, d are the smallest and largest possible weight values for a given product, respectively. We have $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = H$. However, if the space consists of only one product, $H = \{(p, w) : w \in (a, b)\}$ such a decomposition is also not possible, because it would be based on a partition of the interval (a, b), more precisely into (a, c) and (c, b) for any $c \in (a, b)$, but then the pair with weight c does not belong to H. For this case the space is also connected. □

Corollary 2.16. *In the magma H with the linear order topology every point has an arbitrarily small connected neighborhood.*

Proof. Let $E = \{(p, w) : w \in (a - \frac{\epsilon}{k}; a + \frac{\epsilon}{k})\}$ for every $\epsilon > 0$ and for every $a \in \mathbb{R}$ and $k \in \{2, 3, 4, \dots\}$. Assume that E is not a connected set, i.e. it can be decomposed into the union of two nonempty disjoint sets E_1, E_2 such that $E_1 \cup E_2 = E$. Since, in order for E_1, $E_2 \subset E$, the first coordinate of the sets of pairs E_1 and E_2 must contain the same a as in the set E. The set $(a - \frac{\epsilon}{k}, a + \frac{\epsilon}{k})$ can be maximally partitioned into sets $(a - \frac{\epsilon}{k}; c)$ and $(c; a + \frac{\epsilon}{k})$ for each $c \in (a - \frac{\epsilon}{k}; a + \frac{\epsilon}{k})$, but then $c \notin (a - \frac{\epsilon}{k}; a + \frac{\epsilon}{k})$. Otherwise they are not open sets. Therefore E is a connected set. Moreover, $\operatorname{Int} E = E$ and $diam(E) = \sup\{d((p_1, w_1), (p_2, w_2)) : (p_1, w_1), (p_2, w_2) \in E\} < \epsilon$. This means that the space H is a locally connected space, so every point (p, w) has an arbitrarily small connected neighborhood. □

Corollary 2.17. *The space H with the linear order topology is a regular space T_3.*

Proof. We first show that H is a T_1-space. Consider two points (p_i, w_i), $(p_j, w_j) \in H$, which are distinct. If $p_i \neq p_j$, by the definition of our open set we have that $(p_i, w_i) \in U$ and $(p_j, w_j) \notin U$. If $p_i = p_j$ and w_j is an interval (c, d) then an open set V such that $(p_i, w_i) \in U$ and $(p_j, w_i) \notin U$ is the open set $\{(p_i, w_i); w_i \in (a, b)\}$ for any $a < b < c < d$.

Closed sets in H are of the following form:

$$T' = \{(p, w) : \exists c < d : w \in [c, d]\}$$

for some product p.

Now we will show that for any point $(p_i, w_i) \in H$ and a closed subset $A \subset H$ such that $(p_i, w_i) \notin A$, there exist open sets $U, V \subset H$ such that $(p_i, w_i) \in U$ and $A \in V$.

Let us be given a closed set $A = \{(p_j, w_j) : w_j \in [c; d]\}$, then for every (p_i, w_i) this point does not belong to A. There exist open sets U, V such that

$$U = \{(p_i, w_i) : w_i \in (a; b)\}$$

to which (p_i, w_i) belongs, and the open set

$$V = \{(p_j, w_j) : w_j \in (f; g)\}$$

if $c < d < f < g$. Which proves that H is a regular space.
□

Proposition 2.18. *In the space H with linear order topology every point is an accumulation point and $H^d = H$.*

Proof. In our case, the neighborhood of the point $(p, w) \in H$ is described as a set (open set):

$$O = \{(p, w) \in H : \exists_{a<b}\, a < w < b\}.$$

Therefore, a point $(p, w_0) \in O$ if and only if there exist a, b such that $a < w_0 < b$. In linear order, for any point we will always find other points in its neighborhood. More precisely, for every point (p, w_0) and neighborhood O there exists a point (p, w) such that $(p, w) \neq (p, w_0)$, where $(p, w) \in H$. Since each point is an accumulation point, the derivative of this set H^d is equal to this set. □

Remark 2.19. The space H with the linear order topology has no isolated points.

Remark 2.20. The closed sets in H are perfect sets, since every set is everywhere dense.

Remark 2.21. In the set H every open set O satisfies the property $O = \mathrm{Int}(\mathrm{cl}\, O)$. Therefore $D(H)$ is the family of all open sets in H.???

Conjecture 2.22. The set H together with appropriately defined operations of logical addition and multiplication and complement on open sets forms a Boolean algebra.

3. Preference Distribution as Square-Free Factorizations

In this section we show applications of square-free factorizations, which were described in the articles [11,12], to the theory of consumer preferences.

Let P denote a given product. Let $K_1(P), K_2(P), \ldots, K_n(P)$ denote certain features of product P, e.g. price, quality, brand, etc. Let $w_{K_1}, w_{K_2}, \ldots, w_{K_n}$ denote the preference weight for feature K_i $(i = 1, 2, \ldots, n)$ in the interval $[0, 1]$, where $w_{K_1} + w_{K_2} + \cdots + w_{K_n} = 1$. Let $W(P)$ denote the overall weight of the consumer's preference for product P given by the formula:

$$W(P) = K_1(P) \cdot w_{K_1} + K_2(P) \cdot w_{K_2} + \cdots + K_n(P) \cdot w_{K_n}.$$

Let us introduce two simple lemmas.

Lemma 3.1. *The map*

$$W(P) \mapsto W(P)^* = K_1(P)^{w_{K_1}} K_2(P)^{w_{K_2}} \ldots K_n(P)^{w_{K_n}}$$

is isomorphism.

Lemma 3.2. *The map*

$$W(P)^* \mapsto W_{10}(P)^* = K_1(P)^{10 w_{K_1}} K_2(P)^{10 w_{K_2}} \ldots K_n(P)^{10 w_{K_n}}$$

is isomorphism.

Features $K_1(P), K_2(P), \ldots, K_n(P)$ are unique features. Of course, one can forcibly decompose each feature into even finer criteria (if possible), but here let us limit ourselves to those typical product features that have the nature of a square-free element in classical algebra. By Lemma 1.1.3 in [11] the product of two relatively prime square-free elements is also square-free, so the combination of features $K_i(P)K_j(P)$ $(i \neq j)$ can also be a unique feature. In reality, this is exactly what happens, because a person choosing a product may be guided primarily by price, and then quality. But another person may look at both price and quality at the same time - that is, whether the quality of the product is adequate to the price.

Let P be a product considered by some consumer. Let $K_1(P), K_2(P), \ldots, K_n(P)$ be product features. Then, after rescaling feature preferences (Lemma 3.1) and changing from additive to multiplicative version (Lemma 3.2), the product preference weight will be given by:

$$W(P)^* = K_1(P)^{w_{K_1}} K_2(P)^{w_{K_2}} \ldots K_n(P)^{w_{K_n}},$$

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where $w_{K_1} > w_{K_2} > \cdots > w_{K_n}$.

This factorization shows that the most important feature for the consumer is $K_1(P)$ as a separate feature. This is a square-free factorization of type $1s$ (see Section 1.1.5, [11]). By Proposition 1.1.29, [11] the above factorization can be transformed to the remaining types of factorization (type $2s$, $3s$ (see Section 1.1.5, [11]).

$$W(P)^* = L_1(P)^{v_{L_1}} L_2(P)^{v_{L_2}} \ldots L_n(P)^{v_{L_n}},$$

where $L_i(P)$ is the i-th feature of the product (Note: it can be a combined feature, e.g. price and quality at the same time), where for each i we have $L_i(P) \mid L_{i+1}(P)$, v_{L_i} is a given preference for feature L_i, with $i = 1, 2, \ldots, n$. By divisibility of features $L_i(P) \mid L_{i+1}(P)$ we mean that feature $L_{i+1}(P)$ contains feature $L_i(P)$ and possibly another, e.g. price "is a divisor" of price and quality. This factorization shows that for the consumer the most important criteria are such features L_i for which exponents v_{L_i} are the largest. This is a factorization of type $2s$.

$$W(P)^* = M_1(P)^{u_{M_1}} M_2(P)^{u_{M_2}} \ldots L_m(P)^{u_{M_m}},$$

where $u_{M_1} > u_{M_2} > \cdots > u_{M_m}$ are powers of two and $M_i(P)$ is a given product feature (Note: it can be a combined feature), u_{M_i} is a given preference for feature M_i, where $i = 1, 2, \ldots, m$. This factorization shows that the most important feature for the consumer is $M_1(P)$, and he prefers a balance between the other features. This is a $3s$ type factorization.

In theory, there is no equivalence between these factorizations in consumer preferences, but let's look at a simple example.

Example 3.3. Let P be a given product. Let $K_1(P) = C(P) = C$ denote the price criterion of product P, $K_2(P) = J(P) = J$ denote the quality criterion of product and let $K_3(P) = M(P) = M$ denote the brand criterion of product. Let's assume that for the consumer the quality criterion at the level of 0.7 is most important, then the brand criterion at the level of 0.5, and finally the price criterion at the level of 0.3. The consumer states that the price is satisfactory at the level of $w_C = 0.5$, the quality at the level of $w_J = 0.3$, and the brand at the level of $w_M = 0.2$.

Note that $w_C + w_J + w_M = 1$. We calculate $W(PL)$. We have:
$W(P) = C(P) \cdot w_C + J(P) \cdot w_J + M(P) \cdot w_M = 0.7 \cdot 0.5 + 0.5 \cdot 0.3 + 0.3 \cdot 0.2 = 0.46$.

Therefore, the consumer prefers the product at the level of 0.46.

Using Lemmas 3.1 and 3.2 we have the following factorization:

$$W(P)^* = C(P)^5 \cdot J(P)^3 \cdot M(P)^2.$$

By Proposition 1.1.29 of [11] we have the following equivalent preference factorizations (of types $1s$, $2s$, and $3s$):
$W(P)^* = C^5 \cdot J^3 \cdot M^2 = (CJM)^2 \cdot (CJ) \cdot C^2 = C^4 \cdot (JM)^2 \cdot (CJ)$.

Let's look at the highest powers in each of these factorizations.

For the $1s$ type, the most important criterion is the price criterion. For the $2s$ type, the most important criterion is the price-quality-brand and price criterion. For the $3s$ type, the most important criterion is price. In each case, we see that the consumer is primarily guided by the price of the product, and then takes into account the other features of the product.

So the economic analysis can be as follows for the appropriate factorizations:

(1s) Price is the most important feature for the consumer.
(2s) Since price is the most important, the consumer can consider either price alone or price, quality, and brand at the same time.
(3s) Price is very important; quality and brand are important together, price and quality are also important.

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Price is a key factor, but the consumer also values combinations of quality and brand, and price and quality. The consumer prefers a balance between these attributes, but price is the dominant attribute.

From the above considerations it follows that the set of product features with the operation of combining features is a commutative semigroup. Moreover, feature $*$ feature $=$ feature^2 $=$ feature. This means that this semigroup does not have a unique neutral element. The semigroup of product features can be written in the form:

$$< K_1(P), K_2(P), \ldots, K_n(P) >,$$

i.e., the semigroup generated by the features of the product P. Moreover, the ideal generated by the feature $K_i(P)$ contains the feature $K_i(P)$ and all possible products of features "dividing" by $K_i(P)$ for $i = 1, 2, \ldots, n$. The ideal generated by $K_i(P)$ by Proposition 1.1.8, [11] is a square-free ideal. More information on the properties of square-free ideals can be found in [13].

As we mentioned earlier, product features can be treated as a square-free element. In this situation, the ideal generated by one feature is a square-free ideal. Indeed, by definition, we choose feature K from the semigroup of features and assume that $2K$ belongs to this ideal. Of course, the same feature twice is the same feature, so K belongs to this ideal. Moreover, for each feature K, $2K = K$ holds, so the ideal generated by feature K is idempotent. The economic interpretation of the ideal generated by a feature such as price can be considered in terms of the impact of that feature on other product features and on consumer decisions. For example, let the ideal I be generated by feature K, i.e., $I =< K >$. The ideal generated by K includes all combinations of features that contain feature K. That is, feature K is a key element in each of these combinations. Economically, this can be interpreted as follows:

(1) Feature K as a dominant factor: Feature K is the main factor influencing the perception of the product. Regardless of other features, feature K always plays a significant role in consumer decisions.
(2) Feature Interactions K: K feature combined with other features creates different market segments. For example, “price and quality” may represent premium products, while “price and brand” may refer to branded products at different price points.
(3) Square-free: The square-free ideal means that feature K cannot be doubled in the algebraic sense ($K * K = K$). Economically, this may suggest that doubling feature K does not fundamentally change the perception of the product - feature K is already fully factored into the consumer’s decision.
(4) Marketing Strategies: Companies can use this knowledge to create marketing strategies that emphasize the feature K in combination with other features to better segment the market and tailor offerings to different consumer groups.

Example 3.4. Let P be some product that the consumer is considering. Assume that the consumer is guided by three attributes toward product P: price, quality, brand. Furthermore, the consumer rates the importance of the attributes as follows: price $3/10$, quality $7/10$, brand $5/10$.

So the semigroup of product features P is the semigroup generated by price, quality, and brand. Consider the ideal I generated by price. Hence $I = \{$price, price and quality, price and brand, price and quality and brand $\}$.

Let us interpret the previously mentioned aspects.

(1) Price as a dominant factor.
In this case, price is less important $(3/10)$, which means that the consumer does not attach much importance to price when deciding to buy product P.
(2) Price interactions.

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(a) Price and quality.
The consumer rates quality at 7/10, which suggests that quality is important to them. Even if the price is low, high quality can persuade the consumer to buy.

(b) Price and brand.
Brand is of medium importance (5/10). A consumer may be willing to pay more for a branded product, but it is not a decisive factor.

(3) Square-free.
Price is not doubled, which means that its impact on the consumer's decision is already fully factored in. Even if the price is low, other features (quality, brand) have a greater impact on the decision.

Let's assume that product P is a smartphone. The consumer evaluates: price: 3/10 (low importance), quality: 7/10 (high importance), brand: 5/10 (medium importance).

Scenario 1: Budget smartphone:

Price and quality: A consumer can choose a budget smartphone with high quality, even if the brand is not known. The importance of quality (7/10) outweighs the low price (3/10).

Scenario 2: Branded smartphone:

Price and Brand: A consumer may be willing to pay more for a smartphone with a well-known brand, but quality will still be a key factor. Brand (5/10) and quality (7/10) together can convince a consumer to make a purchase.

In summary, the square-free ideal in a semigroup of product features means that no feature can be "duplicated" in the algebraic sense. In practice, this means that the impact of a given feature on consumer decisions is already fully taken into account and cannot be increased by repeating it.

An economist can use square-free ideals to model how different product attributes affect consumer decisions. For example, if price is already factored into a consumer's decision, doubling its importance will not change that decision.

Companies can use this knowledge to optimize their marketing strategies. Knowing that certain features (e.g. price) have limited impact, they can focus on other features (e.g. quality, brand) to better attract consumers.

Square-free ideals can help segment markets. For example, if price is a key factor for one group of consumers and quality for another, companies can tailor their products and marketing campaigns to those segments.

The product feature semigroup is a semigroup in which all elements are square-free. If we introduce the topology $Ł_1$ or $Ł_2$ (see [9]) into this semigroup, then the product feature semigroup becomes a topological space, where the open sets are the sums of idempotent/square-free ideals, i.e. the sums of ideals generated by some features.

4. Social Influence on Individual Preference

In this section, we consider our proposed mathematical model of the impact of society on consumer preference for a given product.

Definition 4.1. Let S be a magma with preference describing society (as defined 2.4). Let $E : H \times S \to H$ be an external operation for the magma H, which influences consumer preference in H, given by the formula

$$E((p, w), (p, s)) = (p, \alpha w + (1 - \alpha)s),$$

where s is the average societal preference for the product p, α is a parameter that determines the strength of society's influence on individual preferences.

Remark 4.2. If $\alpha = 1$, then individual preferences remain unchanged. If $\alpha = 0$, then the individual preference is completely replaced by the societal preference.

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Remark 4.3. With multiple applications of the function E, the values of the function converge to (p, s). This means increasing pressure from society.

Remark 4.4. A fixed point of the function E is given by the pair $(p, w) \in H \times S$, which satisfies the condition $w = s$.

Proof. In our case, a fixed point is a point that satisfies the equation $E((p, w), (p, s)) = (p, w)$. Substituting we get:

$$(p, w) = (p, \alpha w + (1 - \alpha)s)$$

By comparing the components, we obtain the equalities $p = p$, which is always true, and $\alpha w + (1 - \alpha)s = w$. Solving the second equation for s we get: $(1 - \alpha)s = w(1 - \alpha)$, which gives us $s = w$. □

A fixed point of the function E represents a state of balance in which an individual adjusts their preferences to match those of society.

Example 4.5. Let p be a certain product that individual H prefers at the level of 0.7. Whereas society prefers it at the level of 0.1. Let's assume that individual H is influenced by society at a level of 0.3. Then $\alpha = 1 - 0.3 = 0.7$. According to the external operation, we have:
$E((p, 0.7), (p, 0.3)) = (p, 0.7 \cdot 0.7 + (1 - 0.7) \cdot 0.1) = (p, 0.49 + 0.03) = (p, 0.52)$.

Thus, due to the influence of society, individual H prefers product p at the level of 0.52.

Proposition 4.6. *Function E is a surjection, which means that any possible combination of preferences can be represented as the appropriate setting of individual preferences and the average preferences of society.*

Proof. For all $(p', w') \in H$ we can find such a pair $((p, w), (p, s)) \in H \times S$, że $E((p, w), (p, s)) = (p', w')$. In order for the given equality to hold, we must assume:

$$p' = p \text{ and } w = \frac{w' - (1 - \alpha)s}{\alpha},$$

for $\alpha \neq 0$. In case $\alpha = 0$ the pair we are looking for from the set $H \times S$ is (p, s). For $\alpha = 1$ it is sufficient to assume $w' = w$. □

Example 4.7. A function E is not an injective. For $\alpha = 0.5$ and arguments $((p_1, 1), (p_1, 0))$ and $((p_1, 0), (p_1, 1)) \in H \times S$ the values of the function are equal 0.5 In both cases, this means that the function is not injective.

Proposition 4.8. *The function E describing the influence on the preferences of an individual in H is a Lipschitz function with a constant equal to* 1.

Proof. Let $((p_1, w_1), (p_1, s_1))$ and $((p_2, w_2), (p_2, s_2))$ will be arbitrary points in the space $H \times S$. Let us define the norm $||\cdot||$ as $||(a, b)|| = ||a|| + ||b||$. We have:

$$\begin{aligned} &||E((p_1, w_1), (p_1, s_1)) - E((p_2, w_2), (p_2, s_2))|| = \\ &= ||((p_1, \alpha w_1 + (1 - \alpha)s_1)) - (p_2, \alpha w_2 + (1 - \alpha)s_2)|| = \\ &= ||(p_1 - p_2, \alpha(w_1 - w_2) + (1 - \alpha)(s_1 - s_2))|| = \\ &= ||p_1 - p_2|| + ||\alpha(w_1 - w_2) + (1 - \alpha)(s_1 - s_2)|| \end{aligned}$$

which we can estimate as follows:

$$\begin{aligned} &||p_1 - p_2|| + ||\alpha(w_1 - w_2) + (1 - \alpha)(s_1 - s_2)|| \leq \\ &\leq ||p_1 - p_2|| + \alpha||w_1 - w_2|| + (1 - \alpha)||s_1 - s_2||. \end{aligned}$$

We can write out the difference between the arguments as follows:

$$||((p_1, w_1), (p_1, s_1)) - ((p_2, w_2), (p_2, s_2))|| = ||p_1 - p_2|| + ||w_1 - w_2|| + ||s_1 - s_2||.$$

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We can take the following estimate:

$$||p_1 - p_2|| + \alpha||w_1 - w_2|| + (1-\alpha)||s_1 - s_2|| \leq$$
$$\leq L||p_1 - p_2|| + ||w_1 - w_2|| + ||s_1 - s_2||,$$

for the constant $L = 1$, which indicates that the function E satisfies the Lipschitz condition in the form:
$||E((p_1, w_1),(p_1, s_1)) - E((p_1, w_1),(p_2, s_2))|| \leq$
$||((p_1, w_1),(p_1, s_1)) - ((p_2, w_2),(p_2, s_2))||.$ □

In practice, this means that small changes in the individual's preferences or in the average preferences of society lead to proportionally small changes in the values of the function E.

Proposition 4.9. *Let $f, g : H \to H$, be functions such tha $f(p, w) = (p, w)$ and $g(p, w) = (p, s)$. For function $F : H \times S \times [0, 1] \to H$, defined by the formula $F(((p, w),(p, s)), \alpha) = ((p,(1-\alpha)w + \alpha s)$, we have $f \overset{H}{\sim} g$.*

The homotopy indicates that the transition from the function f to g can be carried out continuously. This means that changes in preferences can occur without sudden jumps (f and g belong to the same homotopy class).

Proof. A function f is a constant function that does not change preferences, while the function g sets the individual's preferences to the average value We have:

$$F((p, w),(p, s), 0) = (p, w) \text{ i } F((p, w),(p, s), 1) = (p, s)$$

Function F is a homotopy between the functions f and g, becouse:

$$F((p, w),(p, s), 0) = f(p, w) \text{ i } F((p, w),(p, s), 1) = g(p, w).$$

□

In the following propositions, let us recall from 4.1, that α is a parameter that determines the strength of society's influence on an individual's preferences.

Proposition 4.10. *In the magma H for a given product p the pair (p, α) is the tangent of the angle of inclination of the tangent line to the curve obtained as a result of intersecting the graph of the function E with the plane $y = s$ at the point $((p, w),(p, s), E((p, w),(p, s)))$ with the xOy plane. Whereas, the pair $(p, 1-\alpha)$ is the tangent of the angle of inclination of the tangent line to the curve obtained as a result of intersecting the graph of the function E with the plane $x = w$ at the point $((p, w),(p, s), E((p, w),(p, s)))$ with the xOy plane.*

Proof. Calculate the partial derivatives of the function E:

$$\frac{\partial E}{\partial(p, w)} = (p, \alpha),$$

$$\frac{\partial E}{\partial(p, s)} = (p, 1-\alpha).$$

The thesis statement continues from the geometric interpretation of the partial derivatives of the function E. □

Remark 4.11. A vector $grad\ E = ((p, \alpha),(p, 1-\alpha))$ is the direction of the fastest growth of the function at every point $((p, w),(p, s))$. The gradient $grad\ E$ is perpendicular to the level curve passing through that point. The direction and size of the gradient do not change depending on the location in the

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space $((p,w),(p,s))$. This suggests that the function whose gradient we are considering has a constant direction of fastest growth, independently of the point at which we are located.

Proposition 4.12. *The area of the regular region of the function E is given by:*

$$\alpha s \frac{w^2}{2} + (1-\alpha) w \frac{s^2}{2}.$$

Proof. Calculate the following integral:

$$\int_{(p,w)} \int_{(p,s)} (\alpha w + (1-\alpha)s)\, d(p,s)\, d(p,w) =$$
$$= \int_{(p,w)} \left(\alpha w s + (1-\alpha)\frac{s^2}{2} \right) d(p,w) = \alpha s \frac{w^2}{2} + (1-\alpha) w \frac{s^2}{2}.$$

□

Proposition 4.13. *The change in the value of the function E over the time interval $[t_1, t_2]$ is given by the formula:*

$$\alpha \int_{t_1}^{t_2} w(t)\, dt + (1-\alpha) \int_{t_1}^{t_2} s(t)\, dt,$$

where $w(t)$ is a preference weight function dependent on time t, $s(t)$ is a function of average societal weights dependent on time t.

Proof. Consider the function E over the time t, i.e., $E((p,w)(t), s(t)) = (p, \alpha w(t) + (1-\alpha)s(t))$. We just need to calculate the integral:

$$\int_{t_1}^{t_2} (\alpha w(t) + (1-\alpha)s(t))\, dt = \alpha \int_{t_1}^{t_2} w(t)\, dt + (1-\alpha) \int_{t_1}^{t_2} s(t)\, dt. \quad \square$$

Corollary 4.14. *Using Proposition 4.13 we can simulate the value of the function E at infinite time. It is enough to apply the formula:*

$$\lim_{T \to \infty} \left(\alpha \int_{t_1}^{T} w(t)\, dt + (1-\alpha) \int_{t_1}^{T} s(t)\, dt \right).$$

Proof.

$$\int_{t_1}^{\infty} (\alpha w(t) + (1-\alpha)s(t))\, dt =$$
$$= \lim_{T \to \infty} \left(\alpha \int_{t_1}^{T} w(t)\, dt + (1-\alpha) \int_{t_1}^{T} s(t)\, dt \right).$$

□

Example 4.15. Let's assume that $\alpha = \frac{1}{3}$ and $w(t) = \frac{1}{1+e^{-t}}$, $s(t) = \sin^2 \frac{\pi t}{2}$. We will calculate how the value of the function E changed between the second and third year.

Using the formula from Proposition 4.13 we have:

$$\frac{1}{3}\int_{2}^{3}\left(\frac{1}{1+e^{-t}}\right)dt+\frac{2}{3}\int_{2}^{3}\left(\sin^2\frac{\pi t}{2}\right)dt=$$
$$=\frac{1}{3}(-\ln|1+e^{-3}|+\ln|1+e^{-2}|)+$$
$$+\frac{2}{3}\left(\frac{1}{2}\cdot 3-\frac{1}{2\pi}\sin(3\pi)-\frac{1}{2}\cdot 2-\frac{1}{2\pi}\sin(2\pi)\right)=0.35945.$$

In year 2 we have $w(2)=0.8808$, $s(2)=0$, so the preferences of society did not affect the individual's preferences at all, where the preference level for product p is 0.8808. Whereas in year 3 we have $w(3)=0.9526$, $s(3)=1$, so with a very strong influence of society's preferences, the individual's preference for product p increased. In the time interval between the second and third year, the value of the function E, representing the individual's preference under the influence of societal preferences, changed by 0.35945.

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