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Bang-Bang Property and Time Optimal Control For Caputo Fractional Differential Systems

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Abstract: Fractional differential systems have received much attention in recent decades likely due to its powerful ability in modeling memory processes, which are mostly observed in real world. Fractional models have been widely investigated and applied in many fields such as physics, biochemistry, electrical engineering, continuum and statistical mechanics. It is shown by many researchers that fractional derivatives can provide more accurate models than integer order derivatives do. Obviously, bang-bang property is of a significant importance in optimal control theory. In this paper a Bang-Bang property and time-optimal control problem for time-Fractional Differential System (FDS) is under consideration. First we formulate our problem and we prove the existence theorem. We state and prove the bang-bang theorem. Finally we give the optimality conditions that characterized the optimal control. Some illustrate applications are given to clarify our results.

Keywords: time optimal control problems; bang bang theorem; fractional optimal control; dirichlet parabolic equations; classical control theory; optimality conditions; riemann-liouville and caputo's derivatives

MSC: Primary: 26A33, 49J20; Secondary 35R11, 49J15, 49K20, 45J20, 45D10

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1. Introduction

Fractional calculus is one of the most novel types of calculus which has a wide range of applications in many different scientific and engineering disciplines. Order of the derivatives in the fractional calculus can be any real number which separates the fractional calculus from the ordinary calculus. Therefore, fractional calculus can be considered as an generalization of ordinary calculus. Fractional order differential equations have been successfully appeared in modeling of many different problems for different applications of derivatives and integrals of fractional order in classical mechanics, quantum mechanics, g image processing, earthquake engineering, biomedical engineering, physics, nuclear physics, hadron spectroscopy, viscoelasticity and bioengineering and many others. During the last few decades, the fractional optimal control theory for partial differential equations has a wide range of applications in science, engineering, economics, and some other fields see ([3–15,17–29,32–41] and references therein).

Obviously, bang-bang property is of a significant importance in optimal control theory as stated in [23,31]. Especially, the bang-bang property for certain time optimal controls governed by parabolic equations can be given by using Pontryagin's maximum principle see [42]. In [37], Phung et al. studied the bang-bang property for time optimal controls governed by semi-linear heat equation in abounded domain with control acting locally in a subset. In [20], Chen et al. studied time-varying bang-bang property of time optimal controls for heat equation and showed the applicable side of it.

With the growing number of applications of time-optimal control of fractional problems (TOCFP), it is necessary to establish some (TOCFP), this is the motivation behind this paper.

In this paper a Bang-Bang property and time-optimal control problem for time-(FDS) is under consideration. The fractional time derivative is studied in the Caputo sense. A time optimal control problem is exchanged by an equivalent problem with a performance index in the integral form. Firstly, the existence and the uniqueness of the solution of the fractional differential system in a Hilbert space are investigated. Then we prove that the considered optimal control problem has a unique solution. Constraints on controls are presumed. The Bang Bang theorem is derived. To achieve the optimality conditions for the Dirichlet problem, classical control theorem given by Lions [31] is applied. Some illustrate examples are analyzed.

This paper is organized as follows: In the second section, we introduce some fractional operators and basic definitions which we used in this work. In the third section, we compose the time optimal control problem for fractional systems and then introduce the major results of this paper. In the fourth section, we state and prove the existence theorem. In the fifth section we state and prove the Bang-Bang Theorem. In the sixth section, we prove the optimality conditions. In the seventh section, we introduce some applications and illustrate examples for this problem. Finally, conclusions are formed in the eighth section.

2. Basic Definitions

The fractional derivative is not a new concept. In fact, 24 years after the invention of calculus, it first appears in a letter from Gottfried Wilhelm Leibniz to Guillaume de l'Hopital. It is actually a kind of generalization of the integer derivative. There are many fractional derivatives in the literature, such as Grünwald-Letnikov (GL) derivative, Riemann-Liouville (RL) fractional derivative, Caputo fractional derivative, Katugampola derivative, local fractional operator, Risez fractional derivative and tempered fractional derivative. Amongst all fractional derivatives, the most two widely used generalizations are RL and Caputo fractional derivatives which are powerful enough for modeling most memory processes.

In this section we introduce some basic definitions related to fractional derivatives see [1–6,28,29].

Definition 2.1. *The Left Riemann-Liouville Fractional Integral* and *The Right Riemann-Liouville Fractional Integral* are presented respectively by

$$ {}_aI^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \tag{2.1}$$

$$ I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau, \tag{2.2}$$

where $\alpha > 0,\ n-1 < \alpha < n$. From now on, $\Gamma(\alpha)$ represents the Gamma function.
The Left Riemann-Liouville Fractional Derivative is given by

$$ {}_aD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau. \tag{2.3}$$

The Right Riemann-Liouville Fractional Derivative is defined by

$$ D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)d\tau. \tag{2.4}$$

The fractional derivative of a constant takes the form

$$ {}_aD^\alpha C = C\,\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{2.5}$$

and the fractional derivative of a power of t has the following form

$$ {}_aD^\alpha (t-a)^\beta = \frac{\Gamma(\alpha+1)(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, \tag{2.6}$$

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for $\beta > -1, \alpha \geq 0$.

The Caputo's fractional derivatives are defined as follows:

The Left Caputo Fractional Derivative

$$ {}_{a}^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^{n}f(\tau)d\tau, \tag{2.7}$$

and

The Right Caputo Fractional Derivative

$$ {}^{C}D_{b}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}(\tau-t)^{n-\alpha-1}\left(-\frac{d}{d\tau}\right)^{n}f(\tau)d\tau, \tag{2.8}$$

where α represents the order of the derivative such that $n-1 < \alpha < n$. By definition the Caputo fractional derivative of a constant is zero.

Remark 2.1. The Riemann-Liouville fractional derivatives and Caputo fractional derivatives are connected with each other by the following relations.

$$ {}_{a}^{C}D^{\alpha}f(t) = {}_{a}D^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, \tag{2.9}$$

$$ {}^{C}D_{b}^{\alpha}f(t) = D_{b}^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{(-1)^{k}f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}. \tag{2.10}$$

In [1], a formula for the fractional integration by parts on the whole interval $[a,b]$ was given by the following lemma

Lemma 2.2. ***(Integration by parts)*** *Let $\alpha > 0$, $p,q \geq 1$, and $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p}+\frac{1}{q} = 1+\alpha$)*

(a) If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then

$$\int_{a}^{b}\varphi(t)({}_{a}I^{\alpha}\psi)(t)dt = \int_{a}^{b}\psi(t)(I_{b}^{\alpha}\varphi)(t)dt. \tag{2.11}$$

(b) If $g \in I_{b}^{\alpha}(L_p)$ and $f \in {}_{a}I^{\alpha}(L_q)$, then

$$\int_{a}^{b}g(t)({}_{a}D^{\alpha}f)(t)dt = \int_{a}^{b}f(t)(D_{b}^{\alpha}g)(t)dt, \tag{2.12}$$

where ${}_{a}I^{\alpha}(L_p) := \{f : f = {}_{a}I^{\alpha}g, g \in L_p(a,b)\}$ and $I_{b}^{\alpha}(L_p) := \{f : f = I_{b}^{\alpha}g, g \in L_p(a,b)\}$.

In [1,28,29], other formulas for the fractional integration by parts on the subintervals $[a,r]$ and $[r,b]$ were given by the following lemmas.

Lemma 2.3. *Let $\alpha > 0$, $p,q \geq 1$, $r \in (a,b)$ and $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p}+\frac{1}{q} = 1+\alpha$).*

(a)If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then

$$\int_{a}^{r}\varphi(t)({}_{a}I^{\alpha}\psi)(t)dt = \int_{a}^{r}\psi(t)(I_{r}^{\alpha}\varphi)(t)dt, \tag{2.13}$$

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and thus if $g \in I_r^\alpha(L_p)$ *and* $f \in {}_aI^\alpha(L_q)$*, then*

$$\int_a^r g(t)({}_aD^\alpha f)(t)dt = \int_a^r f(t)(D_r^\alpha g)(t)dt, \tag{2.14}$$

(b)If $\varphi \in L_p(a,b)$ *and* $\psi \in L_q(a,b)$*, then*

$$\int_r^b \varphi(t)({}_aI^\alpha \psi)(t)dt = \int_r^b \psi(t)(I_b^\alpha \varphi)(t)dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^r \psi(t)(\int_r^b \varphi(s)(s-t)^{\alpha-1}ds)dt, \tag{2.15}$$

and hence if $g \in I_b^\alpha(L_p)$ *and* $f \in {}_aI^\alpha(L_q)$*, then*

$$\int_r^b g(t)({}_aD^\alpha f)(t)dt = \int_r^b f(t)(D_b^\alpha g)(t)dt$$
$$-\frac{1}{\Gamma(\alpha)}\int_a^r ({}_aD^\alpha f)(t)(\int_r^b (D_b^\alpha g)(s)(s-t)^{\alpha-1}ds)dt. \tag{2.16}$$

That is

$$\int_r^b g(t)({}_aD^\alpha f)(t)dt = \int_r^b f(t)(D_b^\alpha g)(t)dt$$
$$-\frac{1}{\Gamma(\alpha)}\int_a^r f(t)D_r^\alpha(\int_r^b (D_b^\alpha g)(s)(s-t)^{\alpha-1}ds)dt. \tag{2.17}$$

Lemma 2.4. *Let* $\alpha > 0$, $p,q \geq 1$, $r \in (a,b)$ *and* $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ *(* $p \neq 1$ *and* $q \neq 1$ *in the case when* $\frac{1}{p}+\frac{1}{q} = 1+\alpha$ *).*
(a)If $\varphi \in L_p(a,b)$ *and* $\psi \in L_q(a,b)$*, then*

$$\int_r^b \varphi(t)(I_b^\alpha \psi)(t)dt = \int_r^b \psi(t)({}_rI^\alpha \varphi)(t)dt, \tag{2.18}$$

and thus if $g \in {}_rI^\alpha(L_p)$ *and* $f \in I_b^\alpha(L_q)$*, then*

$$\int_r^b g(t)(D_b^\alpha f)(t)dt = \int_r^b f(t)({}_rD^\alpha g)(t)dt. \tag{2.19}$$

(b)If $\varphi \in L_p(a,b)$ *and* $\psi \in L_q(a,b)$*, then*

$$\int_a^r \varphi(t)({}_bI^\alpha \psi)(t)dt = \int_a^r \psi(t)(I_a^\alpha \varphi)(t)dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_r^b \psi(t)(\int_a^r \varphi(s)(t-s)^{\alpha-1}ds)dt \tag{2.20}$$

and hence if $g \in {}_aI^\alpha(L_p)$ *and* $f \in I_b^\alpha(L_q)$*, then*

$$\int_a^r g(t)(D_b^\alpha f)(t)dt = \int_a^r f(t)({}_aD^\alpha g)(t)dt$$
$$-\frac{1}{\Gamma(\alpha)}\int_r^b (D_b^\alpha f)(t)(\int_a^r ({}_aD^\alpha g)(s)(t-s)^{\alpha-1}ds)dt. \tag{2.21}$$

That is

$$\int_a^r g(t)(D_b^\alpha f)(t)dt = \int_a^r f(t)({}_aD^\alpha g)(t)dt$$

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$$-\frac{1}{\Gamma(\alpha)}\int_r^b f(t)_r D^\alpha\Big(\int_a^r ({}_aD^\alpha g)(s)(t-s)^{\alpha-1}ds\Big)dt. \tag{2.22}$$

Lemma 2.5. *(see [33,34]). Let $0<\alpha<1$. Then for any $\phi\in C^\infty(\overline{Q})$ we have:*

$$\int_0^T\int_\Omega({}_a^CD^\alpha y(x,t)+\mathcal{A}y(x,t))\phi(x,t)dxdt=\int_\Omega\phi(x,T)_aI^{1-\alpha}y(x,T)dx-\int_\Omega\phi(x,0)_aI^{1-\alpha}y(x,0^+)dx$$

$$+\int_0^T\int_{\partial\Omega}y\frac{\partial\phi}{\partial\nu}d\sigma dt-\int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu}\phi d\sigma dt+\int_0^T\int_\Omega y(x,t)({}^RD_b^\alpha\phi(x,t)+\mathcal{A}^*\phi(x,t))dxdt.$$

where $\mathcal{A}^*$ is conjugate of the operator $\mathcal{A}$; which given in the next section and

$$\frac{\partial y}{\partial\nu_{\mathcal{A}}}=\sum_{i,j=1}^n a_{ij}\frac{\partial y}{\partial x_j}cos(n,x_j)\quad on\ \partial\Omega,$$

$cos(n,x_j)$ is the i-th direction cosine of n, n being the normal at $\partial\Omega$ exterior to Ω.

3. Time-Optimal Control Problem For Caputo Fractional Differential System

Let us consider the optimization problem in the following form

$${}_0^CD^\alpha y(t;v)+\mathcal{A}(t)y(t;v)=f+Bv,\qquad x\in\Omega,\quad t\in(0,T), \tag{3.1}$$

$$y(0;v)=y_0, \tag{3.2}$$

where y_0 is a given element in $L^2(\Omega)$. The second order operator operator $\mathcal{A}(t)$ in the state equation (3.1) takes the form:

$$\mathcal{A}(t)y(x,t)=-\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right)+a_0(x,t)y(x,t), \tag{3.3}$$

where $a_{ij}(x,t), i,j=1,...,n$, be given function on Ω with the properties:

$$a_0(x,t),a_{ij}(x,t)\in L^\infty(\Omega)\quad\text{(with real values)},$$

$$a_0(x,t)\geq\delta>0,\quad\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j\geq\delta(\xi_1^2+...+\xi_n^2),\quad\forall\xi\in R^n,$$

a.e. on Ω. i.e., $\mathcal{A}(t)$ is a bounded second order self-adjoint elliptic partial differential operator maps $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$.

Let us denote by $U:=L^2(0,T;L^2(\Omega))=L^2(Q)$, the space of controls and by $Y:=L^2(0,T;H_0^1(\Omega))$ the space of states.

We suppose that U_{ad} is a closed, convex subset of U and y_1 is a given element in $L^2(\Omega)$,

$$B\in L(L^2(Q),L^2(0,T,H^{-1}(\Omega))). \tag{3.4}$$

Assume (Controllability)

$$\left.\begin{array}{l}\text{there exists a } v\in U_{ad}\text{, such that}\\ y(\tau;v)=y_1\text{ for an appropriate }\tau.\end{array}\right\} \tag{3.5}$$

The optimal time is given by

$$\tau_0=\inf\tau,\ \tau\text{ such that (3.5) holds.} \tag{3.6}$$

For the operator $\mathcal{A}(t)$ we define the bilinear form $\pi(t;y,\phi)$ as follows:

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Definition 3.1. On $H_0^1(\Omega)$ we define for each $t \in]0, t[$ the following bilinear form

$$\pi(t; y, \phi) = (\mathcal{A}(t)y, \phi)_{L^2(\Omega)}, \quad y, \phi \in H_0^1(\Omega).$$

Then

$$\begin{aligned}\pi(t; y, \phi) &= \Big(\mathcal{A}y, \phi\Big)_{L^2(\Omega)} \\ &= \Big(-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big) + a_0(x)y, \phi(x)\Big)_{L^2(\Omega)} \\ &= \int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial}{\partial x_i}y(x)\frac{\partial}{\partial x_j}\phi(x)dx + \int_\Omega a_0(x)y(x)\phi(x)dx.\end{aligned} \tag{3.7}$$

Lemma 3.1. *The bilinear form (3.7) is coercive on $H_0^1(\Omega)$ that is*

$$\pi(t; y, y) \geq \lambda \,||y||^2_{H_0^1\{a_\alpha,2\}(\Omega)}, \quad \lambda > 0. \tag{3.8}$$

Proof.

$$\begin{aligned}\pi(t; y, y) &= \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial}{\partial x_i}y(x,t)\frac{\partial}{\partial x_j}y(x,t)dx + \int_\Omega a_0(x,t)y(x,t)y(x,t)dx \\ &\leq \sum_{i,j=1}^{n} a_{ij}(x,t)||\frac{\partial}{\partial x_i}y(x,t)||^2_{L^2(\Omega)} + ||y(x,t)||^2_{L^2(\Omega)} \\ &\geq \lambda||y||^2_{H_0^1(\Omega)}, \ \lambda > 0\end{aligned}$$

□

Also we can assume that

$$\left.\begin{array}{l}\forall y, \phi \in H_0^1(\Omega) \text{ the bilinear form } t \to \pi(t; y, \phi) \text{ is continuously differentiable in }]0, T[, \\ \text{the function } t \to \pi(t; y, \phi) \text{ is measurable on }]0, T[, \quad |\pi(t; y, \phi)| \leq c||y||.||\phi||,\end{array}\right\} \tag{3.9}$$

the bilinear form (3.7) is symmetric,

$$\pi(t; y, \phi) = \pi(t; \phi, y) \qquad \forall y, \phi \in H_0^1(\Omega),$$

and there exists a λ such that

$$\pi(t; \phi, \phi) + \lambda|\phi|^2 \geq \gamma||\phi||^2, \quad \gamma > 0, \quad \forall \phi \in H_0^1(\Omega), t \in]0, T[. \tag{3.10}$$

Equations (3.1)-(3.10) constitute a fractional Dirichlet problem.

Remark 3.2. [33,34]. The operator ${}_0^C D_t^\alpha + \mathcal{A}(t)$ is a second order parabolic operator that maps $L^2(0, T; H_0^1(\Omega))$ onto $L^2(0, T; H^{-1}(\Omega))$.

Remark 3.3. [33,34]. Equations (3.1)-(3.10) have the unique generalized solution

$$y \in W(0, T) := \Big\{ y| \;\; y \in L^2(0, T; H_0^1(\Omega)), \quad {}_0^C D_t^\alpha y \in L^2(0, T; H^{-1}(\Omega)) \Big\}$$

continuously depends on the initial condition (3.2) and the right-hand side of (3.1). Furthermore, $y \in W(0,T)$ is a continuous function $[0,T] \to L^2(\Omega)$ (compare with Theorem 1.1 and 1.2 Chapt. 3 [31]).

We are going to study the following problems:
(i) the existence of an optimal control, i.e., $u \in U_{ad}$ such that

$$y(\tau_0; u) = y_1; \tag{3.11}$$

(ii) properties of the optimal control, if it exists. these problems are treated in sections 4,5.

4. Existence Theorem

Theorem 4.1. *Let $\alpha \in]\frac{1}{2}, 1]$. We assume that (3.4), (3.9), (3.5) hold and that U_{ad} is bounded. Then there exists an optimal control, that is $u \in U_{ad}$, such that (3.11) holds.*

Proof. Let τ_n be such that

$$y(\tau_n; v_n) = y_1,\ v_n \in U_{ad}, \tag{4.1}$$

$$\tau_n \to \tau_0, \tag{4.2}$$

$$\left.\begin{array}{l}\text{Set } y_n = y(v_n). \text{ Since } U_{ad} \text{ is a bounded, we may verify that}\\ y_n \text{ (resp. } {}_0^C D^\alpha y_n) \text{ ranges in a bounded set in } L^2(0,T,H_0^1(\Omega)) \text{ (resp. } L^2(0,T,H^{-1}(\Omega))).\end{array}\right\} \tag{4.3}$$

We may then extract a subsequence, again denoted by $\{v_n, y_n\}$, such that

$$\left.\begin{array}{l} v_n \to v \text{ weakly in } U,\ v \in U_{ad}\\ y_n \to y \text{ (resp } {}_0^C D^\alpha y_n \to {}_0^C D^\alpha y) \text{ weakly in } L^2(0,T,H_0^1(\Omega)) \text{ (resp. } L^2(0,T,H^{-1}(\Omega))).\end{array}\right\} \tag{4.4}$$

We deduce from the equality

$${}_0^C D^\alpha y_n + \mathcal{A} y_n = f + v_n, \qquad x \in \Omega, \quad t \in (0,T),$$

that

$${}_0^C D^\alpha y + \mathcal{A} y = f + v, \qquad x \in \Omega, \quad t \in (0,T),$$

and $y(0) = y_0$ and hence

$$y = y(v). \tag{4.5}$$

But

$$y(\tau_n; v_n) - y(\tau_0; v) = y(\tau_n; v_n) - y(\tau_0; v_n) + y(\tau_0; v_n) - y(\tau_0; v). \tag{4.6}$$

Now from (4.4) $y(\tau_0; v_n) \to y(\tau_0; v)$ weakly in $L^2(0,T,H_0^1(\Omega))$ and

$$\begin{aligned} ||y(\tau_n; v_n) - y(\tau_0; v_n)||_{H^{-1}(\Omega)} &= || \left({}_{\tau_0} I^\alpha\, {}_{\tau_0}^C D^\alpha\right) y(\tau_n; v_n)||_{H^{-1}(\Omega)} \\ &\le \frac{1}{\Gamma(\alpha)} \int_{\tau_0}^{\tau_n} (\tau_n - \tau)^{\alpha-1} ||\, {}_{\tau_0}^C D^\alpha y(\tau; v_n)||_{H^{-1}(\Omega)} d\tau \\ &\le \frac{1}{\Gamma(\alpha)} \left(\int_{\tau_0}^{\tau_n} (\tau_n - \tau)^{2\alpha-2} d\tau\right)^{\frac{1}{2}} \left(\int_{\tau_0}^{\tau_n} ||\, {}_{\tau_0}^C D^\alpha y(\tau; v_n)||^2_{H^{-1}(\Omega)} d\tau\right)^{\frac{1}{2}} \\ &\le C(\tau_n - \tau_0)^{\alpha - \frac{1}{2}}; \end{aligned}$$

hence (4.6) shows that

$$y(\tau_n; v_n) - y(\tau_0; v) \to 0 \text{ weakly in } H^{-1}(\Omega).$$

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But from (4.1) it follows that $y(\tau_0; v) = y_1$. □

Example 4.1. ***(Neumann problem with boundary control)***
Take $U = L^2(\Sigma)$. *Let the state* $y(v)$ *be as follows*

$$
\begin{aligned}
{}_0^C D^\alpha y(v) + \mathcal{A}(t) y(v) &= f, \\
\frac{\partial y}{\partial \nu_{\mathcal{A}}}(v) &= v, \\
y(x, 0, v) &= y_0(x).
\end{aligned}
$$

We may apply Theorem 4.1 to this example. Thus, the theorem deals with the case of boundary control.

Remark 4.2. Theorem 4.1 may be modified easily to deal with the case of systems with Neumann boundary conditions (cf ([31]), section 9) and where the control is exercised through the boundary.

5. Bang-Bang Theorem

We consider the same data and hypotheses as in Section 4 with

$$\mathcal{A} \text{ independent of } t, \tag{5.1}$$

$$U_{ad} = \{v | \ |v(t)| \leq 1 \text{ a.e.}\}. \tag{5.2}$$

Then $-\mathcal{A}$ is the infinitesimal generator of a semi-group $G(t)$ in $L^2(Q)$.

Theorem 5.1. ***(Bang-Bang Theorem)***
We assume that (3.4), (3.9), (5.1), (5.2) and (3.5) hold. Let u be an optimal control, that is, an element of U_{ad} *satisfying (3.11)(we know from Theorem (4.1) that such elements exist). Then*

$$|u(t)| = 1 \ \text{ a.e. on }]0, \tau_0[. \tag{5.3}$$

We shall establish a number of lemmas before giving a proof of the Theorem (5.1). Let us at once isolate.

Corollary 5.2. *Under the hypotheses of Theorem (5.1), there exists an optimal control which is unique.*

Proof. Let $u_1, u_2 \in U_{ad};\ y(\tau_0; u_i) = y_i, i = 1, 2.$. Then

$$y(\tau_0; (1-\theta)u_1 + \theta u_2) = y_1,\ 0 < \theta < 1 \text{ and } u = (1-\theta)u_1 + \theta u_2$$

does not satisfy (3.11) unless $u_1(t) = u_2(t)$ *a.e.* □

Notation

$$\left.\begin{array}{l}
K_s = \{h | h = G * v(s) = \int_0^s G(s-t) v(t) dt, v \in L^\infty(0, s, L^2(\Omega))\} \\
K_s(e) = \{h | h = G * v(s) = \int_0^s G(s-t) v(t) dt, v \in L^\infty(0, s, L^2(\Omega)) \text{ with support in } e \cap [0, s], \\
e = \text{ measurable subset of } [0, T]\}
\end{array}\right\} \tag{5.4}$$

Lemma 5.3. *With the notation of (5.4), we have:*

$$K_s = K'_s,\ \forall s, s' \tag{5.5}$$

We then set

$$K = K_s,\ \forall s \tag{5.6}$$

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We have

$$G(s)H \subset K. \tag{5.7}$$

Proof. 1. Let $\tau > s$. Then we may easily verify

$$\left. \begin{array}{l} G \star v(s) = G \star \overline{v}(\tau), \\ \text{where} \\ \overline{v}(\tau) = \begin{cases} 0 \text{ if } 0 < t < \tau - s, \\ v(t - (\tau - s)) \text{ if } t > \tau - s. \end{cases} \end{array} \right\} \tag{5.8}$$

Consequently

$$K_s \subset K_\tau. \tag{5.9}$$

2. Let $\tau < s$. We may again verify that

$$\left. \begin{array}{l} G \star v(s) = G \star w(s), \\ \text{where} \\ w(s) = v(t + \tau - s)) + \frac{1}{s} G(t) \int_0^{\tau - s} G(\tau - s - t) v(t) dt, \end{array} \right\} \tag{5.10}$$

which proves the reverse inclusion of (5.9), whence (5.5) (and (5.6).

3. We check that

$$G(s)h = G \star (\frac{1}{s} Gh)(s), \text{ whence } (5.7). \tag{5.11}$$

□

The following lemma is fundamental.

Lemma 5.4. *For almost all $t \in e$ we have*

$$K_t(e) = K. \tag{5.12}$$

Proof. 1. Clearly $K_t(e) \subset K_t$ and hence it suffices to prove that for almost all $t \in e$, $K \subset K_t(e)$. Let $h \in K_t$. Then $h \in K_s$ with s arbitrary small and therefore for any $t_1 < t$, we may represent h in the form (cf. (5.8))

$$h = \int_{t_1}^{t} G(t - \sigma) v(\sigma) d\sigma, \; v \in L^\infty(0, T; H). \tag{5.13}$$

Now suppose that we can find a sequence t_n such that

$$\left. \begin{array}{l} t_n < t_{n+1} < \ldots < t, \; t_n \to t, \\ \text{measure}([t_n, t_{n+1}] \cap e) \geq \rho(t_{n+1} - t_n), \;\; \rho > 0, \\ \frac{t_{n+1} - t_n}{t_{n+2} - t_{n+1}} \leq c. \end{array} \right\} \tag{5.14}$$

We shall see in part 2 of the proof that

$$\text{for almost all } t \in e, \;\; \text{we may find a sequence } t_n \;\; \text{such that (5.14) holds.} \tag{5.15}$$

In (5.13) let us choose as t_1 the first element of the sequence $\{t_n\}$. From (5.13) we deduce that we may write

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$$h = \sum_{n=1}^{\infty} G(t-t_{n+1}) \int_{t_n}^{t_{n+1}} G(t_{n+1}-\sigma)v(\sigma)d\sigma$$
$$= \sum_{n=1}^{\infty} \int_{e\cap[t_{n+1},t_{n+2}]} \frac{1}{\text{measure}(e\cap[t_{n+1},t_{n+2}])} G(t-\sigma_1) \tag{5.16}$$
$$\times G(\sigma_1 - t_{n+1})d\sigma_1 \int_{t_n}^{t_{n+1}} G(t_{n+1}-\sigma_2)v(\sigma_2)d\sigma_2 = \int_0^t G(t-\sigma)w(\sigma)d\sigma.$$

where

$$w(s) = \begin{cases} \frac{1}{\text{measure}(e\cap[t_{n+1},t_{n+2}])} G(\sigma - t_{n+1}) \int_{t_n}^{t_{n+1}} G(t_{n+1}-\sigma_2)v(\sigma_2)d\sigma_2 \\ \text{in the set } e\cap[t_{n+1},t_{n+2}],\ n=1,2,...; \\ 0 \text{ otherwise.} \end{cases} \tag{5.17}$$

From (5.14), we have

$$|w(\sigma)| \le \frac{1}{\rho(t_{n+2}-t_{n+1})} C(t_{n+1}-t_n) \le \text{constant}.$$

Hence

$$h = G\star w(t),\ \ w\in L^\infty(0,T,H),\ \ w \text{ has support in } e, \text{ and } h\in K_t(e).$$

The lemma is proved with the exception of

2. Proof of (5.15). This is a result in measure theory. We first define

$$e_m = \left\{\sigma|\sigma\in e, measure(e\cap[\sigma-\frac{1}{k},\sigma]) \ge \frac{1}{2k}, k\ge m\right\}$$

and d_m = set of points of density of e_m. It is known that $measure\left(e - \bigcap_{m\ge1} d_m\right) = 0$ and hence it suffices to prove (5.15) for $t\in d_m$.

But then we may construct $t_n, t_n \in d_m\ \forall n; t_{n+1} = t_n + s_n < t,\ s_n > 0,\ \frac{s_n}{s_{n+1}} \le e$ (in a manner such that the last property of (5.14) holds) and since $t_n\in d_m \subset e_m\ \forall n$ there exists a $\rho > 0$ such that

$$\text{measure}([t_n,t_{n+1}]\cap e) \ge \rho(t_{n+1}-t_n).$$

□

Lemma 5.5. *Let $u\in U_{ad}$ be optimal with respect to $\{y_0,y_1\}$, that is, $y(\tau;u)=y_1$ with τ minimum. Then for any $s<\tau, u$ is optimal with respect to $\{y_0,y(s;u)\}$.*

In other words, if $w\in U_{ad}$ satisfies

$$y(\sigma;w) = y(s;u) \text{ for some appropriate } e, \tag{5.18}$$

we necessarily have $\sigma\ge s$.

Proof. Assume that (5.18) holds with $\sigma<s$. Then define v by

$$v(t) = \begin{cases} w(t)\ \ in\ ]0,\sigma[, \\ u(t+s-\sigma)\ \ in\ ]\sigma, T-(s-\sigma)[. \end{cases}$$

Then from (5.18):

$$y(\sigma;v) = y(\sigma;w) = y(s;u)$$

and
$$ {}_0^C D^\alpha y(t;v) + \mathcal{A}y(t;v) = u(t+s-\sigma), \qquad \text{for } t \geq \sigma. $$
Hence
$$ y(t-(s-\sigma);v) = y(t,u), \;\; t \geq s, $$
and hence
$$ y(\tau-(s-\sigma);v) = y(\tau,u) = y. $$
But since $\tau - (s-\sigma) < \tau$, this contradicts the hypothesis that u is optimal with respect to $\{y_0, y_1\}$. □

Lemma 5.6. *Let $v \in U_{ad}$ be such that*
$$ |v(t)| \leq 1-\varepsilon \text{ almost everywhere, } \varepsilon > 0, \tag{5.19} $$
$$ y(\tau;v) = y_1 \text{ for some appropriate } \tau. \tag{5.20} $$
Then $\tau > \tau_0$.

Proof. To prove this, we will verify that there exists a $s < \tau$ and $w \in U_{ad}$ such that
$$ y(s;w) = y(\tau;v) \tag{5.21} $$
(which proves that τ is not optimal). For this, we note that (5.21) may be written as
$$ \int_0^\tau G(\tau-\sigma)v(\sigma)d(\sigma) + G(\tau)y_0 - G(s)y_0 = \int_0^s G(s-\sigma)w(\sigma)d\sigma \tag{5.22} $$
But it may be easily verified that the left hand side of (5.22) may be written as
$$ \int_0^s G(s-\sigma)\overline{v}(\sigma)d\sigma, $$
with
$$ \overline{v}(\sigma) = v(\sigma+\tau-s) + \frac{1}{s}G(\sigma)\int_0^{\tau-s} G(\tau-s-\sigma_1)v(\sigma_1)d\sigma_1 + \frac{1}{s}[G(\sigma+\tau-s)y_0 - G(\sigma)y_0] $$
and for $\tau - s$ sufficiently small, we have by virtue of (5.19) $|\overline{v}(\sigma)| \leq 1$. Therefore we may take $w = \overline{v}$. □

Proof. Proof of Bang-Bang Theorem (Theorem(5.1)). Assume that (5.3) did not hold. Then there would exists $e \subset [0, \tau_0]$, measure $(e) > 0$. such that
$$ |u(t)| \leq 1-\varepsilon, \;\; \varepsilon > 0, \;\; t \in e. \tag{5.23} $$
It follows from the fundamental Lemma (5.4), that we can find an s such that $K_s(e) = K = K_s$. In other words, there exists a $\overline{g} \in L^\infty(0,T,H)$, with support in e, such that
$$ \int_0^s G(s-\sigma)\overline{g}(\sigma)d\sigma = \int_0^s G(s-\sigma)u(\sigma)d\sigma. \tag{5.24} $$
Let us introduce the control
$$ v = \begin{cases} (1-\delta)u + \delta\overline{g} \;\; in \;\; (0,\tau_0), \;\; \delta > 0 \text{ almost everywhere}, \\ 0 \;\; \text{for} \;\; t > \tau_0. \end{cases} \tag{5.25} $$

We shall choose δ in a manner such that

$$|v(t)| \leq 1 - \varepsilon_1 \text{ almost everywhere}, \varepsilon_1 > 0 \text{ appropriate.} \tag{5.26}$$

This is possible. To see this, we note that on e, we have from (5.23):

$$|v(t)| \leq (1-\delta)(1-\varepsilon) + \delta|\overline{g}(t)| \leq 1 - \varepsilon_2$$

for δ sufficiently small, and outside e,

$$|v(t)| = (1-\delta)|u(t)| \leq (1-\delta) \ (\textit{since } u \in U_{ad}).$$

But

$$y(s;v) = (1-\delta)\int_0^s G(s-\sigma)u(\sigma)d\sigma + \delta \int_0^s G(s-\sigma)\overline{g}(\sigma)d\sigma + G(s)y_0$$

and using (5.24) we have

$$y(s;v) = y(s;u) \tag{5.27}$$

But then from Lemma (5.6)(since we have (5.26) u is not optimal with respect to $\{y_0, y(s;u)\}$ and hence from Lemma (5.5) u is not optimal with respect to $\{y_0, y_1\}$ contradicting the hypotheses. □

Remark 5.7. It is possible to proceed further using much simpler arguments when the semi-group G is a group - in other words when we can reverse time. Indeed we have the following result.

Theorem 5.8. *We assume that $-A$ is the infinitesimal generator of a group $G(t)$. We assume that*

$$U_{ad} = \{v | v(t) \in H_{ad} \subset H, H_{ad} = \textit{neighborhood of } 0 \textit{ in } H\} \tag{5.28}$$

We further suppose that there exists a $v \in U_{ad}$ such that (3.5) holds and that there exists an optimal control u (that is, $y(\tau_0; u) = y_1, \tau_0 =$ optimal time defined in (3.4)).

Then

$$u(t) \in \partial H_{ad} \ (\textit{boundary of } \ H_{ad}), \textit{almost everywhere.} \tag{5.29}$$

Proof. If (5.29) does not hold, there exists an $e \subset [0, \tau_0]$ such that

$$\left.\begin{array}{l} \text{measure}(e) > 0, \\ \text{distance } (u(t), \partial H_{ad}) \geq c_1 > 0, \text{ almost everywhere for } t \in e. \end{array}\right\} \tag{5.30}$$

Let χ_e be the characteristic function of e. Then

$$y(\tau_0; u) = y_1 = G(\tau_0)y_0 + \int_0^{\tau_0} G(\tau_0 - \sigma)u(\sigma)d\sigma$$

may be written as - by virtue of the fact that G is a group:

$$y(\tau_0; u) = G(\tau_1)y_0 + \int_0^{\tau_1} G(\tau_1 - \sigma)\overline{u}(\sigma)d\sigma, \tag{5.31}$$

with

$$\overline{u}(\sigma) = u(\sigma) + \frac{1}{\textit{measure}(e)}\chi_e(\sigma)G(\sigma - \tau_1)[y(\tau_0; u) - y(\tau_1; u)], \tau_1 < \tau_2. \tag{5.32}$$

We may choose τ_1 sufficiently near to τ_0 such that

$$|u(\sigma) - \overline{u}(\sigma)| \leq \frac{c_1}{2} \text{ for } \sigma \in e.$$

and therefore from (5.30), $u \in U_{ad}$. Then, from (5.31)

$$y(\tau_1;\overline{u}) = y(\tau_0;u) = y_1$$

which contradicts the fact that τ_0 is optimal. □

6. Optimality Conditions

Theorem 6.1. *Assume that the hypotheses of Theorem (5.8) hold. We further assume that H_{ad} is convex. Then there exists an $h_0 \in H$ such that*

$$(h_0, y(\tau_0;v) - y(\tau_0;u)) \geq 0 \ \forall v \in U_{ad}. \tag{6.1}$$

Proof. 1. Define the set - which is clearly convex:

$$K = \{h | h \in H, h = y(\tau_0;v) - y(\tau_0;0)), \ v \in U_{ad}. \tag{6.2}$$

Let us verify that

$$0 \text{ is in the interior of } K, \tag{6.3}$$

$$y_1 - y(\tau_0;0) \text{ (which belongs to } K) \text{ is not in the interior of } K, \tag{6.4}$$

To prove (6.3), let $h \in H$ be arbitrary. We have to prove that $\varepsilon h \in K$ for $|\varepsilon|$ sufficiently small. Now

$$h = \int_0^{\tau_0} G(\tau_0 - \sigma)G(\sigma - \tau_0)hd\sigma,$$

and therefore

$$\varepsilon h = y(\tau_0; vG(\sigma - \tau_0)h)) - y(\tau_0;0) \in K \ \ for \ |\varepsilon| \ \text{ sufficiently small.}$$

We prove (6.4) by contradiction. If $y_1 - y(\tau_0;0)$ were in the interior of K, for an appropriate $\varepsilon > 0$ we would have

$$y_1 - y(\tau_0;0) + \varepsilon(y_1 - G(\tau_0)y_0) = y(\tau_0;v) - y(\tau_0;0)$$

whence

$$y_1 = y(\tau_0;0) + \int_0^{\tau_0} G(\tau_0 - \sigma)\frac{1}{1+\varepsilon}v(\sigma)d(\sigma) = y(\tau_0;w)$$

and therefore $w(\sigma) \in$ interior of U_{ad} contradicting (5.29).

2. The result now follows as a consequences of (6.3), (6.4) and of Corollary 5, Section 9 of Dunford-Schwartz [22] □

Remark 6.2. (6.1) may interpreted as follows: let us introduce the adjoint state by

$${}^R D^{\alpha}_{\tau_0} p + \mathcal{A}^* p = 0, \qquad \text{in }]0, \tau_0[, \tag{6.5}$$

$$I^{1-\alpha}_{\tau_0} p(\tau_0) = h_0. \tag{6.6}$$

Then by using Green formula given in Lemma (2.5), (6.1) is equivalent to

$$\int_0^{\tau_0} (p(t), v(t) - u(t))dt \geq 0 \ \ \forall v \in U_{ad}. \tag{6.7}$$

whence we may pass to local conditions in t:

$$(p(t), h - u(t)) \geq 0 \ \text{ almost everywhere in }]0, \tau_0[\ \ \forall h \in H_{ad}. \tag{6.8}$$

7. Application

In this section we state some illustrate examples to explain our abstract conclusions.

Example 7.1. *If H_{ad} = unit ball in H, we would have*

$$u(t) = -\frac{p(t)}{|p(t)|}.$$

Example 7.2. *Let us consider the system with the following state*

$${}_0^R D^\alpha y(t;v) + \mathcal{A}(t) y(t;v) = f + Bv \tag{7.1}$$

$${}_0 I^{1-\alpha} y(0,v) = y_0 \tag{7.2}$$

and the adjoint state is given by

$${}^C D^\alpha_{\tau_0} p + \mathcal{A}^* p = 0, \qquad in \ ]0, \tau_0[, \tag{7.3}$$

$$p(\tau_0) = h_0. \tag{7.4}$$

Then (6.1) is equivalent to

$$\int_0^{\tau_0} (p(t), v(t) - u(t))dt \geq 0 \ \ \forall v \in U_{ad}. \tag{7.5}$$

Remark 7.1. If we take $\alpha = 1$ in the previous sections we obtain the classical results in the optimal control with integer derivatives.

8. Open Problems

1- By a similar manner, we can also study the time-fractional optimal control of the above systems, where the time derivative is considered as the left Atangana-Baleanu fractional derivative in Caputo sense as the following:

$${}_0^{ABC} D^\alpha y(t;v) + \mathcal{A}(t) y(t;v) = f + Bv, \qquad x \in \Omega, \quad t \in (0,T),$$

$$y(0;v) = y_0,$$

where ${}_0^{ABC} D^\alpha$ is the left Atangana-Baleanu fractional derivative in the sense of Caputo. (see [1])

2- The problem can be extended to the time-space fractional derivative as the following:

$${}_0^{ABC} D^\alpha y(t;v) + (-\Delta)^\beta (t) y(t;v) = f + Bv, \qquad x \in \Omega, \quad t \in (0,T),$$

$$y(0;v) = y_0,$$

where $(-\Delta)^\beta$ is is the fractional Laplacian operator for $\beta \in (0,1)$(see [33–35])

9. Conclusions

In this work we considered time-(FDS) with Dirichlet and Neumann boundary conditions and boundary and distributed control using the classical control theory give in Lions [31]. The fractional derivatives were defined in the weak Caputo and Riemann-Liouville senses. We also derived the Bang-Bang theorem (Theorem (5.1) for this fractional deferential systems. The analytical results were obtained in terms of Euler-Lagrange equations for the (TOCFP). The formulation presented and the resultant equations are similar to those for classical optimal control problems. The optimization problem presented in this paper constitutes a generalization of the time-optimal control problems of parabolic systems with Dirichlet and Neumann boundary conditions considered in Lions [31] to fractional time-optimal control problems. In addition the main result of the paper includes necessary conditions of optimality for non-integer order fractional systems that give characterization of optimal control (Theorem (6.1)).

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