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Article

Characterization of Herz-Type Besov-Morrey and Triebel-Lizorkin-Morrey spaces

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Abstract: The main goal of this article is the characterization of Herz-type Besov-Morrey spaces $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ and Triebel-Lizorkin-Morrey spaces $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}$ via ball means of differences, atomic, molecular and wavelet decompositions. There are provided necessary facts for implying these results, which can help to explore such spaces on different domains in further researches. Proposed spaces allow to extend the class of studying functions, containing Leray projection, heat semi-group and other operators, which can be helpful in exploring weak, strong and mild solutions of nonlinear partial differential equations.

Keywords: Herz-type Besov-Morrey spaces; Herz-type Triebel-Lizorkin-Morrey spaces; ball mean of differences; atomic and molecular decomposition; wavelet decomposition

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1. Introduction

Attempts to define function spaces, using different approaches, takes its beginning since the appearance of Hardy, Besov and Triebel-Lizorkin spaces, which help to advance the studying nonlinear PDEs. On example, Besov spaces characterized via ball means of differences allowed to establish higher integrability results for the gradient of local weak solutions to the strongly degenerate or singular elliptic PDE in [1]. Another example is Triebel-Lizorkin spaces defined via Fourier transforms, which allowed to explore mild solutions of Navier-Stokes equation in [2].

There are 3 ways to define any function space [3]: Fourier-analytic, derivatives and differences and bounded mean oscillation. Specifying spaces in various ways allows to extend their properties, which can help in the study of PDE solutions, the behavior of pseudo-differential operators, Riemannian manifolds with the corresponding metric, Lie groups and fractals.

The study of quasi-norms, atomic, molecular and wavelet decompositions remains actual since the definition of local Hardy spaces in [4]. Such research was continued on Lorentz spaces for exploring weak and strong solutions of the nonlinear heat equation in [5]. The similar work was realized for Navier-Stokes equations and Keller-Segel system on Besov and Triebe-Lizorkin spaces in [6]. Afterwards, many researchers came up with the idea of combining function spaces into one function spaces, which can be global or hybrid, well described in [7] and [8], respectively.

Examples of hybrid spaces were demonstrated in [9], where the author researched the Caffarelli-Kohn-Nirenberg inequalities on Herz-type Besov spaces $\dot{K}^{\alpha,p}_{q}B^{s}_{\beta}$ and Triebel-Lizorkin spaces $\dot{K}^{\alpha,p}_{q}F^{s}_{\beta}$ and presented their quasi-norms, applying Fourier transforms and ball means of differences. Such spaces allowed to combine the properties of Besov, Triebel-Lizorkin and Herz spaces. Another example of hybrid spaces is Besov-weak-Herz spaces $BW\dot{K}^{\alpha,s}_{p,q,r}$ introduced in [10], where authors explored mild solutions of the incompressible Navier-Stokes equations. There were provided $K_{\theta,r}$-method real interpolations, embeddings and heat semi-group operator inequalities for obtaining estimates of mild solutions on Besov-weak-Herz spaces $BW\dot{K}^{\alpha,s}_{p,q,r}$. The combining of Besov and weak Herz spaces generalized and extended their properties, what allowed to receive valuable results in the studying the Navier-Stokes equations. The unique maximal strong solution and the local strong

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well-posedness of the Navier–Stokes equations with $f \neq 0$ was constructed on Triebel-Lizorkin-Lorentz spaces in [11], which provided important inequalities containing the Laplace and the Stokes operators.

The main idea of this article arose from the researching quasi-norm characterizations via ball means of differences on Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, which are investigated in [12] and [13], the exploring of the atomic decomposition of Herz-type Besov and Triebel-Lizorkin spaces, and the wavelet decomposition for the corresponding spaces in [14]. The definition of Herz-type Besov and Triebel-Lizorkin spaces via differences in [15] impacted to the topic of this research. In this article, we propose Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey spaces, denoted by $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{r}_{\mu,s}$ and $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{r}_{\mu,s}$, respectively. We establish equivalence between the quasi-norms defined via ball of means differences and Fourier transforms, imply atomic, molecular and wavelet decomposition for corresponding spaces. Taking into account the quasi-norm of Herz spaces, we imply helpful inequalities on Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey spaces, containing Hardy-Littlewood and Peetre maximal operators. The proposed spaces were not present in other papers, then the necessary facts on $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{r}_{\mu,s}$, where $\mathcal{A} \in \{\mathcal{N}, \mathcal{E}\}$, are not provided. Therefore, there are provided some inequalities, containing maximal functions, embeddings and isomorphisms in this article.

The remaining of the paper is organized as follows. Section 2 is devoted to definition of proposed spaces via Fourier-analytic approach. Section 3 contains necessary theorems, lemmas and propositions for further proofs. In section 4 we characterize Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey spaces, utilizing the ball means of differences. Section 5 and section 6 present atomic, molecular and wavelet decompositions, respectively.

2. Definition of Herz-Type Besov-Morrey and Triebel-Lizorkin-Morrey Spaces via Fourier-Analytic Approach

We should provide some helpful notations, which shorten the writing of formulas. Let $\{f_j\}_{j\in\mathbb{N}_0}$ be a sequence of functions, then we note

$$||\{f_j\}_{j\in\mathbb{N}_0}||_{l^q(L^p)} := \left(\sum_{j\in\mathbb{N}_0} ||f_j||^q_{L^p}\right)^{1/q},$$

$$||\{f_j\}_{j\in\mathbb{N}_0}||_{L^p(l^q)} := \left|\left|\left(\sum_{j\in\mathbb{N}_0} |f_j|^q\right)^{1/q}\right|\right|_{L^p},$$

where $0 < p, q \leq \infty$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of all non-negative integers. For definiteness we adopt the following definition as the Fourier transform and its inverse:

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}d\xi, \ \mathcal{F}^{-1}f(x) := \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}dx$$

for every f in the space of tempered distributions $\mathcal{S}'$.

Before giving the definition of the Herz spaces, let us introduce the following notations. Let $k \in \mathbb{Z}$. Then

$$B_k := \{x \in \mathbb{R}^n : |x| \leq 2^k\}, \ R_k := B_k \setminus B_{k-1}, \ \chi_k := \chi_{R_k}.$$

For $m \in \mathbb{N}_0$, we denote

$$\tilde{\chi}_m := \begin{cases} \chi_{R_m}, & m \geq 1 \\ \chi_{B_0}, & m = 0 \end{cases}.$$

Let us recall the definition of homogeneous Herz spaces [9].

Definition 1. *Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. The homogeneous Herz spaces $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are defined as*

$$\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n) := \{f \in L^p_{loc}(\mathbb{R}^n \setminus \{0\}); ||f||_{\dot{K}^{\alpha}_{p,q}} < \infty\}, \tag{1}$$

where

$$||f||_{\dot{K}^{\alpha}_{p,q}} := \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}||f\chi_k||^q_{L^p}\right)^{1/q} = \left|\left|\{2^{k\alpha}|f\chi_k|\}_{k\in\mathbb{Z}}\right|\right|_{l^q(L^p)},$$

with the usual modification when $q = \infty$.

The spaces $\dot{K}^{\alpha}_{p,q}$ are quasi-Banach spaces, and if $\min(p,q) \geq 1$ then $\dot{K}^{\alpha}_{p,q}$ are also Banach spaces. When $\alpha = 0$ and $0 < p = q \leq \infty$ the space $\dot{K}^0_{p,p}$ coincides with the Lebesgue space L^p . In addition $\dot{K}^{\alpha}_{p,p} = L^p(\mathbb{R}^n, |\cdot|^{\alpha p})$, where

$$||f||_{L^p(\mathbb{R}^n,|\cdot|^{\alpha p})} = \left(\int_{\mathbb{R}^n} |f(x)|^p|x|^{\alpha p}\right)^{\frac{1}{p}}.$$

Let $B_r(x_0)$ be an open ball in $\mathbb{R}^n$ centered at x_0 and radius $r > 0$. Next we define the Herz-type Morrey space.

Definition 2. *Let $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$ and $0 < \mu < n$, then Herz-type Morrey spaces $\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(\mathbb{R}^n)$ are defined as the sets of functions $f \in \dot{K}^{\alpha}_{p,q}(B_r(x_0))$ such that*

$$||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} := \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\mu}{p}}||f||_{\dot{K}^{\alpha}_{p,q}(B_r(x_0))} < \infty. \tag{2}$$

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be Schwarz and the tempered distributions spaces, respectively. Suppose, that $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a nonegative radial function such that

$$supp(\phi) \subset \{\xi \in \mathbb{R}^n; 1/2 < |\xi| < 2\}$$

and

$$\sum_{k\in\mathbb{Z}} \phi_k(\xi) = 1, \ \forall \xi \neq 0,$$

where $\phi_k(\xi) = \phi(2^{-k}\xi)$. Using a partition of unity, we can define Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey spaces via Fourier-analytic approach.

Definition 3. *For $0 < p, q \leq \infty$, $0 < \mu < n$, $r \in (0, \infty]$ and $s, \alpha \in \mathbb{R}$,*

(i) the inhomogeneous Herz-type Besov-Morrey space $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^s_{\mu,r}$ is the set of $f \in \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the set of polynomials, such that $\mathcal{F}^{-1}[\phi_k\mathcal{F}f] \in \dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}$ and

$$||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^s_{\mu,r}} = \left(\sum_{k\in\mathbb{N}_0} 2^{ksr}||\mathcal{F}^{-1}[\phi_k\mathcal{F}f]||^r_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}\right)^{1/r} < \infty, \tag{3}$$

with the usual modification when $r = \infty$.

(ii) the inhomogeneous Herz-type Triebel-Lizorkin-Morrey space $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^s_{\mu,r}$ is the set of $f \in \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the set of polynomials, such that $\mathcal{F}^{-1}[\phi_k\mathcal{F}f] \in \dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}$ and

$$||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^s_{\mu,r}} = \left|\left|\left(\sum_{k\in\mathbb{N}_0} 2^{ksr}|\mathcal{F}^{-1}[\phi_k\mathcal{F}f]|^r\right)^{1/r}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} < \infty, \tag{4}$$

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with the usual modification when $r = \infty$.

Let us introduce the Hardy-Littlewood maximal operator:

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n} \int_{B_r(x)} |f(y)|dy, \; \forall x \in \mathbb{R}^n. \tag{5}$$

And we note $\mathcal{M}^{(\tau)}f(x) = (\mathcal{M}|f|^{\tau})^{1/\tau}$, $0 < \tau < \infty$. Suppose that $\{\phi_k\}_{k=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$, $a > 0$, $x \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$, then we introduce the Peetre maximal function defined as

$$(\phi_k^* f)_a(x) = \sup_{y \in \mathbb{R}^n} \frac{|(\mathcal{F}^{-1}[\phi_k \mathcal{F} f])(x-y)|}{1+|2^k y|^a}. \tag{6}$$

These maximal operators are useful for characterization of Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey spaces via ball means differences.

In this section introduced spaces were defined via Fourier analytic approach. This approach was useful in [16], where estimates of mild solutions of the Navier-Stokes equation, containing semi-group heat operator on weak Herz-type Besov-Morrey spaces, were explored. The Fourier-analytic approach on Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey spaces facilitates the studying of mild solutions of nonlinear PDEs, where the heat semi-group operator and the Lerray projection are intensively engaged. For example, a mathematical model of waves on shallow water surfaces described by Korteweg-de Vries equation [17], Keller–Segel System [18] presents a cellular chemotaxis model, or Fokker-Planck equations [19] demonstrates models of anomalous diffusion processes. The developing of atomic, molecular and wavelet decompositions can advance the studying of $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ and $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}$ spaces on smooth and singular manifolds, especially observing them not only using the Fourier-analytic approach, but also using the difference approach.

3. Properties of Herz-Type Besov-Morrey and Triebel-Lizorkin-Morrey Spaces

In this section we provide useful inequalities and properties of spaces $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$, where $\mathcal{A} \in \{\mathcal{N}, \mathcal{E}\}$.

Lemma 1. *Let $0 < p,q < \infty$; $0 < \mu < n$; $\alpha, s \in \mathbb{R}$; $r \in (0,\infty]$ and $f_1, f_2 \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$, then we get the following inequality:*

$$\left(||f_1+f_2||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)} \leq \left(||f_1||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)} + \left(||f_2||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)}. \tag{7}$$

Proof. This inequality implied from Lemma 1 [20] and Minkowski inequality for infinite sums. In case, when $\mathcal{A} = \mathcal{N}$

$$\left(||f_1+f_2||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)} = \left(\sum_{k\in\mathbb{Z}} 2^{ksr}||\mathcal{F}^{-1}[\phi_k\mathcal{F}(f_1+f_2)]||^r_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right)^{\min(1,p,q,r)/r}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} 2^{ksr}||\mathcal{F}^{-1}[\phi_k\mathcal{F}f_1]||^r_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right)^{\min(1,p,q,r)/r} + \left(\sum_{k\in\mathbb{Z}} 2^{ksr}||\mathcal{F}^{-1}[\phi_k\mathcal{F}f_2]||^r_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right)^{\min(1,p,q,r)/r}$$

$$= \left(||f_1||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)} + \left(||f_2||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)}$$

Analogical calculations we apply in case $\mathcal{A} = \mathcal{E}$ and receive the required inequality. □

According to Lemma 1, it is possible to obtain the general inequality for the arbitrary family of functions.

Corollary 1. *Let $0 < p, q < \infty$; $0 < \mu < n$; $\alpha, s \in \mathbb{R}$; $r \in (0, \infty]$ and $\{f_i\}_{i=1}^{l}$ is the family of functions in $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$, then we get the following inequality:*

$$\left(\left|\left|\sum_{i=1}^{l} f_i\right|\right|_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)} \leq \sum_{i=1}^{l}\left(||f_i||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}\right)^{\min(1,p,q,r)}. \tag{8}$$

We also verify the relation between f and $|f|^u$ in the next lemma.

Lemma 2. *Let $0 < p, q < \infty$, $0 < \mu \leq n$ and $\alpha \in \mathbb{R}$.*
(i) If $t \in (0, p)$, then $||f||_{\dot{K}^{\alpha}_{t,q}\mathcal{M}_\mu} \leq ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}$.
(ii) If $0 < u < \infty$, then

$$|||f|^u||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} = \left(||f||_{\dot{K}^{\alpha}_{p/u,uq}\mathcal{M}_{\mu/u}}\right)^{u} \tag{9}$$

holds for all $f \in \dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu$.

Proof. (i) We use the Hölder inequality and represent $f = \delta(x)f$, where $\delta(x)$ is the Dirac δ-function, and there exists some l and t such that $\frac{1}{p} = \frac{1}{t} + \frac{1}{l}$

$$||f\delta(x)||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \geq ||f||_{\dot{K}^{\alpha}_{t,q}\mathcal{M}_\mu}||\delta(x)||_{\dot{K}^{\alpha}_{l,q}\mathcal{M}_\mu} = ||f||_{\dot{K}^{\alpha}_{t,q}\mathcal{M}_\mu}.$$

(ii) Using direct manipulations we receive

$$\begin{aligned} |||f|^u||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} &= \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\mu}{p}}\left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left|\int_{B_r(x_0)} |f(x)\chi_k|^p\right|^{q/p}\right)^{1/q} \\ &= \sup_{x\in\mathbb{R}^n, r>0}\left(r^{-\frac{\mu}{up}}\left(\sum_{k\in\mathbb{Z}} 2^{uk\alpha q}\left|\int_{B_r(x_0)} |f(x)\chi_k|^{p/u}\right|^{uq/p}\right)^{1/uq}\right)^{u}. \end{aligned}$$

□

Now it is necessary to provide helpful inequalities, containing Hardy-Littlewood and Peetre maximal functions.

Theorem 1. *Let $0 < p, q < \infty$, $1 < \eta < \infty$ and $0 < \mu \leq n$, then*
(i) For all $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ we have

$$||\mathcal{M}f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}. \tag{10}$$

In addition, for a sequence $\{f_j\}_{j=1}^{\infty}$ of $\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu$ we have

$$||\sup_{j\in\mathbb{N}} \mathcal{M}f_j||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \lesssim ||\sup_{j\in\mathbb{N}} |f_j|||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}. \tag{11}$$

(ii) For $\{f_j\}_{j=1}^{\infty}$ of $\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu$ we have

$$\left|\left|\left(\sum_{j=1}^{\infty}(\mathcal{M}f_j)^{\eta}\right)^{1/\eta}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \lesssim \left|\left|\left(\sum_{j=1}^{\infty}|f_j|^{\eta}\right)^{1/\eta}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}. \tag{12}$$

Proof. According to Lemma 5 from [21], we obtain (10)-(11). Therefore, from (i) we receive (ii). □

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We should to collect some basic properties of Herz-type Besov- Morrey and Triebel-Lizorkin-Morrey spaces.

Theorem 2. *Let $0 < p,q \leq \infty$, $0 < \mu \leq n$, $0 < r \leq \infty$, $\alpha, s \in \mathbb{R}$, $0 < \nu < n$, $\{\theta_k\}_{k=0}^{\infty} \subset (\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$.*

(i) If $a > \frac{n}{p}$, then we have

$$||\{2^{ks}(\phi_k^* f)_a\}_{k=0}^{\infty}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu})} \lesssim ||\{2^{ks}\mathcal{F}^{-1}[\phi_k \mathcal{F}f]\}_{k=0}^{\infty}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu})}.$$

(ii) If we assume $a > \frac{n}{\min(p,q,r)}$, then we have

$$||\{2^{ks}(\phi_k^* f)_a\}_{k=0}^{\infty}||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(l^r)} \lesssim ||\{2^{ks}\mathcal{F}^{-1}[\phi_k \mathcal{F}f]\}_{k=0}^{\infty}||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(l^r)}.$$

Proof. The estimates (i) and (ii) are the consequences of Theorems 5.4 and 5.5 from [22], μ is a constant function. □

Proposition 1. *Let $0 < p,q \leq \infty$; $0 < \mu \leq n$; $0 < r_1, r_2 \leq \infty$; $\alpha, s \in \mathbb{R}$, then*

(i) If $r_1 \leq r_2$, then we have $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r1}(\mathbb{R}^n) \hookrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r2}(\mathbb{R}^n)$.

(ii) If $\epsilon > 0$, then we have $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}(\mathbb{R}^n) \hookrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-\epsilon}_{\mu,r}(\mathbb{R}^n)$.

(iii) $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,\min(p,q,r)} \hookrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r} \hookrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,\infty}$ for $r \in (0,\infty)$.

Proof. (i) and (ii) have been proved by Proposition 3.3 [20]. We omit the proof of (iii) because it is obtained by an argument similar to [25] Proposition 2 in section 2.3.2. □

Let us imply the Hardy-type inequality, containing the norm of $\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}$, which is a corollary of Lemma 3.9 from [23].

Lemma 3. *Let $0 < p,q \leq \infty$; $0 < \mu \leq n$; $0 < r \leq \infty$ and $\delta > 0$. Let $\{g_v\}_{v=-\infty}^{\infty}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and*

$$G_j(x) := \sum_{v \in \mathbb{Z}} 2^{-(j-v)\delta} g_v(x), \ x \in \mathbb{R}^n, j \in \mathbb{Z}.$$

Then we have

$$||\{G_j]\}_{j=-\infty}^{\infty}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu})} \lesssim ||\{g_v\}_{v=-\infty}^{\infty}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu})}$$

and

$$||\{G_j]\}_{j=-\infty}^{\infty}||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(l^r)} \lesssim ||\{g_v\}_{v=-\infty}^{\infty}||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(l^r)}.$$

From [23] we provide useful inequality.

Lemma 4. *Let $a, b > 0$, $M \in \mathbb{N}$ and $h \in \mathbb{R}^n$. We define*

$$P_{b,a}f(x) := \sup_{z \in \mathbb{R}^n} \frac{|f(x-z)|}{1+|bz|^a}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$ with supp$\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| < b\}$, then we have

$$|\Delta_h^M f(x)| \lesssim \max(1, |bh|^a) \min(1, |bh|^M) P_{b,a}f(x),$$

for all $x \in \mathbb{R}^n$.

Remark, that Δ is defined in section 4.

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For atomic, molecular and wavelet decompositions we need to introduce the following facts. Let $f \in \dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}$. Then for all $\eta > 0$ there holds

$$\sup_{z\in\mathbb{R}^n} \frac{|f(x-z)|}{1+|z|^{\frac{n}{\eta}}} \lesssim \mathcal{M}^{(\eta)} f(x). \tag{13}$$

Now, we consider the lifting properties. If $\sigma \in \mathbb{R}$, then operator I_σ is defined as

$$I_\sigma f = \mathcal{F}^{-1}(1+|x|^2)^{\sigma/2}\mathcal{F}f,\ f \in \mathcal{S}'(\mathbb{R}^n).$$

It is well-known that I_σ is an one to one mapping on $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, respectively. Moreover, the equality $I_\sigma I_\tau = I_{\sigma+\tau}$ is valid for all $\sigma, \tau \in \mathbb{R}$.

Theorem 3. *Let $s,\sigma \in \mathbb{R}$, $r \in (0,\infty]$, $m,n \in \mathbb{N}$, $0 < p,q \le \infty$, $0 < \mu \le n$. Then $I_\sigma : \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r} \to \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-\sigma}_{\mu,r}$. Moreover,*

$$\sum_{|\gamma|\le m} ||\mathcal{F}^{-1}x^\gamma \mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} \ \ and\ \ ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} + \sum_{j=1}^{n} ||\partial_j^m f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}}$$

are equivalent quasi-norms on $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$.

Proof. If $f \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$, by the estimate $|\left(\mathcal{F}^{-1}\phi_k\mathcal{F}f\right)| \le \phi *_k f$, we have

$$\begin{aligned} ||I_\sigma f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-\sigma}_{\mu,r}} &= ||2^{(s-\sigma)k}\mathcal{F}^{-1}(1+|\cdot|)^{\sigma/2}\phi_k\mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}} \\ &= ||2^{sk}\mathcal{F}^{-1}\phi_k\mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}} \le C||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}, \end{aligned} \tag{14}$$

where C does not depend on f. Therefore, I_σ is continuous.

If $g \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$ and

$$||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}} \le C||g||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-\sigma}_{\mu,r}} = C||I_\sigma f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-\sigma}_{\mu,r}}. \tag{15}$$

Hence, by (14) and (15), I_σ is an isomorphism.

Next, we prove that the quasi-norm, containing $\mathcal{F}^{-1}x^\gamma\mathcal{F}$, is an equivalent quasi-norm on $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$. For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ let $x^\gamma = \prod_{j=1}^{n} x_j^{\gamma_j}$, where $\gamma = (\gamma_1, ..., \gamma_n)$. If $|\gamma| \le m$, $f \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$, then we get

$$\begin{aligned} &\sum_{|\gamma|\le m} ||\mathcal{F}^{-1}x^\gamma\mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} \\ &= \sum_{|\gamma|\le m} ||\mathcal{F}^{-1}x^\gamma(1+|x|^2)^{-m/2}\mathcal{F}\mathcal{F}^{-1}(1+|x|^2)^{m/2}\mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} \\ &\le C||I_m f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} \le C||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}, \end{aligned} \tag{16}$$

where the last inequality is obtained by (15).

Finally, we assume that $f \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$ and $\partial^m f \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}$ for $j = 1, ..., n$. We claim that there exist Fourier multipliers $\rho_1(x), ..., \rho_n(x)$ on $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}$, and positive constant C such that $1 + \sum_{j=1}^{n}\rho_j(x)x_j^m \ge C(1+|x|^2)^{m/2}$, for all $x \in \mathbb{R}^n$. Then we have

$$||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} \le C||\mathcal{F}^{-1}(1+|x|^2)^{m/2}\mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}}$$

$$\le C||\mathcal{F}^{-1}(1+|x|^2)^{m/2}\left[1+\sum_{j=1}^{n}\rho_j(x)x_j^m\right]\mathcal{F}\mathcal{F}^{-1}\left[1+\sum_{j=1}^{n}\rho_j(x)x_j^m\right]\mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}}$$

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$$\leq C||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} + C\sum_{j=1}^{n}||\mathcal{F}^{-1}\rho_j(x)x_j^m\mathcal{F}f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}}.$$

However, $x_j^m\mathcal{F}f = C\mathcal{F}\partial_j^m f$. By Fourier multiplier properties of $\rho_j(x)$, we obtain

$$||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}} \leq C||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}} + C\sum_{j=1}^{n}||\partial_j^m f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-m}_{\mu,r}}. \tag{17}$$

By (16) and (17), we prove that the quasi-norms, containing partial derivatives, are equivalent quasi-norms on $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$. □

It is possible to provide the corollary of Theorem 3.

Corollary 2. *Let $0 < p, q \leq \infty$; $0 < \mu \leq n$; $\alpha, s, \sigma \in \mathbb{R}$; $r \in (0, \infty]$ and $m \in \mathbb{N}$. Then*
(i) $(1-\Delta)^{\sigma} : \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r} \longrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-2\sigma}_{\mu,r}$,
(ii) $(1+(-\Delta)^m) : \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r} \longrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-2m}_{\mu,r}$,
(iii) $(1+\sum_{i=1}^{n}\partial_i^{4m}) : \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r} \longrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-4m}_{\mu,r}$
(iv) $1+\sum_{i=1}^{n}(-\partial_i^2)^m) : \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r} \longrightarrow \dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s-2m}_{\mu,r}$
are isomorphic.

Let us provide three useful statements, which can help to prove atomic and molecular decompositions.

Lemma 5. *Let $0 < p, q \leq \infty$; $0 < \mu \leq n$; α, s ,$r \in \mathbb{R}$ and $r \in (0, \infty]$. Any atom belongs to $\dot{k}^{\alpha}_{p,q}\dot{n}^{s}_{\mu,r} \cup \dot{k}^{\alpha}_{p,q}\dot{e}^{s}_{\mu,r}$. Furthermore any compactly supported function a with $K \geq (1+[s])$. Then $a \in \dot{k}^{\alpha}_{p,q}\dot{n}^{s}_{\mu,r} \cup \dot{k}^{\alpha}_{p,q}\dot{e}^{s}_{\mu,r}$.*

Proof. If $a \in C^K$ is a function supported on the unit ball and $||a||_{C^K} \leq 1$, then $||a||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}} \leq c$, where c is a constant.

Let us decompose $|\xi|^{4K}$

$$|\xi|^{4K} = \sum_{|\alpha|=K} P_{\alpha}(\xi)\xi^{\alpha},$$

where $P_{\alpha}(\xi)$ is a homogeneous polynomial of degree $3K$. We set $\tau_j^{\alpha}(\xi) := P_{\alpha}(2^{-j}\xi)|2^{-j}\xi|^{-4K}\phi_j(\xi)$. With this decomposition we obtain

$$\phi_j a = 2^{-jK}\tau_j^{\alpha}\partial^{\alpha}a.$$

From this expression we see that, for every $P \in \mathbb{N}$, there exists a constant $c > 0$ such that the estimate

$$|\phi_j a(x)| \leq c2^{-jK}\sqrt{1+|x|^2}$$

holds for every $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. From this estimate we can get

$$2^{js}||\phi_j a(x)||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} \leq c2^{j(s-K)}$$

and

$$2^{js}||\phi_j a(x)||_{l^r} \leq c2^{j(s-K)},$$

for $j \in \mathbb{N}_0$.

What remains to be dealt with is the assertion that any atom centered at Q_{0m} belongs to $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$. This can be achieved by using the partition of unit and what we have proved for $a \in C^K$ such that $||a||_{C^K} \leq 1$ and $supp(a) \subset Q(1)$, where $Q(r) = \{x \in \mathbb{R}^n, r > 0 : \max(|x_1|, ..., |x_n|) \leq r\}$ means a cube. The strong decay condition gives us the desired convergence. □

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Let us define $k_0 = \phi$ and $k = \Delta^{2N}\phi$, where $N > 0$ is a large integer depending on the parameters α, p, q, μ, r, s. Let $k_j(x) = 2^{jn}k(2^j x)$ for $j \in \mathbb{N}$. Next theorem is a generalization of Theorem 4.1 from [14], where instead of Morrey spaces used Herz-type Morrey spaces.

Theorem 4. *Suppose that $0 < \mu \le p \le \infty$, $0 < q \le \infty$, $\alpha, s \in \mathbb{R}$ and $r \in (0, \infty]$. There exists a constant c such that*
(i) $c^{-1}||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^{s}_{\mu,r}} \le ||2^{jk}k_j * f||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu})} \le c||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^{s}_{\mu,r}}$,
(ii) $c^{-1}||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{E}^{s}_{\mu,r}} \le ||2^{jk}k_j * f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(l^r)} \le c||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{E}^{s}_{\mu,r}}$.

Lemma 6. *Suppose that $0 < p, q \le \infty$; $s, \alpha \in \mathbb{R}$, $0 < \mu < n$ and $r \in (0, \infty]$. Let $K, L \in \mathbb{Z}$ be an integer satisfying*

$$K \ge (1 + [s])_+,\ L \ge \min(-1, [\sigma_{p,q} - s]).$$

For $v \in \mathbb{Z}^n$ we are given an atom centered at $2^{-v}m$.
Assume addition, that a doubly indexed sequence $\lambda = \{\lambda_{v,m}\}_{v\in\mathbb{N}, m\in\mathbb{Z}^n}$ belongs to $\dot{k}^{\alpha}_{p,q}\dot{a}^{s}_{\mu,r}$. Then the series

$$\lim_{p\longrightarrow\infty} \sum_{v=0}^{\infty} \left(\sum_{m\in\mathbb{Z}^n} \lambda_{v,m} m_{v,m} \right)$$

converges in $\mathcal{S}'$.

Proof. The convergence in $\mathcal{S}'$ of the sum $\lim_{p\longrightarrow\infty} \sum_{v=0}^{\infty} (\sum_{m\in\mathbb{Z}^n} \lambda_{v,m} m_{v,m})$ is proved in Lemma 4.5 in [14] for molecules and atoms. Let $\phi \in \mathcal{S}$, then in the case of $\dot{k}^{\alpha}_{p,q}\dot{a}^{s}_{\mu,r}$ with $L > -1 - s$ we obtain

$$\sum_{v=1}^{\infty} \left| \int_{\mathbb{R}^n} \left(\sum_{m\in\mathbb{Z}^n} \lambda_{vm} a_{vm}(x) \right) \phi(x) dx \right| \le c||a||_{\dot{k}^{\alpha}_{p,q}\dot{a}^{s}_{\mu,r}}.$$

This gives us the convergence of $\sum_{v=0}^{\infty} (\sum_{m\in\mathbb{Z}^n} \lambda_{v,m} m_{v,m})$ in $\mathcal{S}'$. □

Proposition 2 ([26]). *Let $\eta > 0$. Then there exists $c > 0$ such that*

$$\sup_{z\in\mathbb{R}^n} \langle z\rangle^{-\frac{n}{\eta}} |f(x - z)| \lesssim \mathcal{M}^{(\eta)} f(x),$$

where $\langle a\rangle = \sqrt{1 + |a|^2}$.

4. Difference and Local Means Approach

Let f be an arbitrary function on $\mathbb{R}^n$ and $x, h \in \mathbb{R}^n$. Then

$$\Delta_h f(x) = f(x + h) - f(x),\ \Delta_h^{M+1} f(x) = \Delta_h(\Delta_h^m f)(x),\ M \in \mathbb{N}.$$

$$\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x + (M - j)h),$$

where $\binom{M}{j}$ – binomial coefficients.

Under the ball means of differences, we mean the quantity

$$d_t^M f(x) = t^{-n} \int_{|h|\le t} |\Delta_h^M f(x)| dh = \int_{\{y\in\mathbb{R}^n: |h|\le 1\}} |\Delta_{th}^M f(x)| dh,$$

which is used to characterize $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$ via the ball means of differences.

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Definition 4. *For $0 < p, q \leq \infty$, $0 < \mu < n$, $r \in (0, \infty]$ and $s, \alpha \in \mathbb{R}$, the homogeneous Herz-type Besov-Morrey space $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ is the set of $f \in \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the set of polynomials, such that*

$$||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} = ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} + \left(\int_0^{\infty} t^{-sr} \left|\left|d^M_t f\right|\right|^r_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} \frac{dt}{t}\right)^{1/r} < \infty,$$

Analogically, we define Herz-type Triebel-Lizorkin spaces $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}$ using the ball means of differences.

Definition 5. *For $0 < p, q \leq \infty$, $0 < \mu < n$, $r \in (0, \infty]$ and $s, \alpha \in \mathbb{R}$, the homogeneous Herz-type Triebel-Lizorkin-Morrey space $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}$ is the set of $f \in \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ is the set of polynomials, such that*

$$||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}} = ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} + \left|\left|\left(\int_0^{\infty} t^{-sr} \left(d^M_t f\right)^r \frac{dt}{t}\right)^{1/r}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} < \infty.$$

Additionally, we provide discrete quasi-norms for $\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$, $\mathcal{A} = \{\mathcal{N}, \mathcal{E}\}$:

$$(i)\ ||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}} = ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} + \left|\left|\left(\sum_{k\in\mathbb{Z}} 2^{ksr} \left(d^M_{2^{-k}} f\right)^r\right)^{1/r}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} < \infty,$$

$$(ii)\ ||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} = ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} + \left(\sum_{k\in\mathbb{Z}} 2^{ksr} \left|\left|d^M_{2^{-k}} f\right|\right|^r_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}\right)^{1/r} < \infty.$$

Afterwards, we establish the equivalence between quasi-norms $||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$, $||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$ and $||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 5. *Let $0 < p, q \leq \infty$, $0 < \mu < n$, $s, \alpha \in \mathbb{R}$, $r \in (0, \infty]$, $M \in \mathbb{N}$ and $M > s$:*

(i) If $s > \sigma_{p,q,r}$, then $f \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}$ holds if and only if $f \in \dot{K}^{\alpha}_{p,q} \cup \mathcal{S}'$ and $||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}} < \infty$. Furthermore, $||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}}$ and $||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}}$ are equivalent. The same statement holds for $||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}}$.

(ii) If $s > \sigma_{p,q,r}$, then $f \in \dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}$ holds if and only if $f \in \dot{K}^{\alpha}_{p,q} \cup \mathcal{S}'$ and $||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} < \infty$. Furthermore, $||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$ and $||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$ are equivalent. The same statement holds for $||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$.

Proof. 1) First we need to prove that quasi-norms $||f||^{*}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}}$ and $||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}}$ are equivalent. It is easy to see that

$$\left[\int_0^{\infty} t^{-sr} \left(\int_{|h|\leq 1} |\Delta^M_{th} f(x)| dh\right)^r \frac{dt}{t}\right]^{1/r} = \left[\sum_{k\in\mathbb{Z}} \int_{2^{-k-1}}^{2^{-k}} t^{-sr} \left(t^{-n} \int_{tB} |\Delta^M_u f(x)| du\right)^r \frac{dt}{t}\right]^{1/r} \tag{18}$$

If $2^{-k-1} \leq t \leq 2^{-k}$, then $2^{ksr} \leq t^{-sr} \leq 2^{(k+1)sr}$ and

$$2^{kn} \int_{2^{(-k+1)}B_1(0)} |\Delta^M_u f(x)| du \lesssim t^{-n} \int_{tB_1(0)} |\Delta^M_u f(x)| du \lesssim 2^{(k+1)n} \int_{2^{-k}B_1(0)} |\Delta^M_u f(x)| du. \tag{19}$$

According to (18) and the right-hand side of (19) we obtain

$$\left[\int_0^{\infty} t^{-sr} \left(\int_{B_1(0)} |\Delta^M_{th} f(x)| dh\right)^r \frac{dt}{t}\right]^{1/r} \lesssim \left[\sum_{k\in\mathbb{Z}} \int_{2^{-k-1}}^{2^{-k}} 2^{ksr} \left(\int_{B_1(0)} |\Delta^M_{2^{-k}} f(x)| du\right)^r \frac{dt}{t}\right]^{1/r}.$$

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It shows that $||f||^*_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^s_{\mu,r}} \lesssim ||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^s_{\mu,r}}$. The inequality $||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^s_{\mu,r}} \lesssim ||f||^*_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^s_{\mu,r}}$ is implied by (18) and the left hand side of (19). The same technique we use for $\dot{K}^{\alpha}_{p,q}\mathcal{N}^s_{\mu,r}$.

2) (i) Before proving the inequality

$$||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^s_{\mu,r}} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}, \tag{20}$$

it is necessary to show that $||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \leq ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}$. By $s > 0$ and Proposition 1, we have

$$||f||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \leq \sum_{k\in\mathbb{Z}} ||\mathcal{F}^{-1}[\phi_k\mathcal{F}f]||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \leq \sum_{k\in\mathbb{Z}} -2^{s\min(1,p,q,r)}||f||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \lesssim ||f||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}.$$

Let $\tau_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\tau_0(x) = 1$ for $|x| \leq 1$ and $\operatorname{supp}(\tau_0) \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$. For $j \in \mathbb{N}$

$$\tau_j(x) = \tau_0(2^{-j}x) - \tau_0(2^{-j+1}x).$$

We use the decomposition $f = \sum_{k\in\mathbb{Z}} f_{(k+l)} = \sum_{k\in\mathbb{Z}} \mathcal{F}^{-1}[\tau_{k+l}\mathcal{F}f]$, where $f_{(k+l)} = 0$ if $k+l < 0$. Hence, for $r \leq 1$ one receives

$$||f(x)||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^s_{\mu,r}} = \sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}f(x)|dh\right)^r = \sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}\mathcal{F}^{-1}[\tau_{k+l}\mathcal{F}f](x)|dh\right)^r$$

$$\leq \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}f_{(k+l)}(x)|dh\right) = \sum_{l=-\infty}^{0}\sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}f_{(k+l)}(x)|dh\right)$$
$$+\sum_{l=1}^{\infty}\sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}f_{(k+l)}(x)|dh\right) = I_1 + I_2.$$

For receiving the desired inequality (20) it suffices to prove $||I_i||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}$, where $i = 1,2$.

If r > 1, by Minkovski inequality, we get

$$\left(\sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}f(x)|dh\right)^r\right)^{1/r} \leq \left(\sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} \sum_{l\in\mathbb{Z}} |\Delta^M_{2^{-k}h}f_{(k+l)}(x)|dh\right)^r\right)^{1/r}$$

$$\leq \sum_{l=-\infty}^{0}\left(\sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}f_{(k+l)}(x)|dh\right)^r\right) + \sum_{l=1}^{\infty}\left(\sum_{k\in\mathbb{Z}} 2^{ksr}\left(\int_{B_1(0)} |\Delta^M_{2^{-k}h}f_{(k+l)}(x)|dh\right)^r\right).$$
$$= I_3 + I_4$$

In this case it suffices to prove $||I_i||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}$, where $i = 3,4$.

We estimate I_1 and I_3, using Lemma 4 in the form

$$|\Delta^M_h f_{(k+l)}(x)| \leq \max(1, |bh|^a)\min(1, |bh|^M)P_{b,a}f_{(k+l)}(x),\ a > 0,\ b = 2^{k+l}$$

and

$$P_{b,a}f(x) = \sup_{x\in\mathbf{R}^n}\frac{|f(x-z)|}{1+|bz|^a}.$$

We use estimate with $2^{-k}h$ instead of h and obtain

$$\int_{B_1(0)} |\Delta^M_{2^{-k}h}f_{(k+l)}(x)|dh \lesssim \int_{B_1(0)} \max(1, |b2^{-k}h|^a)\min(1, |b2^{-k}h|^M)P_{b,a}f_{(k+l)}(x)dh$$
$$\leq 2^{lM}P_{2^{k+l},a}f_{(k+l)}(x). \tag{21}$$

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In fact $\min(1,|b2^{-k}h|^a)\le 1$, $\min(1,|b2^{-k}h|^M)\le w^{lM}$ ($l\le 0$ and $|h|\le 1$).

If $r\le 1$, by $M>s$ and $f_{(k+l)}=0$, where $k+l<0$, we see that

$$I_1\lesssim\sum_{l=-\infty}^{0}\sum_{k\in\mathbb{Z}}2^{ksr}(2^{lM}P_{2^{k+l},a}f_{(k+l)}(x))^r\lesssim\sum_{l=-\infty}^{0}2^{l(M-s)r}\sum_{k\in\mathbb{Z}}2^{(k+l)sr}(P_{2^{k+l},a}f_{(k)}(x))^r$$

$$\lesssim\sum_{k=0}^{\infty}2^{ksr}(P_{2^k,a}f_{(k)}(x))^r.$$

By the boundness of Peetre maximal function (Theorem 2), we get

$$||I_1^{1/r}||_{\dot K^\alpha_{p,q}\mathcal{M}_\mu}\lesssim||\{2^{ks}P_{2^k,a}f_{(k)}\}_{k=0}^\infty||_{\dot K^\alpha_{p,q}\mathcal{M}_\mu(l^r)}\lesssim||\{2^{ks}f_{(k)}\}_{k=0}^\infty||_{\dot K^\alpha_{p,q}\mathcal{M}_\mu(l^r)}\lesssim||f||_{\dot K^\alpha_{p,q}\dot{\mathcal{E}}^s_{\mu,r}}.$$

If $r>1$,

$$I_3\lesssim\sum_{l=-\infty}^{0}\left(\sum_{k\in\mathbb{Z}}2^{ksr}(2^{lM}P_{2^{k+l},a}f_{(k+l)}(x))^r\right)^{1/r}$$

$$=\sum_{l=-\infty}^{0}2^{l(M-s)}\left(\sum_{k\in\mathbb{Z}}2^{(k+l)sr}(P_{2^{k+l},a}f_{(k+l)}(x))^r\right)^{1/r}\lesssim\left(\sum_{k\in\mathbb{Z}}2^{ksr}(P_{2^k,a}f_{(k)}(x))^r\right)^{1/r}.$$

Therefore, we have

$$||I_3||_{\dot K^\alpha_{p,q}\mathcal{M}_\mu}\lesssim||f||_{\dot K^\alpha_{p,q}\dot{\mathcal{E}}^s_{\mu,r}}.$$

We consider the I_2 and I_4. Let $\lambda=1$ when $\min(p,q,r)>1$. Otherwise, there is a $\lambda\in(0,1)$ such that $s>\frac{n}{\min(p,q,r)}(1-\lambda)$ and $s>a(1-\lambda)$. This implies that there exists a real number $a>0$ such that $a>\frac{n}{\min(p,q,r)}$ and $s>a(1-\lambda)$.

By Lemma 4, we see that

$$\int_{B_1(0)}|\Delta^M_{2^{-k}h}f_{(k+l)}|dh=\int_{B_1(0)}|\Delta^M_{2^{-k}h}f_{(k+l)}|^{1-\lambda}dh\int_{B_1(0)}|\Delta^M_{2^{-k}h}f_{(k+l)}|^{\lambda}dh$$

$$\lesssim(2^{la}P_{2^{k+l},a}f_{(k+l)}(x))^{1-\lambda}\int_{B_1(0)}|\Delta^M_{2^{-k}h}f_{(k+l)}|^{\lambda}dh$$

$$\le(2^{la}P_{2^{k+l},a}f_{(k+l)}(x))^{1-\lambda}\sum_{j=0}^{M}((-1)^j\binom{M}{j}\int_{B_1(0)}|f^{(k+l)}(x+j2^{-k}h)|^{\lambda}dh$$

$$\lesssim(2^{la}P_{2^{k+l},a}f_{(k+l)}(x))^{1-\lambda}\sum_{j=0}^{M}(-1)^j\binom{M}{j}\mathcal{M}[|f_{(k+l)}|^{\lambda}],\tag{22}$$

where $\mathcal{M}$ is maximal Hardy-Littlewood operator.

If $r>1$, we denote

$$F(x)=\left(\sum_{k\in\mathbb{Z}}(2^{ks}P_{2^k,a}f_{(k)}(x))^r\right)^{1/r}$$

and

$$B_{k+l}(x)=|2^{(k+l)s}f_{(k+l)}(x)|,\ x\in\mathbb{R}^n.$$

Let $\delta=-(a(1-\lambda)-s)>0$. By Hölder inequality, we obtain

$$I_4=\sum_{l=1}^{\infty}\left(\sum_{k\in\mathbb{Z}}2^{ksr}\left(\int_{B_1(0)}|\Delta^M_{2^{-k}h}f_{(k+l)}(x)|\right)^r\right)^{1/r}$$

$$\lesssim \sum_{l=1}^{\infty} 2^{-l\delta} \left(\sum_{k\in\mathbb{Z}} (2^{(k+l)s} P_{2^k,a} f_{(k+l)}(x))^{(1-\lambda)r} (\mathcal{M}[|B_{(k+l)}|^{\lambda}])^r(x) \right)^{1/r}$$

$$\lesssim F(x)^{1-\lambda} \sum_{l=1}^{\infty} 2^{-l\delta} \left(\sum_{k\in\mathbb{Z}} (\mathcal{M}[|B_{(k+l)}|^{\lambda}])^{r/\lambda}(x) \right)^{\lambda/r}.$$

The Hölder inequality [9] implies that

$$||F_1^{1-\lambda} F_2^{\lambda}||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} \le ||F_1||^{1-\lambda}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} + ||F_2||^{\lambda}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}. \tag{23}$$

By (23), Lemma 1 and Theorem 1,

$$||I_4||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} \lesssim ||F^{1-\lambda} \sum_{l=1}^{\infty} 2^{-l\delta} \left(\sum_{k\in\mathbb{Z}} \mathcal{M}[B^{\lambda}_{(k+l)}] \right)^{\lambda/r} ||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}$$

$$\lesssim \sum_{l=1}^{\infty} 2^{-l\delta \min(1,p,q,r)} \left|\left| F^{1-\lambda} \left(\sum_{k\in\mathbb{Z}} \mathcal{M}[B^{\lambda}_{(k+l)}]^{r/\lambda} \right)^{\lambda/r} \right|\right|^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}$$

$$\lesssim ||f||^{\min(1,p,q,r)(1-\lambda)}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}} \sum_{l=1}^{\infty} 2^{-l\delta \min(1,p,q,r)} \left|\left| \{\mathcal{M}[B^{\lambda}_{(k+l)}]\}_{k\in\mathbb{Z}} \right|\right|^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(l^r)}$$

$$\lesssim ||f||^{\min(1,p,q,r)(1-\lambda)}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}} \sum_{l=1}^{\infty} 2^{-l\delta \min(1,p,q,r)} \left|\left| \{B^{\lambda}_{(k+l)}\}_{k\in\mathbb{Z}} \right|\right|^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}(l^r)} \le ||f||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}}. \tag{24}$$

Next we estimate I_2, when $r \le 1$. By Hölder inequality we have

$$I_2^{1/r} \lesssim F(x)^{(1-\lambda)/r} \sum_{l=1}^{\infty} 2^{(la(1-\lambda)-ls)/2} \left(\sum_{k\in\mathbb{Z}} (\mathcal{M}[|B_{(k+l)}|^{\lambda}])^{r/\lambda} \right)^{\lambda}.$$

Using the similar arguments as in (24), we have

$$||I_2||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}} \lesssim ||f||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{E}}^{s}_{\mu,r}}.$$

(ii) Now we prove $||f||^{**}_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$. According to the definition of quasi-norm $||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$, we consider $l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu})$ quasi-norm of

$$f^{(k)}(x) := 2^{ks} \int_{B_1(0)} |\Delta^M_{2^{-k}h} f(x)| dh = 2^{ks} \int_{B_1(0)} |\Delta^M_{2^{-k}h} \left(\sum_{l\in\mathbb{Z}} f_{(k+l)} \right) | dh$$

We split $f^{(k)}(x)$ into two parts as below, applying $f = \sum_{l\in\mathbb{Z}} f_{(k+l)}$, $k \in \mathbb{Z}$:

$$f^{(k)} = 2^{ks} \int_{B_1(0)} \left| \Delta^M_{2^{-k}h} \left(\sum_{l\in\mathbb{Z}} f_{(k+l)} \right) \right| dh$$

$$\le \sum_{l=-\infty}^{0} 2^{ks} \int_{B_1(0)} |\Delta^M_{2^{-k}h} f_{(k+l)}| dh + \sum_{l=1}^{\infty} 2^{ks} \int_{B_1(0)} |\Delta^M_{2^{-k}h} f_{(k+l)}| dh = f^{(k),I} + f^{(k),II}.$$

Firstly we prove $||\{f^{(k),I}\}_{k\in\mathbb{Z}}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}))} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{N}}^{s}_{\mu,r}}$. Let

$$g^1_u := 2^{us} P_{2^u,a} f_{(u)}(x).$$

Then by (21), it is easy to see that

$$f^{(k),I} \lesssim \sum_{k=-\infty}^{0} 2^{l(M-s)} g^1_{k+l} = \sum_{u=-\infty}^{k} 2^{-2|k-u|(M-s)} g^1_u.$$

Thanks to Lemma 3 and Theorem 2 with $a > \frac{n}{\min(1,p,q)}$, we obtain

$$||\{f^{(k),I}\}_{k\in\mathbb{Z}}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))} \lesssim ||\{g_u\}^{\infty}_{u=0}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^s_{\mu,r}}.$$

Finally, we shall prove $||\{f^{(k),II}\}_{k\in\mathbb{Z}}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^s_{\mu,r}}$.

Due to conditions $s > \sigma_{p,q}$, we can choose such a real number a. Let $g^2_u(x) = |2^{us} f_{(u)}(x)|$, by (22) we have

$$f^{(k),II} \lesssim \sum_{l=1}^{\infty} 2^{ks} \int_{B_1(0)} |\Delta^M_{2^{-k}h} f_{(k+l)}(x)|dh \lesssim \sum_{l=1}^{\infty} 2^{lq(1-\lambda)-ls} (g^1_{k+1})^{1-\lambda} \mathcal{M}[|g^2_{k+l}|^{\lambda}](x).$$

Let $\delta = -(a(1-\lambda)-s) > 0$. By Lemma 1, (23) and Hölder inequality, we see that

$$||\{f^{(k),II}\}_{k\in\mathbb{Z}}||^{\min(1,p,q,r)}_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))} \lesssim \sum_{l=1}^{\infty} 2^{-l\delta\min(1,p,q,r)} ||\{(g^1_{k+l})^{1-\lambda}\mathcal{M}[|g^2_{k+l}|^{\lambda}]\}_{k\in\mathbb{Z}}||^{\min(1,p,q,r)}_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))}$$

$$\lesssim \sum_{l=1}^{\infty} 2^{-l\delta\min(1,p,q,r)} ||\{(g^1_{k+l})^{1-\lambda} ||\mathcal{M}[|g^2_{k+l}|^{\lambda}]||^{\lambda}_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu)}\}_{k\in\mathbb{Z}}||^{\min(1,p,q,r)}_{l^r}$$

$$\lesssim \sum_{l=1}^{\infty} 2^{-l\delta\min(1,p,q,r)} ||\{g^1_{k+l}\}^{(1-\lambda)\min(1,p,q,r)}_{k\in\mathbb{Z}} ||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))} ||\mathcal{M}[|g^2_{k+l}|^{\lambda}]^{1/\lambda}||^{\lambda\min(1,p,q,r)}_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))}.$$

Therefore, by Theorem 1, we obtain

$$||\{f^{(k),II}\}_{k\in\mathbb{Z}}||^{\min(1,p,q,r)}_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))}$$

$$\lesssim \sum_{l=1}^{\infty} 2^{-l\delta\min(1,p,q,r)} ||\{g^1_{k+l}\}^{(1-\lambda)\min(1,p,q,r)}_{k\in\mathbb{Z}} ||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu))} ||\mathcal{M}[|g^2_{k+l}|^{\lambda}]^{1/\lambda}||^{\lambda\min(1,p,q,r)}_{l^{r/\lambda}(\dot{K}^{\alpha}_{p/\lambda,q/\lambda}\mathcal{M}_{\mu/\lambda}))}$$

$$\lesssim ||f||^{\min(1,p,q,r)}_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^s_{\mu,r}}.$$

3) We prove that $||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{A}^s_{\mu,r}} \lesssim ||f||^{**}_{\dot{K}^{\alpha}_{p,q}\mathcal{A}^s_{\mu,r}}$ for $f \in \dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu \cup \mathcal{S}'$. Let $\psi \in C^{\infty}_0(\mathbb{R}^n)$ with $\psi(x) = 1$ whem $|x| \leq 1$ and $\psi(x) = 0$ when $|x| \geq 3/2$. We define

$$\tau_0(x) = (-1)^{M+1} \sum_{\mu=0}^{M-1} (-1)^{\mu} \binom{M}{\mu} \psi((M-\mu)x).$$

Note that $\tau_0 \in C^{\infty}_0(\mathbb{R}^n)$ with $\tau_0(x) = 0$ when $|x| \geq 3/2$ and $\tau_0(x) = 1$ when $|x| \leq 1/M$. We define $\tau_j(x) = \tau_0(2^{-j}x) - \tau_0(2^{-j+1}x)$ for $j \in \mathbb{N}$. The family $\{\tau_j\}^{\infty}_{j=0}$ is a partition of unity, $\tau_0(x) = (-1)^{M+1}(\Delta^M_x \psi(0) - (-1)^M)$ and that

$$\mathcal{F}^{-1}[\tau_j \mathcal{F}f](x) = \begin{cases} (\mathcal{F}^{-1}\Delta^M_{\xi}\psi(0)\mathcal{F}f)(x) + (-1)^{M+1}f(x), j = 0 \\ (\mathcal{F}^{-1}(\Delta^M_{2^{-j}\xi}\psi(0) - \Delta^M_{2^{-j+1}\xi}\psi(0))\mathcal{F}f)(x), j \in \mathbb{N} \end{cases}.$$

Kepka and Vybiral [23] proved that

$$|(\mathcal{F}^{-1}(\Delta^M_{2^{-j}\xi}\psi(0))\mathcal{F}f)(x)| \lesssim \int_{\mathbb{R}^n} |\mathcal{F}\psi(h)||\Delta^M_{2^{-j}\xi}f(x)|dh.$$

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for all $j \in \mathbb{N}_0$. For $\mathcal{A} = \mathcal{E}$, we put $g = \mathcal{F}\psi \in \mathcal{S}(\mathbb{R}^n)$ and obtain

$$||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{E}^s_{\mu,r}} \sim ||\{2^{js}\mathcal{F}^{-1}[\tau_j \mathcal{F}f]\}_{j=0}^{\infty}||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}\mu(l^r)}$$

$$\lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}\mu(l^r)} + \left|\left|\left\{2^{js}\int_{\mathbb{R}^n}|g(h)||\Delta^M_{2^{-j}h}f(x)|dh\right\}_{j=0}^{\infty}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}\mu}.$$

Let $I_0 = B_1(0)$ and $I_k = 2^k B_1(0) \setminus 2^{k-1}B_1(0)$ for $k \in \mathbb{N}$. Take $t > s + n$. By $g = \mathcal{F}\psi \in \mathcal{S}(\mathbb{R}^n)$, $|g(h)| \lesssim 2^{-kt}$ holds for all $h \in I_k$. Then we can estimate

$$\int_{\mathbb{R}^n}|g(h)||\Delta^M_{2^{-j}h}f(x)|dh = \sum_{k=0}^{\infty}\int_{I_k}|g(h)||\Delta^M_{2^{-j}h}f(x)|dh$$

$$\le \sum_{k=0}^{\infty}2^{-kt}\int_{I_k}|\Delta^M_{2^{-j}h}f(x)|dh \lesssim \sum_{k=0}^{\infty}2^{k(n-t)}d^M_{2^{k-j}}f(x).$$

We put $g_l(x) = 2^{ls}d^M_{2^{-j}}f(x)$ for $l \in \mathbb{Z}$. We see that

$$2^{js}\int_{\mathbb{R}^n}|g(h)||\Delta^M_{2^{-j}h}f(x)|dh \lesssim 2^{js}\sum_{k=0}^{\infty}2^{k(n-t)}d^M_{2^{k-j}}f(x) \lesssim \sum_{l\in\mathbb{Z}}2^{|j-l|(s+n-t)}g_l(x). \tag{25}$$

By virtue Lemma 3, we have

$$||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + \left|\left|\{2^{js}\int_{\mathbb{R}^n}|g(h)||\Delta^M_{2^{-j}h}f(x)|dh\}_{j=0}^{\infty}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu(l^r)} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}$$

$$+ \left|\left|\left\{\sum_{k\in\mathbb{Z}}2^{|j-k|(s+n-t)}g_k\right\}_{j=0}^{\infty}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu(l^r)} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + ||\{g_j\}_{j=0}^{\infty}||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu(l^r)} = ||f||^{**}_{\dot{K}^{\alpha}_{p,q}\mathcal{E}^s_{\mu,r}}.$$

Finally $\mathcal{A} = \mathcal{N}$.

$$||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^s_{\mu,r}} \sim ||\{2^{js}\mathcal{F}^{-1}[\tau_j\mathcal{F}f]\}_{j=0}^{\infty}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu)} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}$$

$$+ \left|\left|\left\{2^{js}\int_{\mathbb{R}^n}|g(h)||\Delta^M_{2^{-j}h}f(x)|dh\right\}_{j=0}^{\infty}\right|\right|_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu)}.$$

Using (25) with $t > s + n$ and applying Lemma 3 we have

$$||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^s_{\mu,r}} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + \left|\left|\{2^{js}\int_{\mathbb{R}^n}|g(h)||\Delta^M_{2^{-j}h}f(x)|dh\}_{j=0}^{\infty}\right|\right|_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu)} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}$$

$$+ \left|\left|\left\{\sum_{k\in\mathbb{Z}}2^{|j-k|(s+n-t)}g_k\right\}_{j=0}^{\infty}\right|\right|_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu)} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} + ||\{g_j\}_{j=0}^{\infty}||_{l^r(\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu)} = ||f||^{**}_{\dot{K}^{\alpha}_{p,q}\mathcal{N}^s_{\mu,r}}.$$

□

5. Atomic Decomposition

Let us introduce the following definition.

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Definition 6. *1. Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define*

$$Q_{\nu m} := \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu}\right).$$

2. Let $0 < p \leq \infty$, $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Then we define the p-normalized indicator $\chi_{\nu m}^{(p)}$ by

$$\chi_{\nu m}^{(p)} := 2^{n\nu/p}\chi_{Q_{\nu m}}.$$

3. Let $0 < \mu \leq p \leq \infty$, $0 < q \leq \infty$, $0 < r \leq \infty$. Let double-indexed sequences $\lambda := \{\lambda_{k.m}\}_{k\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C}$ then we denote $\dot{k}^{\alpha}_{p,q}\dot{n}^s_{\mu,r}$ and $\dot{k}^{\alpha}_{p,q}\dot{e}^s_{\mu,r}$ as Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey sequence spaces, which have corresponding quasi-norms:

$$||\lambda||_{\dot{k}^{\alpha}_{p,q}\dot{n}^s_{\mu,r}} = \left(\sum_{k\in\mathbb{Z}} 2^{ksr} \left| | \sum_{m\in\mathbb{Z}^n} \lambda_{k,m}\chi^{(p)}_{Q_{k,m}} \right| |^r_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu}\right)^{1/r} < \infty, \ \ 1 \leq r < \infty$$

$$||\lambda||_{\dot{k}^{\alpha}_{p,q}\dot{e}^s_{\mu,r}} = \left| | \left(\sum_{k\in\mathbb{Z}} 2^{ksr} \left|\sum_{m\in\mathbb{Z}^n} \lambda_{k,m}\chi^{(p)}_{Q_{k,m}}\right|^r\right)^{1/r} \right| |_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_\mu} < \infty, \ \ 1 \leq r < \infty,$$

Now we bring the definition of [K, L]-atoms.

Definition 7. *Let $d > 1$ and $K, L \in \mathbb{Z}$, where $K \geq 0$ and $L \geq -1$.*
1. The function $a \in C^K$ is called an [K, L]-atom centered at $Q_{0,m}$ for $m \in \mathbb{Z}^n$, if $supp(a) \subset dQ_{0,m}$ and $||\partial^\alpha a||_{L^\infty} \leq 1$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq K$.
2. The function $a \in C^K$ is said to an [K, L]-atom centered at $Q_{\nu,m}$ for $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^n$, if $supp(a) \subset dQ_{\nu,m}$, $||\partial^\alpha a/\partial x^\alpha||_{L^\infty} \leq 2^{-\nu\left(s-\frac{n}{p}\right)+\nu|\alpha|}$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq K$ and

$$\int_{\mathbb{R}^n} x^\beta a(x)dx = 0$$

for $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq L$.
3. The function $a \in C^K$ is called an [K, L]-atom centered at $Q_{0,m}$ for $m \in \mathbb{Z}^n$, if

$$|\partial^\alpha a(x)| \leq \langle x - x_0\rangle^{-M-|\alpha|}$$

if $|\alpha| \leq K$.
4. The function $a \in C^K$ is called an [K, L]-atom centered at $Q_{\nu,m}$ for $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^n$, if

$$|\partial^\alpha a(x)| \leq 2^{-(s-n/p)+\nu|\alpha|}\langle 2^\nu(x - x_0)\rangle^{-M-|\alpha|}$$

if $|\alpha| \leq K$ and

$$\int_{\mathbb{R}^n} x^\beta a(x)dx = 0$$

for $|\beta| \leq L$.

Here and below we always assume that $M \geq L + 10n \max(1, 1/q, 1/p, 1/\mu)$. Let us define the following values:

$$\sigma_{p,q} := n\left(\frac{1}{\min(1,p,q)} - 1\right), \ \sigma_{p,q,r} := n\left(\frac{1}{\min(1,p,q,r)} - 1\right),$$

where n is an dimension.

Additionally, we define the [K, L, M]-molecules.

Definition 8. *Let $K, L \in \mathbb{Z}$ and $M \geq L + 10n\max(1, 1/q, 1/p, 1/\mu)$, where $K \geq 0$ and $L \geq -1$. The function $m \in C^K$ is called a [K, L, M]-molecule concentrated in $Q_{\nu,\tau}$, $\nu \in \mathbb{N}$ and $\tau \in \mathbb{Z}^n$, if*

$$|\partial^\tau m(x)| \leq 2^{-(s-n/p)+|\tau|\nu} \langle 2^\nu (x - x_0) \rangle^{M-|\tau|}$$

for $|\tau| \leq K$ and

$$\int_{\mathbb{R}^n} x^\beta m(x) dx = 0$$

for $|\beta| \leq L$.

Theorem 6. *Let $k > 1$ be fixed. Suppose that the parameters $K, L \in \mathbb{Z}$ and $p, q, r, s \in \mathbb{R}$ satisfy $0 < \mu \leq p \leq \infty$, $0 < q \leq \infty$, $0 < r \leq \infty$, $K \geq (1 + [s])_+$, $L \geq \max(-1, [\sigma_{p,q} - s])$ for $\mathcal{N}$-scale and $0 < q \leq p \leq \infty$, $0 < r \leq \infty$, $K \geq (1 + [s])_+$, $L \geq \max(-1, [\sigma_{p,q,r} - s])$ for $\mathcal{E}$-scale.*

1)Assume that $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ is a family of atoms and $\lambda = \{\lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \dot{k}^\alpha_{p,q} a^s_{\mu,r}$. Then for sum $f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$ converges in $\mathcal{S}'$ and belongs to $\dot{K}^\alpha_{p,q} \dot{\mathcal{A}}^s_{\mu,r}$ with the norm estimate

$$||f||_{\dot{K}^\alpha_{p,q} \dot{\mathcal{A}}^s_{\mu,r}} \lesssim ||\lambda||_{\dot{k}^\alpha_{p,q} a^s_{\mu,r}}.$$

2) Conversely, any $f \in \dot{K}^\alpha_{p,q} \dot{\mathcal{A}}^s_{\mu,r}$ admits the following decomposition:

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m},$$

which converges in $\mathcal{S}'$. We can average the family of atoms $\{a_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ such that the family of coefficients $\lambda = \{\lambda_{\nu,m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \dot{k}^\alpha_{p,q} a^s_{\mu,r}$ fulfills the norm estimate

$$||\lambda||_{\dot{k}^\alpha_{p,q} a^s_{\mu,r}} \lesssim ||f||_{\dot{K}^\alpha_{p,q} \dot{\mathcal{A}}^s_{\mu,r}}.$$

Proof. 1) By Lemma 6 we may assume that the coefficients are zero with finite exception. In this case we have that $f \in \dot{K}^\alpha_{p,q} \dot{\mathcal{A}}^s_{\mu,r}$ by Lemma 5. To measure this norm we use Theorem 4. Let $j \geq \nu \geq 0$ with $j \neq 0$. Then

$$k_j * a_{\nu,m}(x) = 2^{jn} \int_{\mathbb{R}^n} k(2^j y) a_{\nu,m}(x - y) dy.$$

We take a homogeneous polynomial $P_\alpha(\xi)$ of degree $3k$ with $|\xi|^{4N} = \sum_{|\alpha|=K} \xi^\alpha P_\alpha(\xi)$, which met in proof of Lemma 5. Then $k(2^j y) = [\Delta^{2N} \phi](2^j y) = 2^{jk} \sum_{|\alpha|=k} \partial^\alpha_y ([P_\alpha(\partial)\phi](2^j y))$. As result we obtain

$$k_j * a_{\nu,m}(x) = 2^{j(n-k)} \sum_{|\alpha|=k} (-1)^k \int_{\mathbb{R}^n} P_\alpha(\partial)\phi(2^j y) \partial^\alpha a_{\nu,m}(x - y) dy. \tag{26}$$

Using Peetre inequality $\langle a + b \rangle \leq \sqrt{2} \langle a \rangle \langle b \rangle$ and taking into account that $j \geq \nu$, we obtain

$$\int_{B(2^{-j+1})} \langle 2^\nu (x - y) - a \rangle^{-M-K} dy \lesssim \langle 2^\nu x - a \rangle^{-M-K} \int_{B(2^{-j+1})} \langle 2^\nu y \rangle^{-M-K} dy$$

$$\lesssim \langle 2^\nu x - a \rangle^{-M-K} |B(2^{-j+1})| = 2^{-jn} \langle 2^\nu x - a \rangle^{-M-K}.$$

As a consequence, we have

$$2^{js} |k_j * a_{\nu,m}(x)| \lesssim 2^{j(n-k+s)} \sum_{|\alpha|=k} (-1)^k \int_{\mathbb{R}^n} |P_\alpha(\partial)\phi(2^j y)| |\partial^\alpha a_{\nu,m}(x - y)| dy$$

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$$\lesssim 2^{j(n-K+s)-\nu(s-\frac{n}{p}-\frac{n}{q}+K)} \int_{B(2^{-j+1})} \langle 2^\nu x - a\rangle^{-M-K} dy$$
$$\lesssim 2^{(j-\nu)(K-s)-n\nu(\frac{1}{q}+\frac{1}{p})} \langle 2^\nu x - a\rangle^{-M-K}.$$

Then we get

$$2^{js}|k_j * a_{\nu,m}(x)| \lesssim 2^{(j-\nu)(K-s)-n\nu(\frac{1}{q}+\frac{1}{p})} \langle 2^\nu x - a\rangle^{-M-K}.$$

Multiplying by $|\lambda_{\nu,m}|$ to both sides and adding the above inequality over $m \in \mathbb{Z}^n$, we obtain

$$\sum_{m\in\mathbb{Z}^n} 2^{js}|\lambda_{\nu,m} k_j * a_{\nu,m}(x)| \lesssim \sum_{m\in\mathbb{Z}^n} 2^{(j-\nu)(K-s)-n\nu(\frac{1}{q}+\frac{1}{p})} |\lambda_{\nu,m}| \langle 2^\nu x - a\rangle^{-M-K}$$
$$= \sum_{k\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} 2^{(j-\nu)(K-s)-n\nu(\frac{1}{q}+\frac{1}{p})} |\lambda_{\nu,m}| \langle 2^\nu x - a\rangle^{-M-K}$$
$$\lesssim 2^{(j-\nu)(K-s)-n\nu(\frac{1}{q}+\frac{1}{p})} \sum_{k\in\mathbb{N}_0} \frac{1}{2^{k(M+K)}} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|.$$

Note that a satisfying $2^k \le \langle 2^\nu x - a\rangle \le 2^{k+1}$ comparable to 2^{kn}. Therefore,

$$2^{n\nu(\frac{1}{p}+\frac{1}{q})} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| \lesssim 2^{\frac{kn}{\eta}} \mathcal{M}^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m}\chi_{\nu,m}\right),$$

where η is sufficiently close to $\min(1, p, q, \mu)$. Since M is assumed sufficiently large, we add

$$\sum_{m\in\mathbb{Z}^n} 2^{js}|\lambda_{\nu,m} k_j * a_{\nu,m}(x)| \lesssim 2^{(j-\nu)(K-s)} \mathcal{M}^{(\eta)}\left(\sum_{m\in\mathbb{Z}^\ltimes} \lambda_{\nu,m}\chi_{\nu,m}\right) \tag{27}$$

Let $\nu > j \ge 0$. Then if $j \ne 0$, we have

$$k_j * a_{\nu,m}(x) = \int_{\mathbb{R}^n} k(x,y) a_{\nu,m}(x - 2^{-j}y) dy,$$

where $N = M + n - L$ and

$$K(x,y) = (\Delta^N \phi)(y) - \sum_{|\beta|\le L} \frac{\partial^\beta \Delta^N \phi(2^j(x-2^{-\nu}a))}{\beta} (y - 2^j(x-2^{-\nu}a))^\beta.$$

If $|2^j(x-2^{-\nu}a) - y| \le \frac{|2^j(x-2^{-\nu}a)|}{2}$, then the mean value theorem gives us

$$|K(x,y)| \lesssim |y - 2^j(x-2^{-\nu}a)|^{L+1} \langle 2^j(x-2^\nu a)\rangle^N$$
$$\lesssim 2^{-(\nu-j)(L+1)} \langle 2^\nu(x-2^{-j}y) - a\rangle^{L+1} \langle 2^j(x-2^\nu a)\rangle^N.$$

Therefore, we obtain

$$2^{js} \int_{B_{2^j(x-2^{-\nu}a)}(\frac{|2^j(x-2^{-\nu}a|}{2})} |K(x,y)| a_{\nu,m}(x-2^{-j}y) dy$$
$$\lesssim \frac{2^{-(\nu-s)(L+1)-\frac{n\nu}{p}-\frac{n\nu}{q}}}{\langle 2^j(x-2^\nu a)\rangle^N} \int_{B_{2^j(x-2^{-\nu}a)}(\frac{|2^j(x-2^{-\nu}a|}{2})} \langle 2^\nu(x-2^{-j}y) - a\rangle^{-N} dy \tag{28}$$
$$\lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}} \langle 2^\nu(x-2^{-j}y) - a\rangle^{-N}.$$

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If $|2^j(x-2^{-\nu}a)-y| \le \frac{|2^j(x-2^{-\nu}a)|}{2}$, then we use

$$|K(x,y)| \lesssim 2^{-(\nu-j)(L+1)}\langle 2^\nu(x-2^{-j}y)-a\rangle^{L+1}.$$

By this estimate, we obtain

$$\begin{aligned} 2^{js}\int_{\mathbb{R}^n\setminus B_{2^j(x-2^{-\nu}a)}(\frac{|2^j(x-2^{-\nu}a|}{2}))}|K(x,y)|a_{\nu,m}(x-2^{-j}y)dy \\ \lesssim 2^{-(\nu-s)(L+1+s)-\frac{n\nu}{q}-\frac{n\nu}{p}}\int_{B_{2^j(x-2^{-\nu}a)}(\frac{|2^j(x-2^{-\nu}a|}{2}))}\langle 2^\nu(x-2^{-j}y)-a\rangle^{-N-n-1}dy \\ \lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}}\langle 2^\nu(x-2^{-j}y)-a\rangle^{-N}. \end{aligned} \tag{29}$$

Combining (28) and (29), we obtain

$$2^{js}|k_j*a_{\nu,m}| \lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}}\langle 2^\nu(x-2^{-j}y)-a\rangle^{-N}.$$

We multiply $\lambda_{\nu,m}$ to the above inequality and add it over $m\in\mathbb{Z}^n$.

$$\sum_{m\in\mathbb{Z}^n}2^{js}|\lambda_{\nu,m}k_j*a_{\nu,m}| \lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu,m}|\langle 2^\nu(x-2^{-j}y)-a\rangle^{-N}$$

$$\lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}}\sum_{k\in\mathbb{N}_0}\frac{1}{2^{kN}}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu,m}|.$$

Let η be a constant smaller than $\min(1,p,q,\mu)$. Then we obtain, by using $(a+b)^\eta \le a^\eta+b^\eta$,

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n}2^{js}|\lambda_{\nu,m}k_j*a_{\nu,m}| &\lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}}\sum_{k\in\mathbb{N}_0}\frac{1}{2^{kN}}\left(\sum_{m\in\mathbb{Z}^n,|2^j(x-2^{-\nu}a)|<2^k}|\lambda_{\nu,m}|^\eta\right)^{1/\eta} \\ &\lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}}\sum_{k\in\mathbb{N}_0}\frac{m_{B_{2^{k-j}}(x)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}\chi_{\nu,m}\right)^{1/\eta}}{2^{k\left(N-\frac{n}{\eta}\right)}} \\ &\lesssim 2^{-(\nu-s)(L+1+s+n)-\frac{n\nu}{p}-\frac{n\nu}{q}}\mathcal{M}^{(\eta)}\left[\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}\chi_{\nu,m}\right](x). \end{aligned}$$

Taking into account (27), we are led to

$$\sum_{m\in\mathbb{Z}^n}2^{js}|\lambda_{\nu,m}k_j*a_{\nu,m}(x)| \lesssim 2^{-2\delta(\nu-j)}\mathcal{M}^{(\eta)}\left[\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}\chi_{\nu,m}\right](x), \tag{30}$$

where $\delta>0$.

As before by the Hölder inequality we obtain

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$$
\begin{aligned}
&\left\{\sum_{j\in\mathbb{N}_0}\left[\sum_{\nu\in\mathbb{N}_0}2^{-2\delta|\nu-j|}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}\chi_{\nu,m}\right)\right]^q\right\}^{1/q}\\
&\lesssim\left\{\sum_{j\in\mathbb{N}_0}\sum_{\nu\in\mathbb{N}_0}2^{-2\delta|\nu-j|}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}\chi_{\nu,m}\right)(x)^q\right\}^{1/q} \qquad (31)\\
&\lesssim\left\{\sum_{\nu\in\mathbb{N}_0}2^{-2\delta|\nu-j|}M^{(\eta)}\left(\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}\chi_{\nu,m}\right)(x)^q\right\}^{1/q}.
\end{aligned}
$$

This inequality and the Fefferman-Stein vector-valued maximal inequality (10) yield (31). Thus, we have the desired result.

2) ($L=-1$) Let $M\in\mathbb{N}$ be constant larger that $K+1$ and $f\in\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$. Assume that $\{\tau_j\}_{j\in\mathbb{N}_0}$ a family $\phi_j(x)=\phi_0(2^{-j}x)-\phi_0(2^{-j+1}x)$ for $j>1$. $\chi_{Q(1)}\le\tau_0\le\chi_{Q(3/2)}$. We also take τ such that $\mathrm{supp}(\tau)\subset Q(1), Q(2)\subset\{\mathcal{F}_k\neq 0\}$. For $j\in\mathbb{N}_0$ set $\tau_j(x)=2^{jn}\tau(2^jx)$ and we define $\psi_j\in\mathcal{S}$ uniquely so that $\phi_j=(2\pi)^{-\frac{n}{s}}\mathcal{F}^{-1}[\psi_j\mathcal{F}\tau_j]$. Then

$$
f=\sum_{j=0}^{\infty}\mathcal{F}^{-1}[\phi_j\mathcal{F}f]=\sum_{j=0}^{\infty}\psi_j*\tau_j*f=\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{Q_{j,m}}\psi_j(x-y)\tau_j*f(y)dy.
$$

Let us set $\lambda_{j,m}=2^{j\left(s-\frac{n}{p}-\frac{n}{q}\right)}\sup_{y\in Q_{j,m}}|\tau_j*f(y)|$ and

$$
a_{j,m}=\begin{cases}0, j=0\\ \frac{1}{\lambda_{\nu,m}}\int_{Q_{j,m}}\psi_j(x-y)\tau_j*f(y)dy,\ otherwise\end{cases}.
$$

Then $f=\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n}\lambda_{j,m}a_{j,m}$. Lemma 4.11 yields $||\{\lambda_{j,m}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^n}||_{\dot{K}^{\alpha}_{p,q}\dot{a}^{s}_{\mu,r}}\lesssim||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}$.
($L\ge 0$) Let $M\in\mathbb{N}$ be constant larger that $K+1$ and $f\in\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}$ can be decomposed as

$$
f=g+(-\Delta)^M g,\ ||f||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s}_{\mu,r}}\simeq||g||_{\dot{K}^{\alpha}_{p,q}\dot{\mathcal{A}}^{s+2M}_{\mu,r}}.
$$

Observe that $g\in C^k$ by virtue of Theorem 3 provided M is large enough. Let $\psi\in\mathcal{S}$- compactly supported function with $\sum_{m\in\mathbb{Z}^n}\psi(x-m)=1$. Suppose that $\mathrm{supp}(\psi)\subset B_{2^r}(0)$.

$$
g(x)=\sum_{m\in\mathbb{Z}^n}\psi(x-m)g(x).
$$

We define coefficients $\lambda_{0,m}$ and functions $a_{0,m}$ by

$$
\lambda_{0,m}=\sup_{|\alpha|\le K}||\partial^{\alpha}(\psi(x-m)g)||_{\dot{K}^{\alpha}_{\infty,q}},\ a_{j,m}=\begin{cases}0,\psi(x-m)g)=0\\ \frac{1}{\lambda_{0,m}}\psi(x-m)g),otherwise\end{cases}.
$$

$$
g=\sum_{m\in\mathbb{Z}^n}\lambda_{0,m}a_{0,m}.
$$

Next we note that

$$
\left|\left|\sum_{m\in\mathbb{Z}^n}\lambda_{0,m}\chi_{0,m}\right|\right|_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}\lesssim\sum_{|\alpha|\le K}||\sup_{x-y\le c}|D^{\alpha}g(y)|||_{\dot{K}^{\alpha}_{p,q}\mathcal{M}_{\mu}}. \qquad (32)
$$

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We decompose

$$\partial^\alpha g(y) = \sum_{j\in\mathbb{N}^0} \mathcal{F}^{-1}[\phi_j \mathcal{F}\partial^\alpha g](y)$$

and use (13) to obtain a pointwise estimate,

$$\sup_{y\in B(x,c)} |\partial^\alpha g(y)| \le \sum_{j\in\mathbb{N}_0} \sup_{y\in B(x,c)} |\mathcal{F}^{-1}[\phi_j \mathcal{F}\partial^\alpha g(y)]| \lesssim \sum_{j\in\mathbb{N}_0} 2^{\frac{jn}{\eta}} \mathcal{M}^{(\eta)} \mathcal{F}^{-1}[\phi_j \mathcal{F}\partial^\alpha g(y)](x).$$

where η lightly less than $\min(1, p, q, \mu)$. Taking into account the triangle inequality

$$\left|\left|\sum_{j\in\mathbb{N}} |h_j|\right|\right|^\eta_{\dot{K}^\alpha_{p,q}\mathcal{M}_\mu} \le \sum_{j\in\mathbb{N}} ||h_j||_{\dot{K}^\alpha_{p,q}\mathcal{M}_\mu}.$$

and with help of Theorem 3, we receive

$$\left|\left|\sup_{|x-y|\le c} |\partial^\alpha g(y)|\right|\right|_{\dot{K}^\alpha_{p,q}\mathcal{M}_\mu} \lesssim \sum_{j\in\mathbb{N}} 2^{jn} ||\mathcal{F}^{-1}[\phi_j \mathcal{F}\partial^\alpha g(y)]||^\eta_{\dot{K}^\alpha_{p,q}\mathcal{M}_\mu} \lesssim ||g||_{\dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}. \quad (33)$$

As a result if M is large enough. Application of Theorem 3 and combination of (32) and (33) give us

$$\left|\left|\sum_{m\in\mathbb{Z}^n} \lambda_{0,m}\chi_{0,m}\right|\right|_{\dot{K}^\alpha_{p,q}\mathcal{M}_\mu} \lesssim ||g||_{\dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^{2n/\eta+|\alpha|}_{\mu,r}} \lesssim ||f||_{\dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}.$$

Next we can see $g \in \dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^{s+2M}_{\mu,r}$ with $s + 2M \ge \sigma_{p,q,M}$, $L = -1$, we obtain $g = \sum_{\nu=1}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} b_{\nu,m}$. Here λ and b satisfy the following conditions:

1. $||\lambda||_{\dot{k}^\alpha_{p,q}\dot{a}^s_{\mu,r}} \lesssim ||g||_{\dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^{s+2M}_{\mu,r}} \lesssim ||f||_{dot K^\alpha_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}$.
2. $\mathrm{supp}(b_{\nu,m}) \subset dQ_{\nu,m}$
3. $|\partial^\alpha b_{\nu,m}| \le 2^{-\nu\left(s+2M-\frac{n}{p}-\frac{n}{q}\right)+|\alpha|\nu}$ for $|\alpha| \le k + 2M$.

Thus, if we set $a_{\nu,m} = (-\Delta)^M b_{\nu,m}$ and $M \ge L$, we get $f = \sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$. □

Corollary 3. *Under the same condition in Theorem 6 with* $M \ge L + 10n\max(1, 1/p, 1/q, 1/\mu)$

1) For the family of [K, L, M]-molecules $\{m_{\nu,\tau}\}_{\nu\in\mathbb{N}_0,\tau\in\mathbb{Z}^n}$ *and* $\lambda = \{\lambda_{\nu,\tau}\}_{\nu\in\mathbb{N}_0,\tau\in\mathbb{Z}^n} \in \dot{k}^\alpha_{p,q}\dot{a}^s_{\mu,r}$ *the sum* $f = \sum_{\nu\in\mathbb{N}_0} \sum_{\tau\in\mathbb{Z}^n} \lambda_{\nu,\tau} m_{\nu,\tau}$ *converges in* $\mathcal{S}'$ *and belongs to* $\dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^s_{\mu,r}$ *with the norm estimate*

$$||f||_{\dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^s_{\mu,r}} \lesssim ||\lambda||_{\dot{k}^\alpha_{p,q}a^s_{\mu,r}}.$$

2) Conversely any $f \in \dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^s_{\mu,r}$ *admits the decomposition* $f = \sum_{\nu\in\mathbb{N}_0} \sum_{\tau\in\mathbb{Z}^n} \lambda_{\nu,\tau} m_{\nu,\tau}$*, which converges in* $\mathcal{S}'$*. We can average the family of [K, L, M]-molecules* $\{m_{\nu,\tau}\}_{\nu\in\mathbb{N}_0,\tau\in\mathbb{Z}^n}$ *and that coefficients* $\lambda = \{\lambda_{\nu,\tau}\}_{\nu\in\mathbb{N}_0,\tau\in\mathbb{Z}^n} \in \dot{k}^\alpha_{p,q}\dot{a}^s_{\mu,r}$ *fulfill the norm estimate*

$$||\lambda||_{\dot{k}^\alpha_{p,q}a^s_{\mu,r}} \lesssim ||f||_{\dot{K}^\alpha_{p,q}\dot{\mathcal{A}}^s_{\mu,r}}.$$

6. Wavelet Decomposition

Let $L \in \mathbb{N}$ and let $\psi_F, \psi_M \in C^L(\mathbb{R})$ are real-valued compactly supported functions with

$$\int_{\mathbb{R}} \psi_F(t)dt = C > 0, \quad \int_{\mathbb{R}} \psi_M(t)t^l dt = 0,\ l < L.$$

The function ψ_F is called scaling function (or father wavelet) and ψ_M is called an associated function (mother wavelet).

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Let $G = (G_1, ..., G_n) \in G^* = \{F, M\}^{d*}$ where $*$ indicates that at least one of the components of G must be an M. Then we set

$$\psi^G_{j,m} = 2^{jn/2} \prod_{r=1}^{n} \psi^{G_r}(2^j x_r - m_r), \quad \psi_m(x) = \prod_{r=1}^{n} \psi_F(x_r - m_r),$$

where $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $G \in G^*$. The family $\{\psi_m, \psi^G_{j,m} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^*\}$ is called (Daubechies) wavelet system.

$$f = \sum_{k\in\mathbb{Z}^n} \lambda_k \psi_k + \sum_{G\in G*} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^n} \lambda^G_{j,k} \psi^G_{j,k}$$

for $\lambda \in \dot{k}^{\alpha}_{p,q}\tilde{a}^s_{\mu,r}$.

Definition 9. *Let $s, \alpha \in \mathbb{R}$, $0 < p, q \le \infty$, $0 < \mu < n$ and $r \in (0, \infty]$.*

(i) Then define

$$\dot{k}^{\alpha}_{p,q}\tilde{a}^s_{\mu,r} := \{\lambda = \{\lambda^G_{v,m}\}_{v\in\mathbb{N}_0, G\in G^*, m\in\mathbb{Z}^n} \subset \mathbb{C} : ||\lambda||_{\dot{k}^{\alpha}_{p,q}\tilde{n}^s_{\mu,r}} \le \infty\},$$

where

$$||\lambda||_{\dot{k}^{\alpha}_{p,q}\tilde{n}^s_{\mu,r}} := \left(\sum_{v\in\mathbb{N}_0} \sum_{G\in G^*} \left|\left| \sum_{m\in\mathbb{Z}^n} |\lambda^G_{v,m}| 2^{vs} \chi_{v,m} \right|\right|^r_{\dot{K}^{\alpha}_{p,q}} \right)^{1/r}.$$

(ii) Let $0 < p, q < \infty$. Define

$$\dot{k}^{\alpha}_{p,q}\tilde{a}^s_{\mu,r} := \{\lambda = \{\lambda^G_{v,m}\}_{v\in\mathbb{N}_0, G\in G^*, m\in\mathbb{Z}^n} \subset \mathbb{C} : ||\lambda||_{\dot{k}^{\alpha}_{p,q}\tilde{e}^s_{\mu,r}} \le \infty\},$$

where

$$||\lambda||_{\dot{k}^{\alpha}_{p,q}\tilde{e}^s_{\mu,r}} := \left|\left| \left(\sum_{v\in\mathbb{N}_0} \sum_{G\in G^*} \sum_{m\in\mathbb{Z}^n} |\lambda^G_{v,m}| 2^{vsr} \chi_{v,m} \right)^{1/r} \right|\right|_{\dot{K}^{\alpha}_{p,q}}.$$

Theorem 7. *Suppose that the parameters $0 < q \le p \le \infty$, $0 < r \le \infty$, $s, \alpha \in \mathbb{R}^n$ and $0 < \mu < n$. Let N be a large integer depending on p, q, r, s, α.*

1. Suppose that we are given $\{\lambda_k\}_{k\in\mathbb{Z}^n}$ and $\{\lambda^G_{j,k}\}_{j\in\mathbb{N}_0, k\in\mathbb{Z}^n}$ in $\dot{k}^{\alpha}_{p,q}\tilde{a}^s_{\mu,r}$ for each $G \in G^*$. Then

$$f = \sum_{k\in\mathbb{Z}^n} \lambda_k \psi_k + \sum_{G\in G*} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^n} \lambda^G_{j,k} \psi^G_{j,k} \tag{34}$$

belongs to $\dot{K}^{\alpha}_{p,q}\dot{A}^s_{\mu,r}$ and satisfies

$$||f||_{\dot{K}^{\alpha}_{p,q}\dot{A}^s_{\mu,r}} \lesssim \left(||\{\lambda_k\}_{k\in\mathbb{Z}^n}||_{\dot{k}^{\alpha}_{p,q}} + \sum_{G\in G*} ||\{2^{j\left(s+\frac{n}{2}-\frac{n}{p}-\frac{n}{q}\right)} \lambda^G_{j,k}\}_{j\in\mathbb{N}_0, k\in\mathbb{Z}^n}||_{\dot{k}^{\alpha}_{p,q}\tilde{a}^s_{\mu,r}} \right) \tag{35}$$

2. Conversely, any $f \in \dot{K}^{\alpha}_{p,q}\dot{A}^s_{\mu,r}$ admits the following decomposition

$$f = \sum_{k\in\mathbb{Z}^n} \langle f, \psi_k \rangle \psi_k + \sum_{G\in G*} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^n} \langle f, \psi^G_{j,k} \rangle \psi^G_{j,k} \tag{36}$$

and the coefficients satisfy

$$||\{\langle f, \psi_k \rangle\}_{k\in\mathbb{Z}^n}||_{\dot{k}^{\alpha}_{p,q}} + \sum_{G\in G*} ||\{2^{j\left(s+\frac{n}{2}-\frac{n}{p}-\frac{n}{q}\right)} \langle f, \psi^G_{j,k} \rangle\}_{j\in\mathbb{N}_0, k\in\mathbb{Z}^n}||_{\dot{k}^{\alpha}_{p,q}\tilde{a}^s_{\mu,r}} \lesssim ||f||_{\dot{K}^{\alpha}_{p,q}\dot{A}^s_{\mu,r}}. \tag{37}$$

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Proof. Fix $G \in G^*$. Choose $\tau, \theta \in \mathcal{S}$ so that

$$supp(\mathcal{F}\tau) \subset B_4(0),\ supp(\mathcal{F}\theta) \subset B_4(0) \setminus B_1(0),\ \mathcal{F}\tau^2 = \sum_{m=0}^{\infty} \mathcal{F}[\theta]^2 = (2\pi)^{-n},$$

where τ and θ are even and real-valued. Let us define θ_m and τ_m by

$$\mathcal{F}[\theta_m] := \mathcal{F}\theta(2^{-m}),\ \mathcal{F}[\tau_m] := \mathcal{F}\tau(2^{-m})$$

for $m \in \mathbb{Z}$. Applying this, we receive

$$\langle f, \psi^G_{j,k} \rangle = \langle f, \tau_j * \tau_j * \psi_k \rangle + \sum_{m=0}^{\infty} \langle f, \theta_{j+m} * \theta_{j+m} * \psi^G_{j,k} \rangle =$$

$$\int_{\mathbb{R}^n} \langle f, \tau_j(x - y + 2^{-j}k) \rangle \tau_j * \psi_j(y) dy + \sum_{m=0}^{\infty} \int_{\mathbb{R}^n} \langle f, \theta_{j+m}(x - y + 2^{-j}k) \rangle \theta_{j+m} * \psi^G_{j,k}(y) dy.$$

Observe that

$$\mathcal{F}[\theta_{j+m} * \psi^G_{j,k}](\xi) = (2\pi)^{\frac{n}{2}} \mathcal{F}[\theta_{j+m}](\xi) \mathcal{F}[\psi^G_{j,k}](\xi) = (2\pi)^{\frac{n}{2}} 2^{\frac{jn}{2}} \mathcal{F}\theta(2^{-j-m}\xi) \mathcal{F}\psi^G_{j,k}(\xi).$$

By virtue of the fact that θ has vanishing moment of any order we see that there exist $M > \frac{n}{\min(1,p,q,\mu)} - s$ and $L > \frac{n}{\min(1,p,q,\mu)} + n + 1$ such that

$$\left| \frac{\partial^\alpha}{\partial \xi^\alpha} \mathcal{F}[\theta_{j+m} * \psi^G_{j,k}](\xi) \right| \lesssim 2^{-mM - \frac{jn}{2}} \langle 2^{-j}\xi \rangle^{-(n+1)},$$

for all α with $|\alpha| \leq n + 1 + L$. As a consequence, it follows that

$$|\theta_{j+m} * \psi^G_{j,k}(x)| \lesssim 2^{-mM - \frac{jn}{2}} \langle 2^{-j}x \rangle^{-L}.$$

Choose $\eta > 0$ so that $\eta < \min(1, p, q)$ when $\mathcal{A} = \mathcal{N}$ and that $\eta < \min(1, p, q, r)$ when $\mathcal{A} = \mathcal{E}$. Inversing this estimate and invoking Proposition 2, we obtain

$$|\langle f, \theta_{j+m}(x - y + 2^{-j}k) \rangle \theta_{j+m} * \psi^G_{j,k}(y)| \lesssim 2^{\frac{jn}{2} - mM} \langle 2^j y \rangle^{-L} |\mathcal{F}^{-1}[\theta_{j+m}\mathcal{F}f(2^{-j}x - y)]|$$

$$\lesssim 2^{\frac{jn}{2} - mM} \langle 2^j y \rangle^{-L} \langle 2^{j+m}(x + y - 2^{-j}k) \rangle^{\frac{n}{\eta}} \mathcal{M}^{(\eta)}[\mathcal{F}^{-1}[\theta_{j+m}\mathcal{F}f]](x),$$

for all $x \in Q_{j,k}$. Then we have

$$\langle 2^{j+m}(x + y - 2^{-j}k) \rangle^{\frac{n}{\eta}} \lesssim 2^{\frac{mn}{\eta}} \langle 2^j(x + y - 2^{-j}k) \rangle^{\frac{n}{\eta}} \lesssim 2^{\frac{mn}{\eta}} \langle 2^j y \rangle^{\frac{n}{\eta}}$$

If we combine the above inequalities, it follows that

$$|\langle f, \theta_{j+m}(x - y + 2^{-j}k) \rangle \theta_{j+m} * \psi^G_{j,k}(y)| \lesssim 2^{\frac{jn}{2} - mM - \frac{mn}{\eta}} \langle 2^j y \rangle^{-L + \frac{n}{\eta}} \mathcal{M}^{(\eta)}[\mathcal{F}^{-1}(\theta_{j+m}\mathcal{F}f)](x).$$

Integrating this inequality, we obtain

$$\left| \sum_{m=0}^{\infty} \int_{\mathbb{R}^n} \langle f, \theta_{j+m}(x - y + 2^{-j}k) \rangle \theta_{j+m} * \psi^G_{j,k}(y) dy \right| \lesssim 2^{\frac{jn}{2} - mM - \frac{mn}{\eta}} \mathcal{M}^{(\eta)}[\mathcal{F}^{-1}(\theta_{j+m}\mathcal{F}f)](x)$$

for all $x \in Q_{j,k}$.

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Now using this pointwise estimate, we obtain

$$\left| \sum_{G \in G*} ||\{2^{j\left(s+\frac{n}{2}-\frac{n}{p}-\frac{n}{q}\right)} \langle f, \psi_{j,k}^{G} \rangle\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n}||_{\dot{k}_{p,q}^{\alpha} \dot{a}_{\mu,r}^{s}} \right|$$

$$\lesssim \sum_{m \in \mathbb{N}} 2^{-mM+\frac{mn}{\eta}-ms} \mathcal{M}^{(\eta)} \left[2^{(j+m)s} \mathcal{F}^{-1}[\theta_{j+m} \mathcal{F} f] \right] (x) + \mathcal{M}^{(\eta)} [2^{js} \mathcal{F}^{-1}[\tau_j \mathcal{F} f]](x).$$

With this estimate and Theorem 1 (ii), we obtain (37). □

7. Conclusion

This article focused on characterization via ball means of differences, atomic, molecular and wavelet decompositions of Herz-type Besov-Morrey and Triebel-Lizorkin-Morrey spaces. There were introduced main definitions in preliminaries and necessary facts for proofs of Theorems 5, 6 and 7 in section 3. They can help to explore Fourier-multipliers, quarkonial decomposition, traces on Riemannian manifolds and fractals. Proposed spaces defined via ball means of differences find their application in the studying of solutions of partial differential equations such as Navier-Stokes, Klein-Gordon, Korteweg–de Vries and Burgers equations. Atomic, molecular and wavelet decompositions can imply useful properties in pseudo-differential operators theory, global and geometric analyses, which include Riemannian manifolds, Lie groups and fractals.

The provided characterizations and decompositions can be useful for determining the Fourier multipliers, as it was done in [24]. Also, they can help us to explore quarkonial decomposition, traces on Riemannian manifolds and fractals. Moreover, it is possible to extend researches on the set of analytic functions on CR-manifolds.

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