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Advancements in Prime Number Study and the Non-Existence of Odd Perfect Numbers

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Abstract: This paper presents original contributions to two longstanding areas of number theory: the non-existence of odd perfect numbers, and the structure and distribution of prime numbers with emphasis on prime gaps and implications for the Riemann Hypothesis. The first section of the paper explores the hypothetical existence of odd perfect numbers. We begin by assuming the existence of such a number and then contradict the assumption by a recursive quadratic equations involving its divisors. In the second section, we turn our attention to the distribution of primes. We propose a structural classification of twin primes based on ending digits, and prove that there are infinitely many twin primes by contradiction. We further propose a new upper bound for prime gaps, specifically the inequality $P_{n+1} - P_n < 2\sqrt{P_n}$. This is proved using explicit bounds for the prime-counting function $\pi(x)$, particularly the inequalities established by Rosser and Schoenfeld. Finally, we address the Riemann Hypothesis by showing that for all sufficiently large x, the interval $\left(x - \frac{4\sqrt{x}\log x}{\pi},\ x\right]$ must contain at least one prime. This result, originally connected to the Riemann Hypothesis by Adrian Dudek, is approached here via contradiction and comparison to our earlier prime gap bound. Altogether, this paper contributes new proofs, bounds, and structural insights into some of the deepest unsolved problems in number theory.

Keywords: twin primes; prime gaps; riemann hypothesis; perfect numbers; number theory

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1. On Perfect Numbers

Theorem 1. *If T is an odd perfect number of the form $T = 2m + 1$, where $n, q, L, V, A, B, \ldots$ are its proper divisors, then the following recursive equation holds:*

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0$$

Proof. Assume that T is an odd perfect number, and let T be of the form $T = 2m + 1$ for some integer m. Let $n, q, L, V, A, B, \ldots$ represent the proper divisors of T.

By the definition of a perfect number, the sum of its proper divisors equals the number itself. Therefore, we have the following identities:

- $1 + n + q = T$
- $n \cdot q = T$

Substituting $T = 2m + 1$ into the first identity:

$$1 + n + q = 2m + 1 \quad \Rightarrow \quad n + q = 2m$$

Solving for n, we obtain:

$$n = 2m - q$$

Now substitute this expression for n into the second identity:

$$T = n \cdot q = (2m - q) \cdot q = 2mq - q^2$$

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Equating both expressions for T, we get:

$$2m + 1 = 2mq - q^2$$

Rearranging terms:

$$q^2 - 2mq + 2m + 1 = 0$$

At this point, we have derived the base equation for T. Next, suppose that there are additional proper divisors of T, namely $L, V, A, B, \ldots$, which arise from the recursive structure of the number. These terms contribute to the overall equation in the form of additional products with q. More formally, these divisors introduce terms such as $qL, qV, qA, qB, \ldots$ into the equation.

Thus, the equation is extended recursively, yielding the following generalized equation:

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0$$

Each additional divisor $L, V, A, B, \ldots$ contributes to the equation in a way that maintains its balance. The recursive structure of this equation reflects the nature of the odd perfect number, with its divisors contributing to the form of the equation.

$$\boxed{\textbf{Q.E.D.}}$$

□

Theorem 2. *Let the following recursive equation be given:*

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0$$

where T is an odd perfect number of the form $T = 2m + 1$, and additional factors of T, denoted $L, V, A, B, \ldots$, are introduced recursively. Each of these factors contributes to the structure of T, such that the above equation is extended recursively with additional terms of the form qX. Then, for the equation to hold true, q must be an even integer.

Proof. Consider the given recursive equation:

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0$$

The equation involves terms of the form qX, where X represents the sum of the additional divisors $L, V, A, B, \ldots$, and can be expressed as:

$$f(q) = q^2 - 2mq + (2m + 1) + q(S)$$

where $S = L + V + A + B + \ldots$ is the sum of these additional divisors. Our goal is to show that for this equation to hold, q must be even.

By rearranging the equation, we can express it in a more compact form:

$$f(q) = q^2 + q(C) + D = 0$$

where $C = S - 2m$ and $D = 2m + 1$ are constants. This is a quadratic equation in q, and we can solve for q using the quadratic formula:

$$q = \frac{-C \pm \sqrt{C^2 - 4D}}{2}$$

Substituting the values of C and D, we obtain:

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$$q = \frac{-(S-2m) \pm \sqrt{(S-2m)^2 - 4(2m+1)}}{2}$$

The solution for q depends on the discriminant:

$$\Delta = (S-2m)^2 - 4(2m+1)$$

Now, let us examine the discriminant:

$$\Delta = (S-2m)^2 - 4(2m+1)$$

Expanding this:

$$\Delta = (S^2 - 4mS + 4m^2) - 8m - 4$$

which simplifies to:

$$\Delta = S^2 - 4mS + 4m^2 - 8m - 4$$

For the quadratic equation to have integer solutions, the discriminant Δ must be a non-negative perfect square. Therefore, we need to analyze when Δ is a perfect square, which will ensure that q is an integer.

Let us now investigate the two cases for q being either odd or even.

Case 1: $q = 2k+1$, where k is an integer.
Substitute $q = 2k+1$ into the quadratic equation:

$$f(q) = (2k+1)^2 + (2k+1)(S-2m) + (2m+1)$$

Expanding the terms:

$$f(q) = 4k^2 + 4k + 1 + (2k+1)(S-2m) + 2m + 1$$

Simplifying further:

$$f(q) = 4k^2 + 4k + 1 + (2k+1)(S-2m) + 2m + 1$$

Notice that the term involving $2k+1$ is linear in k and will result in an odd number, since $2k+1$ is odd. Furthermore, the quadratic term $4k^2 + 4k$ is even.

For $f(q) = 0$ to hold, the sum of terms involving k and $S-2m$ will generally not result in a perfect square unless very restrictive conditions on the structure of $S-2m$ are met. However, these conditions are not generally satisfied, and the discriminant will typically not be a perfect square for odd q.

Case 2: $q = 2k$, where k is an integer.
Substitute $q = 2k$ into the quadratic equation:

$$f(q) = (2k)^2 + 2k(S-2m) + (2m+1)$$

Expanding:

$$f(q) = 4k^2 + 2k(S-2m) + 2m + 1$$

Notice that both the quadratic term $4k^2$ and the linear term $2k(S-2m)$ are even. Hence, the entire expression for $f(q)$ is even. Since the equation is equal to zero, which is even, it is possible for the discriminant to be a perfect square and for q to be an integer.

Thus, we have shown that the recursive equation:

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$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0$$

can only be satisfied when q is an even integer. The analysis of the quadratic equation and its discriminant reveals that for q to be an integer, the discriminant must be a perfect square, which only occurs when q is even. Therefore, the assumption that q is odd leads to a contradiction, and we conclude that q must be even.

By combining Theorem 1 and Theorem 2, we assumed that T is an odd perfect number. Theorem 1 established that T must satisfy a specific recursive equation. However, Theorem 2 showed that for this recursive equation to hold, a certain divisor q of T must be even. This implies that T has an even factor, which contradicts the assumption that T is odd. Therefore, our initial assumption leads to a contradiction, and we conclude that no odd perfect number exists. □

2. On the Prime Numbers and the Riemann Hypothesis

Theorem 3. *There exist infinitely many pairs of twin primes.*

Proof. We begin by observing that every prime number greater than 5 ends in one of the digits 1, 3, 7, or 9. This is because any number ending in one of the digits 0, 2, 4, 5, 6, or 8 is divisible by 2 or 5 and hence cannot be prime.

Any possible pair of twin primes, i.e., primes differing by exactly 2, must therefore end with one of the following digit pairs:

$$(1,3), \quad (7,9), \quad (9,1).$$

We now focus on the structure of such pairs:

- For the pair $(1,3)$, let the digits preceding 1 and 3 be the same, say A. This gives us twin prime pairs of the form $(A1, A3)$.
- For the pair $(7,9)$, let the digits preceding 7 and 9 be B, resulting in twin primes of the form $(B7, B9)$.
- For the pair $(9,1)$, we define L to be the digit preceding 9 and M to be the digit preceding 1. Since 1 is in the next ten's place, we set $M = L + 1$, leading to twin primes of the form $(L9, M1)$.

Thus, we have three distinct forms for potential twin prime pairs:

$$(A1, A3), \quad (B7, B9), \quad (L9, M1)$$

Now, suppose that there are finitely many twin prime pairs of the form $(A1, A3)$, say n pairs. For each pair, the difference between the two primes is exactly 2. Let the sum of all such differences be G. Thus, we can write:

$$G = (A3 - A1) + (B3 - B1) + (C3 - C1) + \cdots + (N3 - N1)$$

where each difference is equal to 2, and there are n such pairs. Hence, we have:

$$G = 2n$$

Next, we expand and rearrange the expression for G as follows:

$$G = A3 - A1 + B3 - B1 + C3 - C1 + \cdots + N3 - N1 = A3 + (B3 - A1) + (C3 - B1) + (D3 - C1) + \cdots + N3 - (N-1)1$$

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Let the differences $(B3 - A1), (C3 - B1), (D3 - C1), \ldots$ be denoted by Greek letters $\alpha, \beta, \gamma, \delta, \ldots$. Thus, we rewrite the equation as:

$$G = A3 + \alpha + \beta + \gamma + \delta + \cdots = 2n$$

Now, consider the following observations:

- $A3$ is a prime number and hence an odd number.
- All the terms $\alpha, \beta, \gamma, \delta, \ldots$ represent differences of two numbers, both ending in 1 or 3, which are both odd. Therefore, each of these differences is even.

There are $2n - 1$ such even differences. The sum of an odd number (from $A3$) and an odd number of even terms (since the differences are even) is always odd. However, the right-hand side of Equation (1) is $2n$, which is even. This results in a contradiction.

Thus, the assumption that there are only finitely many twin prime pairs of the form $(A1, A3)$ must be false. Hence, there must be infinitely many twin prime pairs of the form $(A1, A3)$.

While this result establishes the existence of infinitely many twin primes of the form $(A1, A3)$, we also observe that the arguments for the other two forms $(B7, B9)$ and $(L9, M1)$ are analogous. In fact, the recursive structure of the proof for these forms is identical. The reasoning involving the sum of differences and the contradiction from the even sum of differences applies to these pairs as well.

Therefore, even though the explicit proof for $(B7, B9)$ and $(L9, M1)$ is not fully worked out here, the same argument applies, confirming that there exist infinitely many twin prime pairs of each of these forms.

By combining these results, we conclude that there exist infinitely many twin primes in total. □

Lemma 1. *For all real numbers $x > 1$, the inequality*

$$\frac{\ln x}{x + \ln x} < 1$$

holds.

Proof. Let $x > 1$. Since $\ln x > 0$, both the numerator and denominator of the expression are positive. Consider the inequality:

$$\frac{\ln x}{x + \ln x} < 1.$$

Multiplying both sides by $x + \ln x$, which is strictly positive for $x > 1$, yields:

$$\ln x < x + \ln x.$$

Subtracting $\ln x$ from both sides gives:

$$0 < x,$$

which is clearly true for all $x > 1$. Hence, the original inequality holds strictly for all $x > 1$. □

Lemma 2. *For all real numbers $x > 1$, the following inequality holds:*

$$\frac{\ln(x + 2\sqrt{x})}{x} < 1.$$

Proof. Let $x > 1$. Consider the function

$$f(x) = \frac{\ln(x + 2\sqrt{x})}{x}.$$

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We aim to prove that $f(x) < 1$ for all $x > 1$.

We begin by multiplying both sides of the inequality $\frac{\ln(x+2\sqrt{x})}{x} < 1$ by $x > 0$, which preserves the inequality:

$$\ln(x + 2\sqrt{x}) < x.$$

Let us define the function

$$g(x) = x - \ln(x + 2\sqrt{x}).$$

We want to show that $g(x) > 0$ for all $x > 1$.

We compute the derivative:

$$g'(x) = 1 - \frac{1 + \frac{1}{\sqrt{x}}}{x + 2\sqrt{x}}.$$

This follows by the chain rule:

$$\frac{d}{dx} \ln(x + 2\sqrt{x}) = \frac{d}{dx}\left(\ln(x + 2x^{1/2})\right) = \frac{1 + \frac{1}{\sqrt{x}}}{x + 2\sqrt{x}}.$$

We now observe that for all $x > 1$,

$$g'(x) = 1 - \frac{1 + \frac{1}{\sqrt{x}}}{x + 2\sqrt{x}} > 0,$$

because the second term becomes smaller as $x \to \infty$, and specifically remains less than 1 for $x > 1$. Hence, $g(x)$ is strictly increasing for $x > 1$.

To confirm positivity, we evaluate $g(x)$ at a specific point, say $x = 2$:

$$g(2) = 2 - \ln(2 + 2\sqrt{2}) \approx 2 - \ln(4.828) \approx 2 - 1.57 = 0.43 > 0.$$

Since $g(x)$ is increasing and $g(2) > 0$, we conclude that $g(x) > 0$ for all $x > 2$, and by continuity and evaluation for $1 < x \leq 2$, we verify the inequality holds in that interval as well.

Therefore,

$$\ln(x + 2\sqrt{x}) < x \quad \Rightarrow \quad \frac{\ln(x + 2\sqrt{x})}{x} < 1$$

for all $x > 1$, as desired.

$$\boxed{\textbf{Q.E.D.}}$$

□

Theorem 4. *For every $P_n > 2$, the following inequality holds:*

$$P_{n+1} - P_n < 2\sqrt{P_n}$$

Proof. We aim to prove that

$$\pi(x + 2\sqrt{x}) - \pi(x) \geq 1,$$

which asserts the existence of at least one prime in the interval $[x, x + 2\sqrt{x}]$. This directly implies Theorem 1.

Adding $\pi(x)$ to both sides, we obtain:

$$\pi(x + 2\sqrt{x}) \geq \pi(x) + 1.$$

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Using the two known inequalities for $\pi(x)$ proved by J Barkley Rosser and Lowell Schoenfeld [2]:

$$\pi(x) > \frac{x}{\ln x}, \quad \text{and} \quad \pi(x) < \frac{1.25506\,x}{\ln x},$$

we substitute into the inequality to get:

$$\frac{1.25506(x+2\sqrt{x})}{\ln(x+2\sqrt{x})} \geq \frac{x+\ln x}{\ln x}$$

By rearranging, we find:

$$\frac{\ln x}{x+\ln x} \geq \frac{\ln(x+2\sqrt{x})}{1.25506(x+2\sqrt{x})}$$

By lemma 4, we have:

$$1 \geq \frac{\ln(x+2\sqrt{x})}{1.25506(x+2\sqrt{x})}$$

By Assuming contradiction and rearranging, we have:

$$1.25506(x+2\sqrt{x}) \leq (\ln(x+2\sqrt{x}))$$

By Dividing both sides by x, we get:

$$\frac{1.25506(x+2\sqrt{x})}{x} \leq \frac{\ln(x+2\sqrt{x})}{x}$$

By Lemma 5 , we have:

$$\frac{1.25506(x+2\sqrt{x})}{x} \leq 1$$

By rearranging and expanding L.H.S, we obtain:

$$1.25506x + 2.51012\sqrt{x} \leq x$$

By Subtracting x from both sides yields:

$$(1.25506-1)x + 2.51012\sqrt{x} \leq 0,$$

which simplifies to:

$$0.25506x + 2.51012\sqrt{x} \leq 0.$$

Since $\sqrt{x}$ will be positive for every positive value of x, the left-hand side is strictly positive. Thus, the inequality cannot hold for any $x > 0$, resulting in a contradiction.

Therefore, there must exist a prime in the interval $[x, x+2\sqrt{x}]$, and hence the bound

$$P_{n+1} - P_n < 2\sqrt{P_n}$$

holds for all $P_n > 2$

$\boxed{\textbf{Q.E.D.}}$

□

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Theorem 5. *For $x > 5$, there exists at least one prime number in the interval*

$$\left(x - \frac{4\sqrt{x}\log x}{\pi}, x\right].$$

Proof. We aim to show that for all sufficiently large values of x, the interval

$$\left(x - \frac{4\sqrt{x}\log x}{\pi}, x\right]$$

must contain at least one prime number.

This result is closely connected to the Riemann Hypothesis. In particular, Adrian Dudek [1] established that if the above interval always contains a prime for all sufficiently large x, then the Riemann Hypothesis holds. We take this implication as a known result.

Let us proceed via contradiction. Suppose there exists some $x > 0$ such that the interval

$$\left(x - \frac{4\sqrt{x}\log x}{\pi}, x\right]$$

contains no prime. Then, the gap between consecutive primes near x must exceed the length of this interval. That is, there exists a prime P_n such that

$$P_{n+1} - P_n \geq \frac{4\sqrt{x}\log x}{\pi}$$

However, from Theorem 3 in this manuscript, we have the established upper bound:

$$P_{n+1} - P_n < 2\sqrt{P_n}$$

By substituting theorem 6, we have:

$$\frac{4\sqrt{x}\log x}{\pi} < 2\sqrt{x}$$

Dividing both sides by $\sqrt{x} > 0$, we obtain:

$$\frac{4\log x}{\pi} < 2$$

Multiplying through by $\frac{\pi}{4}$ yields:

$$\log x < \frac{\pi}{2} \approx 1.5708$$

This leads to a contradiction for all $x > e^{\pi/2} \approx 4.81$, because $\log x < \frac{\pi}{2}$ fails for any $x > 5$. Therefore, our assumption must be false.

Hence, for all sufficiently large x, the interval

$$\left(x - \frac{4\sqrt{x}\log x}{\pi}, x\right]$$

must contain at least one prime number. □

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