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A Note on the Mean Square of Riemann Zeta-Function

An-Ping Li *

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Article

A Note on the Mean Square of Riemann Zeta-Function

Li An-Ping

Beijing 100086, P.R.China; apli_1012@sina.cn

Abstract: In this paper, we will give a new proof for a known result of the mean square of the Riemann zeta-function.

Keywords: Riemann zeta-function; mean square

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1. Introduction

Let

$$I = \int_0^T |\zeta(1/2+it)|^2 |A(1/2+it)|^2 dt$$

where $\zeta(s)$ is the Riemann zeta-function, and

$$A(s) = \sum_{1 \le m \le M} a(m) m^{-s}$$

is a finite Dirichlet series. For this type of mean square of $\zeta(s)$, there are a number of researches, which is related to estimate the number of the zeros of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = 1/2$, see [1–6]As before, for two positive integers h, k, denote $h^* = h/(h,k), k^* = k/(h,k)$, and $\overline{h^*}$ is the least positive integer such that $\overline{h^*} h^* \equiv 1 \mod k^*$. For $s = c + it, c = 1 + \eta, 0 < \eta < 1$, denote by

$$\mathscr{M}(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \sum_{h,k \le M} \frac{a(h)\overline{a(k)}}{(hk)^{1-s}} (h,k)^{(1-2s)} e\left(\frac{n\overline{h^*}}{k^*}\right)$$

and $V = \sup_{|t| \le M} \{\mathscr{M}(c+it)\}$.

Balasubramanian, Conrey, and Heath-Brown proved that

For $a(m) \ll m^{\epsilon}$, and $\log(m) \ll \log T$, there is

$$I = T \sum_{h,k \le M} \frac{a(h)}{h} \frac{\overline{a(k)}}{k} (h,k) \left(\log \frac{T(h,k)^2}{2\pi hk} + 2\gamma - 1\right) + \mathscr{E}. \tag{1.1}$$

where $\mathscr{E} \ll VT^{\epsilon} + o(T)$. In the paper [1], it is applied an auxiliary function $w(t, T_1, T_2)$, which plays an important role of "ferrying", Instead, in this paper, we will make use of a new one, i.e. $\omega(t, T_1, T_2)$ defined in Lemma 2.1. Our result is that

Theorem 1.1. *Suppose that* $a(m) \ll m^{\epsilon}$*, and* $\log(m) \ll \log T$*, then*

$$I = T \sum_{h,k \le M} \frac{a(h)}{h} \frac{\overline{a(k)}}{k} (h,k) \left(\log \frac{T(h,k)^2}{2\pi hk} + 2\gamma - 1\right) + \tilde{\mathscr{E}}. \tag{1.2}$$

where $\tilde{\mathscr{E}} \ll VT^{-\eta+\epsilon} + o(T)$.

As it is easy to know that $V \ll M^{2+2\epsilon+2\eta}$, then with a properly re-define ϵ, η, it has

$$\tilde{\mathscr{E}}, \mathscr{E} \ll M^2 T^{\epsilon}. \tag{1.3}$$

The main arguments in this paper are most similar to the ones in [1].

2. The Proof of Theorem 1.1

Lemma 2.1. *Suppose that* $\lambda = T^{2-\epsilon}$, *define*

$$\omega(t, T_1, T_2) = \frac{e^{\lambda}}{2\pi} \int_{T_1}^{T_2} \Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} du. \tag{2.1}$$

Let $\alpha \geq 10$, *denote* $\Delta = (2\alpha\lambda \log T)^{1/2}$, *if* $t \in [T_1 + \Delta, T_2 - \Delta]$, *then*

$$|\omega(t, T_1, T_2) - 1| \ll T^{-\alpha}. \tag{2.2}$$

And if $t \leq T_1 - \Delta$, *or* $t \geq T_2 + \Delta$, *then*

$$|\omega(t, T_1, T_2)| \ll T^{-\alpha}. \tag{2.3}$$

Proof. It is familiar that

$$e^{-\lambda} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\lambda^{-s} ds.$$

Hence,

$$1 - \omega(t, T_1, T_2) = R_1 + R_2,$$

where

$$R_1 = \frac{e^{\lambda}}{2\pi i} \int_{T_2}^{\infty} \Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} du,$$
$$R_2 = \frac{e^{\lambda}}{2\pi i} \int_{-\infty}^{T_1} \Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} du,$$

By Stirling's formula, it has

$$\begin{aligned}\mathrm{Re}(\log \Gamma(\lambda + (u-t)i)) =& \left(\lambda - \frac{1}{2}\right) \frac{\log(\lambda^2 + (u-t)^2)}{2} - \lambda + \frac{1}{2}\log(2\pi) \\ & - (u-t)\arctan\left(\frac{u-t}{\lambda}\right) + O(1/\lambda)\end{aligned}$$

And

$$\begin{aligned}&\mathrm{Re}(\log(\Gamma(\lambda + (u-t)i))) + \mathrm{Re}(\log(\lambda^{-(\lambda+(u-t)i)})) + \lambda \\ =& -\frac{(u-t)^2}{4\lambda^2} - \frac{(u-t)^2}{2\lambda} - \frac{1}{2}\log\lambda + \frac{1}{2}\log(2\pi) + O(1/\lambda)\end{aligned}$$

Hence, if $t \in [T_1 + \Delta, T_2 - \Delta]$, then

$$\begin{aligned}|R_1| &\ll \int_{T_2}^{\infty} \left| e^{\lambda}\Gamma(\lambda + (u-t)i)\lambda^{-(\lambda+(u-t)i)} \right| du \\ &\ll \lambda^{-1/2} \int_{T_2}^{\infty} \exp(-(u-t)^2/2\lambda) du \\ &\ll T^{-\alpha}.\end{aligned}$$

and similarly

$$|R_2| \ll \lambda^{-1/2} \int_{-\infty}^{T_1} \exp(-(u-t)^2/2\lambda) du \ll T^{-\alpha}.$$

if $t \leq T_1 - \Delta$, or $t \geq T_2 + \Delta$, then

$$|\omega(t, T_1, T_2)| \ll \lambda^{-1/2} \int_{T_1}^{T_2} \exp(-(u-t)^2/2\lambda) du \ll T^{-\alpha}.$$

□

Lemma 2.2 (Estermann[2])**.** *Let*

$$S\left(x, \frac{h}{k}\right) = \sum_{n=1}^{\infty} d(n) e\left(\frac{nh}{k}\right) e^{2\pi i n x}$$

Denote by

$$D\left(s, \frac{h}{k}\right) = \sum_{n=1}^{\infty} d(n) e\left(\frac{nh}{k}\right) n^{-s}$$

write $z = -2\pi i x$, *then for* $\operatorname{Im} x > 0, \operatorname{Re} s > 1, k \geq 1$, *then*

$$\begin{aligned} S\left(x, \frac{h}{k}\right) = & \frac{1}{zk^*}(\gamma - \log z - 2\log k^*) + D\left(0, \frac{h^*}{k^*}\right) \\ & - i \int_{(c)} (2\pi)^{-s} \frac{\Gamma(s) k^{*2s-1}}{\sin \pi s} \left(D\left(s, \frac{\overline{h^*}}{k^*}\right) + (cos \pi s) D\left(s, -\frac{\overline{h^*}}{k^*}\right)\right) z^{s-1} ds \end{aligned} \tag{2.4}$$

where $1 < c < 2$, *and it has*

$$\left| D\left(0, \frac{h^*}{k^*}\right) \right| \leq k^* (\log 2k^*)^2$$

As usual, denote

$$\chi(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos(\pi s/2).$$

Lemma 2.3. *Suppose that* $1 < c < 2$, *let*

$$J(y) = \frac{1}{2\pi i} \int_{(c)} \Gamma(s_1 - s) \lambda^{-(s_1 - s)} \chi(1-s) y^{-s} ds,$$

then

$$J(y) = \int_0^{\infty} v^{s_1} e^{-\lambda v} (e^{2\pi i y v} + e^{-2\pi i y v}) \frac{dv}{v}. \tag{2.5}$$

Proof.

$$J(y) = \frac{1}{2\pi i} \int_{(c)} \Gamma(s_1 - s) \lambda^{-(s_1 - s)} \Gamma(s) ((2\pi i y)^{-s} + (-2\pi i y)^{-s}) ds = J_1 + J_2$$

where

$$\begin{aligned} J_1 = & \frac{1}{2\pi i} \int_{(c)} \Gamma(s_1 - s) \lambda^{-(s_1 - s)} \Gamma(s) (2\pi i y)^{-s} ds, \\ J_2 = & \frac{1}{2\pi i} \int_{(c)} \Gamma(s_1 - s) \lambda^{-(s_1 - s)} \Gamma(s) (-2\pi i y)^{-s} ds. \end{aligned}$$

By the theory of Mellin Transforms(refer to [7])

$$\frac{1}{2\pi i} \int_{(c)} G(s) F(s) x^{-s} ds = \int_0^{\infty} f\left(\frac{1}{v}\right) g(xv) \frac{dv}{v}$$

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where

$$f(v) = \frac{1}{2\pi i} \int\limits_{(c)} F(x)x^{-s}ds, \qquad g(v) = \frac{1}{2\pi i} \int\limits_{(c)} G(x)x^{-s}ds.$$

We apply this formula to J_1 and J_2 respectively.

For J_1,

$$\begin{aligned} f(v) =& \frac{\lambda^{-s_1}}{2\pi i} \int\limits_{(c)} \Gamma(s_1 - s)(v/\lambda)^{-s}ds \\ =& \frac{\lambda^{-s_1}}{2\pi i} \int\limits_{(\lambda)} \Gamma(w)(v/\lambda)^{-(s_1-w)}dw \\ =& \lambda^{-s_1}(v/\lambda)^{-s_1}e^{-\lambda/v} = v^{-s_1}e^{-\lambda/v}, \end{aligned}$$

i.e.

$$f(1/v) = v^{s_1}e^{-\lambda v}.$$

And

$$g(v) = \frac{1}{2\pi i} \int\limits_{(c)} \Gamma(s)v^{-s}ds = e^{-v},$$

i.e.

$$g(xv) = e^{-xv}.$$

So

$$J_1 = \int_0^{\infty} v^{s_1}e^{-(\lambda v + 2\pi i y v)}\frac{dv}{v}$$

Similarly,

$$J_2 = \int_0^{\infty} v^{s_1}e^{-(\lambda v - 2\pi i y v)}\frac{dv}{v}$$

□

Denote by L_δ the straight line from 0 to $e^{i\delta}\infty$.

Lemma 2.4. *Let $s_1 = \lambda + 1/2 + iu$, $0 \le \delta \le \pi/2$, then*

$$\int_{L_\delta} \frac{v^{s_1}e^{-\lambda v}}{v(v-1)}dv - \int_{L_{-\delta}} \frac{v^{s_1}e^{-\lambda v}}{v(v-1)}dv = -e^{\lambda}2\pi i, \tag{2.6}$$

And let

$$K = \int_{L_\delta} \frac{v^{s_1}e^{-\lambda v}\log(-i(v-1))}{v(v-1)}dv - \int_{L_{-\delta}} \frac{v^{s_1}e^{-\lambda v}\log(i(v-1))}{v(v-1)}dv,$$

then

$$K = e^{\lambda}2\pi i(\log u + c_0 + c_2 u^{-2}). \tag{2.7}$$

where c_0 and c_2 are two constants.

Proof. Equality (2.6) is followed by the residue theorem for the two integral paths form a contour with a pole at $v = 1$.

To prove (2.7), we change the integral paths L_δ and $L_{-\delta}$ to the positive real axis but with an indentation around $v = 1$ with $\operatorname{Im} v > 0$ and $\operatorname{Im} v < 0$ respectively.

And let $v = e^x$, then

$$\begin{aligned} K = &\int\limits_{\mathscr{L}_+} \exp(\lambda x + ixu - \lambda e^x) \frac{\log(-i(e^x - 1))}{2\sinh(x/2)} dx \\ &- \int\limits_{\mathscr{L}_-} \exp(\lambda x + ixu - \lambda e^x) \frac{\log(i(e^x - 1))}{2\sinh(x/2)} dx \end{aligned}$$

where the integral path $\mathscr{L}_+$ is from $-\infty$ to $-\epsilon$ then along a upper semicircle C_ϵ^+ to $+\epsilon$ and tend to $+\infty$. The integral path $\mathscr{L}_-$ is same but with a lower semicircle C_ϵ^-. Let C_ϵ be the union of C_ϵ^- and the reversal of C_ϵ^+.

Then

$$\begin{aligned} K =& \pi i \int_{-\infty}^{-\epsilon} \frac{\exp(\lambda x + ixu - \lambda e^x)}{2\sinh(x/2)} dx - \pi i \int_{+\epsilon}^{+\infty} \frac{\exp(\lambda x + ixu - \lambda e^x)}{2\sinh(x/2)} dx \\ &- \int_{C_\epsilon} \exp(\lambda x + ixu - \lambda e^x) \frac{\log(i(e^x - 1))}{2\sinh(x/2)} dx \\ &- \frac{\pi i}{2} \int_{C_\epsilon^+ \cup C_\epsilon^-} \frac{\exp(\lambda x + ixu - \lambda e^x)}{2\sinh(x/2)} dx \end{aligned}$$

And then

$$\begin{aligned} K = &- \int_{C_\epsilon} \frac{e^{-\lambda} \log x}{x} dx - R'_\epsilon + \pi i \int_{-\infty}^{-\epsilon} \frac{e^{-\lambda} e^{iux}}{x} dx \\ &+ \pi i P_- - \pi i \int_{+\epsilon}^{+\infty} \frac{e^{-\lambda} e^{iux}}{x} dx - \pi i P_+ - \frac{\pi i}{2} R_\epsilon \end{aligned}$$

where

$$\begin{aligned} R_\epsilon &= \int_{C_\epsilon^+ \cup C_\epsilon^-} \frac{\exp(\lambda x + ixu - \lambda e^x)}{2\sinh(x/2)} dx \\ R'_\epsilon &= \int_{C_\epsilon} \left(\exp(\lambda x + ixu - \lambda e^x) \frac{\log(i(e^x - 1))}{2\sinh(x/2)} - \frac{e^{-\lambda} \log x}{x} \right) dx \\ P_- &= \int_{-\infty}^{-\epsilon} \left(\frac{e^{iux} \exp(\lambda x - \lambda e^x)}{2\sinh(x/2)} - \frac{e^{-\lambda} e^{iux}}{x} \right) dx \\ P_+ &= \int_{+\epsilon}^{+\infty} \left(\frac{e^{iux} \exp(\lambda x - \lambda e^x)}{2\sinh(x/2)} - \frac{e^{-\lambda} e^{iux}}{x} \right) dx \end{aligned}$$

It is easy to know that R_ϵ and $R'_\epsilon \to 0$ as $\epsilon \to 0$. Let $P = \lim\limits_{\epsilon \to 0} (P_- - P_+)$, then

$$K = e^{-\lambda} 2\pi i (\log u + c_0 + P)$$

where

$$\begin{aligned} c_0 =& \int_0^1 \frac{1 - \cos x}{x} dx - \int_1^\infty \frac{\cos x}{x} dx, \\ P =& \frac{1}{2} \int_0^\infty \cos(ux) \left(\frac{1}{x} - \frac{\exp(-\lambda x - \lambda(e^{-x} - 1))}{2\sinh(x/2)} \right) dx \\ &+ \frac{1}{2} \int_0^\infty \cos(ux) \left(\frac{1}{x} - \frac{\exp(\lambda x - \lambda(e^x - 1))}{2\sinh(x/2)} \right) dx \\ &+ \frac{i}{2} \int_0^\infty \sin(ux) \left(\frac{\exp(-\lambda x - \lambda(e^{-x} - 1))}{2\sinh(x/2)} - \frac{1}{x} \right) dx \\ &- \frac{i}{2} \int_0^\infty \sin(ux) \left(\frac{\exp(\lambda x - \lambda(e^x - 1))}{2\sinh(x/2)} - \frac{1}{x} \right) dx \end{aligned}$$

By the partial integration, P can be expressed as

$$P = \sum_{2 \leq n < N} c_n u^{-n} + O(u^{-N}).$$

where N is a any positive integer.

$\square$

The following Lemma is direct.

Lemma 2.5.

$$\int_0^\infty v^{s_1-1} e^{-\lambda x} dx = \lambda^{-s_1} \int_0^\infty x^{s_1-1} e^{-x} dx = \lambda^{-s_1} \Gamma(s_1) \tag{2.8}$$

Lemma 2.6. *Let $c = 1 + \eta, 0 < \eta < 1, s_1 = \lambda + 1/2 + iu, T \leq u \leq 2T, \delta = 1/T$, and let*

$$W = \int_{L_\delta} \int_{(c)} e^{\lambda} v^{s_1} e^{-\lambda v} (1 + |s|) \frac{\Gamma(s)}{\sin \pi s} (\cos \pi s) e^{-\pi i s/2} (v-1)^{s-1} ds \frac{dv}{v}.$$

Then

$$W \ll_{\epsilon,\eta} T^{\epsilon} (\log T/T)^{c}. \tag{2.9}$$

and the estimation is also holds in the cases the term $\cos \pi s$ is removed, or L_δ is replaced by $L_{-\delta}$ and $e^{-\pi i s/2}$ by $e^{\pi i s/2}$.

Proof. Let $v = xe^{i\delta}$, then

$$|\Gamma(s)| \ll (1 + |t|)^{c-1/2} e^{-\pi|t|/2},$$

$$\left| \frac{\cos \pi s}{\sin \pi s} \right| \ll 1,$$

$$|e^{-\pi s/2}| = e^{\pi t/2},$$

$$|v^{s_1}| \ll x^{\lambda+1/2},$$

$$1/|v| \ll 1/x,$$

$$|e^{-\lambda v}| = e^{-\lambda' x},$$

where $\lambda' = \lambda \cos \delta$, and clearly, $\lambda - T^{-\epsilon}/2 < \lambda' < \lambda$.
And

$$|(v-1)^{s-1}| = |v-1|^{c-1} e^{-t \arg(v-1)} = a(x,\delta) \exp(-tb(x,\delta))$$

where

$$a(x,\delta) = ((x-a)^2 + 2x(1-\cos\delta))^{(c-1)/2}$$

$$b(x,\delta) = \arctan \frac{x \sin \delta}{x \cos \delta - 1}$$

Suppose that $\beta \geq 10$, let $\theta = (2\beta \log T/\lambda)^{1/2}$, there are the following estimates

$$a(x,\delta) \ll \begin{cases} 1, & \text{if } x \leq 1 - \theta, \\ \theta^{c-1}, & \text{if } |x-1| \leq \theta, \\ x^{c-1}, & \text{if } x \geq 1 + \theta. \end{cases}$$

and

$$b(x,\delta)\begin{cases} > \delta, & x > 0, \\ \gg \theta^{-1}\delta, & |x-1| \leq \theta, \\ \leq \pi - C\theta^{-1}\delta, & |x-1| \leq \theta, \\ \leq \pi - Cx\delta, & 0 < x \leq 1-\theta, \\ \leq \delta\theta^{-1}, & x \geq 1+\theta. \end{cases}$$

where C is a constant. Hence

$$\int_{-\infty}^{\infty}(1+|t|)^{c+1/2}e^{\pi(t-|t|)/2}e^{-tb(x,\delta)}dt \ll \begin{cases} \delta^{-c-3/2}, & x \geq 1+\theta, \\ (\theta\delta^{-1})^{c+3/2}, & |x-1| \leq \theta, \\ (\delta x)^{-c-3/2}, & 0 < x \leq 1-\theta. \end{cases}$$

Divide the x integral into three pieces W_1, W_2 and W_3 with $0 \leq x \leq 1-\theta$, $1-\theta \leq x \leq 1+\theta$ and $1+\theta \leq x \leq \infty$ respectively, then there are

$$\begin{aligned} &W = W_1 + W_2 + W_3. \\ &W_1 \ll \delta^{-5/2-\eta}e^{\lambda}\int_0^{1-\theta} x^{\lambda-c-2}e^{-\lambda' x}dx \ll \delta^{-5/2-\eta}T^{-\beta} \\ &W_2 \ll T^{\epsilon}\theta^{c} \\ &W_3 \ll \delta^{-5/2-\eta}e^{\lambda}\int_{1+\theta}^{\infty} x^{\lambda+c-3/2}e^{-\lambda' x}dx \ll \delta^{-5/2-\eta}T^{-\beta} \end{aligned}$$

□

Lemma 2.7. *Let* $s_1 = \lambda + 1/2 + iu$*, define*

$$\mathfrak{g}(u) = \frac{e^{\lambda}}{2\pi i}\int\limits_{(1/2)} \Gamma(s_1 - s)\lambda^{-s_1+s}\zeta(s)\chi(1-s)\bar{A}(s)A(1-s)ds.$$

Then for $T \leq u \leq 2T$*, there is*

$$\mathfrak{g}(u) = \sum_{1\leq h,k\leq M} \frac{a(h)\overline{a(k)}}{hk}(h,k)\left(\log\frac{u(h,k)^2}{2\pi hk} + b_0 + O(u^{-2})\right) + \mathscr{E}_0 \tag{2.10}$$

where $\mathscr{E}_0 \leq VT^{-c+\epsilon}$*, and* b_0 *is a constant.*

Proof. We move the integral path from $(1/2)$ to $(c), c = 1+\eta, 0 < \eta < 1$, the residue at $s = 1$ is

$$R = e^{\lambda}\Gamma(\lambda - 1/2 - iu)\lambda^{-(\lambda-1/2-iu)}\zeta(0)A(1)\bar{A}(0) \ll M^{1+\epsilon}T^{-\alpha} \ll T^{-\alpha+1}$$

Hence,

$$\mathfrak{g}(u) = \hat{\mathfrak{g}}(u) - R$$

where $\hat{\mathfrak{g}}(u)$ is the one of $\mathfrak{g}(u)$ moved in the new integral path.

$$\begin{aligned}
\hat{\mathfrak{g}}(u) &= \sum_{1\le h,k\le M} \frac{a(h)\overline{a(k)}}{k} e^{\lambda} \sum_{n=1}^{\infty} d(n) J(\frac{nh}{k}) \\
&= \sum_{1\le h,k\le M} \frac{a(h)\overline{a(k)}}{k} \sum_{n=1}^{\infty} d(n) e^{\lambda} \int_0^{\infty} v^{s_1} e^{-\lambda v} (e^{2\pi inh/k} + e^{-2\pi inh/k}) \frac{dv}{v} \\
&= \sum_{1\le h,k\le M} \frac{a(h)\overline{a(k)}}{k} (I_1 + I_2)
\end{aligned}$$

where

$$I_1 = M_1 + R_1 - iE_1, \quad I_2 = M_2 + R_2 - iE_2.$$

$$\begin{aligned}
M_1 &= e^{\lambda} \int\limits_{L_\delta} v^{s_1} e^{-\lambda v} \frac{\gamma - \log(-2\pi i(v-1)h/k) - 2\log k^*}{-2\pi i h^*} \frac{dv}{v} \\
R_1 &= D\left(0, \frac{h^*}{k^*}\right) e^{\lambda} \int\limits_{L_\delta} v^{s_1} e^{-\lambda v} \frac{dv}{v} \\
E_1 &= e^{\lambda} \int\limits_{L_\delta} v^{s_1} e^{-\lambda v} F_1(v) \frac{dv}{v} \\
F_1(v) &= \int\limits_{(c)} (2\pi)^{-s} \frac{\Gamma(s)}{\sin \pi s} \left(D\left(s, \frac{h^*}{k^*}\right) + (\cos \pi s) D\left(s, -\frac{\overline{h^*}}{k^*}\right)\right) \\
&\qquad \cdot (-2\pi i h^* k^* (v-1))^{s-1} ds
\end{aligned}$$

And

$$\begin{aligned}
M_2 &= e^{\lambda} \int\limits_{L_{-\delta}} v^{s_1} e^{-\lambda v} \frac{\gamma - \log(-2\pi i(v-1)h/k) - 2\log k^*}{2\pi i h^*} \frac{dv}{v} \\
R_2 &= D\left(0, -\frac{h^*}{k^*}\right) e^{\lambda} \int\limits_{L_{-\delta}} v^{s_1} e^{-\lambda v} \frac{dv}{v} \\
E_2 &= e^{\lambda} \int\limits_{L_{-\delta}} v^{s_1} e^{-\lambda v} F_2(v) \frac{dv}{v} \\
F_2(v) &= \int\limits_{(c)} (2\pi)^{-s} \frac{\Gamma(s)}{\sin \pi s} \left(D\left(s, -\frac{h^*}{k^*}\right) + (\cos \pi s) D\left(s, \frac{\overline{h^*}}{k^*}\right)\right) \\
&\qquad \cdot (2\pi i h^* k^* (v-1))^{s-1} ds
\end{aligned}$$

By Lemma 2.4,

$$M_1 + M_2 = \frac{1}{h^*}\left(\log \frac{u(h,k)^2}{2\pi hk} + \gamma + c_0 + O(u^{-2})\right)$$

and by Lemma 2.5

$$R_1 + R_2 \ll \mathrm{Re} D\left(0, \frac{h^*}{k^*}\right) T^{-\alpha} \ll M(\log M)^2 T^{-\alpha} \ll T^{-\alpha+1}$$

So

$$\sum_{1\le h,k\le M} \frac{a(h)\overline{a(k)}}{k} (R_1 + R_2) \ll M^{1+2\epsilon} \log M T^{-\alpha+1} \ll T^{-\alpha+2}$$

The other error term is

$$\sum_{1\le h,k\le M}\frac{a(h)\overline{a(k)}}{k}(E_1+E_2),$$

which may be written as a sum of four terms of a typical one is that

$$Z=\int\limits_{L_\delta}\int\limits_{(c)}G(v,s_1,s)\mathscr{M}(s)dsdv$$

where

$$G(v,s_1,s)=e^{\lambda}v^{s_1}e^{-\lambda v}\frac{\Gamma(s)}{\sin\pi s}(2\pi)^{-2s}(-2\pi i(v-1))^{s-1}v^{-1},$$

$$\mathscr{M}(s)=\sum_{n=1}^{\infty}\frac{d(n)}{n^s}\sum_{h,k\le M}\frac{a(h)\overline{a(k)}}{(hk)^{1-s}}(h,k)^{(1-2s)}e\left(\frac{n\overline{h^*}}{k^*}\right).$$

By Lemma 2.6,

$$Z\ll VT^{\epsilon}\theta^{c}.$$

□

The Proof of Theorem 1.1.

Proof. By Lemma 2.1, it has

$$\begin{aligned}I(T,2T)&=\int_T^{2T}\omega(t,T-\Delta,2T+\Delta)|\zeta A(1/2+it)|^2dt+o(1)\\&=\frac{e^{\lambda}}{2\pi}\int_T^{2T}\int_{T-\Delta}^{2T+\Delta}\Gamma(\lambda+(u-t)i)\lambda^{-(\lambda+(u-t)i)}du|\zeta A(1/2+it)|^2dt+o(1)\\&=\frac{e^{\lambda}}{2\pi}\int_{T-\Delta}^{2T+\Delta}\int_T^{2T}\Gamma(\lambda+(u-t)i)\lambda^{-(\lambda+(u-t)i)}|\zeta A(1/2+it)|^2dtdu+o(1)\\&\le\int_{T-\Delta}^{2T+\Delta}\mathfrak{g}(u)du+o(1)\\&=\int_{T-\Delta}^{2T+\Delta}\sum_{1\le h,k\le M}\frac{a(h)\overline{a(k)}}{hk}(h,k)\left(\log\frac{u(h,k)^2}{2\pi hk}+b_0+O(u^{-2})\right)+O(\mathscr{E}_0T)\end{aligned}$$

$$=T\sum_{1\le h,k\le M}\frac{a(h)\overline{a(k)}}{hk}(h,k)\left(\log\frac{T(h,k)^2}{2\pi hk}+b_0-1+2\log 2\right)+O(\mathscr{E}_0T)$$
$$+O\left(\Delta\log T\sum_{1\le h,k\le M}\frac{a(h)\overline{a(k)}}{hk}(h,k)\right)$$

and

$$\sum_{1\le h,k\le M}\frac{(h,k)}{hk}\le\sum_{1\le t\le M}t\,(\sum_{\substack{h\le M\\ t|h}}h^{-1})^2\ll\log^3 M$$

that is, the last term is $O(\Delta\log^4 MM^{2\epsilon})$.

Then replacing T by $T/2^k, 1\le k\le\log T$, and summing, it follows

$$I\le T\sum_{h,k\le M}\frac{a(h)}{h}\frac{\overline{a(k)}}{k}(h,k)\left(\log\frac{T(h,k)^2}{2\pi hk}+b_0-1\right)+\tilde{\mathscr{E}}$$

On the other hand, similarly it has

$$
\begin{aligned}
I(T,2T) &= \int_T^{2T} \omega(t, T+\Delta, 2T-\Delta)|\zeta A(1/2+it)|^2 dt + o(1) \\
&= \frac{e^\lambda}{2\pi}\int_T^{2T}\int_{T+\Delta}^{2T-\Delta} \Gamma(\lambda+(u-t)i)\lambda^{-(\lambda+(u-t)i)} du |\zeta A(1/2+it)|^2 dt + o(1) \\
&= \frac{e^\lambda}{2\pi}\int_{T+\Delta}^{2T-\Delta}\int_T^{2T} \Gamma(\lambda+(u-t)i)\lambda^{-(\lambda+(u-t)i)} |\zeta A(1/2+it)|^2 dt du + o(1) \\
&\geq \int_{T+\Delta}^{2T-\Delta} \mathfrak{g}(u) du + o(1)
\end{aligned}
$$

Hence, it will lead same bound as the upper bound above. As for the fact $b_0 = 2\gamma$, it may be followed from the known result of Ingham [5]. $\square$

It should be mentioned that the integrant of $\omega(t, T_1, T_2)$ in general is not as the one of $w(t, T_1, T_2)$ a positive real number, nevertheless, its argument is about $\frac{t-u}{2\lambda} - \frac{(t-u)^3}{6\lambda^2}$, which is very small in the context, so can be approximately viewed as a positive real number, and the deduction above is valid as the one in [1].

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