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Results on Cumulative Entropic Properties of Consecutive Systems

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Abstract: This paper reveals some properties of the cumulative entropy of consecutive k-out-of-n:F system's lifetime. First, we introduce a method for calculating the cumulative entropy of their lifetime and explore its preservation properties using established stochastic orders. Additionally, we obtain useful bounds that can used in situations where the distribution function of the system's component lifetimes are complicated or systems have large components. To support practical applications, we present two nonparametric estimators for the cumulative entropy of these systems. The efficiency and performance of these estimators are illustrated with simulated datasets and further validated with real data sets.

Keywords: Consecutive k-out-of-n:F systems; Cumulative entropy; Shannon entropy; Stochastic orders; Real data.

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1. Introduction

Over the past three decades, extensive research has focused on k-out-of-n systems with consecutive structure and their different variations. A consecutive k-out-of-n system can be classified by the arrangement of its components as either linear or circular, and by its functioning principle as either a failure or a good system. A linear consecutive k-out-of-n:F system consisting of n components (with independent and identically distributed (i.i.d.) lifetimes) set out linearly fails if, and only if, at least k consecutive components fail. This classic example illustrates a linear consecutive k-out-of-n:F system. Consider an oil pipeline with n equally spaced pump stations, each 100 km apart, capable of transporting oil up to 400 km. Consequently, the failure of four consecutive pump stations results in system failure, characterizing it as a linear consecutive 4-out-of-n:F system. The consecutive n-out-of-n:F system, requiring all n components to fail, corresponds to a classical system with parallel structure. In contrast, the 1-out-of-n:F system, which requires at least one component to fail, represents a series system. Comprehensive reviews of previous research on this area are available in Jung and Kim [1], Shen and Zuo [2], Kuo and Zuo [3], Chang *et al.* [4], Boland and Samaniego [5], and Eryılmaz [6,7], among others.

The scenario where $2k \geq n$ in linear consecutive k-out-of-n:F systems is particularly significant for researchers in applied probability and reliaiblity due to its mathematical simplicity. In these systems, the lifetime of each component is denoted by X_i, $1 \leq i \leq n$. Each component is assumed to follow a probability density function (pdf) $f(x)$ and a cumulative distribution function (cdf) $F(x)$. The overall system's lifetime is represented by the random variable (rv) $T_{k|n:F}$. A key finding in this area, established by Eryilmaz [8], shows that when $2k \geq n$, the cdf of the consecutive k-out-of-n:F system can be expressed as:

$$F_{k|n:F}(x) = (n-k+1)F^k(x) - (n-k)F^{k+1}(x), \ x > 0. \tag{1}$$

The Shannon differential entropy is given by $H(X) = -\mathbb{E}[\log f(X)]$ for a nonnegative continuous rv X with pdf $f(x)$, is widely utilized for its versatility. This concept originated from Shannon's pioneering

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work [9]. Rao *et al.* [10] initiated another intuitive measure of information using $S(x) = 1 - F(x)$ instead of $f(x)$, as follows:

$$\mathfrak{E}(X) = -\int_0^\infty \overline{F}(x) \log \overline{F}(x) dx. \tag{2}$$

For a detailed study of preliminary aspects of (2), the associated dynamic form and its various generalizations we refer the reader to Asadi and Zohrevand [11], Navarro *et al.* [12], and Toomaj *et al.* [13]. In the spirit of (2), Di Crescenzo and Longobardi [14] initiated cumulative entropy (CE) by replacing $S(x)$ with the cumulative distribution function (cdf) $F(x)$, as

$$\mathfrak{CE}(X) = -\int_0^\infty F(x) \log F(x) dx, \tag{3}$$

$$= \int_0^1 \frac{\xi(u)}{f(F^{-1}(u))} du, \tag{4}$$

where $F^{-1}(u) = \inf\{x; F(x) \geq u\}$ and $\xi(u) = -u\log(u)$, $0 \leq u \leq 1$. A key advantage of the CE measure is its connection to the mean inactivity time (MIT) function, given by $\widetilde{m}(x) = \mathbb{E}(x - X|X \leq x)$. Di Crescenzo and Longobardi showed that the CE is the expected value of the MIT function, expressed as $\mathbb{E}(\widetilde{m}(X)) = \mathfrak{CE}(X)$. This relationship underscores the CE's utility in reliability theory, given that the MIT function is commonly used to characterize the aging properties of systems or components.

Researchers have explored the information properties of the order statistics. This interest extends to reliability engineering, where many studies have examined information properties. For example, Toomaj *et al.* [13] utilized system signatures to analyze the criteria entropy of mixed systems. Alomani and Kayid [15] applied system signature to study the fractional cumulative residual entropy of coherent systems. In another study, Shrahili and Kayid [16] investigated the cumulative entropy of the lifetime of an n-component coherent system under the condition that all components fail at time t. They presented various properties, including formulations, bounds, and orderings for this measure, along with a method to have a sense of a better system based on the cumulative Kullback–Leibler information quantity which is a discriminating tool. Additionally, Kayid and Shrahili [17] examined Rényi entropy for coherent systems with n components provided that all components fail at time t. They presented a number of results which reveal calculative formulas for this entropic measure, obtain some bounds for it, and establish stochastic ordering results. Motivated by the established body of research on information measures in reliability, this paper delves into uncertainty properties of CE specifically within the framework of consecutive k-out-of-n:F systems. By building upon this foundation, we aim to contribute to a deeper understanding of CE properties within this particular system configuration.

This paper is structured as is outlined below:

In Section 2, we derive a representation of the CE for consecutive k-out-of-n:F systems with lifetime $T_{k|n:F}$ based on samples from any continuous distribution function F. This representation is related to the CE from samples drawn from a uniform distribution. We also analyze the preservation of stochastic ordering properties for this system. The section further includes useful bounds for the CE of consecutive k-out-of-n:F systems. Section 3 presents several characterization results, while Section 4 offers computational results that validate our derived outcomes. To this aim, we introduce two nonparametric estimators for the CE of consecutive systems, demonstrating their utility with real and simulated data. In Section 5, we conclude our study by summarizing the key findings and the main contributions.

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2. CE of Consecutive k-out-of-n:F System

This section is divided into two main parts. First, we derive a mathematical expression for the CE of a consecutive k-out-of-n:F system and analyze the preservation properties of its stochastic ordering. In the second part, we establish a set of useful bounds for studying consecutive k-out-of-n:F systems.

2.1. Expression and Stochastic Orders

Hereafter, we derive an explicit expression for the CE of a consecutive k-out-of-n:F system with lifetime $T_{k|n:F}$, where component lifetimes follow a common continuous distribution function F. We employ the probability integral transformation $U_{k|n:F} = F(T_{k|n:F})$ to obtain the useful formula. The transformations of the system's components, $U_i = F(X_i)$ for $i = 1, \ldots, n$, are i.i.d. random rvs uniformly distributed on $[0, 1]$. Using (1), when $2k \geq n$, the cdf of $U_{k|n:F}$ is given by

$$G_{k|n:F}(u) = (n-k+1)u^k - (n-k)u^{k+1}, \tag{5}$$

for all $0 < u < 1$. We are now prepared to present the following theorem based on the previous analysis.

Theorem 1. *For $2k \geq n$, the CE of $T_{k|n:F}$, can be expressed as follows:*

$$\mathfrak{CE}(T_{k|n:F}) = \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))} du, \tag{6}$$

where $\xi(x) = -x \log x, 0 < x < 1$, and $G_{k|n:F}(u)$ is define in (5).

Proof. By employing the change of $u = F(x)$ and referring to (1) and (3), we can derive the following results:

$$\begin{aligned}
\mathfrak{CE}(T_{k|n:F}) &= -\int_0^\infty F_{k|n:F}(x) \log F_{k|n:F}(x) dx \\
&= \int_0^\infty \xi(F_{k|n:F}(x)) dx \\
&= \int_0^\infty \xi((n-k+1)F^k(x) - (n-k)F^{k+1}(x)) dx \qquad (7) \\
&= \int_0^1 \frac{\xi((n-k+1)u^k - (n-k)u^{k+1})}{f(F^{-1}(u))} du \\
&= \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))} du, \qquad (8)
\end{aligned}$$

and this completes the proof. □

Following the representation in equation (6), we now present an illustrative example.

Example 1. Consider a linear consecutive k-out-of-6:F for $k = 3, 4, 5, 6$, system with a lifetime $T_{k|6:F} = \min(X^{1:k}, X^{2:k+1}, \ldots, X^{7-k:6})$ where $X^{[j:m]} = \max(X_j, \ldots, X_m)$ for $1 \leq j < m \leq 6$. Assume further that the lifetimes of the components are i.i.d. following the common Fréchet distribution, also known as inverse Weibull distribution, with the cdf as

$$F(x) = e^{-x^{-\alpha}}, \quad x > 0, \tag{9}$$

where $\alpha > 0$ is a shape parameter. It is worth noting that Fréchet distribution is a special case of the generalized extreme value distribution. The pdf of this distribution is $f(x) = \alpha x^{-(\alpha+1)} e^{-x^{-\alpha}}$, $x > 0$. It is not hard to see that $f(F^{-1}(u)) = \alpha u(-\log(u))^{\frac{\alpha+1}{\alpha}}$ for all $0 < u < 1$. So, we can derive the following expression

$$\mathfrak{CE}(T_{k|6:F}) = \alpha \int_0^1 \xi(G_{k|n:F}(u)) u(-\log(u))^{\frac{\alpha+1}{\alpha}} du. \tag{10}$$

Due to the difficulty of obtaining an explicit analytical expression, we use a computational approach to explore the relationship between $\mathfrak{CE}(T_{k|6:F})$ and the parameter α for values of $k = 3, 4, 5, 6$. This method sheds light on how the Fréchet distribution parameter affects the CE of the consecutive k-out-of-6:F system. Figure 1 summarizes the numerical analysis, displaying the relationship between $\mathfrak{CE}(T_{k|6:F})$ and α for $k = 3, 4, 5, 6$. As the shape parameter α increases, the system's uncertainty regarding the CE initially decreases and then increases. These results reveal the significant influence of the Fréchet distribution parameter α on the CE and the uncertainty of the consecutive k-out-of-6:F system.

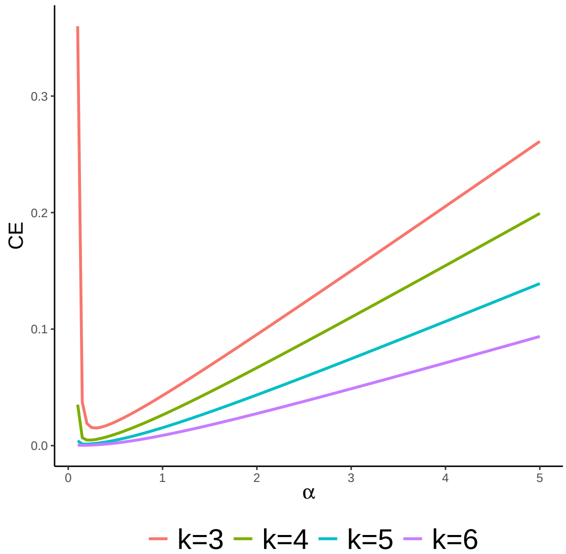

Figure 1. The plot of $\mathfrak{CE}(T_{k|6:F})$ with respect to α as demonstrated in Example 1.

Next, we prove that the CE of the consecutive k-out-of-n:F system preserves the dispersive order and also location-independent riskier order. First, we give the definitions of these stochastic orders before going into their details. Hereafter, $\mathfrak{R}_+ = \{X; X \geq 0\}$ stands as the set of all nonnegative rvs having support $(0, \infty)$ which have an absolutely continuous distribution.

Definition 1. *Let $X \in \mathfrak{R}_+$ and $Y \in \mathfrak{R}_+$ with pdfs f_X and f_Y, cdfs F_X and F_Y, survival functions S_X and S_Y and hazard rate (hr) functions $\lambda_X(x) = \frac{f_X(x)}{S_X(x)}$ and $\lambda_Y(x) = \frac{f_Y(x)}{S_Y(x)}$, respectively. Then,*

1. *X belongs to increasing [resp. decreasing] failure rate (abbreviated by IFR [resp. DFR]) if λ_X is an increasing (a decreasing) function;*
2. *X is said to be less than or equal with Y in the hazard rate order (written as $X \leq_{hr} Y$) whenever $\lambda_X(t) \geq \lambda_Y(t)$ for all $t > 0$;*
3. *X is said to be less than or equal with Y in the dispersive order (written as $X \leq_d Y$) whenever $F_X^{-1}(v) - F_X^{-1}(u) \leq F_Y^{-1}(v) - F_Y^{-1}(u)$, $0 < u \leq v < 1$;*

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4. *X is said to be less than or equal with Y in the location-independent riskier order (written as $X \leq_{lir} Y$) whenever $\int_0^{F_X^{-1}(p)} F_X(x)dx \leq \int_0^{F_Y^{-1}(p)} F_Y(x)dx,\ p \in (0,1)$;*
5. *X is said to be less than or equal with Y in the CE order (written as $X \leq_{CE} Y$) whenever $\mathfrak{CE}(X) \leq \mathfrak{CE}(Y)$.*

It is worth noting that the order $\leq_d$ was firstly utilized by Bickel and Lehmann [18] for some nonparametric inferential purposes, while the order $\leq_{lir}$ was initiated by Jewitt [19] to be used in the theory of expected utility and its applications in insurance. From Shaked and Shanthikumar [20], $X \leq_d Y$ if, and only if,

$$f_Y(F_Y^{-1}(v)) \leq f_X(F_X^{-1}(v)), \text{ for all } 0 < v < 1. \tag{11}$$

The following implications are known:

$$\text{if } X \leq_{hr} Y \text{ and either } X \text{ or } Y \text{ is DFR } \implies X \leq_d Y \Longrightarrow X \leq_{lir} Y. \tag{12}$$

From relations (4) and (11) and since $\xi(u) \geq 0$ for all $0 \leq u \leq 1$, thus $X \leq_d Y$ yields $\mathfrak{CE}(X) \leq \mathfrak{CE}(Y)$ By using (12), the following conclusion is deduced.

Corollary 1. *If $X \leq_{hr} Y$ and X or Y is DFR, then $\mathfrak{CE}(X) \leq \mathfrak{CE}(Y)$.*

We now examine the integrated distribution function of a rv Z with the cdf H, defined as

$$\eta_Z(x) = \int_0^x H(z)dz,\ x > 0.$$

Landsberger and Meilijson [21] showed that

$$X \leq_{lir} Y \Longleftrightarrow \eta_Y^{-1}(x) - \eta_X^{-1}(x) \text{ is increasing in } x > 0. \tag{13}$$

In what follows we consider $T^X_{k|n:F}$ and $T^Y_{k|n:F}$ as the lifetimes of two consecutive k-out-of-n:F systems with i.i.d. absolutely continuous component lifetimes having the common pdfs f_X and f_Y and cdfs F_X and F_Y, respectively. We present a theorem demonstrating that the CE of a series system with k components is less than that of a consecutive k-out-of-n:F system, assuming both have components with the DFR property.

Theorem 2. *Let $Z_{1:m}$ be the lifetime of a series system consisting of i.i.d. components with common hr function h. Let $lim_{t\to\infty} r(t) = \lambda$ and $1 \leq k \leq n$, where r is the common hr of X and let $T_{k|n:F}$ belong to IFR class such that $h(t) \geq \lambda \frac{[\frac{n}{k}]}{m}$ for all $t \geq 0$. If Z belongs to DFR class, then $\mathfrak{CE}(Z_{1:m}) \leq \mathfrak{CE}(T_{k|n:F})$.*

Proof. Since Z is DFR, then $Z_{1:m}$ is also DFR. Moreover, under the conditions $T_{k|n:F}$ is IFR such that $h(t) \geq \lambda[n/k]/m$ for all $t \geq 0$, we have $Z_{1:m} \leq_{hr} T_{k|n:F}$ due to Theorem 3.2 of Eryılmaz and Navarro [22]. Thus, Corollary 1 concludes the proof. □

The next example illustrates the application of Theorem 2.

Example 2. Consider a Gamma distribution with the cdf given by $F(t) = 1 - \lambda t e^{-\lambda t} - e^{-\lambda t}$. We find that $r(t) = \frac{2\lambda t}{1+\lambda t} \to \lambda$ as $t \to \infty$. Letting $n = 4$ and $k = 2$, since X is IFR and a linear consecutive 2-out-of-4:F system preserves the IFR property (see Theorem 4.3.13 of Chang *et al.* [23]), we conclude that $Z_{1:4} \leq_{hr} T_{2|4:F}$ for $h(t) \geq \frac{\lambda}{2}$ for all $t \geq 0$. So, Theorem 2 implies that $\mathfrak{CE}(Z_{1:m}) \leq \mathfrak{CE}(T_{k|n:F})$ provided that $h(t)$ is a decreasing function in t.

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The next theorem establishes the conditions under which the dispersive order is preserved under the formation of consecutive systems.

Theorem 3. *If $X \leq_d Y$, then $\mathfrak{CE}(T^X_{k|n:F}) \leq \mathfrak{CE}(T^Y_{k|n:F})$.*

Proof. The result can be easily derived from Eqs. (6) and (11). □

As an application of Theorem 3, consider the following example.

Example 3. Let us consider two consecutive 4-out-of-5:F systems with lifetimes $T^X_{4|5:F}$ and $T^Y_{4|5:F}$. System $T^X_{4|5:F}$ has i.i.d. component lifetimes X_1, X_2, X_3, X_4, X_5, which follow the Makeham distribution with cdf $F(x) = 1 - e^{-2x+e^{-x}-1}$ for $x > 0$. Moreover, system $T^Y_{4|5:F}$ consists of i.i.d. component lifetimes Y_1, Y_2, Y_3, Y_4 that follow an exponential distribution with cdf $F_Y(x) = 1 - e^{-x}$ for $x > 0$. The hr functions are $\lambda_X(x) = 2 - e^{-x}$ and $\lambda_Y(x) = 1$, showing that $\lambda_X(x) > \lambda_Y(x)$ for $x > 0$ i.e. $X \leq_{hr} Y$. Since Y possesses the DFR property, relation (12) indicates $X \leq_d Y$. Consequently, by Theorem 3, we have $\mathfrak{CE}(T^X_{4|5:F}) \leq \mathfrak{CE}(T^Y_{4|5:F})$, meaning the uncertainty associated with $T^X_{4|5:F}$ is less than or equal to that of $T^Y_{4|5:F}$ in terms of the CE measure.

The next theorem outlines the conditions for preserving location-independent riskier order in consecutive systems.

Theorem 4. *If $X \leq_{lir} Y$, and*

$$\frac{\xi(G_{k|n:F}(t))}{t}, \ 0 \leq t \leq 1, \tag{14}$$

is a decreasing function of t, then $\mathfrak{CE}(T^X_{k|n:F}) \leq \mathfrak{CE}(T^Y_{k|n:F})$.

Proof. First note that Eq. (1) can be rewritten as $F_{k|n:F}(x) = G_{k|n:F}(F(x))$. Assumption $X \leq_{lir} Y$ and Eq. (13) imply

$$\frac{d}{dx}(\eta_Y^{-1}(x) - \eta_X^{-1}(x)) = \frac{1}{F_Y(\eta_Y^{-1}(x))} - \frac{1}{F_X(\eta_X^{-1}(x))} \geq 0,$$

which means

$$F_X(x) \geq F_Y(\eta_Y^{-1}(\eta_X(x))), \tag{15}$$

for all $x > 0$. Now, we get

$$\begin{aligned}
-\int_0^\infty F^X_{k|n:F}(x)\log F^X_{k|n:F}(x)dx &= -\int_0^\infty \frac{F^X_{k|n:F}(x)\log F^X_{k|n:F}(x)}{F_X(x)}F_X(x)\mathrm{d}x \\
&= \int_0^\infty \frac{\xi(F^X_{k|n:F}(x))}{F_X(x)}F_X(x)\mathrm{d}x \\
&= \int_0^\infty \frac{\xi(G_{k|n:F}(F_X(x)))}{F_X(x)}F_X(x)\mathrm{d}x \\
&\leq \int_0^\infty \frac{\xi(G_{k|n:F}(F_Y(\eta_Y^{-1}(\eta_X(x)))))}{F_Y(\eta_Y^{-1}(\eta_X(x)))}F_X(x)\mathrm{d}x,
\end{aligned} \tag{16}$$

where the inequality arises from the fact that $\xi(G_{k|n:F}(t))/t$ decreases for $0 \leq t \leq 1$ and using (15). Setting $u = \eta_Y^{-1}(\eta_X(x))$, we have

$$dx = \frac{F_Y(u)}{F_X(\eta_X^{-1}(\eta_Y(u)))}du.$$

Upon using this, (16) reduces to

$$\begin{aligned}
&\int_{\eta_Y^{-1}(\eta_X(0))}^{\infty} \frac{\xi(G_{k|n:F}(F_Y(u)))}{F_Y(u)} \frac{F_X(\eta_X^{-1}(\eta_Y(u)))F_Y(u)}{F_X(\eta_X^{-1}(\eta_Y(u)))}du \\
= &\int_{\eta_Y^{-1}(\eta_X(0))}^{\infty} \xi(G_{k|n:F}(F_Y(u)))du \\
= &-\int_0^{\infty} F^Y_{k|n:F}(u) \log F^Y_{k|n:F}(u)du.
\end{aligned}$$

The final equality in the above relation is derived by observing that $\eta_Y^{-1}(\eta_X(0)) = 0$, which implies $\mathfrak{CE}(T^X_{k|n:F}) \leq \mathfrak{CE}(T^Y_{k|n:F})$. Thus, the proof is finished. □

2.2. Bounds

Given the absence of closed-form expressions for the CE of consecutive systems for various distributions with complicated distribution functions or having numerous components, it is essential to give some bounds in such situations. Recognizing this challenge, we aim to explore the effectiveness of these bounds in characterizing the CE of consecutive systems. For the first result, we find that the system's CE is bounded by the common CE of its components.

Theorem 5. *For $2k \geq n$, the CE of $T_{k|n:F}$ are bounded as follows:*

$$\mathfrak{B}_1\mathfrak{CE}(X_1) \leq \mathfrak{CE}(T_{k|n:F}) \leq \mathfrak{B}_2\mathfrak{CE}(X_1)$$

where $\mathfrak{B}_1 = \inf_{u\in(0,1)} \frac{\xi(G_{k|n:F}(u))}{\xi(v)}$, $\mathfrak{B}_2 = \sup_{u\in(0,1)} \frac{\xi(G_{k|n:F}(u))}{\xi(u)}$.

Proof. The upper bound can be determined from (6) as shown below

$$\begin{aligned}
\mathfrak{CE}(T_{k|n:F}) &= \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))}du = \int_0^1 \frac{\xi(G_{k|n:F}(u))}{\xi(u)} \frac{\xi(u)}{f(F^{-1}(u))}du \\
&\leq \sup_{u\in(0,1)} \frac{\xi(G_{k|n:F}(u))}{\xi(u)} \int_0^1 \frac{\xi(u)}{f(F^{-1}(u))}du = \mathfrak{B}_2\mathfrak{CE}(X_1).
\end{aligned}$$

The lower bound can be derived using a similar approach. □

The upcoming theorem offers additional straightforward and useful bounds based on the extremes of the pdf and the function $\xi(u)$.

Theorem 6. *Let $T_{k|n:F}$ be the lifetime of consecutive k-out-of-n:F system having the common pdf $f_X(x)$ and cdf $F_X(x)$. If S is the support of f, $m = \inf_{x\in S} f(x)$ and $M = \sup_{x\in S} f(x)$, then*

$$\frac{\mathfrak{CE}(U_{k|n:F})}{M} \leq \mathfrak{CE}(T_{k|n:F}) \leq \frac{\mathfrak{CE}(U_{k|n:F})}{m}, \quad (17)$$

where $\mathfrak{CE}(U_{k|n:F}) = \int_0^1 \xi(G_{k|n:F}(u))du$ *and* $\xi(u) = -u\log(u)$.

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Proof. Since $m \leq f(F^{-1}(u)) \leq M,\ 0 < u < 1$, from (6), we have

$$\mathfrak{CE}(T_{k|n:F}) = \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))} du \geq \frac{1}{M} \int_0^1 \xi(G_{k|n:F}(u)) du.$$

The upper bound can be obtained similarly. □

It should be noted that $\mathfrak{CE}(U_{k|n:F})$ denotes the cumulative entropy of a consecutive k-out-of-n:F system having a common uniform distribution on $(0,1)$. The bounds in Eq. (17) depend on the extremes of the probability density function f. If the lower bound m is zero, there is no upper bound; if the upper bound M is infinite, there is no lower bound. The following example illustrates the application of the bounds from Theorems 5 and 6 in the context of a consecutive k-out-of-n:F system.

Example 4. Assume a linear consecutive 6-out-of-12:F system with lifetime $T_{6|12:F} = \max(X_{[1:6]}, X_{[2:7]}, \ldots, X_{[6:12]})$, where $X_{[j:m]} = \min(X_j, \ldots, X_m)$ for $1 \leq j < m \leq 12$. It is straightforward to calculate that $\mathfrak{CE}(U_{6|12:F}) = 0.1386067$ and $\mathfrak{B}_1 = 0, \mathfrak{B}_2 = 1.783154$. The bounds in Theorems 5 and 6 can be calculated for common component lifetime distributions. To illustrate, we consider the following models as examples.
(a) Assuming a half-normal distribution with pdf

$$f_X(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}},\ x > 0,\ \sigma > 0,$$

it is easy to see that $m = 0$ and $M = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}$. Applying the result from Theorem 6, we can obtain the lower bound $\mathfrak{CE}(T_{6|12:F}) \geq \frac{0.1960195}{\sigma\sqrt{\pi}}$. Furthermore, using the bound provided in Theorem 5, we can derive $\mathfrak{CE}(T_{6|12:F}) \leq 1.783154\mathfrak{CE}(X_1)$. By combining these two bounds, we can conclude that $\frac{0.1960195}{\sigma\sqrt{\pi}} \leq \mathfrak{CE}(T_{6|12:F}) \leq 1.783154\mathfrak{CE}(X_1)$.
(b) Suppose that X follows a Fréchet distribution with cdf given in (9). Then $m = 0$ and

$$M = \alpha\left(\frac{\alpha}{\alpha+1}\right)^{-\frac{\alpha+1}{\alpha}} e^{-(1+\frac{1}{\alpha})}.$$

Furthermore, using the bound provided in Theorem 5, we can derive $\mathfrak{CE}(T_{6|12:F}) \leq 1.783154\mathfrak{CE}(X_1)$. By combining these two bounds, we can conclude that

$$0.1386067\alpha\left(\frac{\alpha}{\alpha+1}\right)^{-\frac{\alpha+1}{\alpha}} e^{-(1+\frac{1}{\alpha})} \leq \mathfrak{CE}(T_{6|12:F}) \leq 1.783154\mathfrak{CE}(X_1).$$

3. Characterization Results

This section aims to present some characterization results based on the cumulative entropy properties of consecutive k-out-of-n:F systems. We begin with a lemma that follows directly from the Stone–Weierstrass Theorem (see Aliprantis and Burkinshaw, [24]), which will be used in proving the main results of this section.

Lemma 1. *If ζ is a continuous function on $[0,1]$ such that $\int_0^1 x^n \zeta(x) dx = 0$ for all $n \geq 0$, then $\zeta(x) = 0$ for any $x \in [0,1]$.*

This lemma allows us to uniquely characterize the parent distribution of a lifetime rv using the CE of $T_{k|n:F}$.

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Theorem 7. *Let $T^X_{k|n:F}$ and $T^Y_{k|n:F}$ be lifetimes of two consecutive k-out-of-n:G systems having the common pdfs $f_X(x)$ and $f_Y(x)$ and cdfs $F_X(x)$ and $F_Y(x)$, respectively. Then F_X and F_Y belong to the same family of distributions if and only if for a fixed n,*

$$\mathfrak{CE}(T^X_{k|n:F}) = \mathfrak{CE}(T^Y_{k|n:F}), \tag{18}$$

for all $2k \geq n$.

Proof. The necessity is trivial, so we must demonstrate the sufficiency. First, observe that Eq. (6) can be rewritten as follows:

$$\mathfrak{CE}(T^X_{k|n:F}) = \int_0^1 u^{2k-n} \frac{\phi(u)}{f_X(F_X^{-1}(u))} du, \tag{19}$$

where $\phi(u) = -u^{n-k}((n-k+1) - (n-k)u) \log(G_{k|n:F}(u)),\ 0 < u < 1$. The same argument applies to $\mathfrak{CE}(T^Y_{k|n:F})$. From (18), we have

$$\int_0^1 \left[\frac{1}{f_X(F_X^{-1}(u))} - \frac{1}{f_Y(F_Y^{-1}(u))} \right] u^{2k-n} \phi(u) du = 0.$$

According to Lemma 1, we can conclude that

$$f_X(F_X^{-1}(u)) = f_Y(F_Y^{-1}(u)), \quad a.e. \quad u \in (0,1).$$

It follows that $F_X^{-1}(u) = F_Y^{-1}(u) + d$, where d is a constant. Since $\lim_{u \to 0} F_X^{-1}(u) = \lim_{u \to 0} F_Y^{-1}(u) = 0$ for all $u \in (0,1)$, we conclude that $F_X^{-1}(u) = F_Y^{-1}(u)$. This indicates that F_X and F_Y have the same family of distributions. □

A consecutive n-out-of-n:F system is a series system, as previously mentioned, the following corollary outlines its characteristics.

Corollary 2. *Under the conditions of Theorem 7, F_X and F_Y belong to the same family of distributions if and only if*

$$\mathfrak{CE}(T^X_{n|n:F}) = \mathfrak{CE}(T^Y_{n|n:F}),$$

for all $n \geq 1$.

The subsequent theorem provides a further characterization.

Theorem 8. *Under the conditions of Theorem 7, F_X and F_Y belong to the same family of distributions, but for a change in scale, if and only if for a fixed n,*

$$\frac{\mathfrak{CE}(T^X_{k|n:F})}{\mathfrak{CE}(X)} = \frac{\mathfrak{CE}(T^Y_{k|n:F})}{\mathfrak{CE}(Y)}, \tag{20}$$

for all $2k \geq n$.

Proof. The necessity is trivial and hence it remains to prove the sufficiency. From (19), we can write

$$\frac{\mathfrak{CE}(T^X_{k|n:F})}{\mathfrak{CE}(X)} = \int_0^1 u^{2k-n} \frac{\phi(u)}{\mathfrak{CE}(X) f_X(F_X^{-1}(u))} du. \tag{21}$$

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The same argument applies to $\mathfrak{CE}(T^Y_{k|n:F})/\mathfrak{CE}(Y)$. From (20) and (21), we have

$$\int_0^1 u^{2k-n}\frac{\phi(u)}{\mathfrak{CE}(X)f_X(F_X^{-1}(u))}du = \int_0^1 u^{2k-n}\frac{\phi(u)}{\mathfrak{CE}(Y)f_Y(F_Y^{-1}(u))}du. \tag{22}$$

Let us set $c = \mathfrak{CE}(Y)/\mathfrak{CE}(X)$. Then, (22) can be expressed as

$$\int_0^1 \left[\frac{1}{f_X(F_X^{-1}(u))} - \frac{1}{cf_Y(F_Y^{-1}(u))}\right]u^{2k-n}\phi(u)du = 0.$$

Applying Lemma 1, we can conclude that

$$f_X(F_X^{-1}(u)) = cf_Y(F_Y^{-1}(u)), \quad a.e. \quad z \in (0,1).$$

It follows that $F_X^{-1}(u) = cF_Y^{-1}(u) + d$, where d is a constant. Since $\lim_{u\to 0} F_X^{-1}(u) = \lim_{u\to 0} F_Y^{-1}(u) = 0$ for all $u \in (0,1)$, we conclude that $F_X^{-1}(u) = cF_Y^{-1}(u)$. This indicates that F_X and F_Y belong to the same family of distributions but for a change of scale. □

Using Theorem 8, we get the following corollary.

Corollary 3. *Suppose the assumptions of Theorem 8 hold. Then, F_X and F_Y belong to the same family of distributions, but for a change in scale, if and only if*

$$\frac{\mathfrak{CE}(T^X_{n|n:F})}{\mathfrak{CE}(X)} = \frac{\mathfrak{CE}(T^Y_{n|n:F})}{\mathfrak{CE}(Y)},$$

for all $n \geq 1$.

4. Nonparametric Estimation

This section develops two nonparametric methods to estimate the cumulative entropy of consecutive k-out-of-n:F systems. Let us assume a sequence of i.i.d. continuous, non-negative rvs $X_1, X_2, \ldots, X_N$, where $X_{1:N} \leq X_{2:N} \leq \cdots \leq X_{N:N}$ denote their order statistics. Applying Eq. (6), the CE of $T_{k|n:F}$ can be reformulated for the case $2k \geq n$ as follows:

$$\begin{aligned}\mathfrak{CE}(T_{k|n:F}) &= \int_0^1 \frac{\xi(G_{k|n:F}(u))}{f(F^{-1}(u))}du = \int_0^1 \xi(G_{k|n:F}(u))\left[\frac{dF^{-1}(u)}{du}\right]du \\ &= \int_0^1 \xi((n-k+1)u^k - (n-k)u^{k+1})\left[\frac{dF^{-1}(u)}{du}\right]du. \end{aligned} \tag{23}$$

Using Eq. (23), we estimate $\mathfrak{CE}(T_{k|n:F})$ by approximating the derivative of the inverse distribution function at sample points. Following Vasicek [25], we estimate this derivative as

$$\frac{dF^{-1}(u)}{du} = \frac{N(X_{i+m:N} - X_{i-m:N})}{2m},$$

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where $X_{i:N} = X_{1:N}$ for $i < 1$ and $X_{i:N} = X_{N:N}$ for $i > N$, N is the sample size and m is a positive integer referred to as the window size, satisfying $m \leq N/2$. Consequently, an estimator for $\mathfrak{CE}(T_{k|n:F})$ is obtained as follows:

$$\begin{aligned}\widehat{\mathfrak{CE}}_1(T_{k|n:F}) &= \frac{1}{N}\sum_{i=1}^{N}\xi\left(G_{k|n:F}\left(\frac{i}{N+1}\right)\right)\left(\frac{N(X_{i+m:N}-X_{i-m:N})}{2m}\right)\\ &= \frac{1}{N}\sum_{i=1}^{N}\xi\left((n-k+1)\left(\frac{i}{N+1}\right)^{k}-(n-k)\left(\frac{i}{N+1}\right)^{k+1}\right)\\ &\times \left(\frac{N(X_{i+m:N}-X_{i-m:N})}{2m}\right).\end{aligned} \tag{24}$$

The second estimator is constructed using the empirical cumulative distribution function associated with $F(x)$ of the sample, as follows:

$$F_N(x) = \sum_{i=1}^{N-1}\frac{i}{N}I_{[x_{i:N},x_{(i+1)}]},\ x \geq 0,$$

where $I_A(x) = 1$ if $x \in A$. Based on Eq. (7), the empirical CE estimator for the consecutive k-out-of-n:F system is given by

$$\begin{aligned}\widehat{\mathfrak{CE}}_2(T_{k|n:F}) &= \int_0^{\infty}\xi((n-k+1)F_N^k(x)-(n-k)F_N^{k+1}(x))dx\\ &= \sum_{i=1}^{N-1}\int_{X_{i:N}}^{X_{(i+1)}}\xi((n-k+1)F_N^k(x)-(n-k)F_N^{k+1}(x))dx\\ &= \sum_{i=1}^{N-1}\xi\left((n-k+1)\left(\frac{i}{N}\right)^{k}-(n-k)\left(\frac{i}{N}\right)^{k+1}\right)D_{i+1},\end{aligned} \tag{25}$$

where $D_{i+1} = X_{i+1:N} - X_{i:N}$, $i = 1, 2, \ldots, N-1$, denotes the sample spacings.

A simulation study using standard exponential distribution is performed to assess the performance of the proposed estimators, $\widehat{\mathfrak{CE}}_1(T_{k|n:F})$ and $\widehat{\mathfrak{CE}}_2(T_{k|n:F})$. The average bias and root mean squared error (RMSE) are computed for different sample sizes ($N = 20, 30, 40, 50, 100$) and various combinations of parameters k and n. The smoothing parameter m is determined using the heuristic formula$m = [\sqrt{N} + 0.5]$, where $[x]$ denotes the integer part of x.

The simulation was run 5,000 times, and the results are shown in Tables 1 and 2. Based on the analysis of these tables, we have reached the following results:

- For all k and n, as the sample size N increases, both bias and RMSE of the estimators decrease.
- For fixed n and N, as the number of consecutive working components k increases, both bias and RMSE of the estimator increase.

In general, the results show that the efficiency of the estimator is influenced by the number of components n and the number of consecutive working components k.

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Table 1. The Bias and RMSE of the first estimator $\widehat{\mathfrak{CE}}_1(T_{k|n:F})$ for different choices of k and n.

		$N=20$		$N=30$		$N=40$		$N=50$		$N=100$	
n	k	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	3	-0.124138	0.230344	-0.100742	0.191860	-0.095708	0.171305	-0.093843	0.160603	-0.094036	0.131869
	4	-0.038756	0.263316	0.000577	0.231150	0.036344	0.218155	0.051543	0.201405	0.093028	0.173277
	5	-0.091323	0.287947	-0.009914	0.253112	0.038630	0.247169	0.066143	0.237489	0.147523	0.237116
6	3	0.072069	0.167615	0.070267	0.138012	0.066130	0.124728	0.061647	0.109277	0.040760	0.075014
	4	-0.031567	0.239911	0.010755	0.209871	0.022254	0.184367	0.035022	0.174554	0.055193	0.134261
	5	-0.151385	0.310038	-0.079245	0.260249	-0.048472	0.232668	-0.015527	0.219851	0.041610	0.175196
	6	-0.272339	0.380990	-0.184579	0.315350	-0.127539	0.277684	-0.088733	0.253952	0.009005	0.199077
7	4	0.021823	0.208960	0.042960	0.183286	0.054940	0.161538	0.054626	0.150306	0.057243	0.113776
	5	-0.092157	0.269532	-0.034616	0.238141	-0.008311	0.214772	0.008427	0.198731	0.051931	0.159579
	6	-0.213470	0.350137	-0.135670	0.289292	-0.086826	0.260511	-0.055488	0.242480	0.031440	0.186683
	7	-0.331163	0.417481	-0.230386	0.349043	-0.161969	0.305570	-0.130153	0.279632	-0.007831	0.216406
8	4	0.075346	0.191621	0.080118	0.161910	0.074436	0.143996	0.076248	0.134626	0.059875	0.092973
	5	-0.035972	0.244008	0.000687	0.214961	0.023995	0.194971	0.038143	0.183108	0.061133	0.145446
	6	-0.168064	0.312048	-0.088598	0.260961	-0.048191	0.238333	-0.020311	0.224606	0.037779	0.178471
	7	-0.291871	0.384639	-0.185520	0.314696	-0.129251	0.281766	-0.093509	0.254625	0.004763	0.198204
	8	-0.385751	0.455766	-0.276726	0.378007	-0.211355	0.332942	-0.163147	0.300208	-0.034566	0.218227

Table 2. The Bias and RMSE of the second estimator $\widehat{\mathfrak{CE}}_2(T_{k|n:F})$ for different choices of k and n.

		$N=20$		$N=30$		$N=40$		$N=50$		$N=100$	
n	k	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	3	-0.025125	0.175512	-0.014116	0.141257	-0.015236	0.123774	-0.012079	0.109911	-0.005279	0.078412
	4	-0.072958	0.267537	-0.050056	0.219368	-0.029108	0.190125	-0.032334	0.172732	-0.012504	0.123541
	5	-0.123044	0.332033	-0.084990	0.272151	-0.055920	0.239984	-0.048001	0.219015	-0.023000	0.156351
6	3	-0.001871	0.132976	0.000809	0.109061	0.000794	0.092947	-0.000688	0.082732	-0.000010	0.058974
	4	-0.045391	0.228448	-0.028728	0.182676	-0.024040	0.157982	-0.016516	0.145011	-0.011101	0.102678
	5	-0.100189	0.299852	-0.064542	0.250162	-0.049838	0.217331	-0.038395	0.195598	-0.018189	0.139785
	6	-0.131134	0.357338	-0.099022	0.297588	-0.077961	0.262055	-0.057160	0.238616	-0.028988	0.170018
7	4	-0.026347	0.182529	-0.018355	0.150178	-0.011043	0.133090	-0.010400	0.117107	-0.006821	0.083850
	5	-0.075611	0.266131	-0.041516	0.221699	-0.037394	0.192767	-0.029404	0.171981	-0.014986	0.122495
	6	-0.124722	0.331439	-0.081209	0.278495	-0.064191	0.245961	-0.050115	0.219265	-0.026817	0.155459
	7	-0.169382	0.377862	-0.106689	0.319331	-0.086256	0.286752	-0.068001	0.256148	-0.032641	0.182052
8	4	-0.002768	0.152239	-0.003317	0.119965	0.000883	0.106233	0.001972	0.094212	-0.000304	0.066692
	5	-0.052062	0.233434	-0.032683	0.190547	-0.027234	0.163031	-0.021617	0.147743	-0.008801	0.104927
	6	-0.096677	0.298313	-0.065127	0.255635	-0.048338	0.221591	-0.041602	0.196996	-0.019996	0.140928
	7	-0.141255	0.362541	-0.094531	0.302222	-0.081784	0.265432	-0.062192	0.240006	-0.026913	0.168080
	8	-0.181689	0.401899	-0.131193	0.343492	-0.091136	0.300505	-0.070395	0.267088	-0.035192	0.200356

Real Data Analysis

We apply the estimator to real data to assess how closely the CE estimators from consecutive k-out-of-n:F systems match the theoretical entropy value. The data includes active repair times (in hours) for an airborne communication transceiver reported by [26]. The actual observations are listed below: Datasets: 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22, 24.5. This data is modeled using the Weibull distribution with the pdf

$$f(x) = \lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}},\ x > 0,$$

where $\lambda > 0$ and $\beta > 0$ are scale and shape parameters, respectively. As noted in [27], the datasets were fitted using the Weibull distribution via the maximum likelihood method for parameter estimation. The

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resulting parameters are $\widehat{\lambda} = 3.391$ and $\widehat{\beta} = 0.899$. The Kolmogorov–Smirnov statistic is 0.120 with a p-value of 0.517, confirming a good fit between the observed data and the fitted exponential distribution.

Table 3 shows various combinations of k and n. The results indicate a correlation between the theoretical entropy value and its estimation when the functioning components are nearly half of the total (n).

Table 3. Comparison of theoretical values and estimates of CE of $T_{k|n:F}$ based on Weibull distribution for active repair times (in hours) for an airborne communication transceiver.

k	$\mathfrak{CE}(T_{k\|5:G})$	$\widehat{\mathfrak{CE}}_1(T_{k\|5:G})$	$\widehat{\mathfrak{CE}}_2(T_{k\|5:G})$	$\mathfrak{CE}(T_{k\|6:G})$	$\widehat{\mathfrak{CE}}_1(T_{k\|6:G})$	$\widehat{\mathfrak{CE}}_2(T_{k\|6:G})$
3	0.188772	3.202371	2.708464	0.139103	2.307699	1.786721
4	0.252975	4.412466	4.098577	0.225504	3.978375	3.511787
5	0.282173	4.786405	4.902895	0.267998	4.650849	4.532230
6				0.289143	4.779245	5.167557
k	$\mathfrak{CE}(T_{k\|7:G})$	$\widehat{\mathfrak{CE}}_1(T_{k\|7:G})$	$\widehat{\mathfrak{CE}}_2(T_{k\|7:G})$	$\mathfrak{CE}(T_{k\|8:G})$	$\widehat{\mathfrak{CE}}_1(T_{k\|8:G})$	$\widehat{\mathfrak{CE}}_2(T_{k\|8:G})$
4	0.191428	3.414607	2.827453	0.151613	2.737446	2.057078
5	0.247997	4.397852	4.073618	0.223027	4.043979	3.538539
6	0.278365	4.738680	4.867858	0.263090	4.605239	4.497665
7	0.294482	4.708580	5.384117	0.285978	4.731164	5.137700
8				0.298730	4.597261	5.563095

5. Conclusions

This study investigated the application of the CE concept to consecutive k-out-of-n:F systems. We established a key finding: a strong relationship exists between the CE of such systems derived from continuous and uniform distributions. This result simplifies CE calculations in many practical scenarios. However, obtaining closed-form CE expressions becomes challenging for systems with large or complex component distributions. To overcome this hurdle, we proposed a set of useful bounds for the CE of consecutive k-out-of-n:F systems. These bounds offer valuable tools for researchers and practitioners to understand and analyze CE behavior. Furthermore, we introduced two nonparametric estimators specifically tailored for consecutive k-out-of-n:F systems. These estimators are designed for real-world applications, as demonstrated by their use in real data sets. The CE estimation provides valuable insights into the uncertainty of the systems, aiding in informed decision-making and meaningful data analysis. In conclusion, this work makes significant contributions to the understanding of CE in consecutive k-out-of-n:F systems. The results of this study can be extended to other information measures including fractional cumulative entropy, cumulative residual Tsallis entropy, and cumulative Tsallis entropy.

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