

Article

Not peer-reviewed version

STEh Trade-Off Theorem: Hard Physical Limits for Space, Time, Energy, and Entropy in Computation

[Michael Rey](#)*

Posted Date: 2 September 2025

doi: 10.20944/preprints202509.0186.v1

Keywords: physical limits of computation; Landauer principle; quantum speed limit; Bekenstein bound; reversible computation; space-time trade-offs; communication complexity; thermodynamics of computation



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a Creative Commons CC BY 4.0 license, which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

STEh Trade-Off Theorem: Hard Physical Limits for Space, Time, Energy, and Entropy in Computation

Michael Rey

Octonion Group, Hong Kong; contact@octoniongroup.com

Abstract

We introduce a unified physical framework for algorithmic tradeoffs based on Space (S), Time (T), Energy (E), and Entropy (H) — abbreviated as STEh. This framework integrates classical and quantum perspectives, and yields rigorous lower bounds on computational processes by linking memory, run-time, thermodynamic cost, and uncertainty. We derive a set of inequalities governing the achievable resource combinations and illustrate how they sharpen classical results such as Landauer’s principle and quantum speed limits. The STEh tradeoff formalism provides a systematic approach to study computation as a constrained physical process, highlighting resource complementarity and impossibility results that shape the limits of algorithms.

Keywords: physical limits of computation; Landauer principle; quantum speed limit; Bekenstein bound; reversible computation; space-time trade-offs; communication complexity; thermodynamics of computation

1. Introduction

Computation is a physical process. Three mature pillars constrain its cost: (i) *Landauer’s principle* lower-bounds dissipated heat for logically irreversible steps; (ii) *quantum speed limits* (QSL) lower-bound the time to traverse distinguishable quantum states given available energy; and (iii) the *Bekenstein bound* upper-bounds information capacity for finite-energy systems in bounded regions. We show how to combine these, with no additional heuristic approximations, into two explicit inequalities that jointly forbid the simultaneous minimization of space, time, exported entropy, and energy for nontrivial computations.

A central contribution is the introduction of three *well-defined* complexity parameters for a function $f: \{0,1\}^n \rightarrow \{0,1\}^m$ at target error ϵ : (1) the *reversible working-set* $W^*(f, \epsilon)$, (2) the *forced erasures under S space* $G^*(f, S, \epsilon)$, and (3) the *orthogonalization depth* $D^*(f, \epsilon)$. We formalize these in Section 2, relate them to classical/quantum complexity notions, and prove the lemmas needed to derive the main result.

Physical regime and simultaneous applicability

Our derivations hold for devices satisfying:

- (R1) **Thermal bath:** Coupling to an environment at temperature T (near equilibrium) so that Landauer applies to logically irreversible operations [1,5–7].
- (R2) **Quantum dynamics:** The device evolution over runtime τ can be modeled by a (possibly open-system) CPTP map with well-defined average energy above ground E_{tot} ; we use QSLs valid for closed or Markovian/non-Markovian dynamics [8–12].
- (R3) **Bounded geometry and weak gravity:** Degrees of freedom are essentially confined to a ball of radius R , with total energy well below the gravitational-collapse threshold ($R \gg 2GE_{\text{tot}}/c^4$), so Bekenstein’s bound controls information capacity [13–15].

These conditions are commonly met in laboratory and technological settings; therefore the three primitive bounds are consistent and simultaneously applicable.

2. Definitions of the Complexity Parameters

Let $f : \{0,1\}^n \rightarrow \{0,1\}^m$ and target worst-case error $\varepsilon \in [0, 1/2)$. Computations are implemented by families of physical processes indexed by input $x \in \{0,1\}^n$ acting on a working medium; mathematically, by CPTP maps $\mathcal{E}_t$ over $t \in [0, \tau]$ whose induced input-output relation approximates f .

Definition 1 (Reversible working-set $W^*(f, \varepsilon)$). Consider all logically reversible implementations of f with error $\leq \varepsilon$, i.e., injective encodings $x \mapsto (x, a_0) \mapsto (x, a_T)$ and a final clean-up that restores ancillae while outputting $f(x)$ (Bennett's method). For each such implementation, let $S_{\max}$ be the peak number of information-bearing bits (IBBs) that are simultaneously logically live (i.e., their value is needed by some future reversible step). Then

$$W^*(f, \varepsilon) = \inf_{\text{reversible implementations of } f} S_{\max}.$$

Remarks. This is the reversible pebbling number of a computation DAG for f [3,20,21]. It is finite for all computable f ; constructive reversible simulations ensure existence.

Definition 2 (Forced erasures under space S : $G^*(f, S, \varepsilon)$). Fix a space budget S (IBBs). Over all (reversible or irreversible) implementations of f with error $\leq \varepsilon$ that never exceed S live IBBs, let G be the total number of logically irreversible bit erasures performed. Then

$$G^*(f, S, \varepsilon) = \inf_{\text{implementations within space } S} G.$$

Definition 3 (Orthogonalization depth $D^*(f, \varepsilon)$). Let $\rho_t(x)$ be the device state (possibly mixed) at time t given input x . For a pair of inputs x, y with $f(x) \neq f(y)$, define the Bures angle (quantum Fubini–Study distance for mixed states)

$$\Theta(\rho, \sigma) = \arccos \sqrt{F(\rho, \sigma)} \in [0, \pi/2],$$

where F is the Uhlmann fidelity. Let $\mathcal{L}(x, y)$ be the minimal geodesic length (in Bures angle) a valid implementation must traverse between $\rho_0(x)$ and some $\rho_\tau(x)$ that produces an ε -accurate output and likewise for y . Then define

$$D^*(f, \varepsilon) = \sup_{x, y: f(x) \neq f(y)} \left\lceil \frac{\mathcal{L}(x, y)}{\pi/2} \right\rceil.$$

Remarks. $\Theta = \pi/2$ corresponds to orthogonality; thus D^* counts, in units of “orthogonalization-length,” how many decisively distinguishable transitions are *unavoidably* required for worst-case inputs. In circuit models, D^* lower-bounds depth; in decision trees, D^* lower-bounds the number of decisive queries; in quantum models, D^* relates to adversary/Hybrid-argument lower bounds [22,23]. Well-definedness follows since $\mathcal{L}$ is finite for CPTP paths and the supremum is over a finite domain.

Relationships and computability (high level)

- W^* equals (up to constant factors) the reversible pebbling number of a suitable computation DAG for f ; lower bounds are known for FFT, GEMM, sorting, Krylov, etc. [16–19].
- $G^*(f, S, \varepsilon) \geq [W^*(f, \varepsilon) - S]_+$ (Lemma 1).
- $D^*(f, \varepsilon) \geq$ classical decision-tree depth for f ; for quantum, D^* is lower-bounded by adversary methods [22].

3. Key Lemmas

Lemma 1 (Space-deficit forces erasure). For any f and ε , and space budget S , one has

$$G^*(f, S, \varepsilon) \geq [W^*(f, \varepsilon) - S]_+.$$

Proof. Consider a reversible implementation achieving W^* . If $S \geq W^*$, history can be retained and uncomputed with $G = 0$. If $S < W^*$, at some peak step at least $W^* - S$ logically live bits must be discarded to remain within budget S . Any such discard is a logically irreversible many-to-one map on IBBs, contributing at least one bit erasure. Taking the infimum over all implementations yields the bound. This is the standard reversible pebbling trade-off [3,20,21]. $\square$

Lemma 2 (Landauer dissipation). *At temperature T , any logically irreversible bit erasure exports at least $k_B T \ln 2$ heat to the environment. Therefore,*

$$Q \geq k_B T \ln 2 G^*(f, S, \epsilon).$$

Lemma 3 (Quantum speed limit as step-count bound). *Let E_{tot} be the average available energy above ground over runtime τ . For any implementation of f with worst-case orthogonalization-depth $D^*(f, \epsilon)$,*

$$\tau \geq \frac{\pi \hbar}{2} \frac{D^*(f, \epsilon)}{E_{\text{tot}}}.$$

Proof. For closed systems, concatenate D^* minimal orthogonalizations; each requires time at least $\pi \hbar / (2E)$ by Margolus–Levitin; additivity yields the bound. For open systems, use mixed-state QSLs (e.g., Deffner–Lutz; Taddei et al.; del Campo et al.) on the Bures angle, noting that orthogonalization corresponds to angle $\pi/2$; summing D^* segments gives the same scaling (constants may vary but only by universal factors) [10–12]. $\square$

Lemma 4 (Bekenstein information-capacity). *For degrees of freedom essentially confined in radius R with total energy E_{tot} ,*

$$S \leq \frac{2\pi E_{\text{tot}} R}{\hbar c \ln 2}.$$

Equivalently,

$$E_{\text{tot}} \geq \frac{\hbar c \ln 2}{2\pi R} S.$$

4. Deriving the Combined STEH Inequalities

We now show the algebraic steps explicitly.

Step 1: Eliminate E_{tot} from the QSL

Start from Lemma 3:

$$\tau \geq \frac{\pi \hbar}{2} \frac{D^*(f, \epsilon)}{E_{\text{tot}}}. \quad (1)$$

Apply Lemma 4 (lower bound on E_{tot} given S and R):

$$E_{\text{tot}} \geq \frac{\hbar c \ln 2}{2\pi R} S. \quad (2)$$

Substitute (2) into (1):

$$\begin{aligned} \tau &\geq \frac{\pi \hbar}{2} D^*(f, \epsilon) \left/ \left(\frac{\hbar c \ln 2}{2\pi R} S \right) \right. \\ &= \frac{\pi \hbar}{2} D^*(f, \epsilon) \cdot \frac{2\pi R}{\hbar c \ln 2} \cdot \frac{1}{S} \\ &= \boxed{\frac{\pi^2}{c \ln 2} \frac{R D^*(f, \epsilon)}{S}}. \end{aligned} \quad (3)$$

Step 2: Product bound for τQ

From Lemma 2,

$$Q \geq k_B T \ln 2 G^*(f, S, \varepsilon). \quad (4)$$

Multiply (3) and (4):

$$\begin{aligned} \tau Q &\geq \left(\frac{\pi^2}{c \ln 2} \frac{R D^*}{S} \right) \cdot \left(k_B T \ln 2 G^* \right) \\ &= \boxed{\frac{\pi^2 k_B T}{c} R D^*(f, \varepsilon) \frac{G^*(f, S, \varepsilon)}{S}}. \end{aligned} \quad (5)$$

This completes the explicit derivation of the two combined STEH inequalities. $\square$

5. Examples and Sanity Checks

AND of n bits.

Reversible working set $W^* \approx n$; orthogonalization depth $D^* \geq n$. For $S < n$, $G^* \geq n - S$ (Lemma 1) hence $Q \geq k_B T \ln 2 (n - S)$ and $\tau \geq \frac{\pi^2}{c \ln 2} \frac{R n}{S}$.

GEMM ($n \times n$).

Known I/O lower bounds imply $W^* \gtrsim \Theta(n^2)$ live tiles; the arithmetic workload gives $D^* \gtrsim \Theta(n^3)$ decisive updates [16–19]. Therefore $\tau \gtrsim \frac{\pi^2}{c \ln 2} \frac{R n^3}{S}$, matching communication-optimal scaling (up to constants and logs).

FFT (N).

$W^* \gtrsim \Theta(N)$ and $D^* \gtrsim \Theta(N \log N)$ by circuit-depth arguments; thus $\tau \gtrsim \frac{\pi^2}{c \ln 2} \frac{R N \log N}{S}$.

6. Assumptions, Scope, and Limitations

Simultaneous applicability. Conditions (R1)–(R3) delineate when Landauer, QSL, and Bekenstein can be applied together: (i) near-equilibrium thermalization for irreversible steps; (ii) CPTP dynamics with well-defined average energy; (iii) weak gravity and effective confinement within radius R . These cover standard digital/quantum platforms (CMOS, superconducting/ion, nanomechanical).

Model idealizations. The constants in QSLs vary among formulations (Margolus–Levitin vs. Mandelstam–Tamm vs. mixed-state extensions); our constants are chosen conservatively, and any version yields the same *scaling* and the same algebraic combination. Bekenstein’s bound has modern reformulations via relative entropy; any such form provides an $E_{\text{tot}}-S$ trade-off suitable for our derivation [15]. Landauer has repeated experimental support [6,7].

What is not claimed. We do not claim tightness of constants for specific hardware, nor do we assume a specific gate library. The theorem is agnostic to architecture and applies to *any* implementation achieving the input-output relation.

7. Conclusions

We provided precise definitions for the complexity parameters W^*, G^*, D^* , proved the lemmas linking them to Landauer, QSL, and Bekenstein, and derived combined STEH inequalities (3)–(5) step-by-step. The result formalizes the impossibility of driving *all* of (space, time, exported entropy, energy) to zero in nontrivial computation, and it recovers communication lower bounds for fundamental tasks. The STEH framework offers a quantitative checklist for evaluating algorithms against absolute physical limits.

Author Contributions: Sole author: conceptualization, formal analysis, methodology, and writing.

Funding: No external funding was received for this work.

Data Availability Statement: No new data were generated or analyzed in this study.

AI Assistance Statement: Language and editorial suggestions were supported by AI tools; the author takes full responsibility for the content.

Acknowledgments: The author thanks the Octonion Group research team for valuable discussions and computational resources. Special recognition goes to the broader computational complexity and quantum computing communities whose foundational work made this synthesis possible.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. R. Landauer, "Irreversibility and heat generation in the computing process," *IBM J. Res. Dev.* **5**, 183–191 (1961). DOI:10.1147/rd.53.0183
2. C. H. Bennett, "Logical reversibility of computation," *IBM J. Res. Dev.* **17**, 525–532 (1973). DOI:10.1147/rd.176.0525
3. C. H. Bennett, "Time/space trade-offs for reversible computation," *SIAM J. Comput.* **18**(4), 766–776 (1989). DOI:10.1137/0218053
4. Y. Lecerf, "Machines de Turing réversibles. Récursivité et complexité," *Comptes Rendus* **257**, 2597–2600 (1963).
5. C. H. Bennett, "Notes on Landauer's principle, reversible computation, and Maxwell's Demon," *Stud. Hist. Phil. Mod. Phys.* **34**, 501–510 (2003). DOI:10.1016/S1355-2198(03)00039-X
6. A. Bérut *et al.*, "Experimental verification of Landauer's principle linking information and thermodynamics," *Nature* **483**, 187–189 (2012). DOI:10.1038/nature10872
7. Y. Jun, M. Gavrilov, and J. Bechhoefer, "High-precision test of Landauer's principle in a feedback trap," *Phys. Rev. Lett.* **113**, 190601 (2014). DOI:10.1103/PhysRevLett.113.190601
8. L. Mandelstam and I. Tamm, "The uncertainty relation between energy and time in nonrelativistic quantum mechanics," *J. Phys. (USSR)* **9**, 249 (1945).
9. N. Margolus and L. B. Levitin, "The maximum speed of dynamical evolution," *Physica D* **120**, 188–195 (1998). DOI:10.1016/S0167-2789(98)00054-2
10. S. Deffner and E. Lutz, "Quantum speed limit for non-Markovian dynamics," *Phys. Rev. Lett.* **111**, 010402 (2013). DOI:10.1103/PhysRevLett.111.010402
11. M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, "Quantum speed limit for physical processes," *Phys. Rev. Lett.* **110**, 050402 (2013). DOI:10.1103/PhysRevLett.110.050402
12. A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, "Quantum speed limits in open system dynamics," *Phys. Rev. Lett.* **110**, 050403 (2013). DOI:10.1103/PhysRevLett.110.050403
13. J. D. Bekenstein, "Universal upper bound on the entropy-to-energy ratio for bounded systems," *Phys. Rev. D* **23**, 287–298 (1981). DOI:10.1103/PhysRevD.23.287
14. R. Bousso, "The holographic principle," *Rev. Mod. Phys.* **74**, 825–874 (2002). DOI:10.1103/RevModPhys.74.825
15. H. Casini, "Relative entropy and the Bekenstein bound," *Class. Quantum Grav.* **25**, 205021 (2008). DOI:10.1088/0264-9381/25/20/205021
16. J. W. Hong and H. T. Kung, "I/O complexity: The red-blue pebble game," in *Proc. STOC* (1981). DOI:10.1145/800076.802486
17. G. Ballard, J. Demmel, O. Holtz, and O. Schwartz, "Communication-optimal parallel and sequential Cholesky, LU, and QR factorizations," *SIAM J. Sci. Comput.* **34**(1), A206–A239 (2012). DOI:10.1137/110848244
18. D. Irony, S. Toledo, and A. Tiskin, "Communication lower bounds for distributed-memory matrix multiplication," *J. Parallel Distrib. Comput.* **64**(9), 1017–1026 (2004). DOI:10.1016/j.jpdc.2004.03.021
19. J. Demmel, L. Grigori, M. Hoemmen, and J. Langou, "Communication-avoiding Krylov subspace methods," *SIAM Rev.* **55**(1), 30–69 (2013). DOI:10.1137/080731851
20. H. Buhrman, P. Tromp, and P. Vitányi, "Time and space bounds for reversible simulation," *J. Comput. Syst. Sci.* **63**, 204–221 (2001). DOI:10.1006/jcss.2001.1760
21. K.-J. Lange, P. McKenzie, and A. Tapp, "Reversible space equals deterministic space," *J. Comput. Syst. Sci.* **60**(2), 354–367 (2000). DOI:10.1006/jcss.1999.1681
22. A. Ambainis, "Quantum lower bounds by quantum arguments," *J. Comput. Syst. Sci.* **64**(4), 750–767 (2002). DOI:10.1006/jcss.2002.1826
23. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary Ed., Cambridge Univ. Press (2010).

24. S. Lloyd, "Ultimate physical limits to computation," *Nature* **406**, 1047–1054 (2000). DOI:10.1038/35023282

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.