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Article

Constructing Physics From Measurements

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Abstract: We present a reformulation of fundamental physics from an enumeration of independent axioms into the solution of a single optimization problem. Any experiment begins with an initial state preparation, involves some physical operation, and ends with a final measurement. Working from this structure, we maximize the entropy of a final measurement relative to its initial preparation subject to a measurement constraint. Solving this optimization problem for the *natural constraint* –the most permissive constraint compatible with said problem– identifies an optimal physical theory. Rather than existing as a collection of postulates, quantum mechanics, general relativity, and Yang-Mills emerge within a unified theory. Notably, mathematical consistency further restricts valid solutions to 3+1 dimensions only. This reformulation reveals that the apparent complexity of modern physics, with its various forces, symmetries, and dimensional constraints, emerges as the solution to an optimization problem constructed over all experiments realizable within the constraint of nature.

Keywords: foundations of physics

1. Introduction

Statistical mechanics (SM), in the formulation developed by E.T. Jaynes [1,2], is founded on an entropy optimization principle. Specifically, the Boltzmann entropy is maximized under the constraint of a fixed average energy $\bar{E}$:

$$\bar{E} = \sum_i \rho_i E_i \tag{1}$$

The Lagrange multiplier equation defining the optimization problem is:

$$\mathcal{L} = -k_B \sum_i \rho_i \ln \rho_i + \lambda \left(1 - \sum_i \rho_i \right) + \beta \left(\bar{E} - \sum_i \rho_i E_i \right), \tag{2}$$

where λ and β are Lagrange multipliers enforcing the normalization and average energy constraints. Solving this optimization problem yields the Gibbs measure:

$$\rho_i = \frac{1}{Z} \exp(-\beta E_i), \tag{3}$$

where $Z = \sum_i \exp(-\beta E_i)$ is the partition function.

For comparison, quantum mechanics (QM) is not formulated as the solution to an optimization problem, but rather consists of a collection of axioms[3,4]:

- | | |
|-----------------|--|
| QM Axiom 1 of 5 | State Space: Every physical system is associated with a complex Hilbert space, and its state is represented by a ray (an equivalence class of vectors differing by a non-zero scalar multiple) in this space. |
| QM Axiom 2 of 5 | Observables: Physical observables correspond to Hermitian (self-adjoint) operators acting on the Hilbert space. |
| QM Axiom 3 of 5 | Dynamics: The time evolution of a quantum system is governed by the Schrödinger equation, where the Hamiltonian operator represents the system's total energy. |

- QM Axiom 4 of 5 **Measurement:** Measuring an observable projects the system into an eigenstate of the corresponding operator, yielding one of its eigenvalues as the measurement result.
- QM Axiom 5 of 5 **Probability Interpretation:** The probability of obtaining a specific measurement outcome is given by the squared magnitude of the projection of the state vector onto the relevant eigenstate (Born rule).

Physical theories have traditionally been constructed in two distinct ways. Some, like quantum mechanics, are defined through a set of mathematical axioms that are first postulated and then verified against experiments. Others, like statistical mechanics, emerge as solutions to optimization problems with experimentally-verified constraints.

We propose to generalize the optimization methodology of E.T. Jaynes to encompass all of physics, aiming to derive a unified theory from a single optimization problem.

To that end, we introduce the following constraint:

Axiom 1 (Nature).

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i$$

where $\mathbf{M}_i$ are $n \times n$ matrices, and $\overline{\mathbf{M}}$ is their average.

This constraint, as it replaces the scalar E_i with the matrix $\mathbf{M}_i$, extends E.T. Jaynes' optimization method to encompass non-commutative observables and symmetry group generators required for fundamental physics.

We then construct an optimization problem:

Definition 1 (Physics). *is the solution to:*

$$\underbrace{\mathcal{L}}_{\text{an optimization problem}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{on the entropy} \\ \text{of a measurement} \\ \text{relative to its preparation} \\ \text{over all}}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr} \left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}}$$

where λ and τ are Lagrange multipliers enforcing the normalization and natural constraints, respectively.

This single definition constitutes our complete proposal for reformulating fundamental physics—no additional principles will be introduced. By replacing the Boltzmann entropy with the relative Shannon entropy, the optimization problem extends beyond thermodynamic variables to encompass any type of experiment. This generalization occurs because relative entropy captures the essence of any experiment: the relationship between a final measurement state and its initial preparation state.

Two key constraints shape our framework. The normalization constraint ensures we are working with a proper predictive theory, while the natural constraint spawns the domain of applicability of the theory. Together, they capture the complete evolution—from initial to final states—that defines any experiment. The crucial insight is that because our formulation maintains complete generality in the structure of experiments while optimizing over all possible predictive theories, the resulting solution holds true, by construction, for all realizable experiments within its domain.

The solution provides a complete set of axioms that automatically satisfy the requirements of a physical theory valid for the constraint: mathematical rigour, internal consistency, optimal predictive power, and automatic validity with all realizable experiments in its domain. This approach reduces our reliance on postulating axioms through trial and error, and simplifies the foundations of physics. Specifically, when we employ the *natural constraint*—the most permissive constraint for this

problem—, the solution spawns its largest domain, pointing towards a unified physics where statistical mechanics, quantum mechanics, general relativity, and Yang-Mills emerge naturally. Importantly, this emergence occurs without additional assumptions and without generating unwanted artifacts like extra dimensions or unobserved gauge symmetries.

Theorem 1. *The general solution of the optimization problem is:*

$$\rho_i = \frac{p_i \det \exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det \exp(-\tau \mathbf{M}_j)}$$

Proof. We solve the maximization problem by setting the derivative of the Lagrangian with respect to ρ_i to zero:

$$\frac{\partial \mathcal{L}}{\partial \rho_i} = -\ln \frac{\rho_i}{p_i} - 1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i = 0. \quad (4)$$

$$\implies \ln \frac{\rho_i}{p_i} = -1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i. \quad (5)$$

$$\implies \rho_i = p_i \exp(-1 - \lambda) \exp(-\tau \operatorname{tr} \mathbf{M}_i). \quad (6)$$

Normalizing the probabilities using $\sum_i \rho_i = 1$, we find:

$$1 = \sum_i \rho_i = \exp(-1 - \lambda) \sum_i p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i), \quad (7)$$

$$\implies \exp(1 + \lambda) = \sum_j p_j \exp(-\tau \operatorname{tr} \mathbf{M}_j). \quad (8)$$

Substituting back, we obtain:

$$\rho_i = \frac{p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i)}{\sum_j p_j \exp(-\tau \operatorname{tr} \mathbf{M}_j)}. \quad (9)$$

Finally, using the identity $\det \exp(\mathbf{M}) = \exp \operatorname{tr} \mathbf{M}$ for square matrices $\mathbf{M}$, we get:

$$\rho_i = \frac{1}{Z} p_i \det \exp(-\tau \mathbf{M}_i). \quad (10)$$

where $Z = \sum_j p_j \det \exp(-\tau \mathbf{M}_j)$. $\square$

The optimization problem can additionally be expressed in the form of an integral:

$$\mathcal{L} = - \int \rho(x) \ln \frac{\rho(x)}{p(x)} dx + \lambda \left(1 - \int \rho(x) dx \right) + \frac{1}{2} \tau \operatorname{tr} \left(\overline{\mathbf{M}} - \int \rho(x) \frac{\mathbf{M}(x)}{\epsilon(x)} dx \right) \quad (11)$$

or in the form of a wavefunctional:

$$\mathcal{L} = - \int \mathcal{D}\phi \rho[\phi] \ln \frac{\rho[\phi]}{p[\phi]} + \lambda \left(1 - \int \mathcal{D}\phi \rho[\phi] \right) + \frac{1}{2} \tau \operatorname{tr} \left(\overline{\mathbf{M}} - \int \mathcal{D}\phi \rho[\phi] \frac{\mathbf{M}[\phi]}{\epsilon[\phi]} \right) \quad (12)$$

whose solutions are:

$$\rho(x) = \frac{1}{Z} \det \exp \left(-\frac{1}{2} \tau \frac{\mathbf{M}(x)}{\epsilon(x)} \right), \text{ and with } Z = \int p(x) \det \exp \left(-\frac{1}{2} \tau \frac{\mathbf{M}(x)}{\epsilon(x)} \right) dx \quad (13)$$

and

$$\rho[\phi] = \frac{1}{Z} \det \exp \left(-\frac{1}{2} \tau \frac{\mathbf{M}[\phi]}{\epsilon[\phi]} \right), \text{ and with } Z = \int \mathcal{D}\phi p[\phi] \det \exp \left(-\frac{1}{2} \tau \frac{\mathbf{M}[\phi]}{\epsilon[\phi]} \right) \quad (14)$$

respectively.

As we will see in the results section, this solution encapsulates three distinct special cases:

1. **Statistical Mechanics:**

To recover statistical mechanics from Equation 10, we consider the case where the matrices $\mathbf{M}_i$ are 1×1 , i.e., scalars. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = [E_i], \quad (15)$$

and take p_i to be a uniform distribution. Then, Equation 10 reduces to the Gibbs distribution:

$$\rho_i = \frac{1}{Z} \exp(-\tau E_i), \quad (16)$$

where τ corresponds to β in traditional statistical mechanics. This demonstrates that our solution generalizes SM, as it recovers it when $\mathbf{M}_i$ are scalars.

2. **Quantum Mechanics:**

By choosing $\mathbf{M}_i$ to generate the U(1) group, we derive the axioms of quantum mechanics from entropy maximization. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}, \quad (17)$$

where E_i are energy levels. In the results section, we will detail how this choice leads to a probability measure that includes a unitarily invariant ensemble and the Born rule, satisfying all five axioms of QM.

3. **Fundamental Physics:**

Extending our approach, we choose $\mathbf{M}_i$ to be 4×4 matrices representing the generators of the $\text{Spin}^c(3,1)$ group. Specifically, we consider multivectors of the form $\mathbf{u} = \mathbf{f} + \mathbf{b}$, where $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudoscalar of the 3+1D geometric algebra $\text{GA}(3,1)$. The matrix representation of $\mathbf{M}_i$ is:

$$\mathbf{M}_i = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix}, \quad (18)$$

where $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the $\text{Spin}^c(3,1)$ group, which includes both Lorentz transformations and U(1) phase rotations. This choice leads to a relativistic quantum probability measure:

$$\rho_i = \frac{p_i \det \exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det \exp(-\tau \mathbf{M}_j)}, \quad (19)$$

where τ emerges as a parameter generating boosts, rotations, and phase transformations.

In the results section, we show how gravity and Yang-Mills emerge naturally from this solution.

4. **Dimensional Obstructions:**

Axiom 1 yields valid probability measures only in specific geometric cases. Beyond the instances of statistical mechanics and quantum mechanics, Axiom 1 yields a consistent solution only in 3+1 dimensions. In other dimensional configurations, various obstructions arise violating the axioms of probability theory. The following table summarizes the geometric cases and their obstructions:

Dimensions	Optimal Predictive Theory of Nature	
GA(0)	Statistical Mechanics	(20)
GA(0,1)	Quantum Mechanics	(21)
GA(1,0)	Obstructed (Negative probabilities)	(22)
GA(2,0)	Quantum Mechanics	(23)
GA(1,1)	Obstructed (Negative probabilities)	(24)
GA(0,2)	Obstructed (Non-real probabilities)	(25)
GA(3,0)	Obstructed (Non-real probabilities)	(26)
GA(2,1)	Obstructed (Non-real probabilities)	(27)
GA(1,2)	Obstructed (Non-real probabilities)	(28)
GA(0,3)	Obstructed (Non-real probabilities)	(29)
GA(4,0)	Obstructed (Non-real probabilities)	(30)
GA(3,1)	Gravity + Yang-Mills	(31)
GA(2,2)	Obstructed (Negative probabilities)	(32)
GA(1,3)	Obstructed (Non-real probabilities)	(33)
GA(0,4)	Obstructed (Non-real probabilities)	(34)
GA(5,0)	Obstructed (Non-real probabilities)	(35)
⋮	⋮	
GA(6,0)	Suspected Obstructed (No observables)	(36)
⋮	⋮	
∞		(37)

where $GA(p,q)$ means the geometric algebra of $p+q$ dimensions. We will first investigate the unobstructed cases in Section 2.1, 2.2 and 2.3 and then demonstrate the obstructions in Section 2.4. These obstructions are desirable because they automatically limit the theory to 3+1D, thus providing a built-in mechanism for the observed dimensionality of our universe.

2. Results

2.1. Quantum Mechanics

In statistical mechanics (SM), the central observation is that energy measurements of a thermally equilibrated system tend to cluster around a fixed average value (Equation 1). In contrast, quantum mechanics (QM) is characterized by the presence of interference effects in measurement outcomes. To capture these features within an entropy maximization framework, we introduce the following special case of Axiom 1:

Definition 2 (U(1) Generating Constraint). *We reduce the generality of Axiom 1 to the generator of the U(1) group. Specifically, we replace*

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}$$

(38)

where E_i are scalar values (e.g., energy levels), ρ_i are the probabilities of outcomes, and the matrices $\mathbf{M}_i$ generate the U(1) group.

The general solution of the optimization problem reduces as follows

$$\rho_i = \frac{1}{\sum_i p_i \det \exp \left(-\tau \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \right)} \det \exp \left(-\tau \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \right) p_i \quad (39)$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

To align our results with conventional quantum mechanical notation, we translate the matrices to complex numbers. Specifically, we consider that:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + ib. \quad (40)$$

Then, we note the following equivalence with the complex norm:

$$\det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \quad (41)$$

$$= r^2 (\cos^2(b) + \sin^2(b)) \quad (42)$$

$$= \|r(\cos(b) + i \sin(b))\| \quad (43)$$

$$= \|r \exp(ib)\| \quad (44)$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial preparation into :

$$\rho_i = \frac{1}{\underbrace{\sum_i p_i \|\exp(-itE_i/\hbar)\|}_{\text{Unitarily Invariant Ensemble}}} \underbrace{\|\exp(-itE_i/\hbar)\|}_{\text{Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \quad (45)$$

The wavefunction emerges by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_i p_i \|\exp(-itE_i/\hbar)\| = Z = \langle \psi | \psi \rangle \quad (46)$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE_1/\hbar) & & \\ & \ddots & \\ & & \exp(-itE_n/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \quad (47)$$

We clarify that p_i represents the probability associated with the initial preparation of the wavefunction, where $p_i = \langle \psi_i(0) | \psi_i(0) \rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle \psi | \psi \rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates [QM Axiom 1 of 5](#).

- In Z , an observable must satisfy:

$$\overline{O} = \sum_i p_i O_i \|\exp(-itE_i/\hbar)\| \quad (48)$$

Since $Z = \langle \psi | \psi \rangle$, then any self-adjoint operator satisfying the condition $\langle O\psi | \phi \rangle = \langle \psi | O\phi \rangle$ will equate the above equation, simply because $\langle O \rangle = \langle \psi | O | \psi \rangle$. This demonstrates [QM Axiom 2 of 5](#).

- Upon transforming Equation 47 out of its eigenbasis through unitary operations, we find that the energy, E_i , typically transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \quad (49)$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{\partial}{\partial t} (\exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle) \quad (50)$$

$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle \quad (51)$$

$$= -i\mathbf{H}/\hbar |\psi(t)\rangle \quad (52)$$

$$\implies \mathbf{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle \quad (53)$$

which is the Schrödinger equation. This demonstrates [QM Axiom 3 of 5](#).

- From Equation 47 it follows that the possible microstates E_i of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from ρ , with the measured state being the occupied microstate i . Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

$$\rho_i(t) = \frac{1}{\langle \psi | \psi \rangle} (\psi_i(t))^\dagger \psi_i(t). \quad (54)$$

In scenarios where the probability measure $\rho_i(\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle \lambda_i | \psi \rangle|^2 \quad (55)$$

Here, $|\langle \lambda_i | \psi \rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates [QM Axiom 4 of 5](#).

- Finally, since the probability measure (Equation 45) replicates the Born rule, [QM Axiom 5 of 5](#) is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the U(1) generating constraint is sufficient to entail the foundations of quantum mechanics through the principle of entropy maximization—in this formulation, QM Axioms 1, 2, 3, 4, and 5 are not fundamental, but the solution to an optimization problem.

2.2. RQM in 2D

In this section, we investigate a model, isomorphic to quantum mechanics, that lives in 2D which provides a valuable starting point before addressing the more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier Equation is:

$$\mathcal{L} = - \sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i \right) + \frac{1}{2} \theta \text{tr} \left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i \right) \quad (56)$$

where λ and θ are the Lagrange multipliers, and where $\mathbf{M}_i$ is the 2×2 matrix representation of the multivectors of $\text{GA}(2)$.

In general a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of $\text{GA}(2)$, where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ a pseudo-scalar, is represented as follows:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y \quad (57)$$

This holds for any 2×2 matrix and any multivectors of $\text{GA}(2)$.

The basis elements are defined as:

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_x \wedge \sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (58)$$

To investigate this case in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 3 (Multivector Conjugate). Let $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ be a multi-vector of the geometric algebra over the reals in two dimensions $\text{GA}(2)$. The multivector conjugate is defined as:

$$\mathbf{u}^\dagger = a - \mathbf{x} - \mathbf{b} \quad (59)$$

The determinant of the matrix representation of a multivector can be expressed as a multivector self-product:

Theorem 2 (The Determinant in Multivector Self-Product Form).

$$\mathbf{u}^\dagger \mathbf{u} = \det \mathbf{M} \quad (60)$$

Proof. Let $\mathbf{u} = a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y$, and let $\mathbf{M}$ be its matrix representation $\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix}$. Then:

$$1: \quad \mathbf{u}^\dagger \mathbf{u} \quad (61)$$

$$= (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y)^\dagger (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) \quad (62)$$

$$= (a - x\sigma_x - y\sigma_y - b\sigma_x \wedge \sigma_y)(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) \quad (63)$$

$$= a^2 - x^2 - y^2 + b^2 \quad (64)$$

$$2: \quad \det \mathbf{M} \quad (65)$$

$$= \det \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \quad (66)$$

$$= (a+x)(a-x) - (y-b)(y+b) \quad (67)$$

$$= a^2 - x^2 - y^2 + b^2 \quad (68)$$

□

This theorem establishes a connection between the determinant and a multivector self-product form. By expressing the determinant in a self-product form, we can later restrict our attention to the even subalgebra of $GA(2)$, where this self-product form becomes positive-definite, thus yielding a proper inner product. This construction will be essential for developing quantum mechanical structures such as probability measures and observables within the geometric algebra framework.

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 4 (Multivector Conjugate Transpose). Let $|V\rangle\rangle \in (GA(2))^n$:

$$|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \quad (69)$$

The multivector conjugate transpose of $|V\rangle\rangle$ is defined as first taking the transpose and then the element-wise multivector conjugate:

$$\langle\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \quad (70)$$

Definition 5 (Bilinear Form). Let $|V\rangle\rangle$ and $|W\rangle\rangle$ be two vectors valued in $GA(2)$. We introduce the following bilinear form:

$$\langle\langle V|W\rangle\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \quad (71)$$

Theorem 3 (Inner Product). Restricted to the even sub-algebra of $GA(2)$, the bilinear form is an inner product.

Proof.

$$\langle\langle V|W\rangle\rangle_{\mathbf{x} \rightarrow 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \quad (72)$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \sigma_x \wedge \sigma_y$. $\square$

Let us now solve the optimization problem for the even multivectors of $GA(2,0)$, whose inner product is positive-definite.

We take $a \rightarrow 0, \mathbf{x} \rightarrow 0$ then $\mathbf{M}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a \rightarrow 0, \mathbf{x} \rightarrow 0} = \mathbf{b} \implies \mathbf{M} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \quad (73)$$

The Lagrange multiplier equation can be solved as follows:

$$0 = \frac{\partial \mathcal{L}[\rho_1, \dots, \rho_n]}{\partial \rho_i} \quad (74)$$

$$= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \quad (75)$$

$$= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \quad (76)$$

$$\implies \ln \frac{\rho_i}{p_i} = -p_i - \lambda - \theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \quad (77)$$

$$\implies \rho_i = p_i \exp(-p_i - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (78)$$

$$= \frac{1}{Z(\theta)} p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (79)$$

The partition function $Z(\theta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (80)$$

$$\implies (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (81)$$

$$Z(\theta) := \sum_i p_i \exp\left(-\theta \operatorname{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (82)$$

Consequently, the optimal probability measure that connects an initial preparation p_i to a final measurement ρ_i , in 2D is:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp\left(-\frac{1}{2}\theta \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det \exp\left(-\frac{1}{2}\theta \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right)}_{\text{Spin(2) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \quad (83)$$

Definition 6 (Spin(2)-valued Wavefunction).

$$|\psi\rangle\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 \\ \vdots \\ \sqrt{\rho_2} R_2 \end{bmatrix} \quad (84)$$

where $\sqrt{\rho_i} = e^{\frac{1}{2}a_i}$ representing the square root of the probability and $R_i = e^{\frac{1}{2}\mathbf{b}_i}$ representing a rotor in 2D.

The partition function of the probability measure can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

Theorem 4 (Partition Function). $Z = \langle\langle \psi | \psi \rangle\rangle$

Proof.

$$\langle\langle \psi | \psi \rangle\rangle = \sum_i \psi_i^\dagger \psi_i = \sum_i \rho_i R_i^\dagger R_i = \sum_i \rho_i = Z \quad (85)$$

□

Definition 7 (Spin(2)-valued Evolution Operator).

$$T = \begin{bmatrix} e^{-\frac{1}{2}\theta\mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta\mathbf{b}_n} \end{bmatrix} \quad (86)$$

Theorem 5. *The partition function is invariant with respect to the Spin(2)-valued evolution operator.*

Proof. We note that:

$$\langle\langle T\mathbf{v}|T\mathbf{v}\rangle\rangle = \langle\langle \mathbf{v}|\mathbf{v}\rangle\rangle = \mathbf{v}^\dagger T^\dagger T \mathbf{v} \implies T^\dagger T = I \quad (87)$$

then, since $\begin{bmatrix} e^{\frac{1}{2}\theta\mathbf{b}_1} & & \\ & \ddots & \\ & & e^{\frac{1}{2}\theta\mathbf{b}_n} \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}\theta\mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta\mathbf{b}_n} \end{bmatrix} = I$, the relation $T^\dagger T = I$ is satisfied. $\square$

We note that the even sub-algebra of $\text{GA}(2)$, being closed under addition and multiplication and constituting an inner product through its bilinear form, allows for the construction of a Hilbert space. In this context, the Hilbert space is Spin(2)-valued. The primary distinction between a wavefunction in a complex Hilbert space and one in a Spin(2)-valued Hilbert space lies in the subject matter of the theory. Specifically, in the latter, the construction governs the change in orientation experienced by an observer (versus change in time), which in turn dictates the measurement basis used in the experiment, consistently with the rotational symmetry and freedom of the system.

The dynamics of observer orientation transformations are described by a variant of the Schrödinger equation, which is derived by taking the derivative of the wavefunction with respect to the Lagrange multiplier, θ :

Definition 8 (Spin(2)-valued Schrödinger Equation).

$$\frac{d}{d\theta} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1 & & \\ & \ddots & \\ & & -\frac{1}{2}\mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} \quad (88)$$

where θ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the Spin(2), locally across the states of the Hilbert space. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction for the Spin(2) group.

Definition 9 (David Hestenes' Formulation). *In 3+1D, the David Hestenes' formulation [5] of the wavefunction is $\psi = \sqrt{\rho} R e^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho} R$, which is the form we have recovered.*

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 10 (Dirac Current). *Given the basis σ_x and σ_y , the Dirac current for the 2D theory is defined as:*

$$J_{(\sigma_x)} \equiv \psi^\dagger \sigma_x \psi = \rho \underbrace{R^\dagger \sigma_x R}_{\text{SO}(2)} = \rho \sigma'_x \quad (89)$$

$$J_{(\sigma_y)} \equiv \psi^\dagger \sigma_y \psi = \rho \underbrace{R^\dagger \sigma_y R}_{\text{SO}(2)} = \rho \sigma'_y \quad (90)$$

where σ'_x and σ'_y are a $SO(2)$ rotated basis vectors.

2.2.1. 1+1D Obstruction

As stated in the introduction, of the dimensional cases, only 2D and 3+1D are free of obstructions. For instance, the 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\mathbf{e}_0 \wedge \mathbf{e}_1)(a + b\mathbf{e}_0 \wedge \mathbf{e}_1)$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. This is why we had to use 2D instead of 1+1D in this two-dimensional introduction. In the following section, we will investigate the 3+1D case, then we will show why all other dimensional cases are obstructed.

2.3. RQM in 3+1D

Extending the framework to relativistic quantum mechanics begins by considering measurements relative to $\text{Spin}^c(3,1)$ symmetries. This allows for transformations that include boosts, rotations, and phases, enabling relativistic consistency within the same entropy-based framework.

Here, we will develop the framework for the continuum case $\Sigma \rightarrow \int$ which can be obtained by solving the following Lagrangian equation:

$$\mathcal{L} = \int_{-\infty}^{\infty} \rho(x) \ln \frac{\rho(x)}{p(x)} dx + \lambda \left(1 - \int_{-\infty}^{\infty} \rho(x) dx \right) + \frac{1}{2} \zeta \text{tr} \left(\overline{\mathbf{M}} - \int_{-\infty}^{\infty} \rho(x) \frac{\mathbf{M}(x)}{\epsilon(x)} dx \right) \quad (91)$$

where $\epsilon(x)$ is a counter-term ensuring the integral remains invariant under coordinate transformations, ζ is a "twisted-phase" rapidity, and

$$\mathbf{M} = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix} \quad (92)$$

Here, $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the $\text{Spin}^c(3,1)$ group, which includes both Lorentz transformations and $U(1)$ phase rotations.

The solution (proof in Annex B) is obtained using the same step-by-step process as the 2D case, and yields a probability density:

$$\rho(x) = \frac{1}{\underbrace{\int_{-\infty}^{\infty} p(x) \det \exp \left(-\frac{1}{2} \zeta \frac{\mathbf{M}(x)}{\epsilon(x)} dx \right)}_{\text{Spin}^c(3,1) \text{ invariant ensemble}}} \underbrace{\det \exp \left(-\frac{1}{2} \zeta \frac{\mathbf{M}(x)}{\epsilon(x)} \right)}_{\text{Spin}^c(3,1) \text{ Born rule}} \underbrace{p(x)}_{\text{Initial preparation}} \quad (93)$$

2.3.1. Probability Measure and Wavefunction

As we did in the 2D case, our goal here will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra $\text{GA}(3,1)$.

Definition 11 (Multivector). Let $\mathbf{u}$ be a multivector of $\text{GA}(3,1)$. Its general form is:

$$\mathbf{u} = a \quad (94)$$

$$+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \quad (95)$$

$$+ f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 + f_{13}\gamma_1 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 \quad (96)$$

$$+ p\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + q\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + w\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \quad (97)$$

$$+ b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \quad (98)$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are the basis vectors in the real Majorana representation.

A more compact notation for $\mathbf{u}$ is

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \quad (99)$$

where a is a scalar, $\mathbf{x}$ a vector, $\mathbf{f}$ a bivector, $\mathbf{v}$ is pseudo-vector and $\mathbf{b}$ a pseudo-scalar.

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation:

Definition 12 (Matrix Representation of $\mathbf{u}$).

$$\mathbf{M} = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix} \quad (100)$$

To manipulate and analyze multivectors in $\text{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 13 (Multivector Conjugate (in 4D)).

$$\mathbf{u}^\dagger = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \quad (101)$$

Definition 14 (Pseudo-Blade Conjugate (in 4D)). The pseudo-blade conjugate of $\mathbf{u}$ is

$$\mathbf{u}^\ddagger = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \quad (102)$$

Lundholm[6] proposes a number the multivector norms, and shows that they are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{u}\mathbf{v}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 15. The self-products associated with low-dimensional geometric algebras are:

$$\text{GA}(0,1) : \quad \varphi^* \varphi \quad (103)$$

$$\text{GA}(2,0) : \quad \varphi^\dagger \varphi \quad (104)$$

$$\text{GA}(3,0) : \quad (\varphi^\dagger \varphi)^* \varphi^\dagger \varphi \quad (105)$$

$$\text{GA}(3,1) : \quad (\varphi^\dagger \varphi)^\dagger \varphi^\dagger \varphi \quad (106)$$

$$\text{GA}(4,1) : \quad ((\varphi^\dagger \varphi)^\dagger \varphi^\dagger \varphi)^* ((\varphi^\dagger \varphi)^\dagger \varphi^\dagger \varphi) \quad (107)$$

where φ^* is a conjugate that reverses the sign of pseudo-scalar blade.

We can now express the determinant of the matrix representation of a multivector via the self-product $(\varphi^\dagger \varphi)^\dagger \varphi^\dagger \varphi$. This choice is not arbitrary, but the unique choice which allows us to represent the determinant of the matrix representation of a multivector within $\text{GA}(3,1)$:

Theorem 6 (Determinant as a Multivector Self-Product).

$$(\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u} = \det \mathbf{M} \quad (108)$$

Proof. Please find a computer assisted proof of this equality in Annex C. $\square$

As can be seen from this theorem, the relationship between determinants and multivector products becomes more sophisticated in 3+1D. Unlike the 2D case where the determinant could be expressed

using a product of two terms, in $GA(3,1)$ the determinant requires two products involving four copies of the multivector. This is reflected in the structure $(\mathbf{u}^\dagger \mathbf{u})^\dagger \mathbf{u}^\dagger \mathbf{u}$, which cannot be reduced to a simpler self-product of two terms. However, when we restrict $\mathbf{u}$ to invertible even-multivectors, this double-product becomes positive-definite as it did in $GA(2)$, making it suitable as a probability measure despite having four terms.

The entropy maximization problem produces the following evolution operator:

Definition 16 ($Spin^c(3,1)$ Evolution Operator).

$$T(x, \zeta) = \exp\left(-\frac{1}{2}\zeta(\mathbf{f}(x) + \mathbf{b}(x))\right) \quad (109)$$

In turn, this leads to a variant of the Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ :

Definition 17 ($Spin^c(3,1)$ -valued Schrödinger equation).

$$\frac{d}{d\zeta} \psi_1(x, \zeta) = -\frac{1}{2}(\mathbf{f}(x) + \mathbf{b}(x)) \psi_1(x, \zeta) \quad (110)$$

In this case ζ represents a one-parameter evolution parameter akin to time, a "twisted-phase" rapidity, which is able to transform the measurement basis of the wavefunction under action of the $Spin^c(3,1)$ group.

This evolution operator acts on the elements of the invertible subset of the even sub-algebra of $GA(3,1)$, which includes the scalar a , the bivector $\mathbf{f}$, and the pseudoscalar $\mathbf{b}$:

Definition 18 ($Spin^c(3,1)$ -valued Wavefunction).

$$\psi(x) = \exp\left(\frac{1}{2}a(x) + \mathbf{f}(x) + \mathbf{b}(x)\right)$$

Theorem 7 (Positive-Definite Probability). *The double-product is positive-definite on ψ .*

Proof.

$$\rho(x) = (\psi(x)^\dagger \psi(x))^\dagger \psi(x)^\dagger \psi(x) \quad (111)$$

$$= e^{\frac{1}{2}(a(x) - \mathbf{f}(x) - \mathbf{b}(x))} e^{\frac{1}{2}(a(x) + \mathbf{f}(x) - \mathbf{b}(x))} e^{\frac{1}{2}(a(x) - \mathbf{f}(x) + \mathbf{b}(x))} e^{\frac{1}{2}(a(x) + \mathbf{f}(x) + \mathbf{b}(x))} \quad (112)$$

$$= e^{2a(x)} \quad (113)$$

Since $e^{2a(x)}$ is in $\mathbb{R}^+$, then it is positive-definite, yielding a value of zero only for the zero-element and positive otherwise. $\square$

Theorem 8 (Local $Spin^c(3,1)$ invariance). *Let $e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}$ be a general element of $Spin^c(3,1)$. Then, the equality:*

$$(\psi^\dagger \psi)^\dagger \psi^\dagger \psi = ((e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger (e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \quad (114)$$

is always satisfied.

Proof.

$$((e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^{\dagger}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^{\dagger}(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^{\dagger}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \quad (115)$$

$$= (\psi^{\dagger}e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^{\dagger}\psi^{\dagger}e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \quad (116)$$

$$= (\psi^{\dagger}e^{\mathbf{b}}\psi)^{\dagger}\psi^{\dagger}e^{\mathbf{b}}\psi \quad (117)$$

$$= (\psi^{\dagger}\psi)^{\dagger}e^{-\mathbf{b}}e^{\mathbf{b}}\psi^{\dagger}\psi \quad (118)$$

$$= (\psi^{\dagger}\psi)^{\dagger}\psi^{\dagger}\psi \quad (119)$$

□

Theorem 9 (David Hestenes' Wavefunction). *The $\text{Spin}^c(3,1)$ -valued wavefunction we have recovered is formulated identically to David Hestenes' [5] formulation of the wavefunction within $\text{GA}(3,1)$.*

Proof.

$$\psi(x) = \underbrace{e^{\frac{1}{2}(a(x)+\mathbf{f}(x)+\mathbf{b}(x))}}_{\text{ours}} = \underbrace{\sqrt{\rho(x)}R(x)e^{-ib(x)/2}}_{\text{Hestenes'}} \quad (120)$$

where $e^{\frac{1}{2}a(x)} = \sqrt{\rho(x)}$, $e^{\frac{1}{2}\mathbf{f}(x)} = R(x)$ and $e^{\frac{1}{2}\mathbf{b}(x)} = e^{-ib(x)/2}$. Here, $\rho(x)$ is a probability density, $R(x)$ is a rotor and $e^{-ib(x)/2}$ is a complex phase. □

2.3.2. Geometry

Definition 19 (Dirac Current). *Using a single-copy of the double-product, the definition of the Dirac current is the same as Hestenes':*

$$J \equiv \overbrace{\psi^{\dagger}\mathbf{e}_0\psi}^{\text{one copy}} \quad (121)$$

$$= \rho R^{\dagger}e^{-ib/2}\mathbf{e}_0e^{-ib/2}R \quad (122)$$

$$= \rho R^{\dagger}\mathbf{e}_0e^{ib/2}e^{-ib/2}R \quad (123)$$

$$= \rho R^{\dagger}\mathbf{e}_0R \quad (124)$$

$$= \rho\mathbf{e}'_0 \quad (125)$$

where $\mathbf{e}'_0$ is a $\text{SO}(3,1)$ rotated basis vector.

Theorem 10 (Metric Tensor). *Taking advantage of the full double-product, we utilize two copies of the Dirac current to obtain the metric tensor:*

$$g_{\mu\nu} = \text{tr}\left(\frac{(\psi^{\dagger}\mathbf{e}_{\mu}\psi)^{\dagger}\psi^{\dagger}\mathbf{e}_{\nu}\psi}{(\psi^{\dagger}\psi)^{\dagger}\psi^{\dagger}\psi}\right) \quad (126)$$

Proof.

$$\text{tr} \left(\frac{(\psi^\dagger \mathbf{e}_\mu \psi)^\dagger \psi^\dagger \mathbf{e}_\nu \psi}{(\psi^\dagger \psi)^\dagger \psi^\dagger \psi} \right) = \text{tr} \left(R^\dagger e^{-ib/2} \mathbf{e}_\mu e^{-ib/2} R R^\dagger e^{-ib/2} \mathbf{e}_\nu e^{-ib/2} R \right) \quad (127)$$

$$= \text{tr} \left(R^\dagger \mathbf{e}_\mu e^{ib/2} e^{-ib/2} R R^\dagger \mathbf{e}_\nu e^{ib/2} e^{-ib/2} R \right) \quad (128)$$

$$= \text{tr} (\tilde{R} e_\mu R \tilde{R} e_\nu R) \quad (129)$$

$$= \text{tr} (\mathbf{e}'_\mu \mathbf{e}'_\nu) \quad (130)$$

$$= \text{tr} (\mathbf{e}'_\mu \cdot \mathbf{e}'_\nu + \mathbf{e}'_\nu \wedge \mathbf{e}'_\mu) \quad (131)$$

$$= \text{tr} (g_{\mu\nu} + \mathbf{e}'_\nu \wedge \mathbf{e}'_\mu) \quad (132)$$

$$= g_{\mu\nu} \quad (133)$$

□

2.3.3. Surprisal

We now change the interpretation of ψ from a probability amplitude, to that of a field. As such, charge conservation will replace probability conservation. The notation will be:

$$\psi \rightarrow \phi \quad (134)$$

$$\sqrt{\rho} R e^{-ib/2} \rightarrow \sqrt{\chi} R e^{-ib/2} \quad (135)$$

Definition 20 (Kinetic Energy). *Starting from the definition of the metric tensor, its kinetic energy is obtained by applying the Dirac operator $\mathbf{e}^\mu D_\mu$ to each bilinear copy:*

$$T = \text{tr} \left(\frac{(\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) \quad (136)$$

where D_μ and D_ν are covariant derivatives.

Theorem 11 (Dirac Equation). *Varying the action yields the Dirac equation as a special case:*

$$\delta \int_{\mathcal{M}} \text{tr} \left(\frac{(\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) \sqrt{-|g|} d^4x = 0 \implies \underbrace{\mathbf{e}^\mu D_\mu \phi = 0}_{\text{Dirac Equation}} \quad (137)$$

Proof.

$$\frac{\delta}{\delta \phi^{\dagger \dagger}} \text{tr} \left(\frac{(\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) = 0 \quad (138)$$

$$\implies \text{tr} \left(\frac{\delta}{\delta \phi^{\dagger \dagger}} \left((\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi \right) \right) = 0 \quad (139)$$

$$\implies \text{tr} \left((\mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi \right) = 0 \quad (140)$$

For the condition to be satisfied, it is sufficient but not necessary that $\mathbf{e}^\nu D_\nu \phi = 0$, which is the Dirac equation. □

This theorem demonstrates that solutions satisfying the Dirac equation are valid critical points of the action. However, the structure of the action allows broader solutions where the trace condition $\text{tr}((\mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi) = 0$ holds without $\mathbf{e}^\nu D_\nu \phi = 0$. We will now investigate these broader solutions.

Theorem 12 (Quantum Action). *Let us investigate a subspace of the field where $R = 1$ and $e^{-ib/2} = 1$, such that $\phi = \sqrt{\chi}$. Due to the non-linearity, the kinetic energy produces a quantum potential in addition to the usual kinetic energy term:*

$$\text{tr} \left(\frac{\sqrt{\chi} \mathbf{e}^\mu D_\mu \chi \mathbf{e}^\nu D_\nu \sqrt{\chi}}{\chi^2} \right) = \underbrace{\frac{1}{2\chi^2} (\partial\chi)^2}_{\text{Quantum Kinetics}} - \underbrace{\left(\frac{1}{4\chi^2} (\partial\chi)^2 - \frac{\partial^2 \chi}{2\chi} \right)}_{\text{Quantum Potential}} \quad (141)$$

The quantum potential herein described is the relativistic version of the quantum potential found in the Bohm-Broglie reformulation of QM, whereas the quantum kinetics can be understood as the standard scalar field kinetic term, as evidenced by a simple change of variable: $\ln \chi \rightarrow \varphi \implies (\partial\chi)^2/\chi^2 = (\partial \ln \chi)^2 = (\partial\varphi)^2$. When integrated, they define a quantity that we refer to as the quantum action:

$$S = \underbrace{\int \left(\frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \right) \sqrt{-|g|} d^4x}_{\text{Quantum Action}} \quad (142)$$

Proof.

$$\text{tr} \left(\chi^{-2} \sqrt{\chi} \mathbf{e}^\mu D_\mu \chi \mathbf{e}^\nu D_\nu \sqrt{\chi} \right) \quad (143)$$

$$= \text{tr} \left(\chi^{-2} \sqrt{\chi} (\mathbf{e}^\mu \partial_\mu \chi) \mathbf{e}^\nu \partial_\nu \sqrt{\chi} + \chi^{-2} \sqrt{\chi} \mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \sqrt{\chi} \right) \quad (144)$$

$$= \text{tr} \left(\chi^{-2} 2^{-1} (\mathbf{e}^\mu \partial_\mu \chi) (\mathbf{e}^\nu \partial_\nu \chi) + \chi^{-1} \sqrt{\chi} \left(-4^{-1} \chi^{-3/2} \mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \chi + 2^{-1} \chi^{-1/2} \mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \chi \right) \right) \quad (145)$$

$$= \text{tr} \left(\frac{(\mathbf{e}^\mu \partial_\mu \chi)(\mathbf{e}^\nu \partial_\nu \chi)}{2\chi^2} - \frac{(\mathbf{e}^\mu \partial_\mu \chi)(\mathbf{e}^\nu \partial_\nu \chi)}{4\chi^2} + \frac{\mathbf{e}^\mu \partial_\mu \mathbf{e}^\nu \partial_\nu \chi}{2\chi} \right) \quad (146)$$

$$= \frac{1}{2\chi^2} (\partial\chi)^2 - \frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \quad (147)$$

$$= \frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \quad (148)$$

□

Theorem 13 (Ricci Scalar). *Let us investigate another subspace of the field where $\sqrt{\chi} = 1$ and $e^{-ib/2} = 1$, such that $\psi = R$. Then the kinetic energy T reduces to the Ricci scalar $\mathcal{R}$.*

$$\text{tr}(\tilde{R} \mathbf{e}^\mu D_\mu R \tilde{R} \mathbf{e}^\nu D_\nu R) = \mathcal{R} \quad (149)$$

Proof.

$$\text{tr}(\tilde{R} \mathbf{e}^\mu D_\mu R \tilde{R} \mathbf{e}^\nu D_\nu R) \quad (150)$$

$$= \text{tr}(\tilde{R} \mathbf{e}^\mu D_\mu \mathbf{e}^\nu D_\nu R) \quad \text{via } \tilde{R} R = 1 \quad (151)$$

$$= \text{tr}(\tilde{R} D^2 R) \quad (152)$$

$$= \text{tr}(R \tilde{R} D^2) \quad (153)$$

$$= \text{tr}(D^2) \quad (154)$$

$$= \mathcal{R} \quad \text{via Lichnerowicz-Weitzenböck identity} \quad (155)$$

□

Definition 21 (Gravity). Let us now consider the full space of the wavefunction $\psi = \sqrt{\rho} e^{-ib/2}$. We are automatically lead into a theory of gravity:

$$S = \int_{\mathcal{M}} \text{tr} \left(\frac{(\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\nu D_\nu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) \sqrt{-|g|} d^4x \quad (156)$$

which expands, via Theorem 12 and 13, as follows:

$$S = \int_{\mathcal{M}} \left(\mathcal{R} + \frac{1}{4\chi^2} (\partial\chi)^2 + \frac{\partial^2 \chi}{2\chi} \right) \sqrt{-|g|} d^4x \quad (157)$$

We note the following equations of motion which must be simultaneously satisfied:

1. Varying with respect to $g_{\mu\nu}$ yields the EFE with the Einstein tensor from $\mathcal{R}$, and is sourced by the quantum action variation yielding the stress-energy tensor.

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = 8\pi G \left[\frac{1}{4\chi^2} \left(\partial_\mu \chi \partial_\nu \chi - \frac{1}{2} g_{\mu\nu} (\partial\chi)^2 \right) + \frac{1}{2\chi} \left(\nabla_\mu \nabla_\nu \chi - \frac{1}{2} g_{\mu\nu} \square \chi \right) \right] \quad (158)$$

2. Varying with respect to χ gives equations of motion that define the flow of χ in spacetime:

$$\chi \square \chi = (\partial\chi)^2 \quad (159)$$

To interpret this theory, let us now introduce the surprisal field and associated definitions.

Definition 22 (Surprisal Field). We define a change of variable:

$$\varphi := -\ln \chi \quad (160)$$

We call φ the surprisal field.

Definition 23 (Surprisal Equation of Motion). We note that the change of variable $\varphi := -\ln \chi$, changes the equation of motion as follows:

$$\chi \square \chi = (\partial\chi)^2 \xrightarrow[\varphi := -\ln \chi]{} \square \varphi = 0 \quad (161)$$

which is the Klein-Gordon equation in curved spacetime, applied to the surprisal field.

Definition 24 (Surprisal Conservation). The following current:

$$\nabla_\mu (\partial^\mu \varphi) = 0 \quad (162)$$

identifies the surprisal as the conserved charge of this action.

Definition 25 (Surprisal Expectation Value). The surprisal expectation value is merely the entropy H of a region of the manifold:

$$H = \underbrace{\langle \ln \chi \rangle}_{\text{expectation value}} = - \int_{\mathcal{M}} \chi(x) \underbrace{\ln \chi(x)}_{\text{observable}} \sqrt{-|g|} d^4x \quad (163)$$

Interpretation

In information theory, the **surprisal** of an event with probability density $\rho(x)$ is defined as $-\ln \rho(x)$, and the entropy $H = -\int \rho \ln \rho d^4x$ represents its expectation value. In our gravitational framework, however, the field χ replaces ρ but differs critically:

- χ is **not a probability density**—it lacks a conserved current ($\nabla_\mu(\chi u^\mu) \neq 0$) and is not normalized.
- Instead, χ is interpreted as an **information density**, encoding spacetime's local information content.

The **surprisal** is defined as $\varphi = -\ln \chi$, which in this theory satisfies the Klein-Gordon equation $\square\varphi = 0$. This ensures:

1. **Conservation:** The current $j^\mu = \partial^\mu \varphi$ is conserved ($\nabla_\mu j^\mu = 0$), making $Q = \int j^\mu \sqrt{-g} d^3x$ a conserved charge.
2. **Causal Propagation:** Surprisal propagates at light speed, enforcing that information (quantified by φ) cannot spread superluminally—a core tenet of relativity.

Thus, while quantum mechanics relies on probabilistic amplitudes ψ , our formulation recasts general relativity as a **deterministic theory of information dynamics**, where spacetime geometry and information flux are dual aspects of $\mathcal{R}$ and φ . The distribution of surprisal in spacetime dictates its geometric structure, which in turns dictates how it propagates.

Probability dynamics, which can exist within a finite region of spacetime via the entropy obtained as the surprisal expectation value, is a special case of information dynamics—which exists globally—and is governed by the Dirac equation (as a special case in Theorem 11).

2.3.4. Yang-Mills

In quantum field theory (QFT), the standard method to identify a local gauge symmetry is to start with a global symmetry of the action or probability measure and then localize it by introducing gauge fields. For example, the U(1) gauge symmetry arises naturally in electromagnetism as the group preserving the probability density (Born rule) under local phase transformations. However, the non-Abelian SU(2) and SU(3) gauge symmetries of the Standard Model are not derived from first principles in this way; their inclusion is empirically motivated by particle physics experiments.

Improvement via Double-Product Structure:

Our framework demonstrates that Yang-Mills theories emerge naturally from constraints on the wavefunction's probability measure and Dirac current. Specifically:

1. **Probability Measure:** The quadratic form $(\psi^\dagger \psi)^\dagger \psi^\dagger \psi = \rho^2$ enforces rotor invariance $\psi \rightarrow R\psi$, restricting transformations to those satisfying $R^\dagger R = 1$, for some rotor R of a geometric algebra of n dimensions.
2. **Dirac Current:** The spacetime current $\psi^\dagger \mathbf{e}_0 \psi = \mathbf{e}_0$ requires gauge generators to commute with $\mathbf{e}_0$, confining them to an internal space.
3. **Spacetime:** The origin of the double-product from STA, associates the resulting internal space, to spacetime.

These constraints limit the allowable symmetry to groups generated by bivector exponentials (which are compact Lie groups), and acting on the internal spaces of spacetime. Since $SU(n) \subset \text{Spin}(2n)$, this framework inherently includes the Standard Model within its landscape but also generalizes to condensed matter systems with emergent $SU(n)$ symmetries.

Mathematical Derivation:

1. **Constraint on Transformations:** For the probability density to remain invariant under a transformation R , we require:

$$(\psi^\dagger R^\dagger R \psi)^\dagger \psi^\dagger R^\dagger R \psi = (\psi^\dagger \psi)^\dagger \psi^\dagger \psi \implies R^\dagger R = 1. \quad (164)$$

Solutions to $R^\dagger R = 1$ are rotor transformations generated by bivectors in the Clifford algebra. For a $2n$ -dimensional algebra, these generate $\text{Spin}(2n)$, whose subgroups include all $SU(n)$.

2. **Dirac Current Invariance:** The spacetime current $\psi^\dagger \mathbf{e}_0 \psi = \mathbf{e}_0$ must remain unchanged under gauge transformations. This implies:

$$\psi^\dagger e^{-\theta^i B_i} \mathbf{e}_0 e^{\theta^i B_i} \psi = \psi^\dagger \mathbf{e}_0 \psi \implies [B_i, \mathbf{e}_0] = 0, \quad (165)$$

where B_i are bivector generators. Thus, B_i act only on **internal degrees of freedom**, orthogonal to spacetime.

Wavefunction and Symmetry Structure:

The total wavefunction is a tensor product of spacetime (STA) and internal space components:

1. For $SU(n)$ Yang-Mills:

$$\psi_{\text{STA}} \otimes \psi_{\mathbb{C}^n}. \quad (166)$$

2. For the Standard Model $SU(3) \times SU(2) \times U(1)$:

$$\psi_{\text{STA}} \otimes \psi_{\mathbb{C}} \otimes \psi_{\mathbb{C}^2} \otimes \psi_{\mathbb{C}^3}. \quad (167)$$

Covariant Derivative and Action (Ex. Standard Model):

The covariant derivative incorporates spacetime curvature (gravity) and gauge fields:

$$D_\mu = \begin{pmatrix} \partial_\mu + \frac{\omega_\mu^{ab}}{2} \gamma_{ab} + ig' Y B_\mu + ig \frac{\sigma^a}{2} W_\mu^a + ig_s \frac{\lambda^a}{2} G_\mu^a & \Phi \\ \Phi^\dagger & \partial_\mu + \frac{\omega_\mu^{ab}}{2} \gamma_{ab} + ig' Y B_\mu + ig_s \frac{\lambda^a}{2} G_\mu^a \end{pmatrix}, \quad (168)$$

where:

1. γ_{ab} : Generators of $\text{Spin}(3,1)$ (gravitational spin connection).
2. B_μ, W_μ^a, G_μ^a : $U(1)$, $SU(2)$, and $SU(3)$ gauge fields.
3. Φ : Higgs field ($SU(2)$ doublet).

It acts on the left/right split of the field.

Our previous gravitational action is reconstructed with a spectral function f :

$$S = \int_{\mathcal{M}} \text{tr} \left(f \left(\frac{1}{\Lambda^2} \frac{(\phi^\dagger \mathbf{e}^\mu D_\mu \phi)^\dagger \phi^\dagger \mathbf{e}^\mu D_\mu \phi}{(\phi^\dagger \phi)^\dagger \phi^\dagger \phi} \right) \right) \sqrt{-|g|} d^4x. \quad (169)$$

Expanding f the field strength term $\text{tr}(f(D^2/\Lambda^2))$ via the Heat kernel yields the Standard Model + gravity (see A. H. Chamseddine and Alain Connes [7] for method):

1. **Leading Terms:**

- (a) Cosmological constant: $\propto \Lambda^4 \int \sqrt{-|g|} d^4x$.
- (b) Einstein-Hilbert term: $\propto \Lambda^2 \int R \sqrt{-|g|} d^4x$.

2. **Yang-Mills and Higgs:**

- (a) Gauge kinetic terms: $\propto \int \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \sqrt{-|g|} d^4x$.
- (b) Higgs kinetic and potential terms:

$$\propto \int \left(|D_\mu \Phi|^2 + \Lambda^2 |\Phi|^2 + \frac{1}{\Lambda^2} |\Phi|^4 \right) \sqrt{-|g|} d^4x. \quad (170)$$

3. **Yukawa Couplings** (from matter fields):

$$\propto \int y_{ij} \bar{\phi}_i \Phi \phi_j \sqrt{-|g|} d^4x. \quad (171)$$

Key Notes:

1. **Higher-Order Terms:** Terms like R^2 or $R_{\mu\nu}R^{\mu\nu}$ appear but are suppressed by Λ^{-2} , making them negligible at low energies.
2. **Uniqueness:** The Standard Model is not uniquely selected but resides within the landscape of allowed Yang-Mills theories.
3. **Experimental Consistency:** The framework resembles Connes’ spectral action (see A. H. Chamseddine and Alain Connes [7]), recovering the Standard Model and general relativity while allowing for testable extensions (e.g., higher-curvature gravity).

This formulation unifies gauge symmetries and gravity within a double-product structure.

2.4. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to resolve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except the previously explored cases are obstructed. By obstructed, we mean that the solution to the entropy maximization problem, ρ , does not satisfy all axioms of probability theory.

<i>Dimensions</i>	<i>Optimal Predictive Theory of Nature</i>	
GA(0)	Statistical Mechanics	(172)
GA(0,1)	Quantum Mechanics	(173)
GA(1,0)	Obstructed (Negative probabilities)	(174)
GA(2,0)	Quantum Mechanics	(175)
GA(1,1)	Obstructed (Negative probabilities)	(176)
GA(0,2)	Obstructed (Non-real probabilities)	(177)
GA(3,0)	Obstructed (Non-real probabilities)	(178)
GA(2,1)	Obstructed (Non-real probabilities)	(179)
GA(1,2)	Obstructed (Non-real probabilities)	(180)
GA(0,3)	Obstructed (Non-real probabilities)	(181)
GA(4,0)	Obstructed (Non-real probabilities)	(182)
GA(3,1)	Gravity + Yang-Mills	(183)
GA(2,2)	Obstructed (Negative probabilities)	(184)
GA(1,3)	Obstructed (Non-real probabilities)	(185)
GA(0,4)	Obstructed (Non-real probabilities)	(186)
GA(5,0)	Obstructed (Non-real probabilities)	(187)
⋮	⋮	
GA(6,0)	Suspected Obstructed (No observables)	(188)
⋮	⋮	
∞		(189)

Let us now demonstrate the obstructions mentioned above.

Theorem 14 (Non-real probabilities). *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. These geometric algebras are classified as follows:

$$\text{GA}(0, 2) \cong \mathbb{H} \quad (190)$$

$$\text{GA}(3, 0) \cong \mathbb{M}_2(\mathbb{C}) \quad (191)$$

$$\text{GA}(2, 1) \cong \mathbb{M}_2^2(\mathbb{R}) \quad (192)$$

$$\text{GA}(1, 2) \cong \mathbb{M}_2(\mathbb{C}) \quad (193)$$

$$\text{GA}(0, 3) \cong \mathbb{H}^2 \quad (194)$$

$$\text{GA}(4, 0) \cong \mathbb{M}_2(\mathbb{H}) \quad (195)$$

$$\text{GA}(1, 3) \cong \mathbb{M}_2(\mathbb{H}) \quad (196)$$

$$\text{GA}(0, 4) \cong \mathbb{M}_2(\mathbb{H}) \quad (197)$$

$$\text{GA}(5, 0) \cong \mathbb{M}_2^2(\mathbb{H}) \quad (198)$$

The determinant of these objects is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. $\square$

Theorem 15 (Negative probabilities). *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable.*

Proof. This category contains three dimensional configurations:

$\text{GA}(1, 0)$: Let $\psi = a + be_1$, then:

$$(a + be_1)^\dagger(a + be_1) = (a - be_1)(a + be_1) = a^2 - b^2e_1e_1 = a^2 - b^2 \quad (199)$$

which is valued in $\mathbb{R}$.

$\text{GA}(1, 1)$: Let $\psi = a + be_0e_1$, then:

$$(a + be_0e_1)^\dagger(a + be_0e_1) = (a - be_0e_1)(a + be_0e_1) = a^2 - b^2e_0e_1e_0e_1 = a^2 - b^2 \quad (200)$$

which is valued in $\mathbb{R}$.

$\text{GA}(2, 2)$: Let $\psi = a + be_0e_2e_1e_2$, where $e_0^2 = -1, e_2^2 = -1, e_1^2 = 1, e_2^2 = 1$, then:

$$\lfloor (a + \mathbf{b})^\dagger(a + \mathbf{b}) \rfloor_{3,4} \lfloor (a + \mathbf{b})^\dagger(a + \mathbf{b}) \rfloor \quad (201)$$

$$= \lfloor a^2 + 2a\mathbf{b} + \mathbf{b}^2 \rfloor_{3,4} (a^2 + 2a\mathbf{b} + \mathbf{b}^2) \quad (202)$$

We note that $\mathbf{b}^2 = b^2e_0e_2e_1e_2e_0e_2e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \quad (203)$$

$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \quad (204)$$

$$= (a^2 + b^2)^2 - 4a^2b^2 \quad (205)$$

which is valued in $\mathbb{R}$.

In all of these cases the probability can be negative. $\square$

Conjecture 1 (No observables (6D)). *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 15 collapse. The research by Acus et al.[8] in 6D geometric algebra concludes that the determinant, so far defined

through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (206)$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[8].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \quad (207)$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \quad (208)$$

$$\langle 74 \text{ monomials} \rangle = 0 \quad (209)$$

$$\langle 74 \text{ monomials} \rangle = 0 \quad (210)$$

From Equation 206, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (211)$$

Any equality of the above type between $b_1 \mathbf{O}$ and $b_2 \mathbf{O}$ is frustrated by the factors b_1 and b_2 , forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) be constructible in 6D, using our method. $\square$

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No observables (above 6D)). *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $GA(1,3)$ even if in relativity, the signature of the metric $(+, -, -, -)$ versus $(-, -, -, +)$ does not influence the physics. However, in geometric algebra, $GA(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $GA(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

3. Discussion

When asked to define what a physical theory is, an informal answer may be that it is a predictive framework of measurements that applies to all possible experiments realizable within a domain, with nature as a whole being the most general domain. While physicists have expressed these theories through sets of axioms, we propose a more direct approach—mathematically realizing this fundamental definition itself. This definition is realized as an optimization problem (Definition 1) that can be solved

directly. The solution to this optimization problem yields precisely those structures that realize the physical theory over said domain. Succinctly, physics is the solution to:

$$\underbrace{\mathcal{L}}_{\text{an optimization problem}} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{on the entropy of a measurement relative to its preparation over all}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{predictive theories}} + \underbrace{\tau \operatorname{tr} \left(\overline{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\text{of nature}} \quad (212)$$

The relative Shannon entropy represents the basic structure of any experiment, quantifying the informational difference between its initial preparation and its final measurement.

The natural constraint is chosen to be the most general structure that admits a solution to this optimization problem. This generality follows from key mathematical requirements. The constraint must involve quantities that form an algebra, as the solution requires taking exponentials:

$$\exp X = 1 + X + \frac{1}{2}X^2 + \dots \quad (213)$$

which involves addition, powers, and scalar multiplication of X . The use of the trace operation further necessitates that X must be represented by square $n \times n$ matrices. Thus Axiom 1 involves $n \times n$ matrices:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \quad (214)$$

The trace operation is utilized because the constraint must be converted back to a scalar for use in the Lagrange multiplier equation; while any function that maps an algebra to a scalar would achieve that, picking the trace recovers quantum mechanics in the $\text{GA}(0,1)$ case.

These mathematical requirements demonstrate that the natural constraint, as it admits the minimal mathematical structure required to solve an arbitrary entropy maximization problem, can be understood as the most general extension of the statistical mechanics average energy constraint which contains QM (as induced by the trace) as a specific solution.

Thus, having established both the mathematical structure and its generality, we can understand how this minimal ontology operates. Since our formulation keeps the structure of experiments completely general, our optimization considers all possible predictive theories for that structure, and the constraint is the most general constraint possible for that structure, the resulting optimal physical theory applies, by construction, to all realizable experiments within its domain.

This ontology is both operational, being grounded in the basic structure of experiments rather than abstract entities, and constructive, showing how physical laws emerge from optimization over all possible predictive theories subject to the natural constraint. Physics is encapsulated not as a pre-defined collection of fundamental axioms but as the optimal solution to a well-defined optimization problem over all experiments realizable within the domain. This represents a significant philosophical shift from traditional physical ontologies where laws are typically taken as primitive.

The next step in our derivation is to represent the determinant of the $n \times n$ matrices through a self-product of multivectors involving various conjugate structures. By examining the various dimensional configurations of geometric algebras, we find that $\text{GA}(3,1)$, representing 4×4 real matrices, admits a sub-algebra whose determinant is positive-definite for its invertible members. All other dimensional configurations fail to admit such a positive-definite structure, with two exceptions: statistical mechanics (found in $\text{GA}(0)$) and quantum mechanics (found in $\text{GA}(0,1)$ and in a sub-algebra of $\text{GA}(2,0)$).

The solution reveals that the 3+1D case harbours a new type of probability amplitude structure analogous to complex amplitudes, one that exhibits the characteristic elements of a quantum mechanical theory. Instead of complex-valued amplitudes, we have amplitudes valued in the invertible subset of the even sub-algebra of $\text{GA}(3,1)$. This probability amplitude is identical to David Hestenes'

wavefunction, but comes with an extended Born rule represented by the determinant, and rather than a complex Hilbert space, it lives in a "double-product structure". This double-product structure automatically incorporates gravity via the $\text{Spin}(3,1)$ connection and local gauge theories as Yang-Mills theories. The square of the Dirac operator, automatically generated by the Lagrangian, then generates the invariants of gravity and of the Yang-Mills theory via a heat kernel expansion, along with the matter fields.

Interpretation: This framework establishes quantum mechanics as the emergent solution to entropy optimization constrained by measurement outcomes, rather than a set of axiomatic entities. The wavefunction arises as a non-fundamental calculational tool, akin to statistical mechanics' probability distributions. Since all spacetime and quantum structures (i.e. the five axioms, gravity, Yang-Mills, $3+1D$, etc.) are in the wavefunction, they are thus revealed as epistemic necessities for describing experiments within the constraint of nature, and are void of ontological commitments.

Measurements: In our interpretation, initial preparations and final measurements constitute the irreducible physical boundaries of an experiment. These empirical endpoints define reality, with the quantum formalism emerging as the least-biased bridge between them through relative entropy maximization under the natural constraint. Traditional interpretations problematically treat measurement as a physical process transforming wavefunctions, necessitating the ad hoc "collapse" postulate. We invert this ontology: the Schrödinger equation and Born rule describe not physical evolution but an inferential relationship connecting preparation to measurement outcomes. The wavefunction exists solely as an epistemic construct encoding this relationship. This eliminates the measurement problem by excising its root cause: the reification of intermediate states (i.e. states between initial preparation and final measurement) as ontological entities.

4. Conclusion

This work suggests that the formal essence of fundamental physics—statistical mechanics, quantum mechanics, general relativity, Yang-Mills and the dimensionality of spacetime—emerges not from axiomatic constructs but as necessity. By reframing physics as the solution to a single optimization problem on relative entropy, constrained only by what measurements nature permits, we derive these structures from first principles. This reformulation dissolves the artificial divide between quantum and classical frameworks: statistical mechanics, quantum theory, gravity and gauge theory all become special cases of information optimization under progressively richer constraints.

The conceptual pivot is ontological inversion. Traditional approaches postulate physical entities (wavefunction, fields) and retroactively justify them through measurement. Our framework begins and ends with measurement outcomes, with the *natural constraint* acting as the keystone. Applied to all possible experiments, this constraint uniquely compels the emergence of Yang-Mills, spacetime, and gravity. These structures dominate not by nature's preference but by mathematical inevitability—they comprise the only solution satisfying the natural constraint for all realizable experiments.

This work suggests a radical re-envisioning of physical law. Concepts we reify as fundamental—Hilbert spaces, gravitational curvature—are revealed as epistemic artifacts: the observer's efficient calculus for experimental prediction under the constraint of nature. Just as statistical mechanics can derive thermodynamics from atomic motions, we derive quantum mechanics from maximizing entropy of initial preparations relative to final measurements. The set of all realizable experiments replaces atoms as the fundamental building block of physics. Future work must test this paradigm's predictive limits, but its mere viability invites reconsideration of long-held ontological assumptions about what physics ultimately describes.

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Appendix A. SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} = \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\bar{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \quad (\text{A1})$$

We solve the maximization problem as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_i, \dots, \rho_n)}{\partial \rho_i} \quad (\text{A2})$$

$$= -\ln \rho_i - 1 - \lambda - \beta E_i \quad (\text{A3})$$

$$= \ln \rho_i + 1 + \lambda + \beta E_i \quad (\text{A4})$$

$$\Rightarrow \ln \rho_i = -1 - \lambda - \beta E_i \quad (\text{A5})$$

$$\Rightarrow \rho_i = \exp(-1 - \lambda) \exp(-\beta E_i) \quad (\text{A6})$$

$$= \frac{1}{Z(\tau)} \exp(-\beta E_i) \quad (\text{A7})$$

The partition function, is obtained as follows:

$$1 = \sum_i \exp(-1 - \lambda) \exp(-\beta E_i) \quad (\text{A8})$$

$$\Rightarrow (\exp(-1 - \lambda))^{-1} = \sum_i \exp(-\beta E_i) \quad (\text{A9})$$

$$Z(\tau) := \sum_i \exp(-\beta E_i) \quad (\text{A10})$$

Finally, the probability measure is:

$$\rho_i = \frac{1}{\sum_i \exp(-\beta E_i)} \exp(-\beta E_i) \quad (\text{A11})$$

Appendix B. RQM in 3+1D

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\substack{\text{Relative} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\frac{1}{2} \zeta \text{tr} \left(\bar{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i\right)}_{\substack{\text{Universal} \\ \text{Measurement} \\ \text{Constraint}}} \quad (\text{A12})$$

The solution is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp(-\frac{1}{2} \zeta \mathbf{M}_i)}}_{\text{Spin}^c(3,1) \text{ Invariant Ensemble}} \underbrace{\det \exp(-\frac{1}{2} \zeta \mathbf{M}_i)}_{\substack{\text{Spin}^c(3,1) \text{ Born Rule}}} \underbrace{p_i}_{\text{Initial Preparation}} \quad (\text{A13})$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_1, \dots, \rho_n)}{\partial \rho_i} \quad (\text{A14})$$

$$= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i \quad (\text{A15})$$

$$= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i \quad (\text{A16})$$

$$\Rightarrow \ln \frac{\rho_i}{p_i} = -p_i - \lambda - \frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i \quad (\text{A17})$$

$$\Rightarrow \rho_i = p_i \exp(-p_i - \lambda) \exp\left(-\frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i\right) \quad (\text{A18})$$

$$= \frac{1}{Z(\zeta)} p_i \exp\left(-\frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i\right) \quad (\text{A19})$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i\right) \quad (\text{A20})$$

$$\Rightarrow (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i\right) \quad (\text{A21})$$

$$Z(\zeta) := \sum_i p_i \exp\left(-\frac{1}{2} \zeta \operatorname{tr} \mathbf{M}_i\right) \quad (\text{A22})$$

□

Appendix C. SageMath Program Showing $[\mathbf{u}^\dagger \mathbf{u}]_{3,4} \mathbf{u}^\dagger \mathbf{u} = \det \mathbf{M}_\mathbf{u}$

```

from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3

```

```

udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

```

```
u2 = u.clifford_conjugate()*u
```

```

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4

```

```

I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

```

```
#MAJORANA MATRICES
```

```

y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
                 [-1, 0, 0, 0]])

```

```

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
                 [0, 0, -1, 0]])

```

```

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])

```

```

y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

```

```

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

```

```
print(u2conj34*u2 == m.det())
```

The program outputs

True

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-product: $\det \mathbf{M}_u = [\mathbf{u}^\dagger \mathbf{u}]_{3,4} \mathbf{u}^\dagger \mathbf{u}$.

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