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Convergence Analysis of the Resummation of a Class of Superfactorially Divergent Stieltjes Series by Weniger's Transformation

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Abstract

The resummation of superfactorially divergent series represents a significant computational challenge in mathematical physics. In the present paper the resummation of a specific class of Stieltjes series characterized by a moment sequence growing as $(2n)!$ will be addressed. Despite the fact that Carleman's condition is satisfied for these series, the practical utility of Padé approximants in resumming them is severely compromised by an extremely slow convergence rate. Weniger's δ-transformation is proposed as a superior resummation tool. Some recently found results on the converging factors of superfactorially divergent Stieltjes series are here used to derive an exact integral representation of the δ truncation error, which allows for a formal proof of convergence and an analytical asymptotic estimate of the corresponding convergence rate. Numerical experiments are carried out to validate our theoretical findings, confirming that the δ transformation offers a robust and computationally efficient framework for decoding this class of wildly divergent expansions.

Keywords: mathematical physics; divergent series; stieltjes series; converging factors

MSC: 40A05; 65B10

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1. Introduction

Divergent series have been a central, if once controversial, topic in mathematics since Euler's time.

Summa cuiusque seriei est valor expressionis illius finitae, ex cuius evolutione illa series oritur

Euler wrote in a letter dated 1745 to C. Goldbach: a series, regardless of its convergent or divergent nature, is fundamentally a *coded representation* of the finite analytical expression that generated it [1]. Today, this is no longer a matter of interpretation, but a computational necessity. In practical terms, a divergent series can be regarded as a useful tool only when paired with a suitable resummation technique. In this context, Borel summation [2,3] and Padé approximants [4] are the most commonly used tools, although they encounter significant shortcomings and limitations when applied to "wildly" divergent expansions [5]. This is a recurring issue in theoretical physics. For example, several perturbation expansions in quantum physics are known to display a structural divergence for any values of the coupling constant and, in this respect, they are only asymptotic. Dyson [6] first argued that perturbation expansions in quantum electrodynamics must diverge factorially. Similarly, Bender and Wu [7,8,9] demonstrated that even *superfactorial* divergence occurs in nonrelativistic quantum mechanics [10]. The resummation of these series is therefore a practical necessity, not just an academic exercise [11].

The specific problem addressed in the present paper concerns a class of Stieltjes series with a moment sequence $\{\mu_n\}_{n=0}^{\infty}$ growing like $(2n)!$ instead of $n!$ (e.g., the Euler series). In principle, such series fall within the reach of Padé approximants, as they satisfy the Carleman condition, which is sufficient for Padé resummability. However, since the divergence of the associated Carleman series is merely logarithmic, the convergence of Padé approximants is extremely slow, as shown for instance in Ref. [11]. Then, the resummation problem is tackled by adopting a particular Levin-type nonlinear sequence transformation, the so-called Weniger, or δ-transformation [12]. Levin-type transformations [12–14] proved to be particularly effective and powerful resummation tools, especially when compared to Padé approximants. Despite the empirical effectiveness of Levin-type transformations, and specifically the δ transformation, rigorous analytical convergence proofs remain notably scarce in the literature. While Padé approximants benefit from a well established theoretical framework, especially in the case of Stieltjes series [15], the mathematical foundation of resummation processes involving Levin-type transformations remains largely underdeveloped. About ten years ago, a rigorous convergence theory for the resummation of the Euler series via δ transformation has been proposed [16].

In the present paper, starting from the same methodology and strategy developed in Ref. [16], we construct an analytical convergence theory for the δ resummation of Stieltjes series exhibiting $(2n)!$ growth. We demonstrate that the δ-transformation is effectively insensitive to the logarithmic limitations that hinder Padé approximants. Furthermore, a formal proof of convergence and an asymptotic estimate of the δ-transformation convergence rate are also provided. The technical challenges encountered in the present study are considerably harder than those solved in [16]. By using some recently published results about the converging factors of the class of Stieltjes series under investigation [17], we derive an integral representation of the δ-transformation error. This representation provides the analytical basis for the subsequent asymptotic analysis aimed at proving both the convergence and at computing the corresponding convergence rate of the resummation process.

Before proceeding, readers should be advised that much of the material presented in Section 2 is drawn from my previous works, specifically Ref. [16] and Refs. [17–19]. This section is intended to provide sufficient self-consistency to the paper, allowing nonspecialists to appreciate the natural derivation of Weniger's transformation. Nevertheless, readers are strongly encouraged to consult the original publications cited throughout the text.

2. Why Should Weniger's Transformation Be Fit for Decoding Stieltjes Series?

Consider a nondecreasing, real-valued function $\mu(t)$ defined for $t \in [0,\infty]$, possessing infinitely many points of increase, so that the associated measure $\mathrm{d}\mu$ turns out to be positive on $[0,\infty)$. Accordingly, all moments

$$\mu_m = \int_0^\infty t^m \,\mathrm{d}\mu\,, \qquad m \geq 0\,, \tag{1}$$

are finite and positive. The formal power series

$$\sum_{m=0}^\infty \frac{(-1)^m}{z^{m+1}}\,\mu_m\,, \tag{2}$$

is called a *Stieltjes series*. Such series turns out to be asymptotic, in the sense of Poincaré, for $z \to \infty$, to the function $f(z)$ defined as

$$f(z) = \int_0^\infty \frac{\mathrm{d}\mu}{z+t}\,, \qquad |\arg(z)| < \pi\,, \tag{3}$$

which is analytic in the whole complex plane cut along the negative real axis (i.e., $\mathbb{C}\backslash(-\infty,0]$), and is called *Stieltjes function* [20].

The most well-known example of Stieltjes series is the Euler series [21], characterized by the moment sequence $\{\mu_m = m!\}_{m=0}^\infty$, and asymptotic to the so-called Euler integral,

$$\int_0^\infty \frac{\exp(-t)\,\mathrm{d}t}{z+t} = \exp(z)\,\Gamma(0,z)\,, \qquad |\arg(z)| < \pi. \tag{4}$$

Decoding the asymptotic series in Eq. (2) to retrieve the correct value of $f(z)$ is known as the Stieltjes moment problem. A sufficient criterion to guarantee unicity to the solution of the moment problem is the so-called Carleman condition, which requires that the following series

$$\sum_{m=0}^{\infty} \mu_m^{-\frac{1}{2m}}, \tag{5}$$

be divergent. Any Stieltjes function $f(z)$ can also be expressed as the sum of the nth-order partial sum of the associated asymptotic series (2) and of a truncation error which has itself the form of a Stieltjes integral (see for example [12] [Theorem 13-1]). More precisely,

$$f(z) = f_n(z) + r_n(z), \tag{6}$$

where $f_n(z)$ denotes the nth-order partial sum,

$$f_n(z) = \sum_{m=0}^{n} \frac{(-1)^m}{z^{m+1}} \mu_m, \tag{7}$$

and the symbol $r_n(z)$ denotes the nth-order remainder, which is formally defined by

$$r_n(z) = \left(-\frac{1}{z}\right)^{n+1} \int_0^\infty \frac{t^{n+1} d\mu}{z+t}, \qquad |\arg(z)| < \pi. \tag{8}$$

The truncation error can be recast as follows:

$$r_n(z) = \frac{(-1)^{n+1}}{z^{n+1}} \mu_{n+1} \varphi_{n+1}(z), \tag{9}$$

where the quantity

$$\varphi_m(z) = \frac{1}{\mu_m} \int_0^\infty t^m \frac{d\mu}{t+z}, \qquad m \in \mathbb{N}_0, \qquad |\arg(z)| < \pi, \tag{10}$$

will henceforth be called the mth-order *converging factor* [22,23] (Actually, the definition of the converging factor φ_n used here differs from the classical definition by a factor z. This has been done for making the subsequent calculations easier).

The search of techniques aimed at estimating convergence factors *without resorting* to the numerical evaluation of the integral in Eq. (10) is pivotal for the development of new Stieltjes series decoding strategies. A powerful sequence transformation aimed at computing Padé approximants is the Wynn ε-algorithm [24]. To obtain approximations of the function $f(z)$, the ε-algorithm needs only the input of the numerical values of a finite substring of the partial sum sequence $\{f_n(z)\}_{n=0}^{\infty}$. In the case of Stieltjes series, however, important a priori additional information on the index dependence of the truncation error are available, as shown for example by Eqs. (8) - (10). Such *structural* information can be employed to improve the efficiency of the transformation process. From Eq. (10), it appears that the value of the Stieltjes function $f(z)$ could be retrieved, in principle, from the knowledge of only a *finite number* of single terms of the associated Stieltjes series, provided that the corresponding converging factor $\varphi_{n+1}(z)$ could be estimated, in some way, starting from the knowledge of the *sole* moment sequence $\{\mu_n\}_{n=0}^{\infty}$. Levin-type transformation theory [13] is ultimately based on the research of approximation models for converging factors. In the following, we shall denote $\varphi_n^{(k)}(z)$ a suitable kth-order approximation (with $k > 1$) of $\varphi_n(z)$ such that, in some limiting sense, it could be possible to write

$$\lim_{k\to\infty} \varphi_n^{(k)}(z) = \varphi_n(z). \tag{11}$$

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A systematic approach for the construction of Levin-type sequence transformations boils down to finding suitable *linear* operators, say $\hat{T}_k$, able to *annihilate* the kth-order converging factor approximant, i.e.,

$$\hat{T}_k\left\{\varphi_n^{(k)}\right\} = 0\,, \tag{12}$$

for fixed k but for all $n \in \mathbb{N}_0$. In [12] [Sections 7–9], E. J. Weniger showed how simple and powerful sequence transformations can systematically be obtained on using annihilation operators based upon the finite difference Δ, and precisely as

$$\hat{T}_k\{\cdot\} = \Delta^k\{P_{k-1}(n)\cdot\}. \tag{13}$$

Here, the symbol $P_{k-1}(n)$ denotes a polynomial of degree $k-1$ with respect to the integer variable n [14] [Section II], while the iterated difference operator Δ^k can be explicitly stated through the help of [24] [Eq. (25.1.1)], i.e.,

$$\Delta^k g(n) \;=\; (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j}\, g(n+j)\,, \qquad k \in \mathbb{N}\,. \tag{14}$$

In other words, the functional form of the convergence factor approximant $\varphi_n^{(k)}(z)$ must be chosen in such a way that the product $P_{k-1}(n)\varphi_n^{(k)}(z)$ reduces itself to a n-polynomial having a *degree less than* k, thus ready to be annihilated by Δ^k.

For the class of Stieltjes series, it has been proved that the converging factor in Eq. (10) can always be represented as an *inverse factorial series* [19]. The proof was ultimately based on the fact that:

(i) the converging factor defined in Eq. (10) satisfies the following first-order difference equation [19]:

$$\varphi_{n+1} \;=\; \frac{\mu_n}{\mu_{n+1}}\,(1 - z\,\varphi_n)\,, \qquad n \geq 0\,, \tag{15}$$

(ii) inverse factorial series constitutes a powerful tool for solving difference equations, similarly as inverse *power* series are customarily used to solve differential equations.

For reader's convenience, the basic definitions and properties of factorial series can be found, for instance, in [16,19,25]. In particular, the solution of Eq. (15) can *always* be set in the following form:

$$\varphi_n(z) \;=\; \sum_{j=0}^{\infty} \frac{a_j}{(n+\beta)_j}\,, \qquad n \in \mathbb{N}_0\,, \tag{16}$$

where $\beta > 0$ and $\{a_k\}_{k=0}^{\infty}$ denotes a sequence which is *independent* of n. The series found in the right side of Eq. (16) is an example of inverse factorial expansion, with the symbol $(n+\beta)_j$ denoting the Pochhammer symbol. From Eqs. (11) and (16), it would then be natural to conclude that a suitable model for representing $\varphi_n^{(k)}(z)$ could be

$$\varphi_n^{(k)}(z) \;=\; \sum_{j=0}^{k-1} \frac{a_j}{(n+\beta)_j}\,, \qquad k > 1, \tag{17}$$

for any $n \in \mathbb{N}$. As a consequence, it would be sufficient to let $P_{k-1}(n) = (n+\beta)_{k-1}$ in Eq. (13) to have

$$\hat{T}_k\{\cdot\} = \Delta^k\{(n+\beta)_{k-1}\cdot\}, \tag{18}$$

as the annihilation operator. Then, from Eqs. (6)–(9) we have

$$f(z) \,-\, f_n(z) \;=\; z\frac{(-1)^{n+1}}{z^{n+2}}\,\mu_{n+1}\,\varphi_{n+1}(z) \;=\; z\,\Delta f_n(z)\,\varphi_{n+1}(z), \tag{19}$$

and, on replacing the quantity $\varphi_{n+1}(z)$ with its kth-order approximant $\varphi^{(k)}_{n+1}(z)$, it follows that

$$z\,\varphi^{(k)}_{n+1} \simeq \frac{f}{\Delta f_n} - \frac{f_n}{\Delta f_n}. \tag{20}$$

The final step is to apply, to both sides of Equation (20), the annihilation operator $\hat{T}_k$ defined in Eq. (18), but written for $n+1$ in place of n. Simple algebra then gives

$$f(z) \simeq \frac{\Delta^k\left\{(n+1+\beta)_{k-1}\dfrac{f_n(z)}{\Delta f_n(z)}\right\}}{\Delta^k\left\{(n+1+\beta)_{k-1}\dfrac{1}{\Delta f_n(z)}\right\}} = \delta^{(n)}_k(\beta+1), \tag{21}$$

where the sequence $\{\delta^{(n)}_k(\gamma)\}^{\infty}_{k=0}$, with

$$\delta^{(n)}_k(\gamma) = \frac{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{f_n(z)}{\Delta f_n(z)}\right\}}{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{1}{\Delta f_n(z)}\right\}}, \qquad \gamma > 0, n \in \mathbb{N}_0, \tag{22}$$

defines the so-called *Weniger*, or δ transformation, of the sequence $\{f_n(z)\}^{\infty}_{n=0}$ [24] [Chapter 3.9(v) Levin's and Weniger's Transformations].

E. J. Weniger first employed his transformation for the evaluation of auxiliary functions in molecular electronic structure calculations [26]. Later, it was successfully used for the evaluation of special functions [12,27–36], the summation of divergent perturbation expansions [11,14,30–32,37–49], as well as for the prediction of unknown perturbation series coefficients [42,43,48,50]. In the last fifteen years, δ-transformation has also been employed in optics in the study of nonparaxial free-space propagation of optical wavefields [51–55], as well as in the numerical evaluation of several types of stable and unstable diffraction catastrophes [56,57].

3. The Integral Representation of the Transformation Error

The importance of Weniger's transformation in the resummation process of Stieltjes asymptotic series has first been conjectured about ten years ago in [16], where E. J. Weniger and I proved that, as far as Euler' series is concerned, it turns out that

$$\lim_{k\to\infty} \delta^{(n)}_k(\gamma) = f(z)\,, \tag{23}$$

when $n=0$ and $\gamma=1$. In other words, the δ-transformed sequence of the partial sum sequence of the Euler series does converge to the Euler integral (4). The proof given in [16] stems on a peculiar integral representation of the Euler series converging factor, thank to which the transformation error, defined as the difference $f(z)-\delta^{(n)}_k(\gamma)$, can be expressed in terms of a simple integral defined on the finite interval $[0,1]$.

The main task of the present paper is to develop a convergence analysis of the Weniger transformation, similar to that developed ten years ago for the Euler series, as far as the resummation of a class of superfactorially divergent Stieltjes series is concerned. To this end, the first step is to emphasize the tight relationship between converging factors and transformation errors. Accordingly, Weniger's transformation is first applied to both sides of Eq. (6), i.e.,

$$f(z) = \frac{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{f(z)}{\Delta f_n}\right\}}{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{1}{\Delta f_n}\right\}} = \frac{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{f_n}{\Delta f_n}\right\}}{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{1}{\Delta f_n}\right\}} + \frac{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{r_n}{\Delta f_n}\right\}}{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{1}{\Delta f_n}\right\}}, \tag{24}$$

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which, on taking Eq. (22) into account, gives at once

$$f(z)\,-\,\delta_k^n(\gamma)\;=\;\frac{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{r_n}{\Delta f_n}\right\}}{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{1}{\Delta f_n}\right\}}\;=\;z\,\frac{\Delta^k\{(n+\gamma)_{k-1}\varphi_{n+1}\}}{\Delta^k\left\{(n+\gamma)_{k-1}\dfrac{1}{\Delta f_n}\right\}}\,, \tag{25}$$

where in the last step, use has been made of Eq. (9).

It is worth recallingthat the whole convergence analysis presented in Ref. [16] was ultimately based on the following integral representation obtained for the nth-order converging factor of the Euler series:

$$\varphi_n(z)\;=\;\int_0^1 t^{n-1}\,\exp\left(z-\frac{z}{t}\right)\mathrm{d}t\,,\qquad\qquad \mathrm{Re}\{z\}>0\,. \tag{26}$$

In [17], it has been shown how such a fundamental representation of the converging factor can be easily extended to a whole class of superfactorially divergent Stieltjes series, which includes the class under investigation in the present paper. E. J. Weniger introduced me to the fascinating field of inverse factorial series fifteen years ago. I would encourage readers to go through his 2010 beautiful paper [25], where a historical account of his re-discovery of factorial series can be found, together with a list of their most important computational features. Extensive reviews can also be found, for instance, in [16,19]. For the scopes of the present paper, it is sufficient to limit ourselves to the following key points.

First of all, it is better to recast the converging factor factorial expansion into Eq. (16) as follows:

$$\varphi_n(z)\;=\;\sum_{k=0}^{\infty}\frac{k!}{(n+\beta)_{k+1}}\,c_k, \tag{27}$$

where now the sequence independent of n is represented by $\{c_k\}_{k=0}^{\infty}$. Equation (27) implies $\lim_{n\to\infty}\varphi_n=0$, which holds for Stieltjes series having null convergence radii, as those we are going to study. Moreover, on letting $x=n+\beta$, we also have the following key relationship:

$$\frac{k!}{(x)_{k+1}}\;=\;\int_0^1 t^{x-1}\,(1-t)^k\,\mathrm{d}t\,,\qquad \mathrm{Re}(x)>0\quad k\in\mathbb{N}_0\,, \tag{28}$$

which, after inserted into Eq. (27) and after interchanging integration and summation, leads to [58, Sec. I on p. 244]

$$\varphi_n(z)\;=\;\int_0^1 t^{n+\beta-1}\,\Phi(t)\,\mathrm{d}t\,,\qquad n+\beta>0\,, \tag{29}$$

where the function $\Phi(t)$ is formally be defined as

$$\Phi(t)\;=\;\sum_{k=0}^{\infty}c_k\,(1-t)^k\,, \tag{30}$$

and, in somewhat sense, it acts as a sort of generating function of the sequence $\{c_k\}_{k=0}^{\infty}$. In Ref. [17], it was shown how to retrieve the explicit expression of $\Phi(t)$, by substituting from Eq. (29) into the fundamental recurrence relation of Eq. (15), which has been converted into a well defined Cauchy problem.

Equation (29) represents the desired generalization of Eq. (26) to a typical Stieltjes series. In particular, on substituting from Eq. (29) into the numerator of Eq. (25), it can be proved that (see Appendix A):

$$\Delta^k\{(n+\gamma)_{k-1}\varphi_{n+1}\}\;=\;(-1)^k\,(n+\gamma)_{k-1}\int_0^1 t^{n+\gamma-1}\,{}_2F_1\left(\begin{matrix}-k,k+n+\gamma-1\\ n+\gamma\end{matrix};t\right)\Phi(t)\,\mathrm{d}t\,, \tag{31}$$

where ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function. Then, on substituting from Eq. (31) into Eq. (25), the following integral representation of the transformation error is obtained:

$$f(z) - \delta_k^{(n)}(\gamma) = z\,(-1)^k\,(n+\gamma)_{k-1}\,\frac{\displaystyle\int_0^1 t^{n+\gamma-1}\,{}_2F_1\left(\begin{matrix}-k, k+n+\gamma-1\\ n+\gamma\end{matrix};t\right)\Phi(t)\,\mathrm{d}t}{\Delta^k\left\{(n+\gamma)_{k-1}\,\dfrac{1}{\Delta f_n}\right\}}, \tag{32}$$

which constitutes our principal weapon for the subsequent convergence analysis.

4. Transformation Error for a Class of Superfactorially Divergent Stieltjes Series

The class of Stieltjes series we are going to investigate is characterised by the following moment sequence:

$$\mu_m = \Gamma(2m+1+q)\,, \qquad m \geq 0\,, \tag{33}$$

where $q \in (-1,1)$. It is not difficult that the measure $\mathrm{d}\mu$ associated to the sequence in Eq. (33) is

$$\mathrm{d}\mu = \frac{t^{\frac{q-1}{2}}\,\exp(-\sqrt{t})}{2}\,\mathrm{d}t, \tag{34}$$

so that the corresponding Stieltjes function defined in Eq. (3) turns out to be

$$f(z) = \int_0^\infty \frac{t^{\frac{q-1}{2}}\,\exp(-\sqrt{t})}{2(z+t)}\,\mathrm{d}t, \tag{35}$$

which can be expressed in closed-form by using for instance Wolfram Mathematica 14.3. For the sake of simplicity, such analytical expression is reported in Appendix B, as it will extensively be employed later. For the moment, it is worth modifying the integral representation of the transformation error by substituting from the explicit expression of $1/\Delta f_n$, i.e.,

$$\frac{1}{\Delta f_n} = \frac{(-)^{n+1}}{\mu_{n+1}}\,z^{n+2}\,, \tag{36}$$

into Eq. (32). After simple algebra we have

$$\begin{aligned}&\Delta^k\left\{(n+\gamma)_{k-1}\,\frac{(-)^{n+1}}{\mu_{n+1}}\,z^{n+2}\right\} = \\ &= -z^2\,(-)^n\,(-1)^k\,\frac{(n+\gamma)_{k-1}}{\Gamma(2n+q+3)}\,{}_2F_3\left(\begin{matrix}-k, k+n+\gamma-1\\ n+\dfrac{q}{2}+\dfrac{3}{2}, n+\dfrac{q}{2}+2, n+\gamma\end{matrix};-\frac{z}{4}\right),\end{aligned} \tag{37}$$

and, once substituted into Eq. (32), the following expression of the transformation error is obtained:

$$\begin{aligned}&f(z) - \delta_k^{(n)}(\gamma) = \\ &= \left(-\frac{1}{z}\right)^{n+1}\Gamma(2n+q+3)\,\frac{\displaystyle\int_0^1 t^{n+\gamma-1}\,{}_2F_1\left(\begin{matrix}-k, k+n+\gamma-1\\ n+\gamma\end{matrix};t\right)\Phi(t)\,\mathrm{d}t}{{}_2F_3\left(\begin{matrix}-k, k+n+\gamma-1\\ n+\dfrac{q}{2}+\dfrac{3}{2}, n+\dfrac{q}{2}+2, n+\gamma\end{matrix};-\dfrac{z}{4}\right)}.\end{aligned} \tag{38}$$

As far as the choice of the function $\Phi(t)$ is concerned, in Ref. [17] it was proved that for the Stieltjes series of Eq. (33) evaluated at $q = 0$, it turns out that

$$\Phi(t) = \frac{1}{2\sqrt{z}} \sin\left[\sqrt{z}\left(\frac{1}{\sqrt{t}} - 1\right)\right], \qquad z > 0, \tag{39}$$

in such a way the integral representation of the nth-order converging factor is obtained by substituting from Eq. (39) into Eq. (29), evaluated at $\beta = 0$. Accordingly, it is not difficult to see that, for $q \neq 0$, the function $\Phi(t)$ in Eq. (39) can still be employed into the integral representation in Eq. (29) simply by choosing $\beta = q/2$. Moreover, similarly as we did in [16], we shall set $n = 0$, representing the customary choice of Weniger's transformation in most applications. Accordingly, the kth-order transformation error we are going to explore will be $\mathcal{E}_k(z,q) = f(z) - \delta_k^{(0)}(1 + q/2)$,

$$\mathcal{E}_k(z,q) = \left(-\frac{1}{z}\right)\Gamma(q+3)\, \frac{\displaystyle\int_0^1 t^{q/2}\,{}_2F_1\left(\begin{matrix} -k, k+q/2 \\ 1+q/2 \end{matrix}; t\right)\Phi(t)\,\mathrm{d}t}{{}_2F_3\left(\begin{matrix} -k, k+\dfrac{q}{2} \\ \dfrac{q}{2}+1, \dfrac{q}{2}+\dfrac{3}{2}, \dfrac{q}{2}+2 \end{matrix}; -\dfrac{z}{4}\right)}. \tag{40}$$

5. Asymptotic Estimate of the Delta Convergence Rate

In the present section, the integral representation in Eq. (40) will be employed to prove that Weniger's transformation is able to sum the Stieltjes series to the corresponding Stieltjes integral in Eq. (35) and also to estimate the convergence rate. To this end, since we were not able to find an explicit expression for the integral in the numerator of Eq. (40), the leading term of the asymptotic expansion of the transformation error $\mathcal{E}_k(z,q)$ in the limit $k \to \infty$ will be found for $z > 0$ and for $q \in (-1,1)$. In doing so, some fundamental approximations of the involved hypergeometric polynomials, that can be found in Ref. [59, Chapter 7.4], will be employed. Due to the extremely technical flavour of the present section, the most technical mathematical steps have been confined into suitable appendices, in order to improve the paper readability. As said above, the subsequent asymptotic analysis will be carried out only for $z > 0$, which is necessary to guarantee the validity of our intermediate steps. However, as it was already put into evidence in [16], manipulations based on the stationary phase approach or related techniques tend to be quite restrictive with respect to the ranges of arguments and parameters for which their validity can be guaranteed. In other words, as it will be illustrated in Sec. 6, the asymptotic estimate of the transformation error we are going to obtain holds in the whole existence domain of the Stieltjes function, thanks to analytical continuation.

Our analysis starts by addressing a first, preliminary problem. Since the δ-transformation is expressed as the ratio of two polynomials, it is a meromorphic function which, with the exception of the zeros of the denominator polynomial, turns out be analytic for all $z \in \mathbb{C}$. We know that the Stieltjes integral $f(z)$ is analytic for all $z \in \mathbb{C}\backslash(-\infty, 0]$. Then, any rational approximant to $f(z)$ must be able to mimic the cut $(-\infty, 0]$. For instance, in the case of Padé approximants $[M + J/M]$, with $J \geq -1$, to a Stieltjes series, it is known that all their poles are simple, that they lie on the negative real semi-axis, and that they have positive residues (see for example [15, Theorem 5.2.1 on p. 201]). In this way, Padé approximants do simulate the cut of Stieltjes functions.

The study of the confinement of Weniger's transformation singularities to the negative real semi-axis has recently been addressed in Refs. [60,61], where it was proved that, for the class of Stieltjes series under investigation, the hypergeometric polynomial in the denominator of Eq. (40) has only real (and negative) zeros. Accordingly, we conclude that the Weniger transformation simulates, for all $k \in \mathbb{N}$ and for all $q \in (-1,1)$, the branch cut $(-\infty, 0]$ of the Stieltjes integral in Eq. (35).

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We are now ready to proceed with the asymptotic estimate of the transformation error in the limit of $k \gg 1$. To this end, denominator and numerator will be treated separately. We begin from the denominator. In Appendix C it is proved that

$$
{}_2F_3\left(\begin{matrix} -k, k+\dfrac{q}{2} \\ \rho_q \end{matrix}; -\frac{z}{4}\right) \sim
\sim \frac{\Gamma\left(\dfrac{q}{2}+1\right)\Gamma\left(\dfrac{q}{2}+\dfrac{3}{2}\right)\Gamma\left(\dfrac{q}{2}+2\right)}{2(2\pi)^{3/2}} \left(\frac{z}{4}\right)^{-\frac{3}{8}(q+2)} k^{-\frac{3}{4}(q+2)} \exp\left\{2\sqrt{2}k^{1/2}z^{1/4}\right\}, \quad k \to \infty. \tag{41}
$$

The integral in the numerator of Eq. (41) is considerably more challenging. First of all, in Appendix D it is proved that

$$
{}_2F_1\left(\begin{matrix} -k, k+\dfrac{q}{2} \\ 1+\dfrac{q}{2} \end{matrix}; t\right) \sim
\sim \frac{\Gamma\left(\dfrac{q}{2}+1\right)}{\sqrt{\pi}} k^{-\frac{1+q}{2}} \sin^{-\frac{1+q}{2}}\frac{\theta}{2}\sqrt{\cos\frac{\theta}{2}}\cos\left(k\theta - \frac{1+q}{4}\pi\right), \qquad k \to \infty, \tag{42}
$$

where $t = \sin^2\dfrac{\theta}{2}$. Then, on substituting from Eqs. (39) and (42) into the numerator of Eq. (40), after changing the integration variable $t \in [0,1]$ into the new variable $\theta \in [0,\pi]$, we have

$$
\int_0^1 t^{q/2}\,{}_2F_1\left(\begin{matrix} -k, k+q/2 \\ 1+q/2 \end{matrix}; t\right)\Phi(t)\,\mathrm{d}t \sim \frac{\Gamma\left(\dfrac{q}{2}+1\right)}{2\sqrt{z}\sqrt{\pi}} k^{-\frac{1+q}{2}}
\times \int_0^\pi \sin\left[\sqrt{z}\left(\frac{1}{\sin\dfrac{\theta}{2}} - 1\right)\right]\cos\left(k\theta - \frac{1+q}{4}\pi\right)\sin^{\frac{1+q}{2}}\frac{\theta}{2}\cos^{3/2}\frac{\theta}{2}\,\mathrm{d}\theta, \quad k \to \infty. \tag{43}
$$

An asymptotic estimate of the θ-integral in Eq. (43) can be found on using standard methods based on stationary phase. The mathematical steps are confined in Appendix E, where it is proved that

$$
\int_0^\pi \sin\left[\sqrt{z}\left(\frac{1}{\sin\dfrac{\theta}{2}} - 1\right)\right]\cos\left(k\theta - \frac{1+q}{4}\pi\right)\sin^{\frac{1+q}{2}}\frac{\theta}{2}\cos^{3/2}\frac{\theta}{2}\,\mathrm{d}\theta
\sim \frac{\sqrt{\pi}\,z^{\frac{2+q}{8}}}{(2k)^{1+\frac{q}{4}}}\sin\left(2\sqrt{2}z^{1/4}k^{1/2} - \sqrt{z} - \frac{q\pi}{4}\right), \qquad k \to \infty, \tag{44}
$$

which, once inserted into Eq. (43) gives at once

$$
\int_0^1 t^{q/2}\,{}_2F_1\left(\begin{matrix} -k, k+q/2 \\ 1+q/2 \end{matrix}; t\right)\Phi(t)\,\mathrm{d}t \sim
\frac{\Gamma\left(\dfrac{q}{2}+1\right)}{2^{2+\frac{q}{4}}} k^{-\frac{3}{4}(2+q)} z^{\frac{1}{8}(q-2)} \sin\left(2\sqrt{2}z^{1/4}k^{1/2} - \sqrt{z} - \frac{q\pi}{4}\right), \quad k \to \infty. \tag{45}
$$

Finally, on substituting from Eqs. (41) and (45) into Eq. (40), the following asymptotics of the transformation error is eventually obtained:

$$\mathcal{E}_k(z,q) \sim -2\pi z^{\frac{q-1}{2}} \exp\left(-2\sqrt{2}\, z^{1/4} k^{1/2}\right) \sin\left(2\sqrt{2}\, z^{1/4} k^{1/2} - \sqrt{z} - \frac{q\pi}{4}\right), \quad k \to \infty. \tag{46}$$

Equation (46) constitutes the main result of the present paper. It predicts an exponential convergence, similarly as it was found in Ref. [16] for the Euler series, with the transformation error being proportional to $\exp\left(-\frac{9}{2} z^{1/3} k^{2/3}\right)$. In the present case, the exponential factor $\exp(-2\sqrt{2}\, z^{1/2} k^{1/2})$ clearly reflects the increased difficulty in resumming the series, due to its superfactorial growth. It should be recalled that Eq. (46) has been derived under the assumption $z > 0$. However, as it was said before, we expect our asymptotic estimate to still work also for complex $z \in \mathbb{C} \backslash (-\infty, 0]$, thanks to analytic continuation. In the next section, this conjecture will be confirmed by carrying out several numerical experiments.

6. Numerical Results

Figure 1 shows a visual comparison between the experimental data obtained by using the δ-transformation and the asymptotic estimate provided by Eq. (46). We refer to the resummation of the Stieltjes series corresponding to the pair $(z, q) = (1, 0)$. In particular, the open circles represent the modulus of transformation error values $\mathcal{E}_k(1, 0)$ obtained through the exact value of $f(z)$ provided by Eq. (35), explicitly given in Appendix B, and the δ sequence values $\delta_k^{(0)}(1)$, with $k \in \{2, 3, 4, \ldots, K\}$, with $K = 100$. The solid curve is the corresponding asymptotic estimate of the transformation error given in Eq. (46).

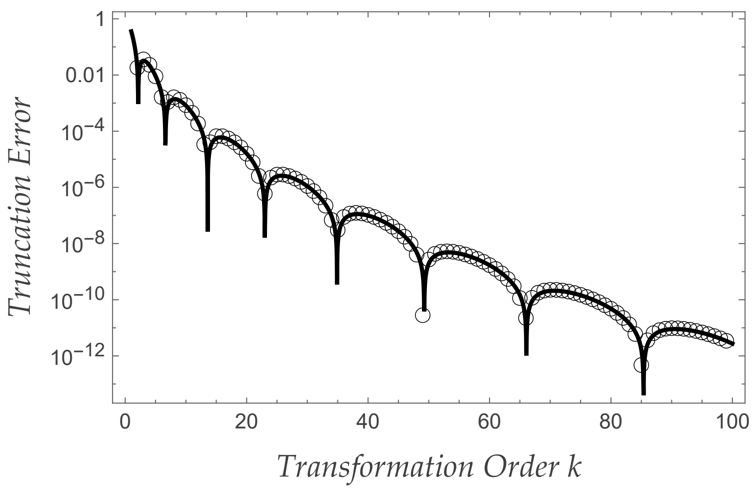

Figure 1. Comparison of the observed transformation errors (open circles) vs the transformation order k of the sequence $\delta_k^{(0)}(1 + q/2)$ for the resummation of the Stieltjes function corresponding to $(z, q) = (1, 0)$. The solid curve represents the corresponding asymptotic estimate given by Eq. (46).

The Stieltjes series (2) is asymptotic to the Stieltjes integral (3) as $z \to \infty$. Therefore, an argument $z = 1$ represents a really challenging summation problem. The result shown in Figure 1 proves that our asymptotic estimate (46) works quite well also for "nonlarge" values of the transformation order k, i.e., that are far away from the asymptotic regime $k \to \infty$. Several other numerical experiments not shown here, carried out on using several different positive values of the argument z, have confirmed our conclusion about the broader validity of the asymptotic estimate.

We have said that, although the asymptotic estimate (46) has been obtained under the hypothesis $z > 0$, analytical continuation should guarantee its validity to be extendable to the whole complex plane, except at the branch cut $z < 0$. To check (numerically) this conjecture, in Figure 2 the same visual comparison carried out in Figure 1 is shown, but now for the complex arguments $z = \exp(i\phi)$, where $\phi = \pi/4, \pi/2, 3\pi/4, 9\pi/10$.

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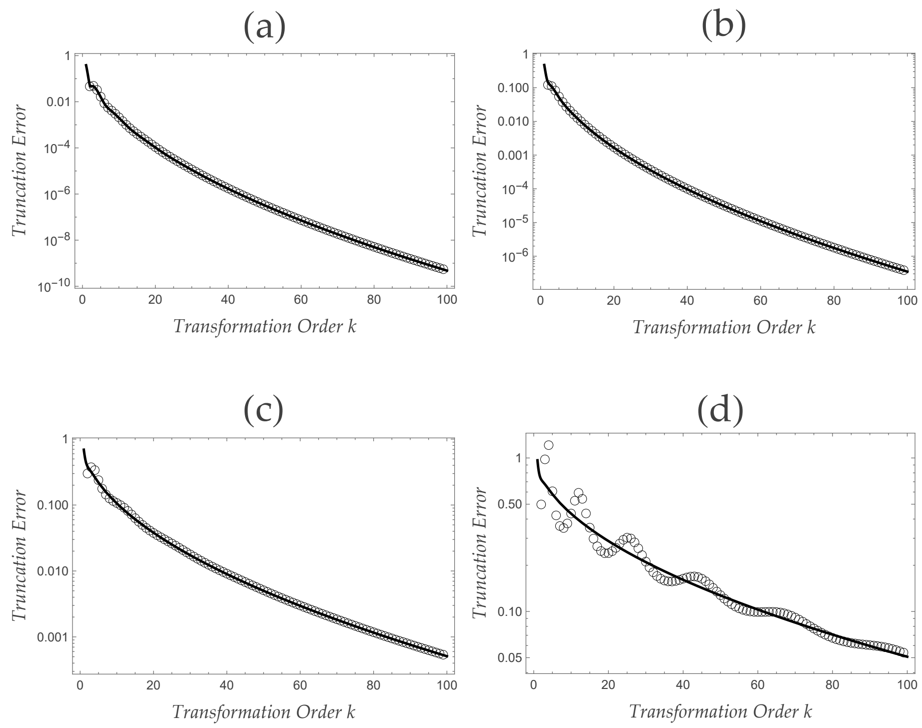

Figure 2. The same as in Figure 1, but for complex $z = \exp(i\phi)$, with $\phi = \pi/4$ (a), $\phi = \pi/2$, (b), $\phi = 3\pi/4$ (c), and $\phi = 9\pi/10$ (d).

The plots in Figure 2 clearly show that our asymptotic estimate (46) continues to work well also for complex arguments the Stieltjes series, provided that they are not too close to the cut of the Stieltjes integral, as it happens in Figure 2(d), for $z = \frac{9\pi}{10}$.

As far as values of q different from zero are concerned, it is worth considering one of the numerical experiments carried out in [11] and already analyzed in Ref. [17] as far as the sole converging factor was concerned. On using, for the sake of simplicity, the same notations employed in Refs. [11] and [17], consider the following Stieltjes integral:

$$\mathcal{J}_3 = \int_0^\infty \frac{t^{-1/2}\exp(-t)}{1 + \dfrac{64}{45\pi^2}t^2}\,\mathrm{d}t = z\int_0^\infty \frac{\mathrm{d}\mu}{z+t}, \tag{47}$$

where $z = \dfrac{45\pi^2}{64}$ and the measure $\mathrm{d}\mu$ is given by Eq. (35) with $q = -1/2$. In [11], it was shown numerically how Weniger's δ transformation (as well as Levin's d) largely outperformed Padé approximants in the resummation of the Stieltjes series which is asymptotic to $\mathcal{J}_3$. To help readers, in Appendix F, Wynn's ϵ algorithm is briefly recalled, together with its tight relationship with Padé approximants.

In Figure 3, the behaviours of the Padé approximant sequences $\{[k/k] =\}_{k=2}^{50}$ (open circles) and $\{[k+1/k] =\}_{k=2}^{50}$ (black squares), obtained through Wynn's ϵ-algorithm as $\{\epsilon_{2k}^{(0)}\}_{k=2}^{50}$ and $\{\epsilon_{2k}^{(1)}\}_{k=2}^{50}$, respectively, are plotted against the transformation order k: the solid line represents the exact value $\mathcal{J}_3 = 1.66017720666804673353089\ldots$ From the figure it is possible to appreciate how the the staircase sequence $[2/2], [3/2], [3/3], \ldots, [50/50], [51/50]$, provides lower and upper bounds of $\mathcal{J}_3$. More quantitative considerations can be done starting from Eqs. (5.6) - (5.10) of Ref. [11]. It follows that, in order to retrieve the Stieltjes integral $\mathcal{J}_3$ with a relative error of the order of 10^{-5}, Wynn's algorithm needs to be fed by the sequence of the first 100 partial sums of the corresponding asymptotic series. The same sequence, when it is δ-transformed, provided a relative error of the order of 10^{-20}.

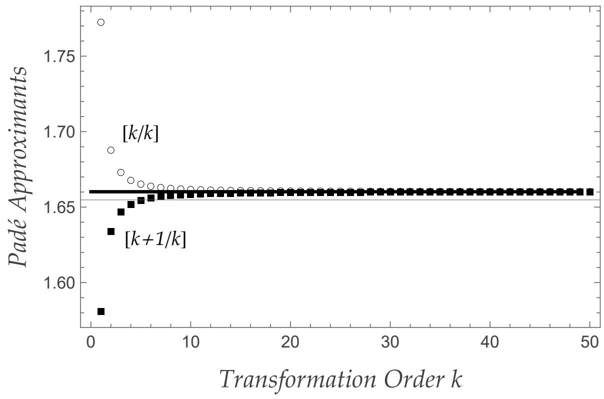

Figure 3. Behaviours of Padé approximant sequences $\{[k/k]\}_{k=2}^{50}$ (open circles) and $\{[k+1/k]\}_{k=2}^{50}$ (black squares) evaluated through Wynn's ϵ-algorithm.

It had already been demonstrated in [16] that Padé approximants were "exponentially inferior" to the δ transformation in the summation of the Euler series, with a Padé approximant transformation error asymptotically ($k \gg 1$) found to be proportional to $\exp(-4z^{1/2}k^{1/2})$. Such analytical result was found on using the fact that Euler series' diagonal Padé approximants can be computed via Drummond's transformation [16], as first proved by A. Sidi [62]. Unfortunately, for the Stieltjes series under investigation, no similar results have been found. For this reason, the resummation performances of the δ transformation with those of Padé approximants will now be compared numerically.

In Figure 4, the relative errors provided by the δ transformation (open circles) and by diagonal Padé approximants (open squares) are shown against the transformation order k, for $z = 45\pi^2/64$. Differently from Figure 1, the sinusoidal factor in the analytical estimate of Eq. (46) has been omitted, in order to emphasize the asymptotic dominance of the exponential factor $\exp\left(-2\sqrt{2}\,z^{1/4}\,k^{1/2}\right)$, which is independent of the value of q. It is worth spending some words about the Padé approximant error behaviour shown in Figure 4 which, as clearly appears, displays the incapacity of Padé approximants to resum our Stieltjes series with high accuracies. Such difficulty has already been put into evidence in Ref. [11], and has been confirmed by several other numerical simulations (not shown here for simplicity), carried out for different (even complex) values of z. In particular, it seems that, differently from what happened for the Euler series, the convergence of Padé approximants is no longer of exponential type: we were not able to fit the behaviour of the relative error in Figure 4 by a function like $A\exp(-Bk^m)$, with (A, B, m) being a triplet of positive reals. While a theoretical investigation about the above Padé inability is beyond the scope of the present paper, some hints for launching future studies on the subject could be done. To this end, it is worth recalling that the divergence of Carleman's series in Eq. (5) can be easily proved by using Stirling's formula, i.e.,

$$\mu_m = \Gamma(2m+q+1) \sim (2m)! \sim \sqrt{4\pi\,m}\left(\frac{2m}{\mathrm{e}}\right)^{2m}, \quad m \to \infty, \tag{48}$$

so that

$$\mu_m^{-\frac{1}{2m}} \sim (4\pi\,m)^{-\frac{1}{4m}}\left(\frac{2m}{e}\right)^{-1} \sim \frac{e}{2m}, \qquad m \to \infty. \tag{49}$$

This, in turn, proves the validity of Carleman's condition, since the series $\sum_m \mu_m^{-\frac{1}{2m}}$ *does diverge*, although logarithmically, i.e., like the harmonic series. In the case of Euler' series ($\mu_m = m!$), the Carleman series diverges as $\sum_m m^{-1/2}$, a strongest divergence condition with respect to the harmonic one. More generally speaking, if $\mu_m = ((2+\epsilon)m)!$, it is not difficult to prove that the Carleman series asymptotically behaves like $\sum_m \frac{1}{m^{1+\epsilon/2}}$, i.e., it converges for $\epsilon > 0$ and diverges for $\epsilon \le 0$. In other words, our Stieltjes series is placed at the edge between divergence and convergence of the associated Carleman series.

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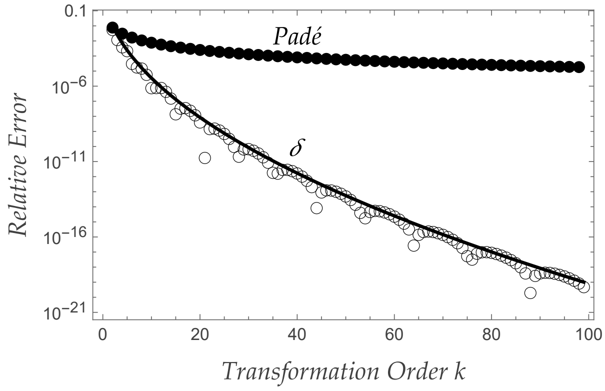

Figure 4. Behaviour of the relative error, against the transformation order k, provided by the sequence $\{\delta_k^{(0)}(1/2)\}_{k=2}^{100}$ (open circles) and by the diagonal sequence of Padé approximants $\{[k/k]\}_{k=2}^{100}$ (open squares), for the resummation of the Stieltjes function corresponding to $(z,q) = \left(\frac{45}{64}\pi^2, -\frac{1}{2}\right)$. The solid curve represents the asymptotic estimate $\exp(-2\sqrt{2}\,z^{1/4}\,k^{1/2})$.

Another example of Stieltjes series whose Carleman series displays a logarithmical divergence is characterized by the moment sequence $\mu_m = (m!)^2$ [20, Corollary 12.11h]. Accordingly, it could be worth comparing the action of δ and Padé in this case, similarly as it was done in Figure 4. To this end, it is not difficult to prove that the associated measure $\mathrm{d}\mu$ turns out to be

$$\mathrm{d}\mu \;=\; 2K_0(2\sqrt{t})\,\mathrm{d}t, \qquad\qquad t \in [0,\infty), \tag{50}$$

where $K_0(\cdot)$ denotes the modified Bessel function of the second kind, and the corresponding Stieltjes integral can expressed (thanks to Wolfram Mathematica 14.3) in terms of G-Meijer functions as follows:

$$\int_0^\infty \frac{\mathrm{d}\mu}{z+t} \;=\; G_{1,3}^{3,1}\left(z \,\middle|\, \begin{matrix} 0 \\ 0,0,0 \end{matrix}\right). \tag{51}$$

Figure 5 shows the same numerical experiment carried out in Fig. 4, but for the Stieltjes series associated to the measure in Eq. (51). From this figure a certain similarity (at least at a visual level) of the behaviours of the relative errror can then be appreciated.

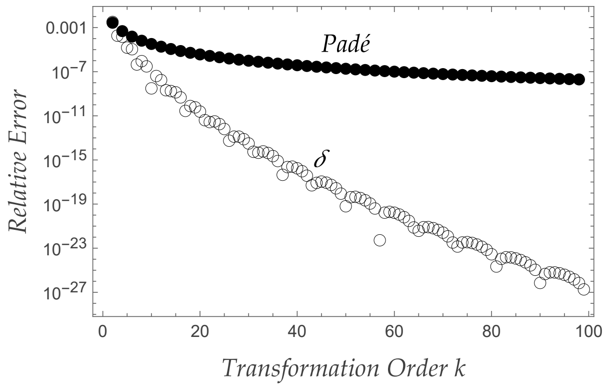

Figure 5. Behaviour of the relative error, against the transformation order k, provided by the sequence $\{\delta_k^{(0)}(1)\}_{k=2}^{100}$ (open circles) and by the diagonal sequence of Padé approximants $\{[k/k]\}_{k=2}^{100}$ (open squares), for the resummation of the Stieltjes function in Eq. (51) when $z = \frac{45}{64}\pi^2$.

7. Conclusions

In this paper, we have constructed an analytical convergence theory for the δ-transformation applied to a class of superfactorially divergent Stieltjes series. An integral representation of the truncation error has been derived, which served as the mathematical basis for an asymptotic proof of convergence as well as of an estimate of the corresponding convergence rate. Our theoretical development addresses a significant gap still present in the literature, where the mathematical understanding of nonlinear Levin-type transformations has remained largely underdeveloped when compared to classical Padé theory. Our results also demonstrate a discrepancy between theoretical resummability and practical computational efficiency. While the series under investigation satisfy the Carleman condition, the logarithmic divergence of the associated Carleman series seems render Padé approximants effectively useless for numerical purposes due to their extremely slow convergence, differently from what happens as far as the δ-transformation is concerned.

It should also be emphasized that the transition from factorial to superfactorial divergence introduces substantial analytical hurdles, which made the technical challenges tackled in the present paper considerably harder than those addressed in Ref. [16]. The present work represents only the second rigorous contribution to a remarkably sparse list of analytical convergence proofs for the δ-transformation. It is hoped that the present findings will stimulate further efforts to expand this theoretical catalog, eventually leading to a comprehensive understanding of the full potential and mathematical reliability of Levin-type nonlinear sequence transformations in the context of those wildly divergent expansions frequently found in quantum mechanics and perturbation theory.

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Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Proof of Eq. (31)

We start on substituting from Eq. (29) into the numerator of Eq. (25), which gives

$$\Delta^k\{(n+\gamma)_{k-1}\varphi_{n+1}\} = \Delta^k\left\{(n+\gamma)_{k-1}\int_0^1 t^{n+\gamma-1}\,\Phi(t)\,\mathrm{d}t\right\} = \\ = \int_0^1 \mathrm{d}t\, t^{\gamma-1}\,\Phi(t)\,\Delta^k\{(n+\gamma)_{k-1}t^n\}, \tag{A1}$$

Now, we have [16]

$$\Delta^k\{(n+\gamma)_{k-1}t^n\} = (-1)^k\,(n+\gamma)_{k-1}\,t^n\,{}_2F_1\left(\begin{matrix}-k, k+n+\gamma-1\\ n+\gamma\end{matrix};t\right), \tag{A2}$$

where ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function. Finally, on substituting from Eq. (A2) into Eq. (A1), after simple algebra Eq. (31) follows □

Appendix B. Analytical expression of the Stieltjes integral in Eq. (35)

Wolfram Mathematica 14.3. gives for the above integral the following expression:

$$f(z) = \begin{cases} \dfrac{2\,\mathrm{Ci}(\sqrt{z})\sin(\sqrt{z}) + (\pi - 2\mathrm{Si}(\sqrt{z}))\cos(\sqrt{z})}{2\sqrt{z}}, & q=0, \\ \Gamma(q-1)\,{}_1F_2\left(1;1-\dfrac{q}{2},\dfrac{3}{2}-\dfrac{q}{2};-\dfrac{z}{4}\right) + \pi z^{\frac{q-1}{2}}\dfrac{\sin(\frac{\pi q}{2}+\sqrt{z})}{\sin \pi q}, & q \in (-1,1)\backslash\{0\}. \end{cases} \tag{A3}$$

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Appendix C. Proof of Eq. (41)

Our proof starts from the following expansion [59, Eq. 7.4.5(6) on p. 263]:

$$
\begin{aligned}
{}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_q\end{matrix}; -z\right) &\sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}}\, \mathcal{L}^{(\alpha_t)}_{p+2,q}(z e^{i\delta\pi}) \\
&+ \frac{(2\pi)^{(1-\beta)/2}\Gamma(\rho_q)}{\beta^{1/2}\Gamma(\alpha_p)}\,(N^{\beta} z)^{\nu}\, \exp\left(N z^{1/\beta}\beta - (az/3) - \Omega(-z)/(N z^{1/\beta}) + O(N^{-2})\right), \\
&|\arg(z)| \le \pi - \epsilon\,, \quad \epsilon > 0\,, \quad \delta = +(-) \quad \text{if} \quad \arg(z) \le (>)\, 0\,.
\end{aligned}
\tag{A4}
$$

In order to apply Eq. (A4) to the present case, we have to extract the right values of the various parameters. In this way we obtain $n = k$, $p = 0$, $q = 3$, $\lambda = q/2$, and $\{\rho_q\} = \left\{\frac{q}{2}+1, \frac{q}{2}+\frac{3}{2}, \frac{q}{2}+2\right\}$. Accordingly, the finite sum is not present (since $p = 0$). The other parameters to be inserted into Eq. (A4) can be derived directly from the prescriptions given in [59, Chapter 7.4]. More precisely, we have $\beta = 4$ and $a = 0$, while

$$
N^4 = k\left(k + \frac{q}{2}\right) \implies N = \left[k\left(k+\frac{q}{2}\right)\right]^{1/4}.
\tag{A5}
$$

Moreover,

$$
\begin{cases}
\Gamma(\rho_q) \,=\, \Gamma\left(\frac{q}{2}+1\right)\Gamma\left(\frac{q}{2}+\frac{3}{2}\right)\Gamma\left(\frac{q}{2}+2\right), \\
B_1 = 0, \\
C_1 = \sum_{t=1}^{3} \rho_t = \frac{q}{2} + 1 + \frac{q}{2} + \frac{3}{2} + \frac{q}{2} + 2 = \frac{3}{2}(3+q), \\
\gamma = \dfrac{\beta - 1 + 2B_1 - 2C_1}{2\beta} = -\dfrac{3}{8}(2+q),
\end{cases}
\tag{A6}
$$

so that Eq. (A4) becomes

$$
\begin{aligned}
{}_2F_3\left(\begin{matrix}-k, k+\frac{q}{2} \\ \rho_q\end{matrix}; -\frac{z}{4}\right) &\sim \frac{(2\pi)^{(1-4)/2}\Gamma\left(\frac{q}{2}+1\right)\Gamma\left(\frac{q}{2}+\frac{3}{2}\right)\Gamma\left(\frac{q}{2}+2\right)}{4^{1/2}} \\
&\times \left[k\left(k+\frac{q}{2}\right)\frac{z}{4}\right]^{-\frac{3}{8}(q+2)} \exp\left\{4\left[k\left(k+\frac{q}{2}\right)\frac{z}{4}\right]^{1/4} + \mathcal{O}(N^{-1})\right\},
\end{aligned}
\tag{A7}
$$

If we now discard, in the exponential factor in Eq. (A7), all contributions that vanish at least like $\mathcal{O}(1/N)$ as $N \to \infty$, after simple algebra the following leading order asymptotic approximation for the denominator in Eq. (41) is obtained:

$$
\begin{aligned}
&{}_2F_3\left(\begin{matrix}-k, k+\frac{q}{2} \\ \rho_q\end{matrix}; -\frac{z}{4}\right) \sim \\
&\sim \frac{\Gamma\left(\frac{q}{2}+1\right)\Gamma\left(\frac{q}{2}+\frac{3}{2}\right)\Gamma\left(\frac{q}{2}+2\right)}{2(2\pi)^{3/2}} \left[k\left(k+\frac{q}{2}\right)\frac{z}{4}\right]^{-\frac{3}{8}(q+2)} \exp\left\{4\left[k\left(k+\frac{q}{2}\right)\frac{z}{4}\right]^{1/4}\right\},
\end{aligned}
\tag{A8}
$$

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or, since $k \gg |q|$,

$$
{}_2F_3\left(\begin{matrix}-k, k+\dfrac{q}{2}\\ \rho_q\end{matrix}; -\frac{z}{4}\right) \sim
$$

$$
\sim \frac{\Gamma\left(\frac{q}{2}+1\right)\Gamma\left(\frac{q}{2}+\frac{3}{2}\right)\Gamma\left(\frac{q}{2}+2\right)}{2(2\pi)^{3/2}} \left(\frac{z}{4}\right)^{-\frac{3}{8}(q+2)} k^{-\frac{3}{4}(q+2)} \exp\left\{2\sqrt{2}k^{1/2}z^{1/4}\right\}, \qquad k \to \tag{A9}
$$

which coincides with Eq. (41) $\square$

Appendix D. Proof of Eq. (42)

First of all, we have to find the asymptotics of the hypergeometric function ${}_2F_1$. To this end, it is worth invoking again the beautiful textbook of Luke, precisely [59, Eq. 7.4.2(8) on p.250]:

$$
\begin{aligned}
{}_{p+2}F_{p+1}\left(\begin{matrix}-n, n+\lambda, \alpha_p\\ \rho_{p+1}\end{matrix}; z\right) &\sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathcal{L}^{(\alpha_t)}_{p+2,p+1}(z) + \frac{\Gamma(\rho_{p+1})\, N^{2\gamma}}{\Gamma(\alpha_p)\,\Gamma(1/2)} \frac{[\sin(\theta/2)]^{2\gamma}}{[\cos(\theta/2)]^{2\gamma+\lambda}}\\
&\times \exp\left\{\frac{\varphi_2(\theta)+a_2}{N^2} + \mathrm{O}\left(\frac{1}{N^4}\right)\right\} \cos\left\{N\theta + \pi\gamma + \frac{\varphi_1(\theta)}{N} + \frac{\varphi_3(\theta)}{N^3} + \mathrm{O}\left(\frac{1}{N^5}\right)\right\},\\
&|\arg(z)| \le \pi - \epsilon, \quad |\arg(1-z)| \le \pi - \epsilon, \quad \epsilon > 0.
\end{aligned} \tag{A10}
$$

Again, we shall set $n = k$ and $\lambda = q/2$, while $N = \sqrt{k(k+q/2)}$ and $z = t \in [0,1]$. In addition, we have $p = 0$, so that, as in Eq. (A4), the finite sum disappears. Moreover, on neglecting all contributions that vanish at least like $\mathrm{O}(1/N)$ as $N \to \infty$, we have that the exponential factor can be approximated by one, while the cosinusoidal factor reduces to $\cos(N\theta + \pi\gamma)$, where θ is related to t by $t = \sin^2(\theta/2)$, so that $\theta \in [0, \pi]$. The remaining unspecified quantities are defined in [59, Eq. 7.4.2(9) on pages 251 and 252.]. More precisely, we have

$$
\begin{cases}
B_1 = 0,\\
C_1 = 1 + \dfrac{q}{2},\\
\gamma = \dfrac{1+2B_1-2C_1}{4} = -\dfrac{1+q}{4},
\end{cases} \tag{A11}
$$

so that

$$
{}_2F_1\left(\begin{matrix}-k, k+\dfrac{q}{2}\\ \rho_1\end{matrix}; t\right) \sim \frac{\Gamma(\rho_1)\, N^{2\gamma}}{\Gamma(1/2)} \frac{[\sin(\theta/2)]^{2\gamma}}{[\cos(\theta/2)]^{2\gamma+\lambda}} \cos\{N\theta + \pi\gamma\}, \qquad k \to \infty,
$$

i.e.,

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}-k, k+\frac{q}{2}\\ \rho_3\end{matrix}; t\right) \sim\\
&\sim \frac{\Gamma\left(\frac{q}{2}+1\right)}{\sqrt{\pi}} [k(k+q/2)]]^{-\frac{1+q}{4}} \times \frac{[\sin(\theta/2)]^{2\gamma}}{[\cos(\theta/2)]^{2\gamma+\lambda}} \cos\{N\theta + \pi\gamma\}, \qquad k \to \infty,
\end{aligned} \tag{A12}
$$

or, on taking into account that $2\gamma + \lambda = -\dfrac{1+q}{2} + \dfrac{q}{2} = -\dfrac{1}{2}$,

$$
{}_2F_1\left(\begin{matrix}-k, k+\frac{q}{2}\\ 1+\frac{q}{2}\end{matrix}; t\right) \sim \frac{\Gamma\left(\frac{q}{2}+1\right)}{\sqrt{\pi}} k^{-\frac{1+q}{2}} \sin^{-\frac{1+q}{2}} \frac{\theta}{2} \sqrt{\cos\frac{\theta}{2}} \cos\left\{k\theta - \frac{1+q}{4}\pi\right\}, \qquad k \to \infty, \tag{A13}
$$

which coincides with Eq. (42) □

Appendix E. Proof of Eq. (44)

We start on recasting the integrand of Eq. (44) as the product $\Psi(\theta)\,\Lambda(\theta)$, where

$$\Psi(\theta) \,=\, \sin\left[\sqrt{z}\left(\frac{1}{\sin\dfrac{\theta}{2}}-1\right)\right]\,\cos\left(k\theta-\eta\frac{\pi}{2}\right), \tag{A14}$$

and

$$A(\theta) \,=\, \sin^{\eta}\frac{\theta}{2}\,\cos^{3/2}\frac{\theta}{2}, \tag{A15}$$

where for the sake of convenience the parameter $\eta = \dfrac{1+q}{2} \in [0,1]$, has been introduced. On applying Werner formulas, Ψ can be rewritten as follows:

$$\Psi(\theta) \,=\, \frac{1}{2}\,\Psi_+(\theta) - \frac{1}{2}\,\Psi_-(\theta), \tag{A16}$$

where

$$\Psi_\pm(\theta) \,=\, \sin\left[\phi_\pm(\theta) - \eta\frac{\pi}{2}\right], \tag{A17}$$

with

$$\phi_\pm(\theta) \,=\, k\theta \pm \sqrt{z}\left(\frac{1}{\sin\dfrac{\theta}{2}}-1\right). \tag{A18}$$

Accordingly, the starting integral, say $\mathcal{I}$, can be written as $\mathcal{I} = \mathcal{I}_+ - \mathcal{I}_-$, where

$$\mathcal{I}_\pm \,=\, \frac{1}{2}\int_0^{\pi} A(\theta)\,\Psi_\pm(\theta)\,\mathrm{d}\theta\,. \tag{A19}$$

The idea consists in estimating (if possible) the two integrals on using standard stationary phase techniques. In particular, the factor $\Psi_\pm(\theta)$ represents the highly oscillating function, due to the presence, for $k\to\infty$, of $k\theta$ as well as to the presence of the term $1/\sin\frac{\theta}{2}$ (in the limit of $\theta\to 0$). The function $A(\theta)$ contains the slowly varying factor. Accordingly, to asymptotically estimate $\mathcal{I}_\pm$, the real solutions of the following equations have to be found:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\,\phi_\pm(\theta) \,=\, 0, \tag{A20}$$

i.e., on taking Eq. (A18) into account,

$$k \,=\, \pm\frac{\sqrt{z}}{2}\,\frac{\cos\dfrac{\theta}{2}}{\sin^2\dfrac{\theta}{2}}\,. \tag{A21}$$

It should be noted that, for $z > 0$ and since $k > 0$, the minus sign has to be neglected. In other words, there are no stationary phase points in the integral $\mathcal{I}_-$ and, accordingly, its contribution will be neglected. Concerning $\mathcal{I}_+$, from Eq. (A21) it is easy to show that, in the limit of large values of k, there is only one stationary point, say θ_s, contributing to the integral, and precisely

$$\theta_s \sim \sqrt{2}\,z^{1/4}\,k^{-1/2}, \qquad\qquad k\to\infty\,. \tag{A22}$$

Accordingly, standard stationary phase theory leads to

$$\mathcal{I}_+ \sim \frac{A(\theta_s)}{2} \sqrt{\frac{2\pi}{|\phi''(\theta_s)|}} \sin\left[\phi(\theta_s) + \frac{\pi}{4}\,\mathrm{sign}\{\phi''(\theta_s)\} - \eta\frac{\pi}{2}\right], \tag{A23}$$

where, from Eq. (A18), we have

$$A(\theta_s) \sim \left(\frac{\theta_s}{2}\right)^\eta \sim 2^{-\eta/2} z^{\eta/4} k^{-\eta/2}, \qquad k \to \infty, \tag{A24}$$

while, from Eq. (A17), we also have

$$\phi''_+(\theta) = \sqrt{z}\,\frac{3 + \cos\theta}{8\,\sin^3\dfrac{\theta}{2}}, \tag{A25}$$

that, when evaluated at θ_s, gives

$$\phi''_+(\theta_s) \sim \frac{4\sqrt{z}}{\theta_s^3} \sim 2^{1/2}\, z^{-1/4}\, k^{3/2}, \qquad k \to \infty. \tag{A26}$$

Moreover,

$$\phi(\theta_s) + \frac{\pi}{4}\,\mathrm{sign}\{\phi''(\theta_s)\} \sim 2\sqrt{2}\, z^{1/4}\, k^{1/2} - \sqrt{z} + \frac{\pi}{4}. \qquad k \to \infty, \tag{A27}$$

Finally, on substituting from Eqs. (A24) - (A27) into Eq. (A23), simple algebra eventually leads to Eq. (44) □

Appendix F. Wynn's ϵ Algorithm

The modern era of sequence transformations started with two seminal articles by Shanks [63] and Wynn [64], respectively. Shanks introduced a powerful sequence transformation that computes Padé approximants, while Wynn showed that Padé approximants can be computed effectively by means the so-called epsilon algorithm, corresponding to the following nonlinear recursive scheme [64, Eq. (4)]:

$$\varepsilon_{-1}^{(n)} = 0, \qquad \varepsilon_0^{(n)} = s_n, \qquad n \in \mathbb{N}_0, \tag{A28a}$$

$$\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + \frac{1}{\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}}, \qquad k, n \in \mathbb{N}_0. \tag{A28b}$$

The elements $\varepsilon_{2k}^{(n)}$ with *even* subscripts provide approximations to the (generalized) limit s of the input sequence $\{s_n\}_{n=0}^{\infty}$, whereas the elements $\varepsilon_{2k+1}^{(n)}$ with *odd* subscripts are only auxiliary quantities which diverge if the whole process converges.

If the elements of the input sequence $\{s_n\}_{n=0}^{\infty}$ were the partial sums of a (formal) power series $f(z)$, then the ϵ algorithm produces Padé approximants to $f(z)$ according to

$$\varepsilon_{2k}^{(n)} = [k + n/k]_f(z), \qquad k, n \in \mathbb{N}_0. \tag{A29}$$

The epsilon algorithm is not restricted to input data that are the partial sums of a (formal) power series. Therefore, it is more general and more widely applicable than Padé approximants. As a recent review we recommend [65].

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