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Symmetric Reverse *n*-Derivations on Ideals of Semiprime Rings

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Abstract: This paper focuses on examining a new type of n-additive maps called the symmetric reverse n-derivations. As implied by its name, it combines the ideas of n-additive maps and reverse derivations, with a 1-reverse derivation being the ordinary reverse derivation. We explore several findings that expand our knowledge of these maps, particularly their presence in semiprime rings and the way rings respond to specific functional identities involving elements of ideals. Also, we provide examples to help clarify the concept of symmetric reverse n-derivations. This study aims to deepen the understanding of these symmetric maps and their properties within mathematical structures.

Keywords: semiprime ring; ideal; symmetric n-derivation; symmetric reverse n-derivation; derivation; trace of symmetric map

MSC: 16W25, 16R50, 16N60

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1. Introduction

In the present paper, we denote T by an associative ring and $\mathcal{Z}(T)$ as its center. This paper focusses on the study of prime and semiprime rings only. By a prime ring we mean whenever $\vartheta T\ell = \{0\}$ it implies either $\vartheta = 0$ or $\ell = 0$, and similarly in case of a semiprime if $\vartheta T\vartheta = \{0\}$ then $\vartheta = 0$ with $\vartheta, \ell \in T$. For any $\vartheta, \ell \in T$, the symbols $[\vartheta, \ell]$ and $\vartheta \; o \; \ell$ represent the commutator $\vartheta\ell - \ell\vartheta$ and the anti-commutator $\vartheta\ell + \ell\vartheta$, respectively. By an n-torsion free ring we mean whenever $n\vartheta = 0$ for some $\vartheta \in T$ then the only choice left for ϑ is 0. An additive mapping $\alpha : T \to T$ is called a derivation if $\alpha(\vartheta\ell) = \alpha(\vartheta)\ell + \vartheta\alpha(\ell)$ holds for all $\vartheta, \ell \in T$. The idea derivation has been translated in many directions and one among those is it define a derivation on Cartesian product of rings. In this direction, Maksa [12] defined what is called a bi-derivation. Yes, the concept is expanded from bi to tri and then to n-derivations. The definition comes very naturally. Let's give the formal definition of these maps. As derivation is additive, while translating this idea to Cartesian product the additivity part refers to the additivity in both the components, called a bi-additive map. A bi-additive map $\mathfrak{D} : T \times T \to T$ is known as a bi-derivation if it is a derivation in both of its components, i.e.,

$$\mathfrak{D}(\vartheta\vartheta', \ell) = \mathfrak{D}(\vartheta, \ell)\vartheta' + \vartheta\mathfrak{D}(\vartheta', \ell),$$

and

$$\mathfrak{D}(\vartheta, \ell\ell') = \mathfrak{D}(\vartheta, \ell)\ell' + \ell\mathfrak{D}(\vartheta, \ell'),$$

for all $\vartheta, \vartheta', \ell, \ell' \in T$. These two conditions can be clubbed in one if the map $\mathfrak{D}$ is also symmetric, i.e., $\mathfrak{D}(\vartheta, \ell) = \mathfrak{D}(\ell, \vartheta)$ for all $\vartheta, \ell \in T$. The idea of bi-derivation was studied extensively by Vukman in [20,21]. Thus, a bi-additive, symmetric $\mathfrak{D}$ which is a derivation in any of its two components is termed as a symmetric bi-derivation. Several authors have studied symmetric bi-derivations on rings (see [3,11,19] and references therein) and produced highly useful outcomes. On similar lines Öztürk [13]

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initiated the study of tri-derivations. Taking this forward Park [14] introduced the notion of permuting n-derivations. The definition goes like:

Definition 1. *Let $n \geq 2$ be a fixed integer, a map $\mathfrak{D} : T^n = \underbrace{T \times T \times \cdots \times T}_{n-times} \to T$ is said to form a symmetric (permuting) n-derivation, if $\mathfrak{D}$ is symmetric, n-additive and in addition to this it is an n-derivation, that is*

$$\begin{aligned}\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{i-1}, \vartheta\ell, \vartheta_{i+1}, \ldots, \vartheta_n) &= \mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{i-1}, \vartheta, \vartheta_{i+1}, \ldots, \vartheta_n)\ell \\ &+ \vartheta\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{i-1}, \ell, \vartheta_{i+1}, \ldots, \vartheta_n)\end{aligned}$$

holds for all $\vartheta_i, \ell \in T, 1 \leq i \leq n$.

In 1957, Herstein [10] introduced the concept of a reverse derivation, defining it as an additive map α that satisfies $\alpha(\omega\theta) = \alpha(\theta)\omega + \theta\alpha(\omega)$ for all $\omega, \theta \in T$. He demonstrated that reverse derivations generally do not exist in case of prime rings. Later, Brešar and Vukman [8] studied reverse derivations in rings with involution. In this vein, Barros et al. [7] examined the additivity of multiplicative of $*$-reverse derivations over alternative algebras and provided a decomposition of Jordan $*$-reverse derivations as the sum of a $*$-reverse derivation and a singular Jordan $*$-reverse derivation. In 2015, Aboubakr and Gonzalez [1] explored the relationship between generalized reverse derivations and generalized derivations on an ideal in semiprime rings. More recently, Sögütcü [17] investigated multiplicative (generalized) reverse derivations in semiprime rings, established some important results and discussed continuous reverse derivations with applications in Banach algebras.

Inspired by these studies, we introduce a new concept called the reverse n-derivation. As the name implies, a reverse n-derivation is essentially a reverse derivation in each of its components. We give the formal definition of reverse n- derivations as follows:

Definition 2. *Let $n \geq 2$ be a fixed integer, a map $\mathfrak{D} : T^n \to T$ is said to be a symmetric (permuting) reverse n-derivation, if $\mathfrak{D}$ is symmetric, n-additive and satisfies,*

$$\begin{aligned}\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{i-1}, \vartheta\ell, \vartheta_{i+1}, \ldots, \vartheta_n) &= \ell\mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{i-1}, \vartheta, \vartheta_{i+1}, \ldots, \vartheta_n) \\ &+ \mathfrak{D}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{i-1}, \ell, \vartheta_{i+1}, \ldots, \vartheta_n)\vartheta\end{aligned}$$

holds for all $\vartheta_i, \ell \in T, 1 \leq i \leq n$.

A reverse 1-derivation is simply a reverse derivation and a reverse 2-derivation is a reverse bi-derivation. The most essential component of a symmetric n-derivation is that of its trace. The trace of a symmetric n-derivation plays an important role as it helps to bridge the gap between an n-derivation and that of an ordinary derivation. It becomes useful while generalizing the results already proved for derivations or bi-derivations to that of n-derivations. Talking about the trace of a reverse n-derivation, we now formally define it as follows:

Definition 3. *Trace of a symmetric map $\mathfrak{D} : T^n \to T$ is defined on T as $f(\vartheta) = \mathfrak{D}(\vartheta, \vartheta, \ldots, \vartheta)$, for all $\vartheta \in T$.*

For a symmetric reverse n-derivation $\mathfrak{D}$, the trace f satisfies the following relation,

$$f(\vartheta + \ell) = f(\vartheta) + f(\ell) + \sum_{t=1}^{n-1} {}^nC_t\, \mathfrak{D}(\underbrace{\vartheta, \ldots, \vartheta}_{(n-t)-\text{times}}, \underbrace{\ell, \ldots, \ell}_{t-\text{times}})$$

for all $\vartheta, \ell \in T$, where ${}^nC_t = \binom{n}{t}$.
The following examples help us to understand symmetric reverse n-derivations clearly and see the obvious difference between symmetric reverse n-derivations and that of symmetric n-derivations.

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Example 31. *Consider the ring* $T = \left\{ \begin{bmatrix} \vartheta & \ell \\ 0 & \vartheta \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$. *Denote* $T_i = \begin{bmatrix} \vartheta_i & \ell_i \\ 0 & \vartheta_i \end{bmatrix} \in T$, $\vartheta_i, \ell_i \in \mathbb{Z}$, $1 \leq i \leq n$, *and let us define* $\mathfrak{D} : T^n \to T$ *by* $\mathfrak{D}(T_1, T_2, \ldots, T_n) = \begin{bmatrix} 0 & \ell_1\ell_2 \cdots \ell_n \\ 0 & 0 \end{bmatrix}$ *with trace* $f : T \to T$ *defined by* $f\left(\begin{bmatrix} \vartheta & \ell \\ 0 & \vartheta \end{bmatrix}\right) = \begin{bmatrix} 0 & \ell^n \\ 0 & 0 \end{bmatrix}$. *It is easy to verify that the above mentioned* $\mathfrak{D}$ *is a symmetric reverse n-derivations.*

Example 32. *Let* $\mathfrak{T}$ *be any commutative ring, define another ring* T *as* $T = \left\{ \begin{bmatrix} \vartheta & \ell \\ 0 & \theta \end{bmatrix} : \vartheta, \ell, \theta \in \mathfrak{T} \right\}$. $\mathfrak{D} : T^n \to T$ *be a map given by* $\mathfrak{D}(A_1, A_2, \ldots, A_n) = \begin{bmatrix} 0 & \ell_1\ell_2 \ldots \ell_n \\ 0 & 0 \end{bmatrix}$, *where* $A_i = \begin{bmatrix} \vartheta_i & \ell_i \\ 0 & \theta_i \end{bmatrix} \in T$, $\vartheta_i, \ell_i \in \mathfrak{T}$, $1 \leq i \leq n$. $\mathfrak{D}$ *so defined forms a symmetric n-derivation but is not a symmetric reverse n-derivation.*

2. Preliminaries

This section comprises of some existing results that prove to be building blocks for the construction of our main results.

Lemma 1. *[14, Lemma 2.2] Let* n *be a fixed positive integer and* T *a* $n!$*-torsion free ring. Suppose that* $\vartheta_1, \vartheta_2, \ldots, \vartheta_n \in T$ *satisfy* $\lambda\vartheta_1 + \lambda^2\vartheta_2 + \cdots + \lambda^n\vartheta_n = 0$ *(or* $\in \mathcal{Z}(T)$*) for* $\lambda = 1, 2, \ldots, n$. *Then,* $\vartheta_i = 0$ *(or* $\in \mathcal{Z}(T)$*) for* $i = 1, 2, \ldots, n$.

Lemma 2. *[9, Lemma 2(b)] If* T *is a semiprime ring, then the center of a nonzero ideal of* T *is contained in the center of* T.

Lemma 3. *[16, Lemma 2.1] Let* T *be a semiprime ring,* I *a nonzero two sided ideal of* T *and* $a \in T$ *such that* $a\vartheta a = 0$ *for all* $\vartheta \in I$. *Then* $a = 0$.

Lemma 4. *[19] Let* T *be a 2-torsion free semiprime ring and* J *be a nonzero ideal of* T. *If* $[J, J] \subseteq \mathcal{Z}(T)$, *then* T *contains a nonzero central ideal.*

Lemma 5. *[18, Lemma 1.4] Let* T *be a semiprime ring. If a nonzero ideal* I *of* T *is in the center of* T. *Then,* T *is a commutative ring.*

Proposition 1. *Let* $n \geq 2$ *be a fixed integer and* T *be a* $n!$*-torsion free semiprime ring,* J *its non-zero ideal. If there exists a symmetric reverse n-derivation* $\mathfrak{D}$ *on* T *with trace* f *such that* $\left[f(\vartheta), \ell\right] \in \mathcal{Z}(T)$ *for all* $\vartheta, \ell \in J$, *then* f *is commuting on* J.

Proof. Since, $\left[f(\vartheta), \ell\right] \in \mathcal{Z}(T)$ for all $\vartheta, \ell \in J$. Replacing ℓ by $\ell\theta$, where $\theta \in J$, we obtain

$$\left[f(\vartheta), \ell\theta\right] \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell, \theta \in J,$$

which gives

$$\ell\left[f(\vartheta), \theta\right] + \left[f(\vartheta), \ell\right]\theta \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell, \theta \in J.$$

Commuting the above expression with $t \in T$, we get

$$\left[\ell\left[f(\vartheta), \theta\right] + \left[f(\vartheta), \ell\right]\theta, t\right] = 0 \text{ for all } \vartheta, \ell, \theta \in J.$$

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On solving further, we get

$$\Big[\ell,t\Big]\Big[f(\vartheta),\theta\Big]+\ell\Big[\Big[f(\vartheta),\theta\Big],t\Big]+\Big[f(\vartheta),\ell\Big]\Big[\theta,t\Big]+\Big[\Big[f(\vartheta),\ell\Big],t\Big]=0.$$

This implies

$$\Big[\ell,t\Big]\Big[f(\vartheta),\theta\Big]+\Big[f(\vartheta),\ell\Big]\Big[\theta,t\Big]=0\ \vartheta,\ell,\theta\in J,\ t\in T.$$

Replace t by θ in the last equation, we get

$$\Big[\ell,\theta\Big]\Big[f(\vartheta),\theta\Big]=0.$$

Again replace ℓ by $t\ell$, $t\in T$, we have

$$\Big[t,\theta\Big]\ell\Big[f(\vartheta),\theta\Big]=0 \text{ for all } \vartheta,\ell,\theta\in J.$$

On substitution of t by $f(\vartheta)$, we obtain

$$\Big[f(\vartheta),\theta\Big]\ell\Big[f(\vartheta),\theta\Big]=0 \text{ for all } \vartheta,\ell,\theta\in J.$$

Therefore, applying Lemma 3 gives,

$$\Big[f(\vartheta),\theta\Big]=0 \text{ for all } \vartheta,\theta\in J.$$

Hence, the trace f is commuting on J. □

Using similar approach with necessary variations, we can prove the following result.

Proposition 2. *Let $n\geq 2$ be a fixed integer and T be a $n!$-torsion free semiprime ring, J its non-zero ideal. If there exists a symmetric reverse n-derivation $\mathfrak{D}$ on T with trace f such that $f(\vartheta)\ o\ \ell\in\mathcal{Z}(T)$ for all $\vartheta,\ell\in J$, then f is commuting on J.*

3. The Main Results

Park in [14] proved various interesting results for symmetric n- derivations like generalization to the famous Posner's result (cf.; [15]) and several other results. Some of his initial work on n-derivations which paved way for future developments in this area are also provided in the same paper. In 2014, Ashraf and Jamal [5] introduced certain interesting identities that help us reveal the structure of any ring, similar to the work of Daif and Bell on semiprime rings [9]. They demonstrated that a ring T is commutative if there exists a symmetric n-additive map $\mathfrak{D}:T^n\to T$ satisfying certain functional identities. Also, in case of n-derivations Ashraf et al. in [4] developed upon the line of inquiry provided by Park in the case of prime rings. Ashraf et al. [6] used properties of an ideal and the trace of symmetric n-derivations to achieve results for the existence of central ideals which eventually helps to see the commutativity of the rings under consideration. Recently, Ali et al. in [2] presented several findings regarding the containment of a nonzero central ideal in a ring T that adheres to specific functional identities involving the traces d and g of symmetric n-derivations D and G, respectively. In addition to proving results about the traces of permuting n-derivations, they studied the relation between n-derivations and n-multipliers and provided a characterization of symmetric n-derivations of prime rings in terms of left n-multipliers. A similar study of characterization of symmetric reverse n-derivations in terms of n-multipliers seems an interesting concept to venture out. Motivated by the existing work on symmetric n-derivations, we wish to explore rings specifically semiprime rings so that to understand the behavior of symmetric reverse n-derivations.

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In this paper, we examine two main aspects of symmetric reverse n-derivations. First, we investigate their behavior when the trace is zero on an ideal. Second, we explore some identities involving trace itself. Previous research has demonstrated the crucial role played by the trace function in the study of n-derivations. Therefore, our focus is directed towards understanding the trace and some associated maps. One of the primary contributions of this paper lies in analyzing how the trace influences the structure of symmetric reverse n-derivations. Additionally, we highlight new properties of trace maps when restricted to ideals. This provides a deeper insight into how reverse n-derivations behave under specific constraints. The results obtained extend few existing findings on symmetric n-derivations to that of symmetric reverse n-derivations. The first important result of this paper is the following:

Theorem 1. *Let $n \geq 2$ be a fixed integer, T be a $n!$-torsion free semiprime ring and J its non-zero ideal. If there exists a symmetric reverse n-derivation $\mathfrak{D}$ on T with trace f such that $f(J) = \{0\}$, then $\mathfrak{D} = 0$.*

Proof. By the hypothesis, we have
$$f(\vartheta) = 0 \text{ for all } \vartheta \in J.$$
Substituting ϑ by $\vartheta + m\ell_1$, where $\ell_1 \in J$ for $1 \leq m \leq n-1$, we obtain

$$f(\vartheta + m\ell_1) = 0 \text{ for all } \vartheta, \ell_1 \in J,$$

which is given by

$$f(\vartheta) + f(m\ell_1) + \sum_{i=1}^{n-1} {}^nC_i \mathfrak{D}(\underbrace{\vartheta, \ldots, \vartheta}_{(n-t)-\text{times}}, \underbrace{m\ell_1, \ldots, m\ell_1}_{t-\text{times}}) = 0 \text{ for all } \vartheta, \ell_1 \in J.$$

Using the given hypothesis, we get

$$\sum_{i=1}^{n-1} {}^nC_i \mathfrak{D}(\underbrace{\vartheta, \ldots, \vartheta}_{(n-t)-\text{times}}, \underbrace{m\ell_1, \ldots, m\ell_1}_{t-\text{times}}) = 0 \text{ for all } \vartheta, \ell_1 \in J.$$

Lemma 1 allows us to equate coefficients to 0, we can write

$$n\mathfrak{D}(\ell_1, \vartheta, \ldots, \vartheta) = 0 \text{ for all } \vartheta, \ell_1 \in J$$

and using torsion restrictions, we have

$$\mathfrak{D}(\ell_1, \vartheta, \ldots, \vartheta) = 0 \text{ for all } \vartheta, \ell_1 \in J.$$

Substitute ϑ again by $\vartheta + m\ell_2$ (where $\ell_2 \in J$ and $1 \leq m \leq n-1$), we get

$$\mathfrak{D}(\ell_1, \vartheta + m\ell_2, \ldots, \vartheta + m\ell_2) = 0 \text{ for all } \vartheta, \ell_1, \ell_2 \in J.$$

Computing further, we obtain

$$\mathfrak{D}(\ell_1, \ell_2, \vartheta \ldots, \vartheta) = 0 \text{ for all } \vartheta, \ell_1, \ell_2 \in J.$$

This process can be continued until we obtain

$$\mathfrak{D}(\ell_1, \ell_2, \ldots, \ell_n) = 0 \text{ for all } \ell_1, \ell_2, \ldots \ell_n \in J. \tag{1}$$

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So, from $f(J) = 0$ we arrive at $\mathfrak{D}(J, J, \ldots, J) = \{0\}$. Now, replace ℓ_1 by $\ell_1 t_1$ in (1), to obtain

$$\mathfrak{D}(\ell_1 t_1, \ell_2, \ldots, \ell_n) = 0 \text{ for all } \ell_1, \ell_2, \ldots \ell_n \in J,\ t_1 \in T.$$

Using Equation (1), we arrive at

$$\mathfrak{D}(t_1, \ell_2, \ldots, \ell_n)\ell_1 = 0 \text{ for all } \ell_1, \ell_2, \ldots \ell_n \in J,\ t_1 \in T.$$

Now, replace ℓ_2 by $\ell_2 t_2$ with $t_2 \in T$, we get

$$\mathfrak{D}(t_1, t_2, \ldots, \ell_n)\ell_2\ell_1 = 0 \text{ for all } \ell_1, \ell_2, \ldots \ell_n \in J,\ t_1, t_2 \in T.$$

Continuing in the same manner, we finally obtain

$$\mathfrak{D}(t_1, t_2, \ldots, t_n)\ell_n\ell_{n-1}\cdots\ell_2\ell_1 = 0 \text{ for all } \ell_1, \ell_2, \ldots \ell_n \in J,\ t_1, t_2, \ldots, t_n \in T.$$

In the above equation replace ℓ_1 by the term $\ell_1\mathfrak{D}(t_1, t_2, \ldots, t_n)\ell_n\ell_{n-1}\cdots\ell_2$, we get

$$\mathfrak{D}(t_1, t_2, \ldots, t_n)\ell_n\ell_{n-1}\cdots\ell_2 J\mathfrak{D}(t_1, t_2, \ldots, t_n)\ell_n\ell_{n-1}\cdots\ell_2 = (0)$$

for all $\ell_2, \ldots \ell_n \in J,\ t_1, t_2, \ldots, t_n \in T$. Since T is semiprime, so by invoking Lemma 3, we obtain

$$\mathfrak{D}(t_1, t_2, \ldots, t_n)\ell_n\ell_{n-1}\cdots\ell_2 = 0$$

for all $\ell_2, \ldots \ell_n \in J,\ t_1, t_2, \ldots, t_n \in T$. Continuing in the similar manner, we can keep on omitting ℓ_i's one by one and obtain

$$\mathfrak{D}(t_1, t_2, \ldots, t_n)\ell_n = 0 \text{ for all } \ell_n \in J,\ t_1, t_2, \ldots, t_n \in T.$$

Lastly, replace ℓ_n by $\mathfrak{D}(t_1, t_2, \ldots, t_n)$ to arrive at,

$$\mathfrak{D}(t_1, t_2, \ldots, t_n)\ell_n\mathfrak{D}(t_1, t_2, \ldots, t_n) = 0 \text{ for all } \ell_n \in J,\ t_1, t_2, \ldots, t_n \in T.$$

This can be written as

$$\mathfrak{D}(t_1, t_2, \ldots, t_n)J\mathfrak{D}(t_1, t_2, \ldots, t_n) = (0) \text{ for all } t_1, t_2, \ldots, t_n \in T.$$

Therefore, using Lemma 3, we get

$$\mathfrak{D}(t_1, t_2, \ldots, t_n) = 0 \text{ for all } t_1, t_2, \ldots, t_n \in T,$$

which is the required conclusion. Hence, $\mathfrak{D} = 0$. □

Theorem 2. *Let T be $n!$-torsion free semiprime ring, J be a non-zero ideal of T and $\mathfrak{D} : T^n \to T$ a symmetric reverse n-derivation on T with trace f. If any one of the following conditions holds in T, then f is commuting on J:*

1. $f(\vartheta \text{ o } \ell) \pm f(\vartheta) \text{ o } \ell \in \mathcal{Z}(T)$ *for all* $\vartheta, \ell \in J$,
2. $f([\vartheta, \ell]) \pm f(\vartheta) \text{ o } \ell \in \mathcal{Z}(T)$, *for all* $\vartheta, \ell \in J$,
3. $f(\vartheta) \text{ o } \ell \pm [f(\ell), \vartheta] \in \mathcal{Z}(T)$, *for all* $\vartheta, \ell \in J$,

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Proof. 1. By hypothesis, we have

$$f(\vartheta\ o\ \ell) \pm (f(\vartheta)\ o\ \ell) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

Taking ℓ in place of $\ell + m\theta, \theta \in J, 1 \leq m \leq n-1$, we get

$$f(\vartheta\ o\ (\ell + m\theta)) \pm (f(\vartheta)\ o\ (\ell + m\theta)) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

$$f(\vartheta\ o\ \ell + \vartheta\ o\ m\theta) \pm (f(\vartheta)\ o\ \ell + f(\vartheta)\ o\ m\theta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J$$

$$f(\vartheta\ o\ \ell) + f(\vartheta\ o\ m\theta) + \sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}(\vartheta\ o\ \ell, \ldots, \vartheta\ o\ \ell, \vartheta\ o\ m\theta, \ldots, \vartheta\ o\ m\theta)$$
$$\pm f(\vartheta)\ o\ \ell \pm f(\vartheta)\ o\ m\theta \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Using the hypothesis, we get

$$\sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}(\vartheta\ o\ \ell, \ldots, \vartheta\ o\ \ell, \vartheta\ o\ m\theta, \ldots, \vartheta\ o\ m\theta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Application of Lemma 1 yields

$$n\mathfrak{D}(\vartheta\ o\ \ell, \ldots, \vartheta\ o\ \ell, \vartheta\ o\ \theta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Since T is $n!$-torsion free, we obtain

$$\mathfrak{D}(\vartheta\ o\ \ell, \ldots, \vartheta\ o\ \ell, \vartheta\ o\ \theta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Replace ℓ by θ to obtain

$$f(\vartheta\ o\ \ell) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

Thus,

$$f(\vartheta)\ o\ \ell \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

Hence, Proposition 2 implies f is commuting on J.

2. We are given with,

$$f([\vartheta, \ell]) \pm f(\vartheta)\ o\ \ell \in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell \in J.$$

Replace ℓ by $\ell + m\theta, \theta \in J, 1 \leq m \leq n-1$, we get the following calculations

$$f([\vartheta, \ell + m\theta]) \pm (f(\vartheta)\ o\ (\ell + m\theta)) \in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell,\ \theta \in J,$$

and

$$f([\vartheta, \ell] + [\vartheta, m\theta]) \pm f(\vartheta)\ o\ \ell \pm f(\vartheta)\ o\ m\theta \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Thus, we have

$$f([\vartheta, \ell]) + f([\vartheta, m\theta]) + \sum_{i=1}^{n-1} \binom{n}{i} {}^nC_i\mathfrak{D}([\vartheta, \ell][\vartheta, \ell] + [\vartheta, m\theta][\vartheta, m\theta])$$
$$\pm f(\vartheta)\ o\ \ell \pm f(\vartheta)\ o\ m\theta \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

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Application of the hypothesis yields

$$\sum_{i=1}^{n-1}\binom{n}{i}\,{}^{n}C_{i}\mathfrak{D}([\vartheta,\ell],[\vartheta,\ell],\ldots,[\vartheta,m\theta][\vartheta,m\theta])\in\mathcal{Z}(T)\text{ for all }\vartheta,\ell,\theta\in J.$$

Using Lemma 1 and torsion restrictions, we have

$$\mathfrak{D}([\vartheta,\ell],[\vartheta,\theta],\ldots[\vartheta,\theta])\in\mathcal{Z}(T),\text{ for all }\vartheta,\ \ell,\ \theta\in J.$$

Substituting θ by ℓ, we get

$$f([\vartheta,\ell])\in\mathcal{Z}(T)\text{ for all }\vartheta,\ell,\theta\in J.$$

So, the given hypothesis boils down to,

$$f(\vartheta)\ o\ \ell\in\mathcal{Z}(T).$$

By Proposition 2, we conclude that f is commuting on J.

3. We are given that

$$f(\vartheta)\ o\ \ell\pm[f(\ell),\vartheta]\in\mathcal{Z}(T)\text{ for all }\vartheta,\ \ell\in J.$$

Replacement of ℓ by $\ell+m\theta$, $\theta\in J$, $1\leq m\leq n-1$, yields

$$f(\vartheta)\ o\ (\ell+m\theta)\pm[f(\ell+m\theta),\vartheta]\in\mathcal{Z}(T)\text{ for all }\vartheta,\ \ell,\ \theta\in J.$$

$$f(\vartheta)\ o\ \ell+f(\vartheta)\ o\ m\theta\pm[f(\ell),\vartheta]\pm[f(m\theta),\vartheta]\pm$$

$$\Big[\sum_{i=1}^{n-1}\binom{n}{i}\,{}^{n}C_{i}\mathfrak{D}(\ell,\ldots,\ell,\ m\theta,\ \ldots,\ m\theta,\ \vartheta)\Big]\in\mathcal{Z}(T)\text{ for all }\vartheta,\ \ell,\ \theta\in J.$$

Using the given condition, we get

$$\left[\sum_{i=1}^{n-1}\binom{n}{i}\,{}^{n}C_{i}\mathfrak{D}(\ell,\ldots,\ell,\ m\theta,\ldots,\ m\theta),\vartheta\right]\in\mathcal{Z}(T)\text{ for all }\vartheta,\ \ell,\ \theta\in J.$$

By applying Lemma 1 and the torsion restrictions, we obtain

$$[\mathfrak{D}(\ell,\theta,\ldots,\theta),\vartheta]\in\mathcal{Z}(T),\text{ for all }\vartheta,\ell,\theta\in J.$$

Again replace θ by ℓ, we then have

$$[f(\ell),\vartheta]\text{ for all }\vartheta,\ell\in J.$$

From the hypothesis, we arrive at

$$f(\vartheta)\ o\ \ell\in\mathcal{Z}(T)\text{ for all }\vartheta,\ell\in J.$$

In view of Proposition 2, we obtain to our desired conclusion. □

Theorem 3. *Let T be $n!$-torsion free semiprime ring, J be a non-zero ideal of T and $\mathfrak{D}:T^n\to T$ a symmetric reverse n-derivation on T with trace f. If any one of the following conditions holds in T, then f is commuting on J:*

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1. $f([\vartheta,\ell]) \pm [f(\vartheta),\ell] \in \mathcal{Z}(T)$, *for all* $\vartheta,\ell \in J$,
2. $f(\vartheta \; o \; \ell) \pm [f(\vartheta),\ell] \in \mathcal{Z}(T)$, *for all* $\vartheta,\ell \in J$,
3. $[f(\vartheta),\ell] \pm [f(\ell),\vartheta] \in \mathcal{Z}(T)$, *for all* $\vartheta,\ell \in J$,

Proof. 1. We have,

$$f([\vartheta,\ell]) \pm [f(\vartheta),\ell] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell \in J.$$

Now, substitute ℓ by $\ell + m\theta$, $1 \leq m \leq n-1$, $\theta \in J$, we get

$$f([\vartheta,\ell + m\theta]) \pm [f(\vartheta),\ell + m\theta] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J$$

$$f([\vartheta,\ell] + [\vartheta,m\theta]) \pm [f(\vartheta),\ell] \pm [f(\vartheta),m\theta] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J$$

$$f([\vartheta,\ell]) + f([\vartheta,m\theta]) + \sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}([\vartheta,\ell],\ldots,[\vartheta,\ell],[\vartheta,m\theta],\ldots,[\vartheta,m\theta]) \pm [f(\vartheta),\ell] \pm [f(\vartheta),m\theta] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J.$$

Application of hypothesis gives,

$$\sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}([\vartheta,\ell],\ldots,[\vartheta,\ell],[\vartheta,m\theta],\ldots,[\vartheta,m\theta]) \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J.$$

On solving further, we get

$$n\mathfrak{D}([\vartheta,\ell],[\vartheta,\theta],\ldots,[\vartheta,\theta]) \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J$$

and

$$\mathfrak{D}([\vartheta,\ell],[\vartheta,\theta],\ldots,[\vartheta,\theta]) \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J.$$

Replacing θ by ℓ in the above expression, we get

$$f([\vartheta,\ell]) \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell \in J.$$

Using the hypothesis, we obtain

$$[f(\vartheta),\ell] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell \in J.$$

Therefore, Proposition 1 implies f is commuting on J.

2. We are given that,

$$f(\vartheta \; o \; \ell) \quad \pm \quad [f(\vartheta),\ell] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell \in J.$$

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Substitute ℓ by $\ell + m\theta, \theta \in J, 1 \leq m \leq n-1$, so that

$$\begin{aligned} f(\vartheta\ o\ (\ell + m\theta)) \pm [f(\vartheta),\ \ell + m\theta] &\in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell,\ \theta \in J \\ f(\vartheta\ o\ \ell + \vartheta\ o\ m\theta) \pm [f(\vartheta), \ell] \pm [f(\vartheta), m\theta] &\in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell,\ \theta \in J \\ f(\vartheta\ o\ \ell) + f(\vartheta\ o\ m\theta) + \sum_{i=1}^{n-1} \mathfrak{D}(\vartheta\ o\ \ell, \ldots \vartheta\ o\ \ell,\ \vartheta\ o\ m\theta, \ldots,\ \vartheta\ o\ m\theta) & \\ \pm [f(\vartheta), \ell] \pm [f(\vartheta), m\theta] &\in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell,\ \theta \in J. \end{aligned}$$

Using the given hypothesis, we get

$$\begin{aligned} \sum_{i=1}^{n-1} \binom{n}{i} =^n C_i \mathfrak{D}(\vartheta\ o\ \ell, \ldots \vartheta\ o\ \ell,\ \vartheta\ o\ m\theta, \ldots,\ \vartheta\ o\ m\theta) & \\ \pm [f(\vartheta), \ell] \pm [f(\vartheta), m\theta] &\in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell,\ \theta \in J. \end{aligned}$$

Application of Lemma 1 and torsion restrictions yields

$$\mathfrak{D}(\vartheta\ o\ \ell,\ \vartheta\ o\ \theta,\ \ldots,\ \vartheta\ o\ \theta) \in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell,\ \theta \in J.$$

Replacing θ by ℓ gives

$$f(\vartheta\ o\ \ell) \in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell \in J.$$

By the hypothesis, we conclude

$$[f(\vartheta), \ell] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ \ell \in J.$$

Hence, f is commuting on J.

3. We have,

$$[f(\vartheta), \ell] \pm [f(\ell), \vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

Replace ℓ by $\ell + m\theta, \theta \in J, 1 \leq m \leq n-1$, we get

$$[f(\vartheta), \ell + m\theta] \pm [f(\ell + m\theta), \vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

This implies,

$$[f(\vartheta), \ell] + [f(\vartheta), m\theta] \pm [f(\ell), \vartheta] \pm [f(m\theta), \vartheta] \pm \left[\sum_{i=1}^{n-1} \mathfrak{D}(\ell, \ldots, \ell, m\theta, \ldots, m\theta), \vartheta\right] \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Using the hypothesis, we get

$$\left[\sum_{i=1}^{n-1} \mathfrak{D}(\ell, \ldots, \ell, m\theta, \ldots, m\theta), \vartheta\right] \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

By using Lemma 1 and the torsion restrictions, we obtain

$$[\mathfrak{D}(\ell, \theta, \ldots, \theta), \vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Replacing θ by ℓ in the above expression, we have

$$[f(\ell), \vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

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In view of Proposition 1, we get the desired conclusion. □

Theorem 4. *Let T be $n!$-torsion free semiprime ring, J be a non-zero ideal of T and $\mathfrak{D} : T^n \to T$ a symmetric reverse n-derivation on T with trace f. If $f([\vartheta,\ell]) - (f(\vartheta)\ o\ \ell) - [f(\ell),\vartheta] \in \mathcal{Z}(T)$ for all $\vartheta,\ell \in J$. Then, f is commuting on J.*

Proof. We are given with

$$f([\vartheta,\ell]) - (f(\vartheta)\ o\ \ell) - [f(\ell),\vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell \in J.$$

Replace ℓ by $\ell + m\theta$, $\theta \in J$, $1 \leq m \leq n-1$, we see that

$$f([\vartheta,\ell + m\theta]) - (f(\vartheta)\ o\ \ell + m\theta) - [f(\ell + m\theta),\vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J.$$

On solving it further, we get

$$\begin{aligned} &f([\vartheta,\ell] + [\vartheta,m\theta]) - f(\vartheta)\ o\ \ell - f(\vartheta)\ o\ m\theta - [f(\ell),\vartheta] - \\ &\qquad [f(m\theta),\vartheta] - \left[\sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}(\ell,\ldots,\ell,m\theta,\ldots,m\theta),\vartheta\right] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J \end{aligned}$$

and

$$\begin{aligned} &f([\vartheta,\ell]) + f([\vartheta,m\theta]) + \sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}\Big([\vartheta,\ell],\ldots,[\vartheta,\ell],[\vartheta,m\theta],\ldots,[\vartheta,m\theta]\Big) - \\ &\qquad f(\vartheta)\ o\ \ell - f(\vartheta)\ o\ m\theta - [f(\ell),\vartheta] - [f(m\theta),\vartheta] - \\ &\qquad\qquad \left[\sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}(\ell,\ldots,\ell,m\theta,\ldots,m\theta),\vartheta\right] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J. \end{aligned}$$

From the given hypothesis the above expression reduces to,

$$\begin{aligned} &\sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}([\vartheta,\ell],\ldots,[\vartheta,\ell],[\vartheta,m\theta],\ldots,[\vartheta,m\theta]) - \\ &\qquad \left[\sum_{i=1}^{n-1} {}^nC_i\mathfrak{D}(\ell,\ldots,\ell,m\theta,\ldots,m\theta),\vartheta\right] \in \mathcal{Z}(T) \text{ for all } x,\ell,\theta \in J. \end{aligned}$$

Application of Lemma 1 yields

$$\mathfrak{D}([\vartheta,\ell],[\vartheta,\theta],\ldots,[\vartheta,\theta]) - [\mathfrak{D}(\ell,\theta,\ldots,\theta),\vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J.$$

Replacing θ by ℓ provides,

$$f([\vartheta,\ell]) - [f(\ell),\vartheta] \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell,\theta \in J.$$

Therefore, we have

$$f(\ell)\ o\ \vartheta \in \mathcal{Z}(T) \text{ for all } \vartheta,\ell \in J$$

and hence from Proposition 2, we conclude that f is commuting on J. This completes the proof. □

Theorem 5. *Let T be $n!$-torsion free semiprime ring, J be a non-zero ideal of T and $\mathfrak{D} : T^n \to T$ a symmetric reverse n-derivation on T with trace f. If any one of the following conditions hold*

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1. *$f(\vartheta\ell) + f(\vartheta)f(\ell) \pm \vartheta\ell \in \mathcal{Z}(T)$, for all $\vartheta, \ell \in J$,*
2. *$f(\vartheta\ell) + f(\vartheta)f(\ell) \pm \ell\vartheta \in \mathcal{Z}(T)$, for all $\vartheta, \ell \in J$,*
3. *$f(\vartheta\ell) - f(\ell\vartheta) \pm [\vartheta, \ell] \in \mathcal{Z}(T)$, for all $\vartheta, \ell \in J$,*

then T contains a non-zero central ideal.

Proof. 1. We have,

$$f(\vartheta\ell) + f(\vartheta)f(\ell) \pm \vartheta\ell \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell \in J.$$

On replacing ℓ by $\ell + m\theta$, $\theta \in J$, $1 \leq m \leq n-1$, we get

$$f(\vartheta(\ell + m\theta)) + f(\vartheta)f(\ell + m\theta) \pm \vartheta(\ell + m\theta) \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell, \theta \in J.$$

This further implies,

$$f(\vartheta\ell) + f(\vartheta m\theta) + \sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta m\theta, \ldots, \vartheta m\theta) + f(\vartheta)(f(\ell) + f(m\theta)) + f(\vartheta) \sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\ell, \ldots, \ell, m\theta, \ldots, m\theta)) \pm \vartheta\ell \pm \vartheta m\theta \in \mathcal{Z}(T).$$

Using the given hypothesis, we obtain

$$\sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta m\theta, \ldots, \vartheta m\theta) + f(\vartheta) \sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\ell, \ldots, \ell, m\theta, \ldots, m\theta) \in \mathcal{Z}(T)$$

for all $\vartheta, \ell, \theta \in J$. The application of Lemma 1 yields,

$$n\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\theta) + nf(\vartheta)\mathfrak{D}(\ell, \ldots, \ell, \theta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Since T is $n!$-torsion free, so we get

$$\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\theta) + f(\vartheta)\mathfrak{D}(\ell, \ldots, \ell, \theta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Replace θ by ℓ in the above equation to obtain

$$\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\ell) + f(\vartheta)\mathfrak{D}(\ell, \ldots, \ell, \ell) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

Hence, we arrive at

$$f(\vartheta\ell) + f(\vartheta)f(\ell) \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell \in J.$$

Using this in the given hypothesis we get

$$\vartheta\ell \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell \in J.$$

On commuting it with any $t \in T$, we obtain

$$[\vartheta\ell, t] = 0 \text{ for all } \vartheta, \ell \in J; t \in T.$$

This can be written as

$$\vartheta[\ell, t] + [\vartheta, t]\ell = 0 \text{ for all } \vartheta, \ell \in J; t \in T$$

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Replacing ℓ by $\ell\theta$ where $\theta \in J$, we see that

$$\vartheta\ell[\theta, t] = 0 \text{ for all } \vartheta, \ell, \theta \in J; t \in T.$$

Now substitute $[\theta, t]$ in place of ϑ to get

$$[\theta, t]\ell[\theta, t] = 0 \text{ for all } \ell, \theta \in J; t \in T.$$

So we can write

$$[\theta, t]\ell T[\theta, t]\ell = \{0\},$$

that gives

$$[\theta, t]\ell = 0 \text{ for all } \ell, \theta \in J; t \in T.$$

Taking ℓ to be $t[\theta, t]$ and using the semiprimeness of T, we get $J \subseteq \mathcal{Z}(T)$. Hence, J is a non-zero central ideal in T.

2. Using similar arguments as done in part 1, we obtain the desired conclusion.

3. We have,

$$f(\vartheta\ell) - f(\ell\vartheta) \pm [\vartheta, \ell] \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell \in J.$$

Substitute $\ell + m\theta$ in place of ℓ, $\theta \in J$, $1 \leq m \leq n-1$ to obtain

$$f(\vartheta(\ell + m\theta)) - f((\ell + m\theta)\vartheta) \pm [\vartheta, \ell + m\theta] \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell, \theta \in J.$$

This implies,

$$f(\vartheta\ell) + f(\vartheta m\theta) + \sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta m\theta, \ldots, \vartheta m\theta) -$$
$$f(\ell\vartheta) - f(m\theta\vartheta) - \sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\ell\vartheta, \ldots, \ell\vartheta, m\theta\vartheta, \ldots, m\theta\vartheta) \pm [\vartheta, \ell] \pm [\vartheta, m\theta] \in \mathcal{Z}(T).$$

Using the given hypothesis, we arrive at

$$\sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta m\theta, \ldots, \vartheta m\theta) - \sum_{i=1}^{n-1} \binom{n}{i} \mathfrak{D}(\ell\vartheta, \ldots, \ell\vartheta, m\theta\vartheta, \ldots, m\theta\vartheta) \in \mathcal{Z}(T)$$

for all $\vartheta, \ell, \theta \in J$. In view of Lemma 1, we have

$$n\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\theta) - n\mathfrak{D}(\ell\vartheta, \ldots, \ell\vartheta, \theta\vartheta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

Since T is $n!$-torsion free, so we have

$$\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\theta) - \mathfrak{D}(\ell\vartheta, \ldots, \ell\vartheta, \theta\vartheta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell, \theta \in J.$$

On replacing θ by ℓ in the above equation, we obtain

$$\mathfrak{D}(\vartheta\ell, \ldots, \vartheta\ell, \vartheta\ell) - \mathfrak{D}(\ell\vartheta, \ldots, \ell\vartheta, \ell\vartheta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

Therefore,

$$f(\vartheta\ell) - f(\ell\vartheta) \in \mathcal{Z}(T) \text{ for all } \vartheta, \ell \in J.$$

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Thus, from the hypothesis we can conclude

$$[\vartheta, \ell] \in \mathcal{Z}(T), \text{ for all } \vartheta, \ell \in T.$$

Application of Lemma 4 gives us the existence of a non-zero central ideal in T. Hence, the desired result. □

The subsequent example illustrates that the requirement of semiprimeness for T in Theorems 5 is indispensable and cannot be overlooked. The following example justifies this fact:

Example 33. *Consider the ring* $T = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$. *Next let* $J = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{Z} \right\}$ *be an ideal of* T. *Denote* $A_i = \begin{bmatrix} a_i & b_i \\ 0 & 0 \end{bmatrix} \in T$ *where* $a_i, b_i \in \mathbb{Z}$, $1 \leq i \leq n$, *and let us define* $\mathfrak{D} : T^n \to T$ *by* $\mathfrak{D}(A_1, A_2, \ldots, A_n) = \begin{bmatrix} 0 & a_1 a_2 \cdots a_n \\ 0 & 0 \end{bmatrix}$ *with trace* $f : T \to T$ *define by* $f\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a^n \\ 0 & 0 \end{bmatrix}$. *One can easily check that* $\mathfrak{D}$ *is a symmetric reverse* n*-derivation such that all the conditions in Theorems 5 are satisfied. However,* J *is non-central ideal. Hence, in Theorem 5, the hypothesis of semiprimeness can not be omitted.*

4. Conclusion

This work embarks on a thorough investigation of a novel class of maps known as symmetric reverse n-derivations, specifically studied on ideals of semiprime rings. The primary objective was to introduced these notions and to analyze the behavior of these maps. Throughout this comprehensive study, we have uncovered various relationships between symmetric reverse n-derivations and their traces, particularly when these traces satisfied specific identities. These findings offer valuable insights into this area of research. By delving into the properties of symmetric reverse n-derivations, we contributed to a deeper understanding of how such maps interact with the underlying algebraic structures. Our results provided a foundation for further future exploration into the behavior of these maps in different contexts. The intricate connections revealed in this work not only advances the study of n-derivations but also opens up new avenues for researchers in this domain.

As we push the boundaries of this area, our findings presented new perspectives on algebraic structures, suggesting a broader application of these maps. The study also raises important questions for future research, particularly in the interaction between symmetric maps and rings. Ultimately, this exploration expands the scope of modern ring theory and paves the way for continued advances in understanding the behavior of these new algebraic tools.

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References

1. Aboubakr, A.; Gonzalez, S. Generalized reverse derivation on semi prime rings. *Sib. Math. J.* **2015**, 56(2), 199-205.
2. Ali, S.; Alsuraiheed, T. M.; Parveen, N.; Varshney V. Action of n-derivations and n-multipliers on ideals of (semi)-prime rings. *AIMS Mathematics* **2023**, 8(7), 17208-17228, doi: 10.3934/math.2023879
3. Ashraf, M. On symmetric bi-derivations in rings. *Rend. Istit. Mat. Univ. Trieste* **1999**, 31, 25-36.
4. Ashraf, M.; Khan, A.; Jamal, M. R. Traces of permuting generalized n-derivations of rings. *Miskolc Math. Notes* **2018**, 19, 731-740.
5. Ashraf, M.; Jamal, M. R. Traces of permuting n-additive maps and permuting n-derivations of rings. *Mediterr. J. Math.* **2014**, 11, 287-297.
6. Ashraf, M.; Jamal, M. R.; Mozumder, M. R. On the traces of certain classes of permuting mappings in rings, *Georgian Math. J.* **2016**, 23(1), 15-23.
7. De Barros, D. A. S.; Ferreira, B. L. M.; Guzzo Jr. H. $*$-Reverse Derivations on Alternative Algebras, *J. Algebra Appl.* **2024**, doi: 10.1142/S0219498825503001.
8. Brešar, M. Centralizing mappings and derivations in prime rings. *J. Algebra* **1993**, 156, 385-394.
9. Daif, M. N.; Bell, H. E. Remarks on derivations on semiprime rings. *Int. J. Math. Math. Sci.* **1992**, 15, 205-206.
10. Herstein, I. N. Jordan derivations of prime rings. *Proc. Amer. Math. Soc.* **1957**, 8, 1104–1110.
11. Maksa, G. On the trace of symmetric biderivations. *C. R. Math. Rep. Acad. Sci. Canada IX* **1987**, 303-308.
12. Maksa, G. A remark on symmetric bi-additive functions having nonnegative diagonalization. *Glas. Math* **1980** 15, 279-282.
13. Öztürk, M. A. Permuting tri-derivations in prime and semi-prime rings. *East Asian Math. J.* **1999**, 15, 177-190.
14. Park, K. H. On prime and semi-prime rings with symmetric n-derivations. *J. Chungcheong Math. Soc.* **2009**, 22, 451-458.
15. Posner, E. C. Derivations in prime rings, *Proc. Amer. Math. Soc.* **1957**, 8, 1093-1100.
16. Samman, M. S.; Thaheem, A. B. Derivations on semiprime rings. *Int. J. Pure. Appl. Math.* **2003**, 5(4), 465-472.
17. Koç Sög*ü*tc*ü*, E. Multiplicative (generalized)-reverse derivations in rings and Banach algebras. *Georgian Math. J.*, **2023**, 30(4), 555-567.
18. Koç Sög*ü*tc*ü*, E; Gölbasi, O. Some result on Lie ideals with symmetric reverse bi-derivations in semiprime rings I. *Facta Univ. Ser. Math. Inform.* **2021**, 36(2), 309-319.
19. Koç Sög*ü*tc*ü*, E.; Huang, S. Note on lie ideals with symmetric bi-derivations in semiprime rings. *Indian J. Pure Appl. Math.* **2023**, 54, 608–618.
20. Vukman, J. Symmetric bi-derivations on prime and semiprime rings. *Aeq. Math.* **1989**, 38, 245-254.
21. Vukman, J. Two results concerning symmetric bi-derivations on prime and semiprime rings. *Aeq. Math.* **1990**, 40, 181-189.