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The Friction Coefficient as a Non-Symmetric Second-Order Tensor

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Abstract

The traditional scalar representation of the friction coefficient has long been challenged by the orthogonal orientation of frictional force ($\boldsymbol{F}$) and normal force ($\boldsymbol{N}$), which violates basic orientational laws of physics. As early as 1972, Hart [1] first proposed that the friction coefficient should be a second-order tensor, but his work lacked a rigorous mathematical formulation of the tensor components and failed to reveal its non-symmetric naturekey limitations that prevented broader acceptance. Here, we address these critical gaps by deriving the explicit form of the friction coefficient tensor via tensor algebra, dimensional analysis, and orientational constraints. We show that the friction coefficient tensor is given by $\boldsymbol{\mu} = N^{-2}\boldsymbol{F} \otimes \boldsymbol{N}$ (where $N = \|\boldsymbol{N}\|$) with non-symmetric components $\mu_{ij} = N^{-2}F_i N_j$, and verify its compatibility with friction shear stress (also a second-order tensor). This formulation resolves the orientational inconsistency of Amontons-Coulomb's law and provides a quantitative framework to describe anisotropic frictional behavior, which is essential for applications ranging from nanotribology to seismic engineering. Our work not only completes Hart's pioneering but incomplete hypothesis but also establishes a physically sound foundation for the tensorial description of friction.

Keywords: friction coefficient; second-order tensor; non-symmetric tensor; Amontons-Coulomb law; anisotropic friction; tensor algebra; nanotribology

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Friction is a universal phenomenon governing interactions between contacting surfaces, with critical implications for engineering, materials science, and nanotechnology [2–7]. The conventional Amontons-Coulomb law describes frictional force as $\boldsymbol{F} = \mu\boldsymbol{N}$, where μ is treated as a scalar proportionality constant between the magnitudes of frictional force ($\boldsymbol{F}$) and normal force ($\boldsymbol{N}$) [5,6]. However, this scalar representation inherently ignores the orthogonality of $\boldsymbol{F}$ and $\boldsymbol{N}$ ($\boldsymbol{F} \cdot \boldsymbol{N} = \boldsymbol{0}$), leading to a fundamental inconsistency with orientational laws of physics [8–12]: a scalar cannot transform a vector (normal force) into another vector (frictional force) of orthogonal direction.

Hart's Pioneering but Incomplete Hypothesis: In 1972, Hart [1] recognized this flaw and proposed that the friction coefficient must be a second-order tensor (dyad) to satisfy vectorial transformation rules, suggesting the form $\boldsymbol{F} = \boldsymbol{\mu} \cdot \boldsymbol{N}$. Despite its conceptual insight, Hart's work suffered from two fatal limitations that precluded academic acceptance:

1. Lack of explicit tensor components: Hart only speculated the tensor nature but did not derive the mathematical expression of $\boldsymbol{\mu}$ or its components;
2. Ignorance of non-symmetry: He did not recognize that the friction coefficient tensor is inherently non-symmetric, a key property arising from the orthogonal relationship between $\boldsymbol{F}$ and $\boldsymbol{N}$;
3. No compatibility verification: Hart failed to connect the tensor to measurable physical quantities (e.g., friction shear stress) to validate its physical relevance.

These gaps left the tensorial hypothesis unsubstantiated, and the scalar model remained dominant despite its orientational inconsistency. To address this, we revisit the problem with rigorous tensor algebra, dimensional analysis, and orientational constraints to complete and validate Hart's unfinished framework.

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Rigorous Formulation of the Friction Coefficient Tensor— To resolve the orientational inconsistency of Amontons-Coulomb's law, we start with the tensorial form proposed by Hart but left unformulated:

$$\boldsymbol{F} = \boldsymbol{\mu} \cdot \boldsymbol{N}, \quad \text{with } \boldsymbol{F} \cdot \boldsymbol{N} = 0, \tag{1}$$

where $\boldsymbol{\mu}$ is a second-order tensor (dyad), and "$\cdot$" denotes the tensor-vector dot product. The key requirement is to determine the explicit form of $\boldsymbol{\mu}$ that satisfies both Eq. (1) and physical constraints (dimensional consistency, orientational orthogonality).

Derivation of Tensor Components: We assume $\boldsymbol{\mu}$ is a dyadic product of $\boldsymbol{F}$ and $\boldsymbol{N}$ (the only vectors defining the frictional interaction) scaled by a constant α:

$$\boldsymbol{\mu} = \alpha\, \boldsymbol{F} \otimes \boldsymbol{N}, \tag{2}$$

where $\otimes$ denotes the dyadic product, and α is a dimensionless constant to be determined. Substituting Eq. (2) into Eq. (1):

$$\boldsymbol{F} = (\alpha\, \boldsymbol{F} \otimes \boldsymbol{N}) \cdot \boldsymbol{N} = \alpha\, \boldsymbol{F}(\boldsymbol{N} \cdot \boldsymbol{N}) = \alpha N^2 \boldsymbol{F}, \tag{3}$$

where $N = \|\boldsymbol{N}\| = \sqrt{\boldsymbol{N} \cdot \boldsymbol{N}}$. For Eq. (3) to hold for non-trivial $\boldsymbol{F}$, we require $\alpha N^2 = 1$, hence $\alpha = N^{-2}$. Substituting back into Eq. (2), the explicit form of the friction coefficient tensor is:

$$\boldsymbol{\mu} = N^{-2}\, \boldsymbol{F} \otimes \boldsymbol{N}. \tag{4}$$

Non-Symmetric Nature of $\boldsymbol{\mu}$: In Cartesian coordinates $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, we expand $\boldsymbol{F} = F_i \boldsymbol{e}_i$ and $\boldsymbol{N} = N_j \boldsymbol{e}_j$ (Einstein summation convention). The dyadic product in Eq. (4) becomes:

$$\boldsymbol{\mu} = \mu_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \quad \text{where } \mu_{ij} = N^{-2} F_i N_j. \tag{5}$$

The tensor components μ_{ij} satisfy $\mu_{ij} \neq \mu_{ji}$ (non-symmetry) because:
- $\boldsymbol{F} \perp \boldsymbol{N}$ implies $F_i N_i = 0$ (orthogonality), but $F_i N_j \neq F_j N_i$ for $i \neq j$ (e.g., $F_1 N_2 \neq F_2 N_1$ in general);
- Symmetry ($\mu_{ij} = \mu_{ji}$) would require $F_i N_j = F_j N_i$ for all i, j, which is only possible if $\boldsymbol{F} \parallel \boldsymbol{N}$ - contradicting the definition of friction.

This non-symmetric property is a critical correction to Hart's work and a key physical feature of the friction coefficient tensor.

Compatibility with Friction Shear Stress: Friction shear stress $\boldsymbol{\tau}^f$ is defined as $\boldsymbol{\tau}^f = \boldsymbol{F}/A$, where A is the contact area (scalar). From Eq. (4):

$$\boldsymbol{\tau}^f = \frac{N}{A} \boldsymbol{\mu}. \tag{6}$$

Since $\boldsymbol{\tau}^f$ is a well-established second-order tensor (shear stress tensor in solid mechanics), Eq. (6) confirms that $\boldsymbol{\mu}$ is also a second-order tensor–providing a critical consistency check absent in Hart's work. For a scalar μ, this compatibility is lost (scalar μ cannot transform to tensor $\boldsymbol{\tau}^f$).

Validation in a Simple Frictional System: To illustrate the utility of our formulation, consider a block sliding on an inclined plane :
- Normal force: $\boldsymbol{N} = -N_3 \boldsymbol{e}_3$ (negative z-direction, contact normal);
- Frictional force: $\boldsymbol{F} = F_1 \boldsymbol{e}_1$ (positive x-direction, along the plane);
- Orthogonality: $\boldsymbol{F} \cdot \boldsymbol{N} = F_1 \cdot 0 + 0 \cdot 0 + 0 \cdot (-N_3) = 0$ (satisfied);
- $N = N_3$, so $N^{-2} = N_3^{-2}$.

From Eq. (5), the tensor components are:
- $\mu_{13} = N_3^{-2} F_1 (-N_3) = -F_1/N_3$ (non-zero);
- All other components $\mu_{ij} = 0$ (no cross terms outside $i = 1, j = 3$).

The tensor simplifies to:

$$\boldsymbol{\mu} = -\frac{F_1}{N_3} \boldsymbol{e}_1 \otimes \boldsymbol{e}_3. \tag{7}$$

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Substituting into Eq. (1):

$$F = \left(-\frac{F_1}{N_3} e_1 \otimes e_3\right) \cdot (-N_3 e_3) = F_1 e_1, \tag{8}$$

which exactly reproduces the frictional force. In contrast, Hart's work only stated that $\mu = \mu_{xx} e_x \otimes e_x + \mu_{xy} e_x \otimes e_y$ (incorrect component indices) without deriving μ_{ij} or verifying consistency–our formulation resolves this by providing a quantifiable, testable tensor.

Implications for Anisotropic Friction: The non-symmetric tensor μ naturally describes anisotropic friction, where frictional behavior depends on the direction of motion— a phenomenon ubiquitous in nanomaterials, textured surfaces, and crystalline solids [2,4]. For example:

- In nanotribology, frictional force between atomic layers varies with sliding direction [3]; our tensor μ_{ij} directly quantifies this direction dependence via $F_i N_j$;
- In seismic engineering, the friction coefficient of fault surfaces exhibits anisotropy [7]; the tensor formulation allows modeling of directional shear resistance.

Hart's incomplete hypothesis could not address anisotropy because it lacked component-specific expressions. Our work provides a quantitative framework to predict and analyze direction-dependent friction, bridging theory and experiment.

Comparison with Conventional and Hart's Models: Table 1 summarizes the key differences between the scalar model, Hart's tensor hypothesis, and our complete formulation:

Table 1. Comparison of Friction Coefficient Models

Model	Formulation	Key Properties	Limitations
Conventional (scalar)	$F = \mu N$	μ is scalar, simple	Violates orientational laws; cannot describe anisotropy
Hart (1972)	$F = \mu \cdot N$	Recognizes tensor nature	No explicit components; ignores non-symmetry; no validation
Our work (complete tensor)	$F = \mu \cdot N$ $\mu_{ij} = N^{-2} F_i N_j$	$\mu = N^{-2} F \otimes N$; non-symmetric; compatible with τ^f	None (resolves orientational inconsistency; describes anisotropy; quantifiable)

We have completed and validated Hart's pioneering but incomplete 1972 hypothesis by deriving the explicit, non-symmetric form of the friction coefficient tensor. Key contributions include:

1. Resolving Hart's critical gaps: deriving $\mu = N^{-2} F \otimes N$ and proving its non-symmetric nature;
2. Establishing compatibility with friction shear stress (a second-order tensor), ensuring physical consistency;
3. Providing a quantitative framework for anisotropic friction, with applications in nanotechnology, materials science, and engineering;
4. Correcting the orientational inconsistency of Amontons-Coulomb's law via rigorous tensor algebra and dimensional analysis.

This work not only addresses why Hart's original proposal was not accepted (lack of formulation and validation) but also provides a physically sound, experimentally testable tensorial model of friction. Future studies can extend this framework to dynamic friction (velocity-dependent μ) and multi-contact surfaces, further expanding its utility.

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References

1. F.X. Hart, The Coefficient of Friction as a Second Rank Tensor, Am. J. Phys. 40, 475-C476 (1972).
2. J. Krim, Friction at the atomic scale, Sci. Am. 275, 74 (1996).
3. M. Urbakh, J. Klafter, D. Gourdon, and J. Israelachvili, The nonlinear nature of friction, Nature 430, 525 (2004).
4. Z.L. Wang, On the expanded Maxwells equations for moving charged media system-general theory, mathematical solutions and applications in TENG, Mater. Today 52, 348-C363 (2021).
5. F. P. Bowden and D. Tabor, The Friction and Lubrication of Solids (Oxford University Press, Oxford, 2001).

 doi:10.20944/preprints202603.2367.v1

6. D. Tabor, History of tribology, Tribol. Int. 12, 301 (1979).
7. S.Y. Huang, et al., Reexamination of damping in sliding friction, Phys. Rev. Lett. 132, 056203 (2024).
8. E. Buckingham, On physically similar systems: illustration of the use of dimensional equation, Phys. Rev. 4, 345-376 (1914).
9. P.W. Bridgman, Dimensional Analysis (Yale University Press, New Haven, 1922).
10. H.E. Huntley, Dimensional Analysis (Dover, New York, 1967).
11. D. Siano, Orientational analysis, J. Franklin Inst. 320, 267-283 (1985).
12. B.H. Sun, Dimensional Analysis and Lie Group (China High Education Press, Beijing, 2016).