

Article

Not peer-reviewed version

A Quasigroup Approach for Conservation Laws in Asymptotically Flat Spacetimes

[Alfonso Zack Robles](#)*, [Alexander I. Nesterov](#), [Claudia Moreno](#)*

Posted Date: 3 September 2025

doi: 10.20944/preprints202509.0356.v1

Keywords: Noether theorem; conservation laws; asymptotic structure of spacetime; Poincaré quasigroup



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a Creative Commons CC BY 4.0 license, which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

A Quasigroup Approach for Conservation Laws in Asymptotically Flat Spacetimes

Alfonso Zack Robles , Alexander I. Nesterov  and Claudia Moreno 

Department of Physics, CUCEI, University of Guadalajara, Guadalajara, 44430, Jalisco, México

* Correspondence: alfonso.robles7829@alumnos.udg.mx (A.Z.B.); claudia.moreno@academico.udg.mx (C.M.)

Abstract

In the framework of the quasigroup approach to conservation laws in general relativity, we show how the infinite-parametric Newman-Unti group of asymptotic symmetries can be reduced to the Poincaré quasigroup. We compute the Noether's charges associated with any element of the Poincaré quasialgebra. The integral conserved quantities of energy-momentum and angular momentum, being linear on generators of Poincaré quasigroup, are identically equal to zero in Minkowski spacetime. We present a definition of the angular momentum free of the supertranslation ambiguity. We provide an appropriate notion of intrinsic angular momentum and a description of the mass reference frame's center at future null infinity. Finally, in the center of mass reference frame, the momentum and angular momentum are defined by the Komar expression.

Keywords: Noether theorem; conservation laws; asymptotic structure of spacetime; Poincaré quasigroup

1. Introduction

The general covariance of the Einstein equations results in differential conservation laws related to the field equations. From Noether's theorem, it follows that for an arbitrary diffeomorphism generated by the vector field ξ , the invariance of the Lagrangian for the Einstein equations leads to the conservation laws of the form, $\mathfrak{J}^\mu(\xi)_{;\mu} = 0$, where the vector density $\mathfrak{J}^\mu$ is defined as $\mathfrak{J}^\mu(\xi) = \mathfrak{h}^{\mu\nu}_{;\nu}$, with $\mathfrak{h}^{\mu\nu} = -\mathfrak{h}^{\nu\mu}$ being the *superpotential*, which is constructed from the densities of spin, bispin, vector field ξ and its derivatives [1–3].

For the Einstein-Hilbert action, we have a simple expression for the Noether current,

$$\mathfrak{J}^\mu = -\frac{1}{4\pi} \left(\sqrt{-g} \xi^{[\mu;\nu]} \right)_{;\nu}, \quad (1)$$

which was first obtained by Komar [4]. Komar's expression provides a fully satisfactory notion of the total mass in stationary, asymptotically flat spacetimes. It's worth noting that the normalization for energy-momentum and angular momentum differs by a factor of 2, and it is impossible simply to "renormalize" $\mathfrak{J}^\mu$. The resolution of this problem was pointed out in [5]. One needs to add to the Einstein-Hilbert action a surface term and apply Noether's theorem to the total action.

In Minkowski space, the existence of the Poincaré group and corresponding Killing vectors leads to the definitions of total momentum and total angular momentum. However, the situation is more complicated in the curved spacetime, even for an isolated system with vanishing curvature tensor at infinity. While we have a well-defined energy-momentum, there is no accordance for the notion of the angular momentum or center of mass [6]. A major difficulty in defining the angular momentum is that the group of asymptotic symmetries is infinite-parametric. Although the asymptotic symmetries group has a unique translation subgroup, there is no canonical Lorentz subgroup [7–10]. The last one emerges as a factor group of the asymptotic symmetries group by the infinite-dimensional subgroup of supertranslations. *Therefore, there does not exist a canonical way of choosing the Poincaré group as a subgroup*

of the group of asymptotic symmetries. There are too many Poincaré subgroups, one for each supertranslation, which is not translation [11].

These circumstances generate the main difficulties in the numerous attempts to find the correct definition of angular momentum in curved spacetime [12–15]. Most definitions suffer from the supertranslation ambiguities [7,8,11–13,16–19] (see [20] for review). The origin-dependence of the angular momentum causes additional difficulties. Hitherto, no satisfactory way of resolving these problems has been found.

In flat spacetime, a total angular momentum can be written as $J^{\mu\nu} = M^{\mu\nu} + S^{\mu\nu}$, where the first term, $M^{\mu\nu} = X^\mu P^\nu - P^\mu X^\nu$, is the angular momentum, with X^μ and P^ν being the position vector and momentum of the particle, respectively, and $S^{\mu\nu}$ is the intrinsic angular momentum [18].

In Minkowski spacetime, one can find a particular trajectory, describing the center of mass motion and having the property that the four-momentum is aligned with the observer's four-velocity [1,2]. Thus, if one wants to generalize this concept to general relativity, then the task would be to find a worldline with similar properties in curved spacetime [21–24]. An alternative approach is to define a preferred section at $\mathcal{I}^+$, which can be associated with the rest reference frame. This approach was developed in [25,26], where the so-called *nice sections* were introduced to study the asymptotic fields of an isolated system.

In our paper, the quasigroup approach to the conservation laws developed in [2,27–29] is applied to asymptotically flat spacetime. The Poincaré quasigroup at future null infinity ($\mathcal{I}^+$) is introduced and compared with other definitions of asymptotic symmetries that have appeared in the literature. We define the complex Noether charge associated with any element of the Poincaré quasialgebra. It may be regarded as a form of linkages by Tamburino and Winicour [16,17] but with the new gauge conditions for asymptotic symmetries.

We present new definitions of the center of mass and intrinsic angular momentum using the available tools on asymptotically flat spacetimes. Our approach is based on the fact that the intrinsic angular momentum is invariant under the Lorentz boosts [30]. It is shown that the momentum and angular momentum are defined by the Komar expression in the center of the mass reference frame.

The metric's signature is $(+, -, -, -)$ throughout the paper, lower case Greek letters range and sum from zero to three.

The paper is organized as follows. In Sec. II, we review the key ideas and tools that are indispensable for further discussions. In particular, we discuss the non-associative generalization of the transformation groups – *quasigroups of transformations* and the structure of the group of asymptotic symmetries at future null infinity. In Sec. III, we explore the reduction of the Newman-Unti (NU) to the Poincaré quasigroup. In Sec. IV, we present conserved quantities at future null infinity based on linkages introduced by Tamburino and Winicour. In Sec. V, we define a cut of $\mathcal{I}^+$ associated with the notion center-of-mass reference frame of isolated systems and show that the Komar expression gives the intrinsic angular momentum. In conclusion, we summarize the obtained results and discuss possible generalizations of our approach. In the appendices, the details of our calculations are presented.

2. Mathematical Background

Quasigroups of Transformations.

The definition of the quasigroup of transformations first was given by Batalin [31]. Below we outline the main facts from the theory of the smooth quasigroups of transformations.

Let $\mathfrak{M}$ be a n -dimensional manifold and the continuous law of transformation is given by $x' = T_a x$, $x \in \mathfrak{M}$, where $\{a^i\}$ is the set of real parameters, $i = 1, 2, \dots, r$. The set of transformations $\{T_a\}$ forms a r -parametric quasigroup of transformations (with right action on $\mathfrak{M}$), if:

1. There exists a unit element which is common for all x^a and corresponds to $a^i = 0$: $T_a x|_{a=0} = x$;

2. The modified composition law holds:

$$T_a T_b x = T_{\varphi(b,a;x)} x; \quad (2)$$

3. The left and right units coincide:

$$\varphi(a, 0; x) = a, \quad \varphi(0, b; x) = b;$$

4. The modified law of associativity is satisfied:

$$\varphi(\varphi(a, b; x), c; x) = \varphi(a, \varphi(b, c; T_a x); x);$$

5. The transformation inverse to T_a exists: $x = T_a^{-1} x'$.

By the other side, the generators of infinitesimal transformations,

$$\Gamma_i = (\partial(T_a x)^\alpha / \partial a^i)|_{a=0} \partial / \partial x^\alpha \equiv R_i^\alpha \partial / \partial x^\alpha,$$

form *quasialgebra* and obey the commutation relations:

$$[\Gamma_i, \Gamma_j] = C_{ij}^p(x) \Gamma_p. \quad (3)$$

Here $C_{ij}^p(x)$ are the structure functions satisfying the modified Jacobi identity

$$C_{ij,\alpha}^p R_k^\alpha + C_{jk,\alpha}^p R_i^\alpha + C_{ki,\alpha}^p R_j^\alpha + C_{ij}^l C_{kl}^p + C_{jk}^l C_{il}^p + C_{ki}^l C_{jl}^p = 0. \quad (4)$$

Theorem. Let the given functions R_i^α , C_{kj}^p obey the Eqs. (3), (4), then locally the quasigroup of transformations is reconstructed as the solution of a set of differential equations:

$$\frac{\partial \tilde{x}^\alpha}{\partial a^i} = R_j^\alpha(\tilde{x}) \lambda_i^j(a; x), \quad \tilde{x}^\alpha(0) = x^\alpha, \quad (5)$$

$$\frac{\partial \lambda_j^i}{\partial a^p} - \frac{\partial \lambda_p^i}{\partial a^j} + C_{mn}^i(\tilde{x}) \lambda_p^m \lambda_j^n = 0, \quad (6)$$

$$\lambda_j^i(0; x) = \delta_j^i.$$

Eq. (5) is an analog of the Lie equation, and Eq. (6) is the generalized Maurer-Cartan equation [31].

Definition. The Poincaré quasigroup is defined as a semi-direct product of the Lorentz quasigroup and the group of translations.

The Poincaré quasialgebra is described by the following set of the commutation relations:

$$[T_a, T_b] = 0, \quad [T_a, L_B] = C_{aB}^b(x) T_b, \quad [L_A, L_B] = C_{AB}^D(x) L_D, \quad (7)$$

C_{AB}^D , C_{aB}^b being the *structure functions*. The last commutation relations mean that the generators of the Lorentz quasigroup form a closed quasialgebra [29].

NP formalism

This section outlines the main facts from the Newman-Penrose (NP) formalism indispensable for future discussions. We follow notations in [32–34].

At each point of spacetime, we introduce a null NP basis $\{l, n, m, \bar{m}\}$ with the following non-vanishing scalar products: $l \cdot n = 1$, $m \cdot \bar{m} = -1$. The metric components are given by

$$g^{\alpha\beta} = l^\alpha n^\beta + l^\beta n^\alpha - m^\alpha \bar{m}^\beta - \bar{m}^\beta m^\alpha, \quad (8)$$

where *bar* means complex conjugate.

To introduce the coordinate system we choose a one-parameter family of two-dimensional space-like cross sections of future null infinity, $\mathcal{I}^+$, which are labeled by a coordinate u . We assume that $l = \partial/\partial r$, being a future directed null vector, is tangent to the surface $u = \text{const}$ and a null vector n is parallelly propagated along the l congruence. An affine parameter r is normalized by the condition $l^\alpha \partial r / \partial x^\alpha = 1$. Since the topology of $\mathcal{I}^+$ is $S^2 \times \mathbb{R}$ then “cuts” of $\mathcal{I}^+$ can be labeled by a complex stereographic coordinate $\zeta = e^{i\varphi} \cot \theta/2$ [35].

From the null tetrad basis, complex Ricci rotation coefficients are defined as follows:

$$\rho = l_{\alpha;\beta} m^\alpha \bar{m}^\beta, \quad \mu = -n_{\alpha;\beta} m^\alpha m^\beta, \quad (9)$$

$$\sigma = l_{\alpha;\beta} m^\alpha m^\beta, \quad \lambda = -n_{\alpha;\beta} \bar{m}^\alpha \bar{m}^\beta, \quad (10)$$

$$\kappa = l_{\alpha;\beta} m^\alpha l^\beta, \quad \nu = -n_{\alpha;\beta} \bar{m}^\alpha n^\beta, \quad (11)$$

$$\tau = l_{\alpha;\beta} m^\alpha n^\beta, \quad \pi = -n_{\alpha;\beta} \bar{m}^\alpha l^\beta, \quad (12)$$

$$\epsilon = \frac{1}{2}(l_{\alpha;\beta} n^\alpha l^\beta - m_{\alpha;\beta} \bar{m}^\alpha l^\beta), \quad (13)$$

$$\gamma = -\frac{1}{2}(n_{\alpha;\beta} l^\alpha n^\beta - \bar{m}_{\alpha;\beta} m^\alpha n^\beta), \quad (14)$$

$$\alpha = \frac{1}{2}(l_{\alpha;\beta} n^\alpha \bar{m}^\beta - m_{\alpha;\beta} \bar{m}^\alpha \bar{m}^\beta), \quad (15)$$

$$\beta = -\frac{1}{2}(n_{\alpha;\beta} l^\alpha m^\beta - \bar{m}_{\alpha;\beta} m^\alpha m^\beta). \quad (16)$$

The Riemann tensor is decomposed into its irreducible parts in a such way, that the corresponding tetrad components are labeled

$$\Psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \quad (17)$$

$$\Psi_1 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta, \quad (18)$$

$$\Psi_2 = -\frac{1}{2}C_{\alpha\beta\gamma\delta}(l^\alpha n^\beta l^\gamma n^\delta - l^\alpha n^\beta m^\gamma \bar{m}^\delta), \quad (19)$$

$$\Psi_3 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta, \quad (20)$$

$$\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta, \quad (21)$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor.

For further, it is convenient to introduce the notation

$$\begin{aligned} D &= l^\alpha \nabla_\alpha, & \Delta &= n^\alpha \nabla_\alpha, \\ \delta &= m^\alpha \nabla_\alpha, & \bar{\delta} &= \bar{m}^\alpha \nabla_\alpha, \end{aligned} \quad (22)$$

for the projections of the covariant derivatives onto the null tetrad. Then one can write

$$\begin{aligned} D &= \frac{\partial}{\partial r}, & \Delta &= \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X^A \frac{\partial}{\partial x^A}, \\ \delta &= \omega \frac{\partial}{\partial r} + \zeta^A \frac{\partial}{\partial x^A}, & A &= 2, 3, \end{aligned}$$

for some U, X^A, ω and ζ^A , where $x^2 = \zeta, x^3 = \bar{\zeta}$ [34,36].

In what follows, employing the coordinate freedom, we will use the Bondi coordinates at $\mathcal{I}^+$. This choice of coordinates implies that a two-dimensional surface S_∞ , being obtained as a cut $u = \text{const}$, is a two-sphere S^2 with the line element written as

$$ds^2 = \frac{4d\bar{\zeta}d\zeta}{1 + \bar{\zeta}\zeta}. \quad (23)$$

We choose the NP-basis at $\mathcal{S}^+$ as follows:

$$\Delta^0 = \frac{\partial}{\partial u}, \quad (24)$$

$$\delta^0 = P \frac{\partial}{\partial \bar{\zeta}} = -\frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right), \quad (25)$$

where $P = \zeta(1 + \zeta\bar{\zeta})/(\sqrt{2}|\zeta|)$. The sign “-” in the definition of δ^0 is introduced to provide an agreement with the standard form of raising and lowering operators (see Appendix A).

The Weyl tensor components and essential for future spin coefficients, α , β , λ , and σ , asymptotically are [18,32,36]:

$$\alpha = \alpha^0 r^{-1} + \mathcal{O}(r^{-2}), \quad (26)$$

$$\beta = \beta^0 r^{-1} + \mathcal{O}(r^{-2}), \quad (27)$$

$$\lambda = \lambda^0 r^{-1} + \mathcal{O}(r^{-2}), \quad (28)$$

$$\sigma = \sigma^0 r^{-2} + \mathcal{O}(r^{-3}), \quad (29)$$

$$\Psi_0 = \Psi_0^0 r^{-5} + \mathcal{O}(r^{-6}), \quad (30)$$

$$\Psi_1 = \Psi_1^0 r^{-4} + \mathcal{O}(r^{-5}), \quad (31)$$

$$\Psi_2 = \Psi_2^0 r^{-3} + \mathcal{O}(r^{-4}), \quad (32)$$

$$\Psi_3 = \Psi_3^0 r^{-2} + \mathcal{O}(r^{-3}), \quad (33)$$

$$\Psi_4 = \Psi_4^0 r^{-1} + \mathcal{O}(r^{-2}). \quad (34)$$

In the Bondi coordinates, relationships between functions on $\mathcal{S}^+$ are [18,22,37]:

$$\alpha^0 = \frac{\bar{P}}{2} \frac{\partial \ln P}{\partial \bar{\zeta}}, \quad (35)$$

$$\beta^0 = -\bar{\alpha}^0, \quad \lambda^0 = \dot{\sigma}^0, \quad (36)$$

where “dot” denotes the derivative $\partial/\partial u$.

The raising and lowering operators $\bar{\delta}$ and δ , respectively, are defined by the following expressions:

$$\bar{\delta}\eta = \delta\eta + s(\bar{\alpha}^0 - \beta)\eta, \quad (37)$$

$$\delta\eta = \bar{\delta}\eta - s(\bar{\alpha}^0 - \beta)\eta, \quad (38)$$

where s is the spin weight of the function η [18,35].

The Weyl tensor [22]:

$$\Psi_2^0 - \bar{\Psi}_2^0 = \bar{\sigma}^0 \bar{\lambda}^0 - \sigma^0 \lambda^0 + \bar{\delta}^2 \sigma^0 - \delta^2 \bar{\sigma}^0, \quad (39)$$

$$\Psi_3^0 = -\bar{\delta} \lambda^0, \quad (40)$$

$$\Psi_4^0 = -\dot{\lambda}^0. \quad (41)$$

The mass aspect function [9,26,34]

$$\Psi = \Psi_2^0 + \sigma^0 \lambda^0 + \bar{\delta}^2 \bar{\sigma}^0, \quad (42)$$

satisfies the reality condition, $\Psi = \bar{\Psi}$.

The evolution equations (Bianci identities) have the following asymptotic form:

$$\dot{\Psi}_0^0 = \bar{\partial}\Psi_1^0 + 3\sigma^0\Psi_2^0, \quad (43)$$

$$\dot{\Psi}_1^0 = \bar{\partial}\Psi_2^0 + 2\sigma^0\Psi_3^0, \quad (44)$$

$$\dot{\Psi}_2^0 = \bar{\partial}\Psi_3^0 + \sigma^0\Psi_4^0. \quad (45)$$

Comments. The relations Eqs. (40) – (45) have the opposite sign in comparison with similar expressions in Ref. [32]. This is due the difference in the sign of δ^0 . In [32], it is defined as $\delta^0 = -P\partial_{\bar{\zeta}}$.

3. Asymptotic Symmetries and the Poincaré Quasigroup

The asymptotic symmetries of the asymptotically flat at future null infinity spacetime are described by the infinite-parametric Newman-Unti (NU) group [32,36,38,39]. The latter is defined by the transformation $\mathcal{I}^+ \rightarrow \mathcal{I}^+$, having the form

$$\begin{aligned} u &\rightarrow u' = f(u, \zeta, \bar{\zeta}), \quad \partial f / \partial u > 0, \\ \zeta &\rightarrow \zeta' = (\alpha\zeta + \beta) / (\gamma\zeta + \delta), \quad \alpha\delta - \beta\gamma = 1, \end{aligned}$$

where $f(u, \zeta, \bar{\zeta})$ is an arbitrary function.

The infinite-dimensional Bondi-Metzner-Sachs (BMS) group preserving strong conformal geometry, is a subgroup of the NU-group. The BMS-group is defined as follows [9,10,14,21,26]:

$$u \rightarrow u' = K(\zeta, \bar{\zeta})(u + a(\zeta, \bar{\zeta})), \quad (46)$$

$$\zeta \rightarrow \zeta' = (\alpha\zeta + \beta) / (\gamma\zeta + \delta), \quad (47)$$

$$\alpha\delta - \beta\gamma = 1, \quad (48)$$

where $\alpha, \beta, \gamma, \delta$ are complex constants, and $a(\zeta, \bar{\zeta})$ is an arbitrary regular function on S^2 , besides

$$K(\zeta, \bar{\zeta}) = \frac{(1 + \zeta\bar{\zeta})}{(|\alpha\zeta + \beta|^2 + |\gamma\zeta + \delta|^2)}. \quad (49)$$

The infinite-parameter normal subgroup of BMS-group

$$\zeta' = \zeta, \quad u' = u + a(\zeta, \bar{\zeta}), \quad (50)$$

is called the subgroup of *supertranslations* and contains a four-parameter normal translation subgroup

$$a = \frac{p + q\zeta + \bar{q}\bar{\zeta} + g\zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}}, \quad \Im p = \Im g = 0. \quad (51)$$

The quotient (factor) group of the BMS-group by the supertranslations consists from the conformal transformations $S^2 \rightarrow S^2$ and it is isomorphic to the *proper orthochronous Lorentz group*.

Since the BMS group is the semi-direct product of the Lorentz group and supertranslation's group, there is no canonical way to embed the Poincaré group in the BMS group. One has an infinite number of alternatives to extract the Poincaré group. However, at least in the Minkowski spacetime, one can elucidate which additional structure on $\mathcal{I}^+$ the Poincaré group preserves. It turns out to be that the Poincaré group transforms the so-called *good cuts* – cuts with vanishing shear σ^0 , to the good cuts [40].

It can be easily seen by considering the transformation of shear under supertranslations Eq. (50). We obtain $\sigma^0 \rightarrow \sigma'^0 = \sigma^0 - \bar{\partial}^2 a$, where $\bar{\partial}$ is the “eth” operator on $\mathcal{I}^+$. Thus, a supertranslation transforms a good cut to a bad cut – a cut with nonvanishing shear. If we impose the condition $\sigma'^0 = \sigma^0 = 0$, we obtain $\bar{\partial}^2 a = 0$. The solution of this equation with a real a yields a four-parametric subgroup of translations defined by Eq. (51).

3.1. Reduction of the NU-Group to the Poincaré Quasigroup

The infinitesimal NU-group is obtained from the asymptotic Killing equations [38]:

$$\mathcal{L}_{\xi} g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0(r^{-n}), \quad (52)$$

$$l^{\nu} \mathcal{L}_{\xi} g_{\mu\nu} = (\xi_{\mu;\nu} + \xi_{\nu;\mu}) l^{\nu} = Q(u, \zeta, \bar{\zeta}) l_{\mu}, \quad (53)$$

where $Q(u, \zeta, \bar{\zeta})$ is an arbitrary function on $\mathcal{S}^+$.

One can write a general element of NU-algebra as follows:

$$\begin{aligned} \xi &= B(u, \zeta, \bar{\zeta}) \Delta^0 + C(u, \zeta, \bar{\zeta}) \bar{\delta}^0 + \bar{C}(u, \zeta, \bar{\zeta}) \delta^0, \\ \bar{\partial} C &= 0, \end{aligned} \quad (54)$$

where $\bar{\partial}$, Δ^0 , δ^0 are the standard NP operators, “eth”, Δ and δ , restricted on $\mathcal{S}^+$.

The generators of the four-parameter translation subgroup are given by

$$\xi_a = B_a(\zeta, \bar{\zeta}) \Delta^0, \quad (55)$$

where the function B_a assumed to be a real function and is the solution of the following equation:

$$\bar{\partial}^2 B_a = 0, \quad \Im B_a = 0 \quad (a = 0, 1, 2, 3). \quad (56)$$

The generators of “Lorentz group” are determined as follows:

$$\xi_A = B_A(u, \zeta, \bar{\zeta}) \Delta^0 + C_A \bar{\delta}^0 + \bar{C}_A \delta^0, \quad (57)$$

$$\bar{\partial} C_A = 0, \quad (A = 1, 2, \dots, 6), \quad (58)$$

where $B_A(u, \zeta, \bar{\zeta})$ is an arbitrary real function.

The generators of the NU-group obey the commutation relations:

$$[\xi_a, \xi_b] = 0, \quad [\xi_a, \xi_B] = C_{aB}^b(u, \zeta, \bar{\zeta}) \xi_b, \quad (59)$$

$$[\xi_A, \xi_B] = C_{AB}^D(u, \zeta, \bar{\zeta}) \xi_D, \quad (60)$$

where C_{aB}^b , C_{AB}^D are the *structure functions*. This points out that the NU group is a *quasigroup* with the closed Lorentz quasialgebra.

To reduce the NU-group to the particular Poincaré quasigroup one needs to impose the constraints on a function $B_A(u, \zeta, \bar{\zeta})$ and, thus, fix the supertranslational ambiguity in the definition of the Lorentz quasigroup.

In our approach, we use the fact that a group of isometries transforms an arbitrary geodesic to a geodesic one, and the Killing vectors satisfy the geodesic deviation equation for any geodesic [28,29]. In the construction below, only *null* geodesics passing inward are transformed to the geodesics under the transformations of the Poincaré quasigroup. Instead of using the approximate Killing equations, we propagate the asymptotic generators ξ defined on $\mathcal{S}^+$ inward along the null surface Γ intersecting $\mathcal{S}^+$ in Σ^+ employing the geodesic deviation equation.

$$\nabla_l^2 \xi + R(\xi, l)l = 0. \quad (61)$$

Since the geodesic deviation equation is the second-order ordinary differential equation for obtaining the unique solution, we need to impose the initial conditions on the vector ξ and its first derivatives on $\mathcal{S}^+$. We use the asymptotic Killing equations for determining them.

The key idea behind our approach is to use the geodesic deviation equation only for a null geodesic congruence passing inward $\mathcal{S}^+$ to define the generators of the Poincaré quasigroup. It

implies that the Poincaré quasigroup transforms not an arbitrary geodesic to a geodesic, but only the null geodesics belonged to the null congruence defined above.

3.1.1. Minkowski spacetime

We demonstrate our approach in the Minkowski spacetime. The key idea is to reduce the NU group to the ten-parametric Poincaré group, imposing the appropriate conditions on an arbitrary function B_A , and thus fixing the supertranslational freedom (See for details Refs. [27–29]).

Let us write the Killing vector as

$$\xi = AD + B\Delta + \bar{C}\delta + C\bar{\delta}. \quad (62)$$

Using the asymptotic expansion

$$A = A_1 r + A_0 + A_{-1} r^{-1} + O(r^{-2}), \quad (63)$$

$$B = B_1 r + B_0 + B_{-1} r^{-1} + O(r^{-2}), \quad (64)$$

$$C = C_1 r + C_0 + C_{-1} r^{-1} + O(r^{-2}), \quad (65)$$

we obtain the solution of the geodesic deviation equation in the following form:

$$A_{-n} = A_{-n}(u, \zeta, \bar{\zeta}), \quad C_{-n} = C_{-n}(u, \zeta, \bar{\zeta}), \quad B_{-n} = 0 \quad (n \geq 1). \quad (66)$$

The explicit dependence of $A_{-n}(u, \zeta, \bar{\zeta})$, and $C_{-n}(u, \zeta, \bar{\zeta})$ is not important for study the structure of asymptotic symmetries.

To obtain the unique solution of the geodesic deviation equation, we have to impose the conditions on the functions A, B, C and its first derivatives at $\mathcal{I}^+$. This implies that $A_0, A_1, B_0, B_1, C_0, C_1$ should be determined. We adapt the asymptotic Killing equations to determine these coefficients:

$$\begin{aligned} \lim_{r \rightarrow \infty} l^\mu l^\nu \mathcal{L}_\xi g_{\mu\nu} &= 0, \quad \lim_{r \rightarrow \infty} m^\mu n^\nu \mathcal{L}_\xi g_{\mu\nu} = 0, \\ \lim_{r \rightarrow \infty} m^\mu \bar{m}^\nu \mathcal{L}_\xi g_{\mu\nu} &= 0, \quad \lim_{r \rightarrow \infty} r m^\mu \bar{m}^\nu \mathcal{L}_\xi g_{\mu\nu} = 0, \\ \lim_{r \rightarrow \infty} r l^\mu m^\nu \mathcal{L}_\xi g_{\mu\nu} &= 0. \end{aligned} \quad (67)$$

After some algebra we obtain

$$A_0 = -\frac{1}{2}(\partial\bar{C}_0 + \bar{\partial}C_0) - \frac{B_0}{2}(\mu^0 + \bar{\mu}^0) + \frac{1}{2}(\tau^0\bar{C}_1 + \bar{\tau}^0C_1), \quad (68)$$

$$A_1 = -\frac{1}{2}(\partial\bar{C}_1 + \bar{\partial}C_1), \quad (69)$$

$$B_1 = 0, \quad (70)$$

$$C_0 = -\partial B_0 + \sigma^0 \bar{C}_1, \quad (71)$$

$$\partial C_0 = \partial\sigma^0 C_1 + \frac{\sigma^0}{2}(\partial\bar{C}_1 - \bar{\partial}C_1), \quad (72)$$

$$\dot{C}_1 = 0, \quad \bar{\partial}C_1 = 0, \quad (73)$$

where “dot” denotes the derivative with respect to the retarded time u . Substituting C_0 from Eq. (71) in Eq. (72), we get

$$\dot{C}_1 = 0, \quad \bar{\partial}C_1 = 0, \quad (74)$$

$$\partial^2 B_0 - \frac{\sigma^0}{2}(3\partial\bar{C}_1 - \bar{\partial}C_1) - \bar{C}_1 \partial\sigma^0 - C_1 \bar{\partial}\sigma^0 = 0. \quad (75)$$

A general solution of this system can be written as

$$\begin{aligned} B_0 &= B_t + \bar{\partial}\eta\bar{C}_1 + \bar{\partial}\eta C_1 + \frac{u-\eta}{2}(\bar{\partial}\bar{C}_1 + \bar{\partial}C_1), \\ \bar{\partial}^2 B_t &= 0, \quad \bar{\partial}C_1 = 0, \end{aligned} \quad (76)$$

where $\sigma^0 = \bar{\partial}^2\eta$.

The system of differential constraints, Eqs. (74)-(75), is the unique one that determines the functions B_0, C_1 and restricts the NU group to a particular Poincaré group. Thus in the Minkowski spacetime, one can reduce the NU-group to the Poincaré group even for “bad” cuts ($\sigma^0 \neq 0$).

Now an arbitrary Killing vector at $\mathcal{S}^+$, $\xi^0 = \xi|_{\mathcal{S}^+}$, can be wrtitten as

$$\xi^0 = B_0(u, \zeta, \bar{\zeta})\Delta^0 + C_1(\zeta, \bar{\zeta})\bar{\delta}^0 + \bar{C}_1(\zeta, \bar{\zeta})\delta^0. \quad (77)$$

To specify the generators of the Poincaré group one should impose the additional conditions on the functions B_0 and C_1 .

Translations. – The generators of translations are

$$l_a = B_a\Delta^0, \quad a = 0, 1, 2, 3, \quad (78)$$

with $\Im B_a = 0$, and B_a being solutions of the differential equation $\bar{\partial}^2 B_a = 0$. With real B_a we obtain four independent solutions of this equation yielding

$$l_a = (1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \quad (79)$$

Boosts and rotations. – The generators of boosts and rotations are given by

$$\xi_A = B_A\Delta^0 + C_A\bar{\delta}^0 + \bar{C}_A\delta^0, \quad (80)$$

where $\bar{\partial}C_A = 0$ and

$$B_A = \bar{\partial}\eta\bar{C}_A + \bar{\partial}\eta C_A + \frac{u-\eta}{2}(\bar{\partial}\bar{C}_A + \bar{\partial}C_A), \quad (81)$$

with $\Im B_A = \Im\eta = 0$ and $\sigma^0 = \bar{\partial}^2\eta$. Imposing the additional conditions:

$$\bar{\partial}C_A + \bar{\partial}\bar{C}_A = 0 \quad (\text{rotations}), \quad (82)$$

$$\bar{\partial}C_A - \bar{\partial}\bar{C}_A = 0 \quad (\text{boosts}), \quad (83)$$

we obtain six independent solutions of the equation $\bar{\partial}C_A = 0$. It is convenient to divide them in two groups, writing $C_A = \{L_i, R_i\}$, where L_i and R_i describe the rotations and boosts, respectively. We denote the generators of the complex Lorentz group as $\Gamma_A = \bar{C}_A\delta^0 = \xi_A\partial_\zeta$. Using the results of Ref. [39], we obtain the generators of the boost and rotations. They are shown in Table I.

Table 1. Generators of the complex Lorentz group. (Note that L_1, L_2, L_3 are defined from L_x, L_y, L_z . (see Appendix B))

Generator	$\Gamma_A = \zeta_A \frac{\partial}{\partial \bar{\zeta}}$
$\Gamma_1 = L_1$	$\frac{i}{2}(1 - \zeta^2) \frac{\partial}{\partial \bar{\zeta}}$
$\Gamma_2 = L_2$	$-\frac{1}{2}(1 + \zeta^2) \frac{\partial}{\partial \bar{\zeta}}$
$\Gamma_3 = L_3$	$i\zeta \frac{\partial}{\partial \bar{\zeta}}$
$\Gamma_4 = R_1$	$\frac{1}{2}(1 - \zeta^2) \frac{\partial}{\partial \bar{\zeta}}$
$\Gamma_5 = R_2$	$\frac{i}{2}(1 + \zeta^2) \frac{\partial}{\partial \bar{\zeta}}$
$\Gamma_6 = R_3$	$\zeta \frac{\partial}{\partial \bar{\zeta}}$

Let us introduce a complex vector ζ_c at $\mathcal{S}^+$ such that an arbitrary element of the infinitesimal NU group is written as $\zeta = \zeta_c + \bar{\zeta}_c$, where

$$\zeta_c = \zeta_s \Delta^0 + \zeta_1 \delta^0. \tag{84}$$

(Hereafter we omit the index “0” in ζ^0 .) We specify the vector ζ_s as follows:

$$\zeta_s = \zeta_t + \bar{\partial} \eta \zeta_1 + \frac{u - \eta}{2} \bar{\partial} \zeta_1, \tag{85}$$

$$\bar{\partial}^2 \zeta_t^0 = 0, \quad \bar{\partial} \zeta_1 = 0, \tag{86}$$

where $\Im \zeta_t^0 = \Im \eta = 0$ and $\bar{\partial}^2 \eta = \sigma^0$. Comparing this expression with Eq. (77), we find $B_0 = \zeta_s + \bar{\zeta}_s$ and $C_1 = \bar{\zeta}_1$.

A straightforward computation shows that ζ_s obeys the following differential equation:

$$\begin{aligned} \bar{\partial}^2 \zeta_s &= \frac{3}{2} \sigma^0 \bar{\partial} \zeta_1 + \bar{\partial} \sigma^0 \zeta_1, \\ \bar{\partial} \zeta_1 &= 0. \end{aligned} \tag{87}$$

Thus, instead of employing Eqs. (74)-(75) to reduce the NU group to the Poincaré group in the Minkowski spacetime, one can consider the equivalent differential constraints Eq. (87).

3.1.2. General case: asymptotically flat spacetime with radiation

As was mentioned above, the NU group is an infinite-dimensional group and therefore there is no exist an unique way to reduce the NU group to the finite-dimensional group even in the Minkowski spacetime. All attempts suffer on the supertranslational ambiguity. To overcome this issue, we impose on ζ_c the differential constraints restricting the NU group to a particular Poincaré quasigroup [29]:

$$\bar{\partial}^2 \zeta_s = \frac{3}{2} \sigma^0 \bar{\partial} \zeta_1 + \bar{\partial} \sigma^0 \zeta_1, \tag{88}$$

$$\bar{\partial} \zeta_1 = 0. \tag{89}$$

Since the spin weight of the asymptotic shear σ^0 is two, one can write $\sigma^0 = \bar{\partial}^2 \eta$, where $\eta = \eta(u, \zeta, \bar{\zeta})$ is a complex function. Then a general solution of the Eq. (88) can be written as

$$\zeta_s = \zeta_t + \bar{\partial} \eta \zeta_1 + \frac{u - \eta}{2} \bar{\partial} \zeta_1, \tag{90}$$

where $\bar{\partial}^2 \zeta_t = 0$.

We consider Eq. (88) as the differential constraint restricting the NU group to a particular Poincaré quasigroup [29]. In the absence of radiation, the differential constraint Eq. (88) is compatible with

the Killing equations, and the Poincaré quasigroup becomes the Poincaré group. Note that the same constraint was obtained in [15,41] in the twistor theory framework.

The structure of the Poincaré quasialgebra is as follows:

Translations. – The generators of translations are given by

$$\xi_a = \xi_{ta} + \bar{\xi}_{ta} = B_a \Delta^0, \quad (91)$$

where $B_a = \bar{B}_a$ with B_a being solution of the equation

$$\partial^2 B_a = 0. \quad (92)$$

There are four independent solutions of this equation, if we assume that $\Im B_a = 0$.

Lorentz quasialgebra. – The generators of boosts and rotations are given by

$$\xi_A = \xi_{cA} + \bar{\xi}_{cA}, \quad (93)$$

where

$$\xi_{cA} = (\partial\eta \bar{C}_A + \frac{u-\eta}{2} \partial \bar{C}_A) \Delta^0 + \bar{C}_A \delta^0, \quad (94)$$

and $\partial C_A = 0$. There are six independent solutions of the equation $\partial C_A = 0$, if we impose additional constraints:

$$\partial C_A + \partial \bar{C}_A = 0 \quad (\text{rotations}), \quad (95)$$

$$\partial C_A - \partial \bar{C}_A = 0 \quad (\text{boosts}), \quad (96)$$

The straightforward computation shows that the generators of the Poincaré quasigroup obey at $\mathcal{S}^+$ the commutation relations:

$$[\xi_a, \xi_b] = 0, \quad [\xi_a, \xi_B] = C_{aB}^b(u, \zeta, \bar{\zeta}) \xi_b, \quad (97)$$

$$[\xi_A, \xi_B] = C_{AB}^D(u, \zeta, \bar{\zeta}) \xi_D, \quad (98)$$

C_{AB}^D, C_{aB}^b being the *structure functions*. The last commutation relations mean that the generators of Lorentz quasigroup form a closed algebra.

4. Energy-Momentum and Angular Momentum at $\mathcal{S}^+$

As well known, the general covariance of the Einstein equations results in differential conservation laws related to the field equations. From Noether's theorem, it follows that for an arbitrary diffeomorphism generated by the vector field ξ , the invariance of the Lagrangian for the Einstein equations leads to the conservation laws of the form

$$\mathfrak{J}^\mu(\xi)_{;\mu} = 0, \quad (99)$$

where the vector density $\mathfrak{J}^\mu$ is defined as $\mathfrak{J}^\mu(\xi) = \mathfrak{h}^{\mu\nu}_{;\nu}$, with $\mathfrak{h}^{\mu\nu} = -\mathfrak{h}^{\nu\mu}$ being the *superpotential*, which is constructed from the densities of spin, bispin, vector field ξ and its derivatives [1–3].

For the Einstein-Hilbert action, one obtains up to a factor of 2 a simple expression

$$\mathfrak{J}^\mu = -\frac{1}{4\pi} \left(\sqrt{-g} \xi^{[\mu;\nu]} \right)_{;\nu} \quad (100)$$

which was first given by Komar [4]. It is impossible simply to “renormalize” $\mathfrak{J}^\mu$ by a factor of 2 because the normalization for energy-momentum and angular momentum differs by a factor of 2. The resolution of this problem is known as pointed out in [5]. One needs to add to the Einstein-Hilbert

action I a surface term I_s and apply Noether's theorem to $I + I_s$. Komar's expression provides a fully satisfactory notion of the total mass in stationary, asymptotically flat spacetimes.

As is known, the Komar integral is not invariant under a change of the choice of the generators of time translations in the equivalence class associated with the given BMS translation. Besides, the resulting energy would not be the monotonically decreasing Bondi energy but the less physical Newman-Unti energy [38].

For an asymptotically flat at future null infinity spacetime the modified "gauge invariant" Komar integral (linkage) was introduced by Tamburino and Winicour [16,17]. The computation leads to the following coordinate independent expression [8,12,13]:

$$\mathcal{L}_{\xi} = -\frac{1}{4\pi} \oint \left[B \left(\Psi_2^0 + \sigma^0 \lambda^0 - \bar{\partial}^2 \bar{\sigma}^0 \right) + \bar{C} \left(\Psi_1^0 - \sigma^0 \bar{\partial}^2 \bar{\sigma}^0 - (1/2) \bar{\partial}(\sigma^0 \bar{\sigma}^0) \right) \right] d\Omega, \quad (101)$$

where $\lambda^0 = \bar{\sigma}^0$, and ξ is an arbitrary generator of the NU group at $\mathcal{I}^+$,

$$\xi = B\Delta^0 + \bar{C}\delta^0 + C\bar{\delta}^0, \quad \bar{\partial}C = 0. \quad (102)$$

A complex generator, $\xi_c = \xi_s \Delta^0 + \xi_1 \delta^0$, of the Poincaré quasigroup yields the complex Noether charge that can be written as [11]:

$$Q_{\xi_c} = -\frac{1}{8\pi} \oint \left\{ \xi_s \left(\Psi_2^0 - \sigma^0 \lambda^0 - \bar{\partial}^2 \bar{\sigma}^0 \right) + \xi_1 \left(\Psi_1^0 - \sigma^0 \bar{\partial}^2 \bar{\sigma}^0 - (1/2) \bar{\partial}(\sigma^0 \bar{\sigma}^0) \right) \right\} d\Omega. \quad (103)$$

Setting $B = (\xi_s + \bar{\xi}_s)/2$ and $\bar{C}_1 = \xi_1$, we obtain

$$\mathcal{L}_{\xi} = Q_{\xi_c} + \bar{Q}_{\xi_c} = Q_{\xi_c} + Q_{\bar{\xi}_c}. \quad (104)$$

The factor $1/8\pi$ in Eq. (103) is determined by calculating the intrinsic angular momentum, $J = ma$, for the Kerr metric with the rotational Killing vector, L_z , yielding $\xi_1 = -L_z \cdot \bar{m} = i/\sqrt{2} \sin \theta$. The Weyl scalar, $\psi_1^0 = \sigma/\sqrt{2}$ is computed with respect to a Bondi frame with shear-free cross sections [11,42].

As in the previous section, to reduce the NU group to the Poincaré quasigroup, we impose the following differential constraints:

$$\bar{\partial}^2 \xi_s = \frac{3}{2} \sigma^0 \bar{\partial} \xi_1 + \bar{\partial} \sigma^0 \xi_1, \quad (105)$$

$$\bar{\partial} \xi_1 = 0. \quad (106)$$

One can show that the following integral identity is valid:

$$\oint \bar{\partial}^2 \xi_s \bar{\sigma}^0 d\Omega = \oint \xi_1 \left(\bar{\partial} \sigma^0 \bar{\sigma}^0 - \frac{3}{2} \bar{\partial}(\sigma^0 \bar{\sigma}^0) \right) d\Omega. \quad (107)$$

Using this identity in Eq. (103), one can rewrite it in the equivalent form introduced in [15]

$$Q_{\xi_c} = -\frac{1}{8\pi} \oint \left\{ \xi_s \left(\Psi_2^0 + \sigma^0 \lambda^0 + \bar{\partial}^2 \bar{\sigma}^0 \right) + \xi_1 \left(\Psi_1^0 + \sigma^0 \bar{\partial}^2 \bar{\sigma}^0 + (1/2) \bar{\partial}(\sigma^0 \bar{\sigma}^0) \right) \right\} d\Omega. \quad (108)$$

Now employing Eqs. (105)-(106) and performing integration by parts, we find that Eq. (108) can be recast as

$$Q_{\xi_c} = -\frac{1}{8\pi} \oint \left(\xi_s (\Psi_2^0 + \sigma^0 \lambda^0) + \xi_1 \Psi_1^0 \right) d\Omega. \quad (109)$$

We adopt this as the definition of the conserved quantities on $\mathcal{I}^+$ associated with the generators of the Poincaré quasigroup, writing [11,15,29,43]

$$\mathcal{L}_{\xi} = Q_{\xi_c} + \overline{Q}_{\xi_c}. \quad (110)$$

The integral four-momentum is given by

$$P_a = -\frac{1}{4\pi} \Re \oint l_a (\Psi_2^0 + \sigma^0 \lambda^0) d\Omega. \quad (111)$$

Using the Bianchi identities, we compute the loss of energy-momentum and get the standard expression (see, e. g. [8])

$$\dot{P}_a = -(1/4\pi) \oint l_a |\lambda^0|^2 d\Omega. \quad (112)$$

The angular momentum is given by

$$M_i = -\frac{1}{4\pi} \Re \oint \left(\xi_i (\Psi_2^0 + \sigma^0 \lambda^0) + \bar{L}_i \Psi_1^0 \right) d\Omega, \quad (113)$$

where ξ_i is the solution of the Eq. (105),

$$\xi_i = \eth \eta \bar{L}_i + \frac{u - \eta}{2} \eth L_i. \quad (114)$$

Here L_i is the solution of the equation $\eth L_i = 0$ such that $\eth L_i + \eth \bar{L}_i = 0$, and $\sigma^0 = \eth^2 \eta$.

Substituting ξ_i in Eq. (113) and performing integration by parts, we find

$$M_i = -\frac{1}{4\pi} \Re \oint \bar{L}_i \mathcal{J} dS\Omega, \quad (115)$$

where

$$\mathcal{J} = \Psi_1^0 + \frac{3}{2} \eth \eta (\Psi_2^0 + \sigma^0 \lambda^0) + \frac{\eta}{2} (\eth \Psi_2^0 + \eth (\sigma^0 \lambda^0)). \quad (116)$$

It yields the following expression for the angular momentum loss:

$$\dot{M}_i = -\frac{1}{4\pi} \Re \oint \bar{L}_i \frac{\partial \mathcal{J}}{\partial u} d\Omega. \quad (117)$$

Comments. Geroch and Winicour have given a list of properties which conserved quantities $P(\xi, \Sigma)$ defined at $\mathcal{I}^+$ should have [12]:

- $P(\xi, \Sigma)$ should be linear in the generators of the asymptotic symmetry group.
- $P(\xi, \Sigma)$ should be invariant with respect to the conformal transformations $\tilde{g}_{\mu\nu} = \tilde{\Omega}^2 g_{\mu\nu}$.
- The expression $P(\xi, \Sigma)$ should depend on the geometry of $\mathcal{I}^+$ and behavior of generators in the neighbourhood of $\mathcal{I}^+$.
- $P(\xi, \Sigma)$ should be proportional to the corresponding Komar integral for the exact symmetries and coincide with the Bondi four-momentum when ξ is a BMS translation.
- $P(\xi, \Sigma)$ should be defined also for the system with radiation on $\mathcal{I}^+$.
- There should exist a flux integral $\mathcal{I}$ which is linear in ξ and which gives the difference $P(\xi, \Sigma') - P(\xi, \Sigma)$, for Σ' and Σ closed two-surfaces on $\mathcal{I}^+$.
- In Minkowski spacetime $P(\xi, \Sigma)$ should vanish identically.

Our definition of the conserved quantities, $\mathcal{L}_{\xi} = Q_{\xi_c} + \overline{Q}_{\xi_c}$, where the complex Noether charge is given by Eq. (109), is free from the supertranslation ambiguity and satisfies all these conditions.

5. Center of Mass and Intrinsic Angular Momentum

In special relativity, total angular momentum is given by $J^{\mu\nu} = X^\mu P^\nu - P^\mu X^\nu + S^{\mu\nu}$, where the two first terms are the angular momentum, and $S^{\mu\nu}$ denotes intrinsic angular momentum, with X^μ and P^ν being the position vector and momentum of the particle, respectively. Using space-like translation freedom, one can transform the given reference frame to the center-of-mass reference frame (CMRF), where P^ν is aligned with the observer's four-velocity. In the CMRF one has $J^{\mu\nu} P_\nu = 0$. Applying the transformation $X^\mu = P^\mu / P_\nu P^\nu$, we obtain the Dixon condition on the intrinsic angular momentum, $S^{\mu\nu} P_\nu = 0$ [44].

The Dixon condition implies that the boost generators do not contribute to the internal part of the total angular momentum [30]. At future null infinity, this condition reduces to the requirement that the linkage $L_{\bar{\zeta}} = 0$ for the boost generators. We adopt this requirement for the general case by imposing the condition that the Noether charge $K_i = 0$ for the boost generators, given by

$$K_i = -\frac{1}{4\pi} \Re \oint \left(\bar{\zeta}_i^R (\Psi_2^0 + \sigma^0 \lambda^0) + \bar{R}_i \Psi_1^0 \right) d\Omega, \quad (118)$$

here

$$\bar{\zeta}_i^R = \bar{\partial} \eta \bar{R}_i + \frac{u - \eta}{2} \bar{\partial} \bar{R}_i, \quad (119)$$

and R_i is the solution of the equation $\bar{\partial} R_i = 0$ such that $\bar{\partial} \bar{R}_i - \bar{\partial} \bar{R}_i = 0$.

As one can see, the quantities R_i and L_i are related by the relations $R_i = \pm i L_i$ (See Table I in Sec. III.). Then the condition $K_i = 0$ can be written as

$$\Im \oint \left(\bar{\zeta}_i (\Psi_2^0 + \sigma^0 \lambda^0) + \bar{L}_i \Psi_1^0 \right) d\Omega = 0, \quad (120)$$

where $\bar{\zeta}_i$ are defined by Eq. (114). In terms of the complex Noether charge Eq. (120) can be recast as $\Im Q_{\bar{\zeta}_c} = 0$.

To transform the given reference frame at null infinity to the CMRF frame, we use supertranslation freedom to align the total momentum with the observer's four-velocity. To explain in detail, let us consider the Noether charge associated with the mass aspect,

$$\mathcal{Q}_a = \oint \bar{\zeta}_a \Psi d\Omega = \oint \bar{\zeta}_a (\Psi_2^0 + \sigma^0 \lambda^0 + \bar{\partial}^2 \bar{\sigma}^0) d\Omega. \quad (121)$$

In the CMRF, only the component $\mathcal{Q}_0$ is different from zero. Thus, to align the total momentum with the observer's four-velocity, one should require $\mathcal{Q}_i = 0$ for $i = 1, 2, 3$. Using this condition in Eq. (120) we obtain

$$\Im \oint (\bar{L}_i \Psi_1^0 - \bar{\zeta}_i \bar{\partial}^2 \bar{\sigma}^0) d\Omega = 0, \quad (122)$$

where the integration is performed over the section defined by equation $u - \eta_0 = 0$. Further simplification can be made by employing Eq. (105). The computation yields

$$\Im \oint \bar{L}_i (2\Psi_1^0 - \bar{\partial} \sigma^0 \bar{\sigma}^0 - 3\sigma^0 \bar{\partial} \bar{\sigma}^0) d\Omega = 0. \quad (123)$$

The obtained results are an integral expression of the condition defining the center of mass deduced in [22]. It corresponds to the Dixon condition imposed on the intrinsic angular momentum, $S^{\mu\nu} P_\nu = 0$.

To proceed further, we expand ξ_i , σ^0 and η in terms of spin-weighted spherical harmonics:

$$\xi_i = \sum_{l=0}^{\infty} \sum_{m=-l}^l \xi_{lm}^i Y_{lm}, \quad (124)$$

$$\sigma^0 = \sum_{l=2}^{\infty} \sum_{m=-l}^l h_{lm}(u) {}_2Y_{lm}, \quad (125)$$

$$\eta = \eta_0 + \sum_{l=1}^{\infty} \sum_{m=-l}^l \eta_{lm} Y_{lm}. \quad (126)$$

From the relation $\sigma^0 = \bar{\partial}^2 \eta$ it follows

$$\eta_{lm} = \frac{2h_{lm}}{\sqrt{l(l^2-1)(l+2)}}, \quad l \geq 2. \quad (127)$$

We use the freedom in the choice of η to eliminate in Eq. (124) the contribution of the term with $l = 0$, choosing the cut as $u - \eta_0 = 0$. We obtain

$$\xi_i = \sum_{l=1}^{\infty} \sum_{m=-l}^l \xi_{lm}^i Y_{lm}. \quad (128)$$

Using Eq. (128) in Eq. (121), we have

$$\mathcal{J}_i = \sum_{l=1}^{\infty} \sum_{m=-l}^l \xi_{lm}^i \oint Y_{lm} (\Psi_2^0 + \sigma^0 \lambda^0 + \bar{\partial}^2 \bar{\sigma}^0) d\Omega. \quad (129)$$

To proceed further, we consider the supermomenta $P_{lm}(\Sigma)$ introduced in [25],

$$P_{lm}(\Sigma) = -\frac{1}{\sqrt{4\pi}} \int_{\Sigma} Y_{lm} \Psi d\Omega, \quad (130)$$

where Σ is an arbitrary section of $\mathcal{S}^+$. A nice section is defined by the requirement that the supermomentum, $P_{lm}(\Sigma) = 0$ for $l \geq 1$. Thus only the component P_0 of the total momentum is nonvanishing, providing us with a geometric notion of a reference frame at rest. This, in general, involves the need to make a Lorentz boost, which keeps Σ fixed and aligns the generator of time translations with the total momentum [26]. As one can see, for the nice sections the quantity $\mathcal{J}_i = 0$.

Employing the condition $\mathcal{J}_i = 0$ in Eq. (113), after some computations we find that the intrinsic angular momentum, J_i , can be written as the Komar angular momentum:

$$J_i = -\frac{1}{4\pi} \Re \oint \bar{L}_i (\Psi_1^0 + \sigma^0 \bar{\partial} \bar{\sigma}^0) d\Omega. \quad (131)$$

An independent calculation by Gallo and Moreschi [26] confirms Eq. (131).

Hence the angular momentum loss is given by

$$\dot{J}_i = -\frac{1}{4\pi} \Re \oint \bar{L}_i (\dot{\Psi}_1^0 + \bar{\lambda}^0 \bar{\partial} \bar{\sigma}^0 + \sigma^0 \bar{\partial} \lambda^0) d\Omega, \quad (132)$$

Using the Bianchi identities [33,37]

$$\Psi_2^0 - \bar{\Psi}_2^0 = \bar{\sigma}^0 \bar{\lambda}^0 - \sigma^0 \lambda^0 + \bar{\partial}^2 \sigma^0 - \bar{\partial}^2 \bar{\sigma}^0, \quad (133)$$

$$\dot{\Psi}_1^0 = \bar{\partial} \Psi_2^0 - 2\sigma^0 \bar{\partial} \lambda^0, \quad (134)$$

Eq. (132) reduces to

$$\dot{J}_i = -\frac{1}{8\pi} \Re \oint \bar{L}_i (3\bar{\lambda}^0 \bar{\partial} \bar{\sigma}^0 - 3\sigma^0 \bar{\partial} \lambda^0 + \bar{\sigma}^0 \bar{\partial} \bar{\lambda}^0 - \lambda^0 \bar{\partial} \sigma^0) d\Omega. \quad (135)$$

This expression agrees with the results obtained in [45].

To compute the angular momentum loss, we express σ^0 and $\bar{L}_i$ in terms of spin-weighted spherical harmonics:

$$\sigma^0 = \sum_{l=2}^{\infty} \sum_{m=-l}^l h_{lm}(u) {}_2Y_{lm}, \quad (136)$$

$$\bar{L}_i = \sum_{m=-1}^1 L_{im-1} Y_{1m}. \quad (137)$$

The computation yields (see for details Appendix B)

$$\begin{aligned} \dot{J}_i &= \frac{1}{8\pi} \Re \sum_{l,m,l',m'} (h_{lm} \dot{h}_{l'm'} - \dot{h}_{lm} \bar{h}_{l'm'}) c_{lm l' m'}^i \\ &= \frac{1}{8\pi} \Re \sum_{l,m,l',m'} h_{lm} \dot{h}_{l'm'} (c_{lm l' m'}^i - \bar{c}_{l' m' l m}^i), \end{aligned} \quad (138)$$

where the coefficients $c_{lm l' m'}^i$ are given in terms of the Wigner 3- j symbols:

$$\begin{aligned} c_{lm l' m'}^i &= (-1)^{m'} \sqrt{\frac{3(2l+1)(2l'+1)}{8\pi}} \left\{ 3\sqrt{(l'+2)(l'-1)} \begin{pmatrix} l & l' & 1 \\ -2 & 1 & 1 \end{pmatrix} \right. \\ &\quad \left. + \sqrt{(l+3)(l-2)} \begin{pmatrix} l & l' & 1 \\ -3 & 2 & 1 \end{pmatrix} \right\} \sum_{m''=-1}^1 L_{im''} \begin{pmatrix} l & l' & 1 \\ m & -m' & m'' \end{pmatrix}. \end{aligned} \quad (139)$$

In Appendix B, it shall be demonstrated that, for the specific case $l = 2$, the loss of the z -component of angular momentum is represented by the equation,

$$\dot{J}_z = \frac{1}{4\pi} \Im \sum_{m=-2}^{m=2} m h_{2m} \dot{h}_{2m}. \quad (140)$$

Finally, in the last equation, we obtain the momenta carried away by the terms of the gravitational waves polarizations h from gauge-invariant perturbations derived by Thorne in [6] and [46].

6. Discussion and Conclusions

In this paper, we have addresses the task of identifying conserved quantities, such as energy-momentum and angular momentum and linear moment radiate by an isolated system, within asymptotically flat spacetimes as governed by general relativity. Our investigation demonstrates that the application of quasigroup symmetries facilitates a systematic and unified representation of the conserved quantities.

By employing the quasigroup methodology, we have developed a framework that effectively resolves the conventional challenges presented by supertranslation ambiguity at null infinity. Our method facilitates the establishment of a geometric and algebraic structure invariant under supertranslations; in particular, we derive the intrinsic angular momentum and delineate with the center of the mass frame reference. In conjunction with methodologies for modulating supertranslation freedom, this highlights the pivotal importance of selecting suitable sections of null infinity for rigorous physical analysis.

Utilizing methodologies such as the spin coefficient formalism and Newman-Penrose scalar quantities, in conjunction with an in-depth analysis of the Bondi-Metzner-Sachs group, we have

derived explicit formulae that describe the loss of angular momentum and energy conveyed by gravitational radiation by the Weyl tensor Ψ_4 in terms of the spin-weighted spherical harmonics. It is important to remark that our results for conservative quantities are well known and in concordance with established results using other methodologies.

This investigation enhances our understanding of the asymptotic structure of spacetime and offers a dependable methodology for distinctly identifying conserved quantities within gravitational systems. Subsequent research endeavors might extend these concepts to encompass more generalized frameworks, such as spacetimes affected by gravitational forces and intricate matter configurations, thereby augmenting both theoretical insights and their astrophysical significance.

Funding: This research received no external funding

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: A. Z. R. acknowledges CONACYT scholarship. C. M. and A. I. N. want to thank SNII-SECIHTI, PROINPEP-UDG and PROSNII-UDG.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Spin-Weighted Spherical Harmonics

A quantity η is said to have a spin weight s (s.w. s), if η transforms as $\eta \rightarrow e^{is\theta}\eta$ under a rotation of $m^\alpha \rightarrow e^{i\theta}m^\alpha$, with θ a real function of position. The complex conjugate $\bar{\eta}$ has spin weight $-s$. The spin coefficients of the set $\{\rho, \mu, \epsilon + \bar{\epsilon}, \gamma + \bar{\gamma}\}$ have s.w. zero, elements of $\{\kappa, \tau, \bar{\alpha} + \beta, \nu, \bar{\pi}\}$ have s.w. +1 and elements of $\{\sigma, \bar{\lambda}\}$ have s.w. +2 [18].

For a function $\eta(u, \zeta, \bar{\zeta})$ of the spin weight s , the raising and lowering operators $\eth$ and $\bar{\eth}$, respectively, are defined on $\mathcal{S}^+$ as follows [22,35]:

$$\begin{aligned}\eth\eta &:= \delta^0 f + 2\bar{\alpha}^0 s f = P\bar{P}^{-s} \frac{\partial(\bar{P}^s \eta)}{\partial \zeta}, \\ \bar{\eth}\eta &:= \bar{\delta}^0 f - 2\alpha^0 s f = \bar{P}P^s \frac{\partial(P^{-s} \eta)}{\partial \bar{\zeta}},\end{aligned}$$

where

$$\delta^0 = P \frac{\partial}{\partial \zeta}, \quad \alpha^0 = \frac{P}{2} \frac{\partial \ln \bar{P}}{\partial \zeta}. \quad (\text{A1})$$

Here P is a conformal factor emerged in two-dimensional metric of cuts on $\mathcal{S}^+$:

$$ds^2 = \frac{2d\zeta d\bar{\zeta}}{|P(u, \zeta, \bar{\zeta})|^2}. \quad (\text{A2})$$

In a particular choice $P = P_0 = (1 + \zeta\bar{\zeta})/\sqrt{2}$ the metric above will be the metric of two-sphere, and the coordinates referred to as Bondi coordinates (BC).

The commutator $[\eth, \bar{\eth}]$ is given by

$$[\eth, \bar{\eth}]\eta = s\eta\eth\bar{\eth}\ln(P\bar{P}) = (K + \bar{K})s\eta, \quad (\text{A3})$$

where $(K + \bar{K})$ is the Gaussian curvature of two-dimensional cut. The following useful lemma holds:

Lemma. If s.w. A + s.w. $B = -1$, then

$$\oint A\eth B d\Omega = - \oint B\eth A d\Omega. \quad (\text{A4})$$

where s.w. denotes spin weight [11].

In what follows we make a choice of the conformal factor as $P = P_0 \sqrt{\zeta/\bar{\zeta}}$. This yields

$$\delta^0 = \frac{1 + \zeta\bar{\zeta}}{\sqrt{2}} \sqrt{\frac{\zeta}{\bar{\zeta}}} \frac{\partial}{\partial \bar{\zeta}}. \quad (\text{A5})$$

In the spherical coordinates we obtain

$$\delta^0 = -\frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi} \right). \quad (\text{A6})$$

This gives, up to the factor $1/\sqrt{2}$, the standard form of raising and lowering operators [35,36,47]

$$\bar{\partial} \eta = -\frac{1}{\sqrt{2}} \left(\partial_\theta + \frac{i}{\sin \theta} \partial_\phi - s \cot \theta \right) \eta = -\frac{1}{\sqrt{2}} \sin^s \theta \left(\partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right) (\eta \sin^{-s} \theta), \quad (\text{A7})$$

$$\bar{\partial} \eta = -\frac{1}{\sqrt{2}} \left(\partial_\theta - \frac{i}{\sin \theta} \partial_\phi + s \cot \theta \right) \eta = -\frac{1}{\sqrt{2}} \sin^{-s} \theta \left(\partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right) (\eta \sin^s \theta), \quad (\text{A8})$$

A direct computation shows

$$\bar{\partial} \bar{\partial} \eta = \frac{1}{2} \left(\frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + \frac{2is \cos \theta}{\sin^2 \theta} \partial_\phi - \frac{s^2}{\sin^2 \theta} + s(s+1) \right) \eta, \quad (\text{A9})$$

$$\bar{\partial} \bar{\partial} \eta = \frac{1}{2} \left(\frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + \frac{2is \cos \theta}{\sin^2 \theta} \partial_\phi - \frac{s^2}{\sin^2 \theta} + s(s-1) \right) \eta, \quad (\text{A10})$$

Let ${}_s Y_{jm}$ be a normalized spin-weighted spherical harmonic of order j and spin weight s , then we have, ${}_s \bar{Y}_{lm} = (-1)^{s+m} {}_{-s} Y_{l-m}$, and

$$\bar{\partial} {}_s Y_{jm} = \sqrt{\frac{j(j+1) - s(s+1)}{2}} {}_{s+1} Y_{jm}, \quad (\text{A11})$$

$$\bar{\partial} {}_s Y_{jm} = -\sqrt{\frac{j(j+1) - s(s-1)}{2}} {}_{s-1} Y_{jm}. \quad (\text{A12})$$

When $s_1 + s_2 + s_3 = 0$ the triple integral is given in terms of the Wigner 3- j symbols:

$$\begin{aligned} & \int_{S^2} {}_{s_1} Y_{j_1 m_1} {}_{s_2} Y_{j_2 m_2} {}_{s_3} Y_{j_3 m_3} d\Omega \\ &= \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{4\pi}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ -s_1 & -s_2 & -s_3 \end{pmatrix} \end{aligned} \quad (\text{A13})$$

For the references, we present analytic expressions for the first few orthonormalized spin-weighted spherical harmonics [48]:

$$-1 Y_{10} = -\sqrt{\frac{3}{8\pi}} \sin \theta \quad (\text{A14})$$

$$-1 Y_{1\pm 1} = -\sqrt{\frac{3}{16\pi}} (1 \pm \cos \theta) e^{\pm i\varphi} \quad (\text{A15})$$

$$-2 Y_{20} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta \quad (\text{A16})$$

$$-2 Y_{2\pm 1} = \sqrt{\frac{5}{16\pi}} \sin \theta (1 + \cos \theta) e^{\pm i\varphi} \quad (\text{A17})$$

$$-2 Y_{2\pm 2} = \sqrt{\frac{5}{64\pi}} (1 + \cos \theta)^2 e^{\pm 2i\varphi} \quad (\text{A18})$$

Using the relation $s\bar{Y}_{lm} = (-1)^{s+m} {}_{-s}Y_{l-m}$, we obtain

$${}_1Y_{10} = \sqrt{\frac{3}{8\pi}} \sin \theta \quad (\text{A19})$$

$${}_1Y_{1\pm 1} = -\sqrt{\frac{3}{16\pi}} (1 \mp \cos \theta) e^{\pm i\varphi} \quad (\text{A20})$$

$${}_2Y_{20} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta \quad (\text{A21})$$

$${}_2Y_{2\pm 1} = -\sqrt{\frac{5}{16\pi}} \sin \theta (1 \mp \cos \theta) e^{\pm i\varphi} \quad (\text{A22})$$

$${}_2Y_{2\pm 2} = \sqrt{\frac{5}{64\pi}} (1 \mp \cos \theta)^2 e^{\pm 2i\varphi} \quad (\text{A23})$$

Appendix B. Angular Momentum Loss

The loss of the intrinsic angular momentum is given by

$$\dot{J}_i = -\frac{1}{8\pi} \Re \oint \bar{L}_i (3\bar{\lambda}^0 \bar{\partial} \bar{\sigma}^0 - 3\sigma^0 \bar{\partial} \lambda^0 + \bar{\sigma}^0 \bar{\partial} \bar{\lambda}^0 - \lambda^0 \bar{\partial} \sigma^0) d\Omega. \quad (\text{A24})$$

To proceed further, we express σ^0 and $\bar{L}_i$ in terms of spin-weighted spherical harmonics:

$$\sigma^0 = \sum_{l=2}^{\infty} \sum_{m=-l}^l h_{lm} {}_2Y_{lm}, \quad (\text{A25})$$

$$\bar{L}_i = \sum_{m=-1}^1 L_{im} {}_{-1}Y_{1m}. \quad (\text{A26})$$

Taking the derivative, we obtain

$$\bar{\partial} \sigma^0 = \sum_{l=2}^{\infty} \sum_{m=-l}^l h_{lm} \sqrt{\frac{(l-2)(l+3)}{2}} {}_3Y_{lm}, \quad (\text{A27})$$

$$\bar{\partial} \bar{\sigma}^0 = -\sum_{l=2}^{\infty} \sum_{m=-l}^l \bar{h}_{lm} \sqrt{\frac{(l+2)(l-1)}{2}} {}_1\bar{Y}_{lm}. \quad (\text{A28})$$

The computation yields

$$\dot{J}_i = \frac{1}{8\pi} \Re \sum_{l,m,l',m'} (h_{lm} \dot{\bar{h}}_{l'm'} - \dot{h}_{lm} \bar{h}_{l'm'}) c_{lm'l'm'}^i = \frac{1}{4\pi} \Re \sum_{l,m,l',m'} h_{lm} \dot{\bar{h}}_{l'm'} (c_{lm'l'm'}^i - \bar{c}_{l'm'lm}^i), \quad (\text{A29})$$

where

$$c_{lm'l'm'}^i = \quad (\text{A30})$$

$$\sum_{m''=-1}^1 L_{im''} \oint d\Omega {}_{-1}Y_{1m''} \left(\sqrt{\frac{(l+3)(l-2)}{2}} {}_3Y_{lm} {}_2\bar{Y}_{l'm'} - 3\sqrt{\frac{(l'+2)(l'-1)}{2}} {}_2Y_{lm} {}_1\bar{Y}_{l'm'} \right) \quad (\text{A31})$$

Performing the integration, we obtain

$$c_{lm'l'm'}^i = (-1)^{m'} \sqrt{\frac{3(2l+1)(2l'+1)}{8\pi}} \left\{ 3\sqrt{(l'+2)(l'-1)} \begin{pmatrix} l & l' & 1 \\ -2 & 1 & 1 \end{pmatrix} + \sqrt{(l+3)(l-2)} \begin{pmatrix} l & l' & 1 \\ -3 & 2 & 1 \end{pmatrix} \right\} \sum_{m''=-1}^1 L_{im''} \begin{pmatrix} l & l' & 1 \\ m & -m' & m'' \end{pmatrix} \quad (\text{A32})$$

The recurrence relations for the Wigner 3- j symbols:

$$\begin{aligned} & -\sqrt{(l_3 \mp s_3)(l_3 \pm s_3 + 1)} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \pm 1 \end{pmatrix} \\ & = \sqrt{(l_1 \mp s_1)(l_1 \pm s_1 + 1)} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 \pm 1 & s_2 & s_3 \end{pmatrix} \\ & + \sqrt{(l_2 \mp s_2)(l_2 \pm s_2 + 1)} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 \pm 1 & s_3 \end{pmatrix} \end{aligned} \quad (\text{A33})$$

For the standard rotational Killing vectors about the x, y and z axes given by

$$L_x = \sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi \quad (\text{A34})$$

$$L_y = \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi \quad (\text{A35})$$

$$L_z = \partial_\varphi. \quad (\text{A36})$$

the computation yields

$$\bar{L}_1 = -L_x \cdot \bar{m} = \frac{1}{\sqrt{2}} (i \cos \theta \cos \varphi - \sin \varphi) = i \sqrt{\frac{2\pi}{3}} (-{}_1Y_{1-1} - {}_{-1}Y_{11}), \quad (\text{A37})$$

$$\bar{L}_2 = -L_y \cdot \bar{m} = \frac{1}{\sqrt{2}} (i \cos \theta \sin \varphi + \cos \varphi) = -\sqrt{\frac{2\pi}{3}} (-{}_1Y_{11} + {}_{-1}Y_{1-1}), \quad (\text{A38})$$

$$\bar{L}_3 = -L_z \cdot \bar{m} = \frac{i}{\sqrt{2}} \sin \theta = -2i \sqrt{\frac{\pi}{3}} {}_{-1}Y_{10}. \quad (\text{A39})$$

Using these results in (A32), after some algebra we obtain

$$\begin{aligned} c_{lm'l'm'}^x &= i(-1)^{m'} \sqrt{\frac{(2l+1)(2l'+1)}{4}} \left\{ 3\sqrt{(l'+2)(l'-1)} \begin{pmatrix} l & l' & 1 \\ -2 & 1 & 1 \end{pmatrix} \right. \\ & \left. + \sqrt{(l+3)(l-2)} \begin{pmatrix} l & l' & 1 \\ -3 & 2 & 1 \end{pmatrix} \right\} \left\{ \begin{pmatrix} l & l' & 1 \\ m & -m' & -1 \end{pmatrix} - \begin{pmatrix} l & l' & 1 \\ m & -m' & 1 \end{pmatrix} \right\} \end{aligned} \quad (\text{A40})$$

$$\begin{aligned} c_{lm'l'm'}^y &= (-1)^{m'+1} \sqrt{\frac{(2l+1)(2l'+1)}{4}} \left\{ 3\sqrt{(l'+2)(l'-1)} \begin{pmatrix} l & l' & 1 \\ -2 & 1 & 1 \end{pmatrix} \right. \\ & \left. + \sqrt{(l+3)(l-2)} \begin{pmatrix} l & l' & 1 \\ -3 & 2 & 1 \end{pmatrix} \right\} \left\{ \begin{pmatrix} l & l' & 1 \\ m & -m' & 1 \end{pmatrix} + \begin{pmatrix} l & l' & 1 \\ m & -m' & -1 \end{pmatrix} \right\} \end{aligned} \quad (\text{A41})$$

$$\begin{aligned} c_{lm'l'm'}^z &= i(-1)^{m'+1} \sqrt{(2l+1)(2l'+1)} \left\{ 3\sqrt{(l'+2)(l'-1)} \begin{pmatrix} l & l' & 1 \\ -2 & 1 & 1 \end{pmatrix} \right. \\ & \left. + \sqrt{(l+3)(l-2)} \begin{pmatrix} l & l' & 1 \\ -3 & 2 & 1 \end{pmatrix} \right\} \begin{pmatrix} l & l' & 1 \\ m & -m' & 0 \end{pmatrix} \end{aligned} \quad (\text{A42})$$

In particular for $l = 2$, the computation of the angular momentum z -component loss yields

$$\dot{J}_z = \frac{1}{4\pi} \Im \sum_{m=-2}^{m=2} m h_{2m} \dot{h}_{2m}. \quad (\text{A43})$$

In the final equation, we derive the momenta associated with the polarizations h of gravitational waves, based on gauge-invariant perturbations, as outlined by Ruiz [6], Thorne [45] and Lousto [46].

References

1. Mitskievich, N.V. *Physical Fields in General Relativity*. Nauka: Moscow, Russia, 1969.
2. Mitskievich, N.V.; Efremov, A.P.; Nesterov, A.I. *Dynamics of Fields in General Relativity*. Energoatomizdat: Moscow, Russia, 1985.
3. Bak, D. and Cangemi, D. and Jackiw, R. Energy-Momentum Conservation in Gravity Theories. *Phys. Rev. D* **1994**, *49*, 5173–5181. [[CrossRef](#)]
4. Komar, A. Covariant conservation laws in general relativity, *Phys. Rev.* **1959**, *113*(3), 934–936. [[CrossRef](#)]
5. Wald, R.M. Black Hole entropy is the Noether charge, *Phys. Rev. D* **1993**, *48*(8), R3427–R3431. [[CrossRef](#)]
6. Ruiz, M. et al. Multiple expansions for energy and momenta carried by gravitational waves, *Gen. Relativ. Grav.* **2007**, *40*(8), 1705–1729. [[CrossRef](#)]
7. Bramson, B.D. Relativistic angular momentum for asymptotically flat Einstein-Maxwell manifolds, *Proc. R. Soc. Lond. A Math. Phys. Sci.* **1975**, *341*(1627), 463–490. [[CrossRef](#)]
8. Winicour, J. Angular momentum in general relativity, In *General Relativity and Gravitation*; Held, A.; Plenum Press: New York, USA, 1980.
9. Bondi, H. and Van der Burg, M. G. J. and Metzner, A. W. K. Gravitational waves in general relativity, VII. waves from AXI-symmetric isolated system, *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1962**, *269*(1336), 21–52. [[CrossRef](#)]
10. Sachs, R. Asymptotic symmetries in gravitational theory, *Phys. Rev.* **1962**, *128*(6), 2851–2864. [[CrossRef](#)]
11. Dray, T. and Streubel, M. Angular momentum at Null Infinity, *Class. Quantum Grav.* **1984**, *1*(1), 15–26. [[CrossRef](#)]
12. Geroch, R. and Winicour, J. Linkages in general relativity, *J. Math. Phys.* **1981**, *22*(4), 803–812. [[CrossRef](#)]
13. Winicour, J. Some total invariants of asymptotically flat space-times, *J. Math. Phys.* **1968**, *9*(6), 861–867. [[CrossRef](#)]
14. Sachs, R. K. Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1962**, *270*(1340), 103–126. [[CrossRef](#)]
15. Moreschi, O.M. On angular momentum at future null infinity, *Class. Quantum Grav.* **1986**, *3*(4), 503–525. [[CrossRef](#)]
16. Winicour, J. and Tamburino, L. Lorentz-covariant gravitational energy-momentum linkages, *Phys. Rev. Lett.* **1965**, *15*(17), 720–720. [[CrossRef](#)]
17. Tamburino, L.A. and Winicour, J.H. Gravitational fields in finite and conformal Bondi frames, *Phys. Rev.* **1966**, *150*(4), 1039–1053. [[CrossRef](#)]
18. Prior, C.R. Angular momentum in general relativity I. Definition and asymptotic behaviour, *Proc. R. Soc. Lond. A Math. Phys. Sci.* **1977**, *354*(1679), 379–405. [[CrossRef](#)]
19. Penrose, R. Quasi-local mass and angular momentum in general relativity, *Proc. R. Soc. Lond. A Math. Phys. Sci.* **1982**, *381*(1780), 53–63. [[CrossRef](#)]
20. Szabados, L.B. Quasi-local energy-momentum and angular momentum in general relativity. *Living Rev. Relativ.* **2009**, *12*(1). [[CrossRef](#)]
21. Moreschi, O.M. and Dain, S. Rest frame system for asymptotically flat space-times, *J. Math. Phys.* **1998**, *39*(12), 6631–6650. [[CrossRef](#)].
22. Kozameh, C.N. and Quiroga, G.D. Center of mass and spin for isolated sources of gravitational radiation, *Phys. Rev. D* **2016**, *93*(6). [[CrossRef](#)]
23. Kozameh, C.N. and Nieva, J.I. and Quiroga, G.D. Relativistic center of mass in general relativity, *Phys. Rev. D* **2020**, *101*(2). [[CrossRef](#)]
24. Woodford, C.J. and Boyle, M. and Pfeiffer, H.P. Compact binary waveform center-of-mass corrections, *Phys. Rev. D* **2019**, *100*(12). [[CrossRef](#)]
25. Moreschi, O.M. Supercentre of mass system at future null infinity, *Class. Quantum Grav.* **1988**, *5*(3), 423–435. [[CrossRef](#)]
26. Gallo, E. and Moreschi, O.M. Intrinsic angular momentum for radiating spacetimes which agrees with the Komar integral in the axisymmetric case, *Phys. Rev. D* **2014**, *89*(8). [[CrossRef](#)]
27. Nesterov, A.I. Asymptotic symmetries and conservation laws, *Grav. Cosmology (Moscow, VINITI)* **1992**.
28. Nesterov, A.I. Quasigroups, asymptotic symmetries, and conservation laws in general relativity, *Phys. Rev. D* **1997**, *56*(12). [[CrossRef](#)]
29. Nesterov, A.I. On angular momentum of gravitational radiation, *Phys. Lett. A* **1998**, *250*(1–3), 55–61. [[CrossRef](#)]
30. Lorcé, C. The relativistic center of mass in field theory with spin, *Eur. Phys. J. C* **2018**, *78*(9), 785. [[CrossRef](#)]

31. Batalin, I.A. Quasigroup construction and first class constraints, *J. Math. Phys.* **1981**, 22(9), 1837–1850. [[CrossRef](#)]
32. Newman, E.T. and Tod, K. P. Asymptotically Flat Space-Times, In *General Relativity and Gravitation* Held, A.; Plenum Press: New York, USA, 1980.
33. Penrose, R., Rindler, W. and Goldberg, J.N. Spinors and space-time, volume I: Two-spinor calculus and relativistic fields, *Phys. Today* **1986**, 39(2), 72–73. [[CrossRef](#)]
34. Adamo, T.M., Newman, E.T. and Kozameh, C. Null geodesic congruences, asymptotically-flat spacetimes and their physical interpretation, *Living Rev. Relativ.* **2012**, 15(1). [[CrossRef](#)]
35. Goldberg, J.N. et al. Spin-s spherical harmonics and $\bar{\partial}$, *J. Math. Phys.* **1967**, 8(11), 2155–2161. [[CrossRef](#)]
36. Newman, E. and Penrose, R. An approach to gravitational radiation by a method of spin coefficients, *J. Math. Phys.* **1962**, 3(3), 566–578. [[CrossRef](#)]
37. Penrose, R. and Rindler, W. and Goldberg, J.N. Spinors and space-time, volume II: Spinor and twistor methods in space-time geometry, *Phys. Today* **1986**, 39(12), 79–81. [[CrossRef](#)]
38. Newman, E.T. and Unti, T.W. Behavior of asymptotically flat empty spaces, *J. Math. Phys.* **1962**, 3(5), 891–901. [[CrossRef](#)]
39. Held, A. and Newman, E.T. and Posadas, R. The Lorentz group and the sphere, *J. Math. Phys.* **1970**, 11(11), 3145–3154. [[CrossRef](#)]
40. Penrose, R. Relativistic symmetry groups, In *Group Theory in Non-Linear Problems*; Barut, A. O.; Springer Netherlands: Dordrecht, Netherlands, 1974, 1–58. [[CrossRef](#)]
41. Shaw, W.T. Symplectic geometry of Null Infinity and two-surface twistors, *Class. Quantum Grav.* **1984**, 1(4). [[CrossRef](#)]
42. Gong, X. et al. Newman-Penrose Constants of the Kerr-Newman metric, *Phys. Rev. D* **2007**, 76(10). [[CrossRef](#)]
43. Streubel, M. "Conserved" quantities for isolated gravitational systems, *Gen. Relat. Gravit.* **1978**, 9(6), 551–561. [[CrossRef](#)]
44. Dixon, W. G. and Bondi, H. Dynamics of extended bodies in general relativity. I. Momentum and angular momentum, *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1970**, 314(1519), 499–527. [[CrossRef](#)]
45. Lousto, C.O. and Zlochower, Y. Practical formula for the radiated angular momentum, *Phys. Rev. D* **2007**, 76(4). [[CrossRef](#)]
46. Thorne, K.S. Multipole expansions of gravitational radiation, *Rev. Modern Phys.* **1980**, 52(2), 299–339. [[CrossRef](#)]
47. Newman, E.T. and Penrose, R. Note on the bondi-metzner-sachs group, *J. Math. Phys.* **1966**, 7(5), 863–870. [[CrossRef](#)]
48. Ajith, P. et al. Data formats for numerical relativity waves, *arXiv* **2007**. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.