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Posted Date: 1 July 2025

doi: 10.20944/preprints202506.2497.v1

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On the Definability Problem of First-Order Sentences by Propositional Intuitionistic Formulas

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Abstract

We consider restricted forms of the algorithmic problem of definability of first-order sentences by propositional formulas with intuitionistic Kripke frames semantics. We demonstrate positive resolutions for classes of intuitionistic Kripke frames based on linear orders and conversely show that a few natural first-order definable classes give rise to undecidable definability problems by applying the model-theoretic in nature technique of stable classes of Kripke frames.

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MSC: 03B55; 03C40; 03B25

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1. Introduction

A classical topic in modal logic (and more generally in nonclassical logic) is the comparison of expressiveness of formulas compared to formulas of other logics, notably first-order logic. In general the classes of frames definable by first-order sentences and those definable by propositional formulas (of a nonclassical logic) are different and some properties are only expressible by one of the languages. This naturally led to the study of correspondence theory (see [1]) and while there are notable cases where a decidable class of formulas of one language has the property that each of its elements is guaranteed to express a property definable by a formula of the other language (such a case are Sahlqvist formulas), in general comparing expressivity of formulas is not algorithmically feasible. A classical result is that of Chagrova [2,3] stating that it is undecidable to check whether a first-order sentence defines a class of frames definable by a propositional formula with intuitionistic Kripke semantics.

We further study the topic in two directions. First, while Chagrova's theorem gives a negative answer for the general correspondence problem for the class of all Kripke frames, one can reduce the complexity of the problem by only considering correspondence with respect to smaller classes of frames. In this direction, we examine a few classes of frames based on linear orders due to their simple structure and rich properties. Second, even if we consider restricted correspondence problems, in general this gives us no guarantee that they become simpler, even if we restrict ourselves to simple first-order definable classes of frames. While Chagrova's results were groundbreaking, the original constructions are involved and based on reductions of undecidable problems for Minsky machines. We reiterate a technique developed by Balbiani and Tinchev [4] to the context of intuitionistic propositional logic to obtain a means of proving undecidability by model-theoretic means and reductions of problems inherent to the classes of consideration, more precisely the first-order validity problem for the classes.

The text is structured as follows:

- Section 2 consists of a listing of preliminary facts where we introduce notation and recall basic properties about the topics of first- and second-order logic and intuitionistic propositional logic.

 doi:10.20944/preprints202506.2497.v1

- Section 3 deals with the definability problem with respect to certain classes based on linear orders. We obtain positive results about the classes and show effective means of finding propositional definitions of first-order sentences with respect to the restricted problem. The main result follows by reducing the decidability of the monadic second-order theory of the class of at most countable disjoint unions of linear orders to the monadic second-order theory of at most countable linear orders, [6].
- Section 4 points out a few classes of frames which give rise to undecidable instances of the definability problem. The main result is obtained by applying the technique of showing stability of the classes in the sense of [4] and then proving undecidability of the first-order validity problem. The latter we achieve by obtaining a chain of reductions of the undecidable problem of validity of sentences for the first-order theory of a symmetric and reflexive relation.

2. Preliminaries

We will briefly outline any relevant notation and related fundamental results. We work in the framework of Zermelo–Fraenkel set theory with the axiom of choice, ZFC, denote by ω the set of all natural numbers, and use α as a variable for ordinals.

We will say that that a chain has length $n < \omega$ if it contains exactly n elements, and that a partial order P is of depth $n < \omega$ if every chain in P is of length at most n and there is at least one chain in P of length n.

2.1. First-Order Languages and Logic

The first-order languages we discuss will have a countably infinite set of individual variables and will contain a symbol $\doteq$ which will be interpreted as formal equality. We will only be interested in relational languages, i.e. containing no function (or constant) symbols. We use standard first-order logic semantics based on variable assignments, using gothic letters to denote structures (we might sometimes alternatively say models for the language, [5]) and capital latin letters to denote formulas.

To recall, atomic formulas of a relational first-order language $\mathcal{L}$ are of the form $p(x_1, \ldots, x_n)$ or $x_1 \doteq x_2$ where $x_1, x_2, \ldots, x_n$ are individual variables and p is a relation symbol in $\mathcal{L}$; the formulas of $\mathcal{L}$ are either atomic formulas or of the form $(A \wedge B)$, $\neg A$, $\exists x A$. Other propositional connectives and the universal quantifier are introduced as abbreviations in terms of the symbols already introduced; as usual, we sometimes omit parentheses for brevity. We define free and bound variables in a formula in the usual manner and say that A is a sentence if none of its variables are free. We denote by $qr(A)$ the quantifier rank of A, i.e. the largest depth of nesting of quantifiers within A.

A structure $\mathfrak{A}$ for $\mathcal{L}$ consists of a set $|\mathfrak{A}|$ (the universe of $\mathfrak{A}$) together with a corresponding relation $p^{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$ of an appropriate arity for each relation symbol p of $\mathcal{L}$. Given a variable assignment V that maps individual variables of $\mathcal{L}$ to elements of $|\mathfrak{A}|$, the Tarski satisfaction relation $\models$ is defined by recursion on formulas:

- $\mathfrak{A}, V \models x \doteq y$ iff $V(x) = V(y)$
- $\mathfrak{A}, V \models p(x_1, \ldots, x_n)$ iff $p^{\mathfrak{A}}(V(x_1), \ldots, V(x_n))$
- $\mathfrak{A}, V \models \neg A$ iff it is not the case that $\mathfrak{A}, V \models A$ (we write $\mathfrak{A}, V \not\models A$)
- $\mathfrak{A}, V \models A \wedge B$ iff $\mathfrak{A}, V \models A$ and $\mathfrak{A}, V \models B$
- $\mathfrak{A}, V \models \exists x A$ iff there exists an assignment V' such that $V(y) = V'(y)$ for all variables except maybe x, such that $\mathfrak{A}, V' \models A$

Whenever $\mathfrak{A}$ is a structure, A is a first-order formula with free variables among $x_1, \ldots, x_n$, and $a_1, \ldots, a_n$ are elements of $|\mathfrak{A}|$, we shall write $\mathfrak{A} \models A[\![a_1, \ldots, a_n]\!]$ for $\mathfrak{A}, V \models A$ where V is any variable assignment such that $V(x_i) = a_i$. If A is a sentence, then we write $\mathfrak{A} \models A$ for $\mathfrak{A}, V \models A$ where V is any variable assignment.

We say that $\mathfrak{A}$ validates a sentence A if $\mathfrak{A} \models A$. The theory of a class $\mathcal{K}$ of structures is the set of those sentences valid in each structure in $\mathcal{K}$.

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We assume that the reader is acquainted with fundamental notions and results such as the compactness theorem and the Löwenheim-Skolem theorem, substructures and elementary substructures, isomorphisms and embeddings. The reader may consult [5] for further references.

For a formula A with free variables among $x_1, \ldots, x_n, y$ and a formula B having no common variables with A, we define the relativization B with respect to A and y and denote it by $(B)^{A,y}$, by recursion on B:

- $(B)^{A,y} = B$ if B is atomic
- $(\neg B)^{A,y} = \neg(B)^{A,y}$
- $(B_1 \wedge B_2)^{A,y} = (B_1)^{A,y} \wedge (B_2)^{A,y}$
- $(\exists z B_1)^{A,y} = \exists z(A[y/z] \wedge (B_1)^{A,y})$ where $A[y/z]$ is the formulas obtained by replacing every free occurence of y in A by z.

Given structures $\mathfrak{A}$ and $\mathfrak{B}$, formula A with free variables among $x_1, \ldots, x_n, y$ and elements $a_1, \ldots, a_n \in |\mathfrak{A}|$, we say that $\mathfrak{B}$ is the relativized reduct of A with respect to A and $a_1, \ldots, a_n$ if $\mathfrak{B}$ is the substructure of $\mathfrak{A}$ with universe those elements $b \in |\mathfrak{A}|$ such that $\mathfrak{A} \models A[\![a_1, \ldots, a_n, b]\!]$. The relativized reduct of A with respect to A and $a_1, \ldots, a_n$ exists iff $\mathfrak{A} \models \exists y A[\![a_1, \ldots, a_n]\!]$.

The relativization theorem connects the notion of relativized reducts and relativizations of formulas:

Theorem 1. *(Relativization theorem)*

If $\mathfrak{B}$ is the relativized reduct of $\mathfrak{A}$ with respect to A and $a_1, \ldots, a_n$ and B is a formula with free variables among $z_1, \ldots, z_k$ that has no common variable with A, then for every $b_1, \ldots b_k \in |\mathfrak{B}|$ it holds that $\mathfrak{B} \models B[\![b_1, \ldots, b_k]\!]$ iff $\mathfrak{A} \models (B)^{A,y}[\![a_1, \ldots, a_n, b_1, \ldots, b_k]\!]$.

Proof. The reader may consult Theorem 5.1.1 of [7]. □

2.2. Monadic Second-Order Languages and Logic

We obtain a relational monadic second-order language by extending a first-order language $\mathcal{L}$ with a countably infinite set of set variables which we shall denote by capital latin letters, and a new logical symbol $\in$. We introduce new atomic formulas of the form $x \in Z$ for each individual variable x and set variable Z, and we obtain the formulas of $\mathcal{L}$ by extending the first-order definition with the additional clause that if X is a set variable and A is a formula, then $\exists X A$ is a formula.

Structures for a monadic second-order language are defined in the same way as in the first-order case. For a structure $\mathfrak{A}$ we extend variable assignments to map set variables to subsets of $|\mathfrak{A}|$. We then define the satisfaction relation by adding the following clauses to the first-order definition:

- $\mathfrak{A}, V \models x \in Z$ iff $V(x) \in V(Z)$
- $\mathfrak{A}, V \models \exists X A$ iff there exists an assignment V' such that $V(y) = V'(y)$ for all individual variables and $V(Z) = V'(Z)$ for all set variables except maybe X, such that $\mathfrak{A}, V' \models A$

Similarly to the first-order case, if A is a monadic second-order formula with free variables among $x_1, \ldots, x_n, X_1, \ldots, X_k$, and $a_1, \ldots, a_n \in |\mathfrak{A}|$ and $T_1, \ldots, T_k \subseteq |\mathfrak{A}|$, then we write $\mathfrak{A} \models A[\![a_1, \ldots, a_n, T_1, \ldots, T_k]\!]$ for $\mathfrak{A}, V \models A$ where V is any variable assignment such that $V(x_i) = a_i$ and $V(X_j) = T_j$.

We shall use abbreviations for usual set-theoretic properties, e.g. we write $X \subseteq Y$ for $\forall z(z \in X \leftrightarrow z \in Y)$ and write $X \doteq Y$ for $X \subseteq Y \wedge Y \subseteq X$.

2.3. Intuitionistic Propositional Logic

The formulas of the intuitionistic propositional logic are the same syntactic objects as the formulas of classical propositional logic, i.e., we have a countably infinite set of propositional variables, together with the propositional connectives $\wedge, \vee, \rightarrow$ and the propositional constant $\bot$. Any propositional constant or variable is a propositional formula and if φ and ψ are propositional formulas, then so are

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$(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \to \psi)$; we often omit parentheses. We introduce the connective $\neg$ and the constant $\top$ as abbreviations, i.e. $\neg\varphi$ is an abbreviation for $\varphi \to \bot$ and $\top$ is an abbreviation for $\neg\bot$.

We recall the standard relational Kripke semantics (see [8]). A Kripke frame is $\mathfrak{F} = \langle F, \leq^{\mathfrak{F}} \rangle$ where F is a non-empty set (the universe of the frame) and $\leq^{\mathfrak{F}}$ is a partial order on F. A variable assignment for $\mathfrak{F} = \langle F, \leq^{\mathfrak{F}} \rangle$ maps propositional variables to upward closed subsets of F, i.e. if $x \in V(p)$ and $x \leq^{\mathfrak{F}} y$, then $y \in V(p)$ for every propositional variable p and elements $x, y \in F$. A Kripke model is a Kripke frame together with a variable assignment. For a Kripke model consisting of a frame $\mathfrak{F}$ together with a variable assignment V the pointwise satisfaction relation $\models$ at points $x \in F$ is defined by recursion on formulas:

- $(\mathfrak{F}, V, x) \not\models \bot$
- $(\mathfrak{F}, V, x) \models p$ iff $x \in V(p)$
- $(\mathfrak{F}, V, x) \models \varphi \wedge \psi$ iff $(\mathfrak{F}, V, x) \models \varphi$ and $(\mathfrak{F}, V, x) \models \psi$
- $(\mathfrak{F}, V, x) \models \varphi \vee \psi$ iff $(\mathfrak{F}, V, x) \models \varphi$ or $(\mathfrak{F}, V, x) \models \psi$
- $(\mathfrak{F}, V, x) \models \varphi \to \psi$ iff for every $y \in F$ such that $x \leq^{\mathfrak{F}} y$ it holds that if $(\mathfrak{F}, V, y) \models \varphi$ then $(\mathfrak{F}, V, y) \models \psi$

We say that a frame $\mathfrak{F}$ validates a propositional formula φ and write $\mathfrak{F} \models \varphi$ if $(\mathfrak{F}, V, x) \models \varphi$ for every variable assignment V and every $x \in |\mathfrak{F}|$; if $\mathcal{K}$ is a class of frames we write $\mathcal{K} \models \varphi$ if $\mathfrak{F} \models \varphi$ for every $\mathfrak{F} \in \mathcal{K}$. We denote by Int the minimal intuitionistic logic. We say that $\mathfrak{F}$ validates a superintuitionistic logic L and write $\mathfrak{F} \models L$ if $\mathfrak{F}$ validates each of its axioms. We say that a logic L is complete with respect to a class of frames $\mathcal{K}$ if $\varphi \in L$ iff $\mathfrak{F} \models \varphi$ for every $\mathfrak{F} \in \mathcal{K}$. The logic of a class of frames $\mathcal{K}$ is the logic $Log(\mathcal{K}) = \{\varphi \mid \mathcal{K} \models \varphi\}$.

We define a certain set of formulas $\varphi_{depth \leq n}$ for each natural number n by recursion:

- $\varphi_{depth \leq 0} = \bot$
- $\varphi_{depth \leq n+1} = p_{n+1} \vee (p_{n+1} \to \varphi_{depth_{\leq n}})$

For each natural number n, a frame $\mathfrak{F}$ validates the formula $\varphi_{depth \leq n}$ iff the depth of $\mathfrak{F}$ is at most n.

We will assume the reader is acquainted with standard constructions and properties of Kripke frames, such as generated subframes, disjoint unions of frames and p-morphic images, isomorphisms. For further reference the reader may consult [8].

2.4. Definability by Intuitionistic Formulas

There is a natural duality between Kripke frames and first-order models for the language of order, i.e. the language containing a single nonlogical relation symbol $\leq$ which is interpreted as the partial order of the frame.

Every Kripke frame is a model for the first-order theory of partial orders. Consider a non-empty class $\mathcal{K}$ of Kripke frames. We say that a propositional formula φ defines a first-order sentence A with respect to $\mathcal{K}$ if for every Kripke frame $\mathfrak{F} \in \mathcal{K}$ we have that $\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \models A$. If φ defines A with respect to $\mathcal{K}$ we also say φ is a propositional definition of A with respect to $\mathcal{K}$ and that A defines φ with respect to $\mathcal{K}$.

The problem $Int - def$ is the following algorithmic task: when given as input a first-order sentence A, output 'yes' if there exists a propositional definition φ of A, and output 'no' otherwise.

For our purposes, we shall deal with algorithms by describing them in natural language pseudocode, and assume that the reader is able to translate it to either a formal mathematical model such as a Turing machine, or to a program in a desired programming language.

3. Decidable Instances of Definability

We consider classes based on linear orders, in particular:

1. The class LIN of all linear orders.
2. The class LIN^{fin} of all finite linear orders.
3. The class $DLIN$ of all unions of families of pairwise disjoint linear orders.

4. The class $DLIN^{fin}$ of all finite frames in $DLIN$.

We note that the classes LIN and $DLIN$ are finitely axiomatizable, while the classes LIN^{fin} and $DLIN^{fin}$ are not. A possible axiomatization for $DLIN$ is to take as its axiom the conjunction of the axiom for partial orders and the sentence

$$\forall x \forall y (\exists z ((x \leq z \wedge y \leq z) \vee (z \leq x \wedge z \leq y)) \rightarrow (x \leq y \vee y \leq x)).$$

The above classes are prospective candidates for a positive resolution to the algorithmic definability problem since the linear intuitionistic Kripke frames have simple structure. We denote by LC the superintuitionistic logic obtained by adding to Int the additional axiom $(p \rightarrow q) \vee (q \rightarrow p)$. This logic is well studied, and in the following proposition we remind the reader of some of its properties that will be useful later:

Proposition 1. *The logic LC has the following properties:*

- *(i)* *A frame $\mathfrak{F}$ validates LC iff every generated subframe of $\mathfrak{F}$ is linear. Since a frame validates a formula iff each of its generated subframes validate it, LC is complete with respect to LIN and with respect to DLIN.*
- *(ii)* *For any propositional formula φ with $vars(\varphi) \subseteq \{p_1, \ldots, p_n\}$, $LC \models \varphi$ iff $\mathfrak{F}_{n+1} \models \varphi$ where $\mathfrak{F}_{n+1}$ is a linear order with $n+1$ elements.*
- *(iii)* *LC is complete with respect to LIN^{fin} and with respect to $DLIN^{fin}$.*
- *(iv)* *Any finite linear order is a p-morphic image of any infinite linear order.*
- *(v)* *LC is complete with respect to any infinite linear order.*
- *(vi)* *For any propositional formula φ, either $LC \models \varphi$ or there is a natural number n such that for every frame $\mathfrak{F}$ for LC it holds that $\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \models \varphi_{depth \leq n}$.*

Proof. The properties in (i) are well-known standard results.

(ii) The left to right direction is immediate, for right to left:
Suppose that $\mathfrak{F}_{n+1} \models \varphi$ and take any linear order $\mathfrak{F} = \langle F, \leq \rangle$, variable assignment V in $\mathfrak{F}$ and point $x \in \mathfrak{F}$, we will show that $(\mathfrak{F}, V, x) \models \varphi$.
For a point $z \in F$ denote by $p(z)$ the set of those propositional variables among $\{p_1, \ldots, p_n\}$ such that $(\mathfrak{F}, V, z) \models p_i$. Consider the finite partial order $\mathfrak{G} = \langle G, \subseteq \rangle$, where $G = \{p(z) \mid x \leq z\}$. Since variable assignments are upward closed, $\mathfrak{G}$ is a linear order. Moreover, G contains at most $n+1$ elements since any $P \in G$ contains at most n elements and the sets in G are ordered by inclusion.
One verifies by straightforward induction on the formula ψ with $vars(\psi) \subseteq \{p_1, \ldots, p_n\}$ that for every $z \in \mathfrak{F}$ such that $x \leq z$ we have that $(\mathfrak{F}, V, z) \models \psi \iff (\mathfrak{G}, V', p(z)) \models \psi$ where $V'(p_i) = \{P \in G \mid p_i \in P\}$. But $\mathfrak{G}$ is isomorphic to a generated subframe of $\mathfrak{F}_{n+1}$ and since $\mathfrak{F}_{n+1} \models \varphi$ we conclude that $(\mathfrak{G}, V', p(x)) \models \varphi$, hence $(\mathfrak{F}, V, x) \models \varphi$.

(iii) Immediate corollary to (i) and (ii).

(iv) Take any infinite linear order $\mathfrak{F} = \langle F, \leq^{\mathfrak{F}} \rangle$ and natural number $n \geq 1$, we will show that $\mathfrak{F}_n = \langle \{0, 1, \ldots, n-1\}, \leq \rangle$ is a p-morphic image of $\mathfrak{F}$. Since $\mathfrak{F}$ is infinite, we can pick $t_0, \ldots, t_{n-1} \in \mathfrak{F}$ such that $t_i <^{\mathfrak{F}} t_{i+1}$ for $0 \leq i < n-1$. Now the following function $f : F \rightarrow \{0, \ldots, n-1\}$ is a p-morphism of $\mathfrak{F}$ onto $\mathfrak{F}_n$:

- For $t \in F$ such that $t >^{\mathfrak{F}} t_{n-1}$ define $f(t) = n-1$.
- For $t \in F$ such that $t \leq^{\mathfrak{F}} t_{n-1}$ define $f(t) = m$, where m is the least natural number such that $0 \leq m \leq n-1$ and $t \leq t_m$.

(v) Take an infinite linearly ordered frame $\mathfrak{F}$. For any propositional formula φ, if $\mathfrak{F} \models \varphi$ we have by (iv) and the properties of p-morphic images that φ is valid in all finite linearly ordered frames, therefore $LC \models \varphi$ by (iii).

(vi) Suppose that $LC \not\models \varphi$. Then by (ii), $\mathfrak{F}_{n+1} \not\models \varphi$ where $vars(\varphi) \subseteq \{p_1, \ldots, p_n\}$ and $\mathfrak{F}_{n+1}$ is a linear order with $n+1$ elements. If $\mathfrak{F}_k$ is a finite linear order with $k \geq n+1$ elements, then $\mathfrak{F}_k \not\models \varphi$ since $\mathfrak{F}_{n+1}$ is isomorphic to a generated subframe of $\mathfrak{F}_k$. Therefore either φ is valid in no finite linear frame (in which case $\bot = \varphi_{depth \leq 0}$ satisfies the desired property) or there is a greatest natural number m such that $1 \leq m \leq n$ and $\mathfrak{F}_m \models \varphi$. Since $\mathfrak{F}_k$ is isomorphic to a generated subframe of $\mathfrak{F}_m$ for $1 \leq k \leq m$, we have that the finite linear frames that validate φ are exactly those of depth at most m.
Now if $\mathfrak{F} \models \varphi_{depth \leq m}$, then every chain in $\mathfrak{F}$ contains at most m elements, in particular every generated subframe of $\mathfrak{F}$ is a linear order with at most m elements and thus validates φ, so $\mathfrak{F} \models \varphi$.
Conversely, if $\mathfrak{F} \models \varphi$, then every generated subframe of $\mathfrak{F}$ is a linear order that validates φ, therefore contains at most m elements. Therefore any chain in $\mathfrak{F}$ must have at most m elements, thus $\mathfrak{F} \models \varphi_{depth \leq m}$.

□

Notice that property (vi) is quite illuminating: it states that the formulas $\top$ and $\varphi_{depth \leq n}$ for natural n exhaust all possible definitions expressible in the propositional language. More broadly, Proposition 1 provides enough clarity about the properties of the logic LC to be able to devise a general procedure that can be used to resolve the propositional definability problem for each of the classes $LIN, DLIN, LIN^{fin}, DLIN^{fin}$.

Throughout the remainder of this section, unless explicitly specified, we shall write $\mathcal{K}$ for any of the classes mentioned above. We now describe the procedure for solving the $Int - def$ problem with respect to $\mathcal{K}$ and argue about the correctness of the steps involved:

(1) (*Validity*) `Decide whether` $\mathcal{K} \models A$.
If `true`, then $\top$ clearly is a propositional definition of A with respect to $\mathcal{K}$.
If `false`:

(2) (*Finiteness*) `Decide whether` A `is valid in a structure from` $\mathcal{K}$ `with an infinite chain.`
If `true`, then A is undefinable with respect to $\mathcal{K}$.

Proof. Suppose that there is some $\mathfrak{F} \in \mathcal{K}$ such that $\mathfrak{F} \models A$ and $\mathfrak{F}$ contains an infinite chain. Assume for contradiction that φ defines A with respect to $\mathcal{K}$. Then in particular $\mathfrak{F} \models \varphi$. By property (v), this means that $LC \models \varphi$ so $\mathcal{K} \models \varphi$. But then since φ defines A, this means that $\mathcal{K} \models A$, which we ruled out in the previous step and is therefore a contradiction. □

If `false`:

(3) (*Boundedness*) `Decide whether there exists a uniform bound` m `of the depth of all models of` A `in` $\mathcal{K}$.
If `false`, then A is undefinable with respect to $\mathcal{K}$.

Proof. Assume for contradiction that A has a propositional definition φ. Then since (*Boundedness*) gives a negative answer, there are frames $\mathfrak{F} \in \mathcal{K}$ of arbitrary depth validating φ. In particular, there are arbitrarily long linear orders (as generated subframes of the frames in $\mathcal{K}$) validating φ and therefore by property (3) we have that $LC \models \varphi$, thus $\mathcal{K} \models \varphi$. But since φ defines A, this means that $\mathcal{K} \models A$ which we ruled out in step (*Validity*). □

If `true`:

(4) (*Least bound*) `Find the least uniform bound` m `of the depth of all models of` A `in` $\mathcal{K}$.

Proof. Such uniform bound exists by the positive answer given in the previous step. □

(5) (*Bound-completeness*) `Decide whether` $\mathfrak{F} \models A$ `for all frames` $\mathfrak{F} \in \mathcal{K}$ `of depth at most` m.
If `true`, then $\varphi_{depth \leq m}$ defines A with respect to $\mathcal{K}$.

 doi:10.20944/preprints202506.2497.v1

Proof. Take $\mathfrak{F} \in \mathcal{K}$.

Suppose first that $\mathfrak{F} \models A$. Then by (*Boundedness*), all chains in $\mathfrak{F}$ are of length at most m. Therefore $\mathfrak{F} \models \varphi_{depth \leq m}$.

Now suppose that $\mathfrak{F} \models \varphi_{depth \leq m}$. Then by the positive answer of (*Bound-completeness*), $\mathfrak{F} \models A$. □

If `false`, then A is undefinable with respect to $\mathcal{K}$.

Proof. Assume for contradiction that there exists a propositional definition φ of A with respect to $\mathcal{K}$. The uniform bound m cannot be 0, since then $\mathcal{K} \models \neg A$ hence (*Bound-completeness*) gives a positive answer. Since m is the least uniform bound, this means that there is frame $\mathfrak{F}_A \in \mathcal{K}$ of depth m such that $\mathfrak{F}_A \models A$, otherwise m would not be least. Since φ defines A, $\mathfrak{F}_A \models \varphi$. But then the linear order with m elements $\mathfrak{F}_m$ is a generated subframe of $\mathfrak{F}$ so $\mathfrak{F}_m \models \varphi$ and therefore $\mathfrak{F}_k \models \varphi$ for $1 \leq k \leq m$. Take any frame $\mathfrak{F} \in \mathcal{K}$ of depth at most m. Then any generated subframe of $\mathfrak{F}$ is isomorphic to $\mathfrak{F}_k$ for some $1 \leq k \leq m$. So any generated subframe of $\mathfrak{F}$ validates φ, so $\mathfrak{F} \models \varphi$. But since $\mathfrak{F} \in \mathcal{K}$ was arbitrary of depth at most m, (*Bound-completeness*) gives a positive answer — contradiction. □

Observe that the procedure we just described not only recognizes the definable first-order sentences but also explicitly points out a propositional definition when it exists. Now, in order to prove that the propositional definability problem with respect to $\mathcal{K}$ is decidable, it is sufficient to show the problems (*Validity*), (*Finiteness*), (*Boundedness*), (*Least bound*) and (*Bound-completeness*) can be computably solved for the class $\mathcal{K}$. This will allow us to prove the main result of this section, namely the following theorem:

Theorem 2. *The instances of the definability problem $Int-def$ with respect to any of the classes $LIN, LIN^{fin}, DLIN, DLIN^{fin}$ is decidable.*

Proof. The result follows from the procedure described above and the following lemmas:

- Lemma 2 shows that (*Validity*) is decidable.
- Lemma 1 deals with (*Bound-completeness*) and (*Least bound*).
- Lemma 3 shows that (*Finiteness*) is decidable.
- Lemmas 4 and 5 show that (*Boundedness*) is decidable.

□

We now proceed to show that the above problems are decidable. First, we notice that solving (*Validity*) and (*Boundedness*) immediately allows us to solve (*Bound-completeness*) and (*Least bound*).

Lemma 1. *If the first-order theory of $\mathcal{K}$ is decidable, then the problem (Bound-completeness) and the restriction of the problem (Least bound) to only those input sentences for which (Boundedness) gives a positive answer are decidable.*

Proof. Suppose that the first-order theory of $\mathcal{K}$ is decidable. For any natural number m denote by B_m the sentence $\neg \exists x_1 \exists x_2 \ldots \exists x_{m+1} \left(\bigwedge_{i=1}^{m} (x_i < x_{i+1}) \right)$ which is valid in exactly the frames of depth at most m.

We now have that $\mathfrak{F} \models A$ for all frames $\mathfrak{F} \in \mathcal{K}$ of depth at most m iff $\mathcal{K} \models B_m \rightarrow A$. Since the sentence $B_m \rightarrow A$ can be computed from the parameters A and m and the first-order theory of $\mathcal{K}$ is decidable, this gives us an effective procedure to solve (*Bound-completeness*).

Now suppose in addition that the instance of (*Boundedness*) for the class $\mathcal{K}$ gives a positive answer for the sentence A. Then there exists a least uniform bound m on the depth of the the frames from $\mathcal{K}$ that validate A. This m is the least natural number such that $\mathcal{K} \models A \rightarrow B_m$ (this sentence is again computable from the parameters A and m). Since such a natural number is guaranteed to exist, an effective solution to (*Least bound*) is to check in increasing order for each natural number t whether the sentence $A \to B_t$ is valid in $\mathcal{K}$, halting with answer m when in step m we get a positive answer for the first time. □

Of the problems we now need to solve, in some sense the most challenging is (*Finiteness*) because it deals with a property which is not expressible in the first-order language. This naturally leads us to consider the monadic second-order language which is better suited for internally analyzing this property. When dealing with second-order theories we shall be interested in the class $\mathcal{K}^{\to}$ consisting of those frames in $\mathcal{K}$ whose universe is at most countable. This is necessitated by the fact that the monadic second-order theory of the full class LIN is undecidable (a result due to Shelah [9]), which immediately means that the monadic second-order theory of the full class $DLIN$ is also undecidable. Working with the restricted class $\mathcal{K}^{\omega}$ will allow us to obtain decidability results on the monadic second-order side and then we will make use of the Löwenheim-Skolem theorem to reduce problems for $\mathcal{K}$ to problems for $\mathcal{K}^{\omega}$ which we can solve. A direct corollary to Rabin's theorem [6] on the decidability of the theory $S2S$ is the following theorem, which we will be key to our results:

Theorem 3. *(Rabin [6]) The monadic second-order theory of the class LIN^{ω} in the language of order is decidable.*

Note that an immediate corollary of the above theorem is that the monadic second-order theory of LIN^{ω} in the language of order expanded by a predicate FIN for finite sets is decidable. The reason is that $FIN(X)$ can be defined by the formula

$$\forall Y((Y \subseteq X \wedge \exists x(x \in Y)) \to ((\exists x \in Y)(\forall z \in Y)(x \leq z) \wedge (\exists y \in Y)(\forall z \in Y)(z \leq y)),$$

which states that X is finite iff every non-empty subset of X has a least and a greatest element.

Now, our main task is to obtain the corresponding result for the class $DLIN^{\omega}$. We will do this by showing that we can embed frames $\mathfrak{F} \in DLIN^{\omega}$ inside frames $\mathfrak{L} \in LIN^{\omega}$ in a way that allows us to translate properties of $\mathfrak{F}$ into properties of $\mathfrak{L}$. To further elaborate, since $\mathfrak{F}$ is the disjoint union of some family $\mathcal{F}$ of linear orders, we clearly have to embed the linear orders of $\mathcal{F}$ inside $\mathfrak{L}$, but in such a way that we can easily distinguish which elements of $\mathfrak{L}$ belong to the same chain of $\mathfrak{F}$ and which do not. We accomplish this by adding to $\mathfrak{L}$ new auxiliary elements which we will call indices of $\mathfrak{F}$. To each linear order $L \in \mathcal{F}$ we designate an index i_L and then obtain a "segment" from L by adding to it the index i_L as a greatest element. Then we obtain $\mathfrak{L}$ from $\mathfrak{F}$ by gluing one after another all the "segments" obtained from all the linear orders in $\mathcal{F}$.

This clearly results in an at most countable linear order $\mathfrak{L}$ which we will call a linearization of $\mathfrak{F}$. A linearization is inherently dependent on the order in which we glue together the segments. We can think of this order as an ordering on the set of indices of $\mathfrak{F}$ and different orderings on the set of indices may result in non-isomorphic linearizations of $\mathfrak{F}$. Additionally, if we are given a linearization $\mathfrak{L}$ and know the set $D \subset |\mathfrak{L}|$ of all indices, we want to be able to distinguish which elements of $\mathfrak{L}$ belong to the same segment. The most natural way to do this is to look at the position in $\mathfrak{L}$ of the index i_L and declare that its corresponding linear order L consists of all elements in $\mathfrak{L}$ which are less than i_L and bigger than any other index $i \in D$.

Those considerations lead us to consider only sets of indices which are isomorphic to initial segments of ω (with the inherited order from the linearization). Such sets avoid pathological situations in which non-indices from $\mathfrak{L}$ cannot be assigned to an index, while also somewhat limiting the variance of possible linearizations of a frame $\mathfrak{F} \in DLIN^{\omega}$, making them simpler to reason about. Clearly, we do not lose any expressiveness by imposing this restriction as any frame $\mathfrak{F} \in DLIN^{\omega}$ can be obtained by taking the union of a family of linear orders $\mathcal{F}$ with an index set which is an initial segment of ω.

Definition 1. *Let $\mathfrak{F}$ be a frame in $DLIN^\omega$. Without loss of generality, we will assume that (i) $\mathfrak{F}$ is the union of a family of pairwise disjoint linear orders $\{\mathfrak{F}_n \mid n < \alpha\}$ where α is an ordinal, $\alpha \leq \omega$; and $(ii)|\mathfrak{F}| \cap \alpha = \emptyset$. We will say that α is the set of indices of $\mathfrak{F}$.*

We define a linearization of $\mathfrak{F}$ *(with respect to the set of indices α) as the frame $\mathfrak{L} \in LIN^\omega$ with universe $|\mathfrak{L}| = |\mathfrak{F}| \cup \{n \mid n < \alpha\}$ and whose order $\leq^\mathfrak{L}$ is the unique linear order on $|\mathfrak{L}|$ with the following properties:*

- $\leq^\mathfrak{L}$ *extends* $\leq^\mathfrak{F}$
- $x <^\mathfrak{L} m$ *for each* $n \leq m < \alpha$ *and each* $x \in \mathfrak{F}_n$
- $m <^\mathfrak{L} x$ *for each* $m < n < \alpha$ *and each* $x \in \mathfrak{F}_n$

Let R be a monadic set variable and define the following mapping tr — translation *— between formulas of the monadic second-order language of order expanded by the predicate $FIN(X)$ for finite sets:*

- $tr(x \doteq y) = x \doteq y$
- $tr(FIN(X)) = FIN(X)$
- $tr(x \in X) = x \in X$
- $tr(x \leq y) = x \leq y \wedge \neg(\exists d \in R)(x \leq d \wedge d \leq y)$
- $tr(\neg A) = \neg tr(A)$
- $tr(A \wedge B) = tr(A) \wedge tr(B)$
- $tr(\exists x A) = \exists x(x \notin R \wedge tr(A))$
- $tr(\exists X A) = \exists X(\neg \exists x(x \in X \wedge x \in R) \wedge tr(A))$

Now the translation tr transforms a formula A expressing a monadic second-order property of $\mathfrak{F}$ into the formula $tr(A)$ which expresses a monadic second-order property of $\mathfrak{L}$. The key consideration in the translation is that the order of $\mathfrak{F}$ is interpreted inside $\mathfrak{L}$ as discussed before Definition 1 and that this interpretation is monadic second-order definable in terms of the set variable R which will be interpreted as the set of indices of $\mathfrak{F}$.

Proposition 2. *Let $\mathfrak{F} \in DLIN^\omega$ and $\mathfrak{L} \in LIN^\omega$ be a linearization of $\mathfrak{F}$. For each monadic second-order sentence A such that the variable R does not occur in A we have that $\mathfrak{F} \models A$ iff $\mathfrak{L} \models tr(A)[\![\alpha]\!]$ where α is the set of indices of $\mathfrak{F}$.*

Proof. By induction on the formulas A with free variables among $x_1, \ldots, x_n, X_1, \ldots, X_n$ and having no occurrences of R we can prove that for each tuple $a_1, \ldots, a_n$ of elements of $\mathfrak{F}$ and each tuple $S_1, \ldots, S_m$ of subsets of $\mathfrak{F}$ we have that $\mathfrak{F} \models A[\![a_1, \ldots, a_n, S_1, \ldots, S_n]\!]$ iff $\mathfrak{L} \models tr(A)[\![a_1, \ldots, a_n, S_1, \ldots, S_n, \alpha]\!]$. In particular, when A is a sentence we obtain the required property. □

Proposition 3. *For a frame $\mathfrak{L} \in LIN^\omega$ and a subset $D \subseteq |\mathfrak{L}|$, $\mathfrak{L}$ is isomorphic to a linearization of a frame $\mathfrak{F} \in DLIN^\omega$ with the set of indices of $\mathfrak{F}$ mapped to the set D precisely when D satisfies the following condition, expressible in the monadic second-order language by a formula $indices(R)$:*

(1) *D is either finite or isomorphic to ω under $\leq^\mathfrak{L}$.*
(2) *For every $d_1, d_2 \in D$ such that $d_1 <^\mathfrak{L} d_2$ there exists some $x \in |\mathfrak{L}| \setminus D$ such that $d_1 <^\mathfrak{L} x <^\mathfrak{L} d_2$.*
(3) *$|\mathfrak{L}| \setminus D \neq \emptyset$ and for every $x \in |\mathfrak{L}| \setminus D$ there is some $d \in D$ such that $x <^\mathfrak{L} d$.*

Proof. The listed conditions are immediately verified and follow the discussion before Definition 1.

We define the formula $indices(R)$ as the conjunction of the following formulas, each corresponding to one of the listed properties:

(1) $(\forall y \in R)(\forall Y \subseteq R)((\forall x \in R)(x \leq y \leftrightarrow x \in Y) \to FIN(Y))$ stating that all initial segments of elements of the interpretation of R are finite. This happens precisely when the interpretation of R is either a finite linear order or has the same order type as ω.
(2) $(\forall d_1 \in R)(\forall d_2 \in R)(d_1 < d_2 \to \exists x(x \notin R \wedge d_1 < x \wedge x < d_2))$
(3) $\exists x(x \notin R) \wedge \forall x(x \notin R \to (\exists d \in R)(x < d))$

□

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Theorem 4. *The monadic second-order theory of $DLIN^{\omega}$ in the language of order expanded by the predicate FIN for finite sets is decidable.*

Proof. We reduce to the monadic second-order theory of LIN^{ω}. For a monadic second-order sentence A, we first replace all occurrences of the set variable R in A with a new set variable not occurring in A. We will show that $DLIN^{\omega} \models A$ iff $LIN^{\omega} \models \forall R(indices(R) \to tr(A))$ where $indices(R)$ is the formula from Proposition 3. This immediately shows how to obtain the desired reduction.

Suppose first that $DLIN^{\omega} \models A$. Let $\mathfrak{L} \in LIN^{\omega}$ and $D \subseteq |\mathfrak{D}|$ be such that $\mathfrak{L} \models indices(R)[\![D]\!]$. Then by Proposition 3 we know that $\mathfrak{D}$ is isomorphic to a linearization of some frame $\mathfrak{F} \in DLIN^{\omega}$ with the set of indices of $\mathfrak{F}$ mapped to the set D. Since $DLIN^{\omega} \models A$, this means that $\mathfrak{F} \models A$ so by Proposition 2 this means that $\mathfrak{L} \models tr(A)[\![D]\!]$. Since D was an arbitrary subset of $|\mathfrak{L}|$ such that $\mathfrak{L} \models indices(R)[\![D]\!]$, this means that $\mathfrak{L} \models \forall R(indices(R) \to tr(A))$. Since $\mathfrak{L}$ was arbitrary, this in turn means that $LIN^{\omega} \models \forall R(indices(R) \to tr(A))$.

Now suppose that $LIN^{\omega} \models \forall R(indices(R) \to tr(A))$. Let $\mathfrak{F} \in DLIN^{\omega}$ be arbitrary and $\mathfrak{L} \in LIN^{\omega}$ be a linearization of $\mathfrak{F}$. Since $\mathfrak{L} \in LIN^{\omega}$ this means that $\mathfrak{L} \models \forall R(indices(R) \to tr(A))$. Let D be the set of indices of $\mathfrak{F}$. Then $\mathfrak{L} \models indices(R)[\![D]\!]$ hence $\mathfrak{L} \models tr(A)[\![D]\!]$. Then by Proposition 2 this means that $\mathfrak{F} \models A$. Since $\mathfrak{F}$ was arbitrary, this means that $DLIN^{\omega} \models A$. □

Lemma 2. *Let $\mathcal{K}$ be any of the classes LIN, LIN^{fin}, $DLIN$, $DLIN^{fin}$. Then (Validity) for $\mathcal{K}$, i.e. the following problem:*

`input:` *first-order sentence A*

`output:` `true`, *if $\mathcal{K} \models A$; and* `false`, *otherwise*

is decidable.

Proof. Consequence of Theorems 3 and 4.

For a first-order sentence A we have that $DLIN^{fin} \models A$ precisely when $DLIN^{\omega} \models \forall X(FIN(X)) \to A$. This reduces the first-order theory of $DLIN^{fin}$ to the monadic second-order theory of $DLIN^{\omega}$. We reduce the first-order theory of LIN^{fin} to the monadic second-order theory of LIN^{ω} in a similar way.

By the downward Löwenheim-Skolem theorem we can argue that the first order theories of $DLIN$ and $DLIN^{\omega}$ coincide because the class is axiomatizable. But the first-order theory of $DLIN^{\omega}$ is decidable because it is a restriction of the dedidable monadic second-order theory of the class. The same argument goes for the decidability of the first-order theory of the class LIN. □

Lemma 3. *Let $\mathcal{K}$ be any of the classes LIN, LIN^{fin}, $DLIN$, $DLIN^{fin}$. Then (Finiteness) for $\mathcal{K}$, i.e. the following problem:*

`input:` *first-order sentence A*

`output:` `true`, *if A is valid in some frame $\mathfrak{F} \in \mathcal{K}$ which has an infinite chain; and* `false`, *otherwise*

is decidable.

Proof. Clearly (*Finiteness*) is trivial for LIN^{fin} and $DLIN^{fin}$.

We now argue for $DLIN$. The sentence

$$B = \exists X(\neg FIN(X) \wedge (\forall y \in X)(\forall z \in X)(y \leq z \vee z \leq y))$$

is valid in a frame iff the frame contains an infinite chain. We will prove that (*Finiteness*) for $DLIN$ should output `true` on input A iff $DLIN^{\omega} \not\models A \to \neg B$. The latter condition can be computably checked in view of Theorem 4 and the fact that the sentence $A \to \neg B$ can be computed from A.

Suppose first that $\mathfrak{F} \in DLIN$ has an infinite chain and $\mathfrak{F} \models A$. Let C be a countably infinite chain in $\mathfrak{F}$ and obtain by the downward Löwenheim-Skolem theorem a countable elementary substructure $\mathfrak{F}_c$ of $\mathfrak{F}$ such that $C \subseteq |\mathfrak{F}_c|$. Then, since $\mathfrak{F}_c$ is an elementary substructure of $\mathfrak{F}$ and is countable, we

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have that $\mathfrak{F}_c$ belongs to $DLIN^\omega$ and $\mathfrak{F}_c \models A$. Moreover, $\mathfrak{F}_c$ contains the infinite chain C so $\mathfrak{F}_c \models B$. Therefore, $\mathfrak{F}_c \not\models A \to \neg B$, hence $DLIN^\omega \not\models A \to \neg B$.

Conversely, suppose that $DLIN^\omega \not\models A \to \neg B$. Then there is a frame $\mathfrak{F} \in DLIN^\omega$, such that $\mathfrak{F} \not\models A \to \neg B$. But this means that $\mathfrak{F} \in DLIN$, $\mathfrak{F} \models A$ and $\mathfrak{F} \models B$, i.e. $\mathfrak{F} \in DLIN$ validates A and has an infinite chain.

The argument for LIN is similar to that for $DLIN$. □

Lemma 4. *Let $\mathcal{K}$ be any of LIN, $DLIN$. Then (Boundedness) for $\mathcal{K}$, i.e. the following problem:*

`input:` *first-order sentence A*

`output:` `true`, *if there exists a uniform bound $m < \omega$ on the depth of the frames from $\mathcal{K}$ that validate A; and* `false`, *otherwise*

is decidable. More specifically, on input A (Boundedness) for $\mathcal{K}$ always outputs the answer (Finiteness) did not, so (Boundedness) is decidable as the complement of a decidable problem.

Proof. Let A be a first-order sentence.

First, if (*Finiteness*) outputs `true` on input A, this means that there is a frame $\mathfrak{F} \in \mathcal{K}$ with an infinite chain such that $\mathfrak{F} \models A$. This clearly means that (*Boundedness*) should output `false`.

Conversely, suppose that (*Boundedness*) outputs `false` on input A. Then no natural number m uniformly bounds the depth the frames from $\mathcal{K}$ that validate A. By application of a standard compactness argument we can then show that the set

$$T = \{K, A\} \cup \{c_i < c_{i+1} \mid i < \omega\}$$

is satisfiable where K is the axiom for $\mathcal{K}$ and the c_i for $i < \omega$ are distinct constant symbols. Let $\mathfrak{F} \models T$, then we can conclude that $\mathfrak{F} \models A$ because $A \in T$, $\mathfrak{F} \in \mathcal{K}$ because $K \in T$, and $\mathfrak{F}$ has an infinite chain because the set $\{c_i^{\mathfrak{F}} \mid i < \omega\}$ is a chain. Therefore (*Finiteness*) outputs `true` on input A. □

Remark 1. *Observe that in the procedure for solving the propositional definability problem for $\mathcal{K}$ where $\mathcal{K}$ is LIN or $DLIN$, the step in which we need to query (Boundedness) is only reached if (Finiteness) has previously output* `false` *for A. Therefore, in view of the lemma we just proved, we can skip the step that involves (Boundedness) in the procedure for $\mathcal{K}$*

Lemma 5. *(Boundedness) for LIN^{fin} and for $DLIN^{fin}$ is decidable.*

Proof. Let A be a first-order sentence and $qr(A) = n$.

A classical application of Ehrenfeucht-Fraïssé games shows that any two linear orders with more at least 2^n elements agree on the validity of A (the reader may consult [10] for information on Ehrenfeucht-Fraïssé games and basic results). Therefore, in LIN^{fin} there are frames of arbitrarily large depth that validate A precisely when the unique up to isomorphism linear order with 2^n elements validates A. Since $n = qr(A)$ and a linear order with 2^n elements are computable from the parameter A, and satisfiability in the obtained structure is computable, the result follows for LIN^{fin}.

We will now obtain a similar property for $DLIN^{fin}$. First, from each frame $\mathfrak{F} \in DLIN^{fin}$ obtain the frame $\mathfrak{F}' \in DLIN^{fin}$ by replacing each maximal chain of depth more than 2^n with its initial segment of length 2^n and keeping the other maximal chains intact. We claim that $\mathfrak{F}$ and $\mathfrak{F}'$ agree on the validity of A and prove it by describing a winning strategy for Duplicator for the n-turn Ehrenfeucht-Fraïssé game for $\mathfrak{F}$ and $\mathfrak{F}'$. On each step:

- If Spoiler picks an element from a chain that has not been modified in the construction, Duplicator picks the same element from the other frame.
- If Spoiler picks an element from a maximal chain C_S in $\mathfrak{F}$ that has been shrinked to C_D in $\mathfrak{F}'$ or an element from a maximal chain C_S in $\mathfrak{F}'$ that that has been obtained by shrinking the chain C_D in $\mathfrak{F}$, Duplicator chooses an element in C_D by consulting the winning strategy for the n-turn

Ehrenfeucht-Fraïssé game for the linear orders C_S and C_D (both chains contain at least 2^n elements so such strategy exists).

We will use the property to prove that there is no uniform bound m of the depth of the frames from $DLIN^{fin}$ that validate A iff A has a model $\mathfrak{F} \in DLIN^{fin}$ which has a chain C with at least 2^n elements.

Suppose that A has a model $\mathfrak{F} \in DLIN^{fin}$ which contains a chain C with at least 2^n elements and assume for contradiction that there is a uniform bound m of the depth of the frames from $DLIN^{fin}$ that validate A. Consider the frame $\mathfrak{F}_m$ obtained by extending with m new elements the maximal chain in $\mathfrak{F}$ which contains C. Now, since $\mathfrak{F} \models A$ we obtain that $\mathfrak{F}' \models A$. Clearly, $(\mathfrak{F}_m)'$ is the same as $\mathfrak{F}'$, so $(\mathfrak{F}_m)' \models A$. And finally $\mathfrak{F}_m \models A$ since $(\mathfrak{F}_m)' \models A$. But $\mathfrak{F}_m$ contains a chain with more than m elements which contradicts the property of m being a uniform bound for A.

Conversely, if A has no uniform bound of the depth of its models from $DLIN^{fin}$, this means that in particular 2^n is not a uniform bound so A must have a model from $DLIN^{fin}$ which contains a chain with at least 2^n elements.

Combining the above, we conclude that (*Boundedness*) for $DLIN^{fin}$ reduces to checking whether A has a model $\mathfrak{F} \in DLIN^{fin}$ with a chain of depth (at least) 2^n. This in turn reduces to (*Validity*), as such a model exists iff $A \wedge \exists x_1 \exists x_2 \ldots \exists x_{2^n} (\bigwedge_{i=1}^{2^n-1} (x_i < x_{i+1}))$ is satisfiable in $DLIN^{fin}$. □

Remark 2. *Note that we can further employ the technique of Ehrenfeucht-Fraïssé games used in the above proof to show alternative decision methods for the validity problem for LIN^{fin} and $DLIN^{fin}$. Let A be a first-order sentence and $qr(A) = n$.*

For finite linear orders, we have that $LIN^{fin} \models A$ precisely when all (finitely many up to isomorphism) linear orders of depth at most 2^n validate A because all linear orders with at least 2^n elements agree on the validity of A.

Let $\mathfrak{F} \in DLIN^{fin}$ be arbitrary and obtain $\mathfrak{F}'$ as in the proof of 5. We then obtain the subframe $\mathfrak{F}''$ of $\mathfrak{F}'$ by removing some (possibly none) of the maximal chains in $\mathfrak{F}'$ in the following way: for each $k \leq 2^n$ such that there are more than n maximal chains of depth k in $\mathfrak{F}'$ we remove all but n of those chains in $\mathfrak{F}''$. There is again a simple winning strategy for Duplicator for the n-turn Ehrenfeucht-Fraïssé game for $\mathfrak{F}'$ and $\mathfrak{F}''$.

This means that $DLIN^{fin} \models A$ precisely when all frames of the form $\mathfrak{F}''$ for some $\mathfrak{F} \in DLIN^{fin}$ validate A. But the frames of type $\mathfrak{F}''$ are precisely those (finitely many up to isomorphism) frames whose chains are of depth at most 2^n and for each $k \leq 2^n$ contain at most n maximal chains with k elements.

Now, to decide validity for either class we can generate and check the validity of A in a finite number of finite frames (in which satisfaction of sentences is computable) which can be computed from A.

4. Some Classes of Partial Orders with Respect to Which the Problem $Int - def$ Is Undecidable

In this section we obtain negative solutions to the definability problem for some natural classes of partial orders. A classical result in correspondence theory is a theorem by Chagrova [2,3] stating that the definability problem with respect to the class of all partial orders is undecidable. Her method is based on reductions of undecidable problems for Minsky machines. Here we present a technique developed by Balbiani and Tinchev [4] which allows for a model theoretic approach for proving negative results for the definability problem for modal logics. First we present the technique in its straightforward adaptation for the intuitionistic case.

Definition 2. *We say that a class $\mathcal{K}$ of partial orders is* stable *if there is a first-order sentence B and a first-order formula A with free variables among $\{y, x_1, \ldots, x_n\}$ which satisfy the following conditions:*

(1) For every frame $\mathfrak{F} \in \mathcal{K}$ and every tuple $a_1, \ldots, a_n$ of points in $\mathfrak{F}$, the relativized reduct of $\mathfrak{F}$ with respect to A and $a_1, \ldots a_n$, if it exists, is a frame in $\mathcal{K}$.

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(2) *For every frame* $\mathfrak{F} \in \mathcal{K}$ *there are frames* $\mathfrak{F}_1 \in \mathcal{K}$ *and* $\mathfrak{F}_2 \in \mathcal{K}$ *such that* $Log(\mathfrak{F}_2) \subseteq Log(\mathfrak{F}_1)$, $\mathfrak{F}_1 \not\models B$, $\mathfrak{F}_2 \models B$ *and* $\mathfrak{F}$ *is a relativized reduct of* $\mathfrak{F}_2$ *with respect to* A *and some parameters* $a_1, \ldots, a_n \in \mathfrak{F}_2$.

Theorem 5. *If* $\mathcal{K}$ *is a stable class of partial orders, then the first-order validity problem reduces to the problem* $Int - def$ *with respect to* $\mathcal{K}$.

Proof. The proof is almost the same as the proof for the modal version of the theorem in [4], only replacing modal with intuitionistic semantics for the propositional formulas. □

The above theorem provides a very useful reduction since it allows to prove undecidability of the definability problem by showing stability of the class in consideration and undecidability of its first-order theory.

We shall apply the theorem to show that the problem $Int - def$ with respect to the following classes is undecidable:

- The class PO of all partial orders (a model-theoretic proof of Chagrova's result).
- For each $n \geq 2$ the class $PO_{depth \leq n}$ of all partial orders of depth at most n.
- The class DPO of all dense partial orders.
- The classes PO^{fin} and $PO^{fin}_{depth \leq n}$ for $n \geq 2$ (where the superscript fin denotes the restriction of the class to finite frames).

We begin with stability of the classes:

Lemma 6. *The classes* PO, $PO_{depth \leq n}$ *for* $n \geq 2$ *and* DPO *are stable.*

Proof. Denote by $\mathcal{K}$ any of the above classes and by K its first-order axiom (all of the above classes are finitely axiomatizable). We choose the formula $A = \neg(y \doteq x_1) \wedge \neg(y \doteq x_2)$ and the sentence $B = \exists z_1 \exists z_2 (\neg(z_1 \doteq z_2) \wedge isolated(z_1) \wedge isolated(z_2))$ where $isolated(z)$ is an abbreviation for the formula $\forall x (x \leq z \vee z \leq x \to x \doteq z)$ which states that z is incomparable with any other point.

We show that A and B witness the stability of $\mathcal{K}$.

For property (1) in the definition of stability: suppose that $\mathfrak{F} \in \mathcal{K}$ and $a_1, a_2 \in \mathfrak{F}$ are such that the relativized reduct $(\mathfrak{F})^{A,a_1,a_2}$ of $\mathfrak{F}$ with respect to A and a_1, a_2 exists. Then $(\mathfrak{F})^{A,a_1,a_2}$ is the subframe of $\mathfrak{F}$ with universe $|\mathfrak{F}| \setminus \{a_1, a_2\}$. If $\mathcal{K}$ is PO, then obviously $(\mathfrak{F})^{A,a_1,a_2}$ is a partial order so $(\mathfrak{F})^{A,a_1,a_2} \in \mathcal{K}$. If $\mathcal{K}$ is $PO_{depth \leq n}$ then the depth of the chains in $(\mathfrak{F})^{A,a_1,a_2}$ is also bounded by n since every chain in $(\mathfrak{F})^{A,a_1,a_2}$ is a chain in $\mathfrak{F}$ so $(\mathfrak{F})^{A,a_1,a_2} \in \mathcal{K}$. If $\mathcal{K}$ is DPO then to show that $(\mathfrak{F})^{A,a_1,a_2} \in \mathcal{K}$ it suffices to check that if $b_1, b_2 \in |(\mathfrak{F})^{A,a_1,a_2}|$ and $b_1 < a_1 < b_2$ or $b_1 < a_2 < b_2$ then there is some $c \in (\mathfrak{F})^{A,a_1,a_2}$ such that $b_1 < c < b_2$. But this is immediate since if $b_1 < a_i < b_2$ for $i \in \{1, 2\}$ then by density there is some c_1 such that $b_1 < c_1 < a_i < b_2$. If $c_1 \neq a_1, a_2$ we are done and otherwise by density again there exists some c such that $b_1 < c < c_1 < b_2$ and then $c \neq a_1, a_2$.

We now turn to property (2) in the definition of stability. Suppose that $\mathfrak{F} \in \mathcal{K}$ and take $a_1, a_2 \notin \mathfrak{F}$. Take the following frames:

- $\mathfrak{F}_1$ with universe $\{a\}$ and $\leq^{\mathfrak{F}_1} = \{\langle a, a \rangle\}$
- $\mathfrak{F}_2$ with universe $|\mathfrak{F}| \cup \{a, b\}$ and $\leq^{\mathfrak{F}_2} = \leq^{\mathfrak{F}} \cup \{\langle a, a \rangle, \langle b, b \rangle\}$.

Obviously $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are partial orders, if the chains in $\mathfrak{F}$ are bounded in depth by $n \geq 2$ then so are the chains in $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and if $\mathfrak{F}$ is dense then so are $\mathfrak{F}_1$ and $\mathfrak{F}_2$; so $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{K}$. Since $\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}_2$ we have that $Log(\mathfrak{F}_2) \subseteq Log(\mathfrak{F}_1)$. $\mathfrak{F}_1$ contains a single point so it certainly does not have two isolated points, i.e. $\mathfrak{F}_1 \not\models B$. On the other hand, $\mathfrak{F}_2$ contains the isolated points a and b so $\mathfrak{F}_2 \models B$. Finally, $\mathfrak{F}$ is obtained from $\mathfrak{F}_2$ by omitting the new points a and b so $\mathfrak{F}$ is the relativized reduct of $\mathfrak{F}_2$ with respect to A and a, b. □

Lemma 7. *The classes* PO^{fin} *and* $PO_{depth \leq n}$ *for* $n \geq 2$ *are stable.*

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Proof. The construction for the unrestricted classes in the previous lemma produces finite frames when $\mathfrak{F}$ is finite. □

We are now left to show undecidability of the first-order theories of the classes. A theorem of Tarski [11] gives undecidability of the first-order theories of the classes PO and PO^{fin}. In order to prove that the other classes of frames have undecidable first-order theories, we will consider a certain class of frames which we will call connectivity maps. We will later see that such frames are intrinsically related to undirected graphs without loops (which we think are represented by reflexive and symmetric relations).

Definition 3. *The class of connectivity maps, CM, contains all partial orders that satisfy the following conditions:*

- *Any point is either maximal or has exactly two incomparable points strictly above it.*
- *Any two distinct points either have no common point strictly below them or they have exactly one such point.*

In the context of partial orders we will use the following abbreviations:

- $min(x)$ stands for the formula $\neg\exists y(y < x)$ which states that x is a minimal element.
- $max(x)$ stands for the formula $\neg\exists y(x < y)$ which states that x is a maximal element.
- $mid(x)$ stands for the formula abbreviated by $\neg min(x) \wedge \neg max(x)$ which states that x is neither minimal nor maximal.

Proposition 4. *The class CM has the following properties:*

(1) *Each connectivity map is a partial order of depth at most* 2.
(2) *CM is finitely axiomatizable.*
(3) *The first-order theory of CM reduces to the first-order theory of $PO_{depth\leq n}$ for $n \geq 2$. The same relationship holds for CM^{fin} and $PO^{fin}_{depth\leq n}$ for $n \geq 2$.*

Proof. (1) Assume for contradiction that there is $\mathfrak{F} \in CM$ and $x, y, z \in \mathfrak{F}$ are such that $x <^{\mathfrak{F}} y <^{\mathfrak{F}} z$. Since y has a point above it, it must have exactly two points z_1, z_2 above it. But then both x and y are strictly below z_1 and z_2 which violates the second condition for membership to CM.

(2) We take the axioms for partial orders together with the following axioms corresponding to the two membership conditions for CM:
$\forall x(\neg\exists z(x < z) \vee \exists z_1\exists z_2(x < z_1 \wedge x < z_2 \wedge z_1 \not\doteq z_2 \wedge \forall z(x < z \rightarrow (z \doteq z_1 \vee z \doteq z_2))))$
$\forall x\forall y(x \not\doteq y \rightarrow \forall z_1\forall z_2((z_1 < x \wedge z_1 < y \wedge z_2 < x \wedge z_2 < y) \rightarrow z_1 \doteq z_2))$

(3) CM is an axiomatizable subclass of $PO_{depth\leq n}$ for $n \leq 2$ so for a first-order sentence A we have that $CM \models A$ iff $PO_{depth\leq n} \models C \rightarrow A$ where C is the axiom for CM. The same reduction holds for the classes of finite frames.
□

The essence of the above proposition is that undecidability of the first-order theory of the class CM (and CM^{fin}) propagates to the other classes we have listed with the exception of the class DPO. For DPO we will need to take an intermediate step by considering "dense variants" of connectivity maps.

Definition 4. *Let $\mathfrak{C}$ be a connectivity map. We say that the partial order $\mathfrak{D}$ is a* densification of *$\mathfrak{C}$ if $\mathfrak{D}$ is obtained by inserting between each pair $x <^{\mathfrak{C}} y$ a fresh dense linear order with least element x and greatest element y. Denote by DCM the class of all densifications of connectivity maps.*

Proposition 5. *The class DCM has the following properties:*

(1) *DCM is finitely axiomatizable.*

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(2) *There is a translation tr_D of first-order formulas such that for each first-order sentence A it holds that $\mathfrak{C} \models A$ iff $\mathfrak{D} \models tr_D(A)$ where $\mathfrak{C}$ is any connectivity map and $\mathfrak{D}$ is any of its densifications.*

(3) *The first-order theory of CM reduces to the first-order theory of DCM*

Proof. (1) The following sentences (we use $\exists! x$ as the usual abbreviation for "there exists a unique x") provide an axiomatization for DCM:

(a) The axiom for dense partial orders.

(b) $\forall x \forall y (min(x) \wedge max(y) \rightarrow \forall z_1 \forall z_2 ((x \leq z_1 \wedge z_1 \leq y \wedge x \leq z_2 \wedge z_2 \leq y) \rightarrow (z_1 \leq z_2 \vee z_2 \leq z_1)))$ stating that any interval between a minimal and a maximal element is linearly ordered.

(c) $\forall z (mid(z) \rightarrow \exists! x (min(x) \wedge x \leq z) \wedge \exists! y (max(y) \wedge z \leq y))$ stating that each nonextremal element is comparable with a unique maximal element and a unique minimal element.

(d) $\forall x (min(x) \rightarrow (max(x) \vee \exists y_1 \exists y_2 (max(y_1) \wedge max(y_2) \wedge x < y_1 \wedge x < y_2 \wedge \forall y ((max(y) \wedge x < y) \rightarrow (y \doteq y_1 \vee y \doteq y_2)))))$ stating that each minimal element is either maximal or is below exactly two maximal elements (the axiom roughly corresponds to the first membership condition for CM).

(e) $\forall y_1 \forall y_2 ((y_1 \neq y_2 \wedge max(y_1) \wedge max(y_2)) \rightarrow \forall x_1 \forall x_2 ((min(x_1) \wedge min(x_2) \wedge x_1 < y_1 \wedge x_2 < y_1 \wedge x_1 < y_2 \wedge x_2 < y_2) \rightarrow x_1 \doteq x_2)$ stating that each pair of distinct maximal elements have at most one common minimal element below them (roughly corresponds to the second membership condition for CM).

An immediate verification shows that each frame from DCM satisfies the axioms. Conversely, suppose that $\mathfrak{F}$ satisfies the axioms. We can then obtain $\mathfrak{C}$ as the subframe of $\mathfrak{F}$ whose universe consists of the extremal elements of $\mathfrak{F}$. $\mathfrak{F}$ is a partial order so $\mathfrak{C}$ is too. Moreover, $\mathfrak{C}$ satisfies the two membership conditions for CM because the last two axioms above force the extremal elements of $\mathfrak{F}$ to be in an appropriate configuration, so $\mathfrak{C} \in CM$.
By axiom (c) we know that the extremal elements of $\mathfrak{F}$ together with the intervals between minimal and maximal elements exhaust all of $\mathfrak{F}$ and that the interiors of such intervals are two by two disjoint. Axiom (b) together with density of $\mathfrak{F}$ means that each such interval is a dense linear order. Therefore, $\mathfrak{F}$ can be obtained as a densification of $\mathfrak{C}$ by replacing each pair $x <^{\mathfrak{C}} y$ in $\mathfrak{C}$ by the dense and linearly ordered interval $[x, y] \subseteq |\mathfrak{F}|$, hence $\mathfrak{F} \in DCM$.

(2) Consider the formula $U = min(y) \vee max(y)$. Define the translation tr_{DCM} that transforms a first-order formula A into its relativization with respect to the formula U and the variable y, i.e. $tr_{DCM}(A) = (A)^{U,y}$. Now consider a connectivity map $\mathfrak{C}$ and any of its densifications $\mathfrak{D}$. $\mathfrak{C}$ consists of the extremal elements of $\mathfrak{D}$ so $\mathfrak{C}$ is the relativized reduct of $\mathfrak{D}$ with respect to U and y. Therefore, by the relativization theorem it follows that $\mathfrak{C} \models A$ iff $\mathfrak{D} \models tr_{DCM}(A)$.

(3) Denote by D the axiom for the class DCM and consider the translation tr_{DCM} from (2). Then for any sentence A we have that $CM \models A$ iff $\mathfrak{C} \models A$ for each connectivity map $\mathfrak{C} \in CM$ iff (by (2)) $\mathfrak{D} \models tr_{DCM}(A)$ for each densification $\mathfrak{D} \in DCM$ of any connectivity map $\mathfrak{C} \in CM$ iff $DCM \models tr_{DCM}(A)$.
□

Corollary 1. *The first-order theory of CM reduces to the first-order theory of DPO*

Proof. The first-order theory of CM reduces to the first-order theory of DCM by the previous proposition. But if D is the axiom for DCM and A is an arbitrary sentence, then $DCM \models A$ iff $DPO \models D \rightarrow A$. □

We now turn to proving that the first-order theory of CM is undecidable. As we previously eluded, the class CM is closely connected to the class of graphs. A theorem by Rogers [12] shows the undecidability of the first-order theory of a reflexive and symmetric relation (which we think of as encoding a graph) and we will use it to anchor the chain of reductions.

 doi:10.20944/preprints202506.2497.v1

Definition 5. *A* graph *is a frame for the language with sole non-logical symbol E which is interpreted as a reflexive and symmetric relation.*

If $\mathfrak{G}$ is a graph (without loss of generality we assume that the elements in $|\mathfrak{G}|$ are not two-element sets), its canonical connectivity map *is the frame $\mathfrak{C} \in CM$ with the following definition:*

- $|\mathfrak{C}| = |\mathfrak{G}| \cup \{\{u, v\} \subseteq |\mathfrak{G}| \mid u \neq v, E^{\mathfrak{G}}(u, v)\}$
- $\leq^{\mathfrak{C}}$ *is the least partial order on $|\mathfrak{C}|$ such that $\{u, v\} \leq^{\mathfrak{C}} u$ and $\{u, v\} \leq^{\mathfrak{C}} v$ for each $\{u, v\} \in |\mathfrak{C}|$*

We briefly elaborate on the above definition. In traditional terms, the graphs we consider are unordered and without loops, and $E(u, v)$ says that there is an edge between u and v when u and v are distinct vertices. As we do not allow loops in our graphs, $E(u, u)$ does not encode any significant information and for simplicity, we take E to be reflexive.

The canonical connectivity map $\mathfrak{C}$ for $\mathfrak{G}$ is then essentially a different representation of the graph: the vertices are maximal elements in $\mathfrak{C}$ and an edge $E(u, v)$ is encoded in $\mathfrak{C}$ by the configuration of $\{u, v\}$ and u, v. An immediate but useful property is that $\mathfrak{G}$ is finite iff its canonical connectivity map $\mathfrak{C}$ is finite. The following Definition and Proposition show how we translate first-order formulas for graphs into first-order formulas for connectivity maps which have the same meaning.

Definition 6. *The translation tr_M that transforms first-order formulas for the language $\{E\}$ to first-order formulas for the language of order is defined by recursion as follows:*

- $tr_M(x \doteq y) = x \doteq y$
- $tr_M(E(x, y)) = \exists z(z < x \land z < y)$
- $tr_M(\neg A) = \neg tr_M(A)$
- $tr_M(A \land B) = tr_M(A) \land tr_M(B)$
- $tr_M(\exists x A) = \exists x(max(x) \land tr_M(A))$

Proposition 6. *If $\mathfrak{G}$ is a graph and $\mathfrak{C} \in CM$ is its canonical connectivity map, then for each sentence A in the language $\{E\}$ it holds that $\mathfrak{G} \models A$ iff $\mathfrak{C} \models tr_M(A)$.*

Proof. By straightforward induction on formulas A in the language of graphs with free variables $x_1, \ldots, x_n$ we can prove that for each tuple $g_1, \ldots, g_n$ of elements of $\mathfrak{G}$ we have that $\mathfrak{G} \models A[\![g_1, \ldots, g_n]\!]$ iff $\mathfrak{C} \models tr(A)[\![g_1, \ldots, g_n]\!]$. In particular, when A is a sentence we obtain the desired property of the translation. □

Proposition 7. *Any connectivity map $\mathfrak{C}$ is isomorphic to the canonical connectivity map of some graph $\mathfrak{G}$.*

Proof. It is immediately verified that $\mathfrak{C}$ is isomorphic to the canonical connectivity map of the graph $\mathfrak{G}$ defined as follows:

- $|\mathfrak{G}|$ is the set of all maximal elements in $\mathfrak{C}$
- For each distinct $c_1, c_2 \in |\mathfrak{G}|$ we define $E^{\mathfrak{G}}(c_1, c_2)$ iff c_1 and c_2 have a lower bound in $\mathfrak{C}$

□

Theorem 6. *The first-order theories of the classes CM, $PO_{depth \leq n}$ for $n \geq 2$ and DPO are undecidable. The first-order theories of the classes CM^{fin} and $PO^{fin}_{depth \leq n}$ for $n \geq 2$ are undecidable (not even semidecidable).*

Proof. The first-order theory of graphs is undecidable by a theorem of Rogers [12]. We claim that the sentence A about graphs is valid iff $CM \models tr_M(A)$. Indeed, all graphs validate A iff (by Proposition 6) the class of all canonical connectivity maps validates $tr_M(A)$ iff (by Proposition 7) all connectivity maps validate $tr_M(A)$. Therefore, tr_M reduces the first-order theory of graphs to the first-order theory of CM, which we then conclude is undecidable.

Now undecidability of the first-order theory of $PO_{depth \leq n}$ for $n \geq 2$ follows from (3) in Proposition 4 and undecidability of the first-order theory of DPO follows from Corollary 1.

 doi:10.20944/preprints202506.2497.v1

For the classes of finite frames we follow the parallel chain of reductions starting from the first-order theory of finite graphs (the relevant constructions for the reductions construct finite frames when applied to finite frames). Its undecidability follows from a theorem in [13] which states that the first-order theory of graphs and the complement of the first-order theory of finite graphs are recursively inseparable, in particular this means that the first-order theory of finite graphs is undecidable. A finer analysis shows that the first-order theory of finite graphs is not even semidecidable since its complement is semidecidable (given a sentence A start enumerating all finite graphs and search for a countermodel for A). Then untractability propagates down the chain of reductions. □

Theorem 7. *The problem $Int - def$ with respect to each of the classes PO, PO^{fin}, DPO, $PO_{depth \leq n}$ and $PO^{fin}_{depth \leq n}$ for $n \geq 2$ is undecidable.*

Proof. By Lemma 6 and Lemma 7 each of the listed classes is stable. The first-order theory of PO and PO^{fin} are undecidable by [11] and the first-order theories of the other classes are undecidable by Theorem 6. By Theorem 5 the first-order theories reduce to the respective instances of $Int - def$ with respect to the classes hence the result follows. □

5. Conclusion and Further Research

We have shown that the propositinal definability problem with respect to the class $\mathcal{K}$ is decidable, where $\mathcal{K}$ is among $LIN, LIN^{fin}, DLIN$ and $DLIN^{fin}$. A related question is the decidability of the first-order definability problem and the correspondence problem with respect to $\mathcal{K}$. The first-order definability problem with respect to $\mathcal{K}$ asks whether an input propositional formula can be defined with respect to $\mathcal{K}$ by a first-order sentence. Proposition 1 *(vi)* shows that a propositional formula is validated either by all frames in $\mathcal{K}$ or by all frames from $\mathcal{K}$ of depth at most n for some natural number n. Those classes are all first-order axiomatizable relative to $\mathcal{K}$, i.e. every propositional formula is first-order definable. This means that the first-order definability problem has a trivial solution, namely for every input propositional formula output `true`. Remark that first-order definability with respect to the class of all partial orders is undecidable, [14]. The correspondence problem with respect to $\mathcal{K}$ asks whether the input propositional formula and first-order sentence are validated by the same frames from $\mathcal{K}$. The argument in Proposition 1 *(vi)* can be refined into an algorithm that computes a first-order definition for an input propositional formula. This allows to reduce the correspondence problem with respect to $\mathcal{K}$ to equivalence of first-order sentences with respect to $\mathcal{K}$. The latter reduces to the first-order theory $\mathcal{K}$, which we have seen is decidable.

There are a few further directions that can be worked on with regard to the results in Section 3. Although we have shown that the problems are decidable, the question of their complexity is left open. In addition, the fruitful setting of working with frames for the logic LC can be expanded on and the questions of algorithmic definability and correspondence can be posed with respect to the classes of frames which validate superintuitionistic extensions of LC.

In Section 4 we have highlighted the model-theoretic approach for proving undecidability results for the propositional definability problem with respect to a class $\mathcal{K}$ by showing that $\mathcal{K}$ is stable and has an undecidable first-order theory. This allows for such proofs to be carried out purely in the context of frame constructions without having to rely on objects (notably any kind of machine) which are external to the setting. An interesting and important result would be the development of a similar technique for proving undecidability of the dual problem of first-order definability that is easy to apply to a broad range of classes.

Author Contributions: Writing—original draft, G.K.; writing—review and editing, T.T. authors contributed equally to the manuscript. Both authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

doi:10.20944/preprints202506.2497.v1

Acknowledgments: The work of Grigor Kolev is supported by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project no. BG-RRP-2.004-0008-C01.

Conflicts of Interest: The authors declare no conflicts of interest.

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