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Article

Generator Idempotents in Semi-Simple Ring FC_{16p^n} and the Codes

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Abstract: In semi-simple ring $R_{16p^n} \equiv \frac{GF(q)[x]}{<x^{16p^n}-1>}$, where p is prime and q is some prime or prime power (of type $16k+3$), n is a positive integer, subject to order of q modulo p^n is $\frac{\phi(p^n)}{2}$, expressions for primitive idempotents are obtained. Generating polynomials, dimensions and minimum distance bounds for the cyclic codes generated by these idempotents are also obtained.

Keywords: cyclotomic cosets; primitive idempotents; generating polynomials; minimum distance

MSC: 11T71; 11T55; 22D20

1. Introduction

The group algebra FC_{16p^n}, F is field of order q and C_{16p^n} is cyclic group of order $16p^n$ such that $g.c.d.(q, 16p) = 1$, is semi-simple having finite cardinality of collection of primitive idempotents which equals the cardinality of collection of q-cyclotomic cosets modulo $16p^n$ [?] The primitive idempotents of minimal cyclic codes of length m in case, when order of q modulo m is $\phi(m)$ for $m = 2, 4, p^n, 2p^n$ were computed in [6,9]. The primitive idempotents of length p^n with order of q modulo p^n is $\frac{\phi(p^n)}{2}$ were obtained in [10] and minimal quadratic residue codes of length p^n in [7]. Cyclic codes of length $2p^n$ over F, where order of q modulo $2p^n$ is $\frac{\phi(2p^n)}{2}$ were discussed in [8]. Minimal cyclic codes of length p^nq, where p and q are distinct odd primes were derived in [1,3]. Further, when order of q modulo p^n is $\phi(p^n)$, the minimal cyclic codes of length $8p^n$ were discussed in [4,5]. Irreducible cyclic codes of length $4p^n$ and $8p^n$, where $q \equiv 3(mod\ 8)$ and $p/(q-1)$ were obtained in [2].
In present paper, we obtained cyclic codes of length $16p^n$ over F where q is some prime power of the form $16k+3$ and order of q modulo p^n is $\frac{\phi(p^n)}{2}$. We considered the case for p being a prime of the form $8k+1$ and $8k+3$. However, in other cases whenever p is a prime of the form $8k+5$ and $8k+7$ the expressions for the idempotents can be obtained by using some permutation on the set A_2. The q-cyclotomic cosets modulo p^n are obtained in section 2 and termed those as Ω_{tp^n} for $t \in B' = \{0, 1, 2, 4, 5, 8, 10\}$ and Ω_{ap^i} for a belongs to different sets $A = \{1, 2, 4, 8, 16, \lambda, 2\lambda, 4\lambda, \mu, 2\mu, \nu, 2\nu, \eta, \xi, \rho, \chi\}$, $A_2 = \{2, 4, 8, 16, 2\lambda, 4\lambda, 2\mu, 2\nu, \}$, $B_2 = \{1, 2, 4, 8, \lambda, 2\lambda, \mu, \nu\}$. Primitive idempotents corresponding to Ω_{tp^n} for $t \in B'$ are obtained in section 3. In section 4, we derived the expression of primitive idempotents corresponding to $\Omega_{tg^kp^i}$ for $t = 8, 16$, $k = 0, 1$ and those for $t = 4, 4\lambda$ in section 5. In section 6, we derived the expressions of primitive idempotents corresponding to $\Omega_{2tg^kp^i}$ for $t \in B_1 = \{1, \lambda, \mu, \nu\}$ and remaining expressions are obtained in section 7. Section 8, consists of generating polynomials and dimensions for the corresponding cyclic codes of length $16p^n$. The minimum distance or bounds for minimum distance of these codes are obtained in section 9. At the end, an example is discussed to illustrate the various parameters for these codes in section 10.

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2. Cyclotomic Cosets

Let $S = \{1, 2, ..., 16p^n\}$. For $a, b \in S$, consider $a \sim b$ iff $a \equiv bq^i(mod\ 16p^n)$ for some integer $i \geq 0$. This is an equivalence relation on S. The equivalence classes due to this relation are called q-cyclotomic cosets modulo $16p^n$. The q-cyclotomic coset containing $s \in S$ is $\Omega_s = \{s, sq, sq^2, ..., sq^{t_s-1}\}$, where t_s is the smallest positive integer such that $sq^{t_s} \equiv s(mod\ 16p^n)$.

Lemma 1. *[8], If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, then the order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, $0 \leq i \leq n-1$.*

Lemma 2. *If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n then for $0 \leq i \leq n-1$, following holds:*
(i)For $p \equiv 1(mod 8)$ order of q modulo $2tp^{n-i}$ where $t = 1, 2, 4, 8$, is $\frac{\phi(p^{n-i})}{2}$.
(ii)For $p \equiv 5(mod 8)$ order of q modulo $2tp^{n-i}$ where $t = 1, 2, 4$, is $\frac{\phi(p^{n-i})}{2}$ and modulo $16p^{n-1}$ is $\phi(p^{n-i})$.
(iii)For $p \equiv 3(mod 4)$ order of q modulo $2p^{n-i}$ is $\frac{\phi(p^n)}{2}$, order of q modulo $4tp^{n-i}$ where $t = 1, 2$, is $\phi(p^{n-i})$, order of q modulo $16p^{n-1}$ is $2\phi(p^{n-i})$.

Proof. (i) Since $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, therefore by lemma 2.1, order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, $1 \leq i \leq n-1$. Hence

$$q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ p^{n-i}) \tag{1}$$

Since q is of the form $16k + 3$, therefore $q \equiv 1(mod\ 2)$. Hence, $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2)$. As $gcd(2, p^{n-i}) = 1$ and order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, so $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2p^{n-i})$. This implies that $\frac{\phi(p^{n-i})}{2}$ is the smallest integer for which (2.1) holds. Hence order of q modulo $2p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$. Again $q^2 = 256k^2 + 96k + 9 \equiv 1(mod 4)$, since $gcd(4, p^{n-i}) = 1$, so $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod 4p^{n-i})$. Since order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, so $\frac{\phi(p^{n-i})}{2}$ is the smallest integer for which equation 2.1 holds. Hence order of q modulo $4p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$. On similar lines, we can obtain that the order of q modulo $8p^{n-i}$ and $16p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$.
(ii)As by lemma 2.1 and $q^4 = (16k + 3)^4 \equiv 1(mod 16)$, since $gcd(16, p^{n-i}) = 1$

$$q^{\phi(p^{n-i})} \equiv 1(mod 16p^{n-i}). \tag{2}$$

However, as by lemma 2.1 and $\phi(p^{n-i})$ is the least integer that satisfies the equation 2. So, $\phi(p^{n-i})$ is the order of q modulo $16p^{n-i}$. Proof of other parts are similar to (i).
(iii) As by lemma 2.1 and $q^4 \equiv 1(mod 16)$, since $gcd(16, p^{n-i}) = 1$, so

$$q^{2\phi(p^{n-i})} \equiv 1(mod 16p^{n-i}). \tag{3}$$

However, as by lemma 2.1 and $2\phi(p^{n-i})$ is the least integer that satisfies the equation 3. Thus $2\phi(p^{n-i})$ is the order of q modulo $16p^{n-i}$. Proof of other parts are similar to part (i) and (ii). □

Lemma 3. *For any odd prime p, there exist an integer g, $1 < g < 16p$ which is a primitive root modulo p, further when p is of the form $4k + 1$ then order of g modulo 4, modulo 8 is 2 and modulo 16 is 4. and when p is of the form $4k + 3$ then order of g mod 4, mod 8 and modulo 16 is 2. Also, if q is any prime or prime power and $g.c.d.(p, q) = 1$, then $g \notin \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$.*

Proof. Since p is an odd prime some $v \in T_p = \{0, 1, 2, 3, ..., p-1\}$ will exist with $2v - p = 1$. Consider a primitive root a in T_p. For $p \equiv 1(mod 8)$, define $g \equiv (2av + tp + 6ap)(mod\ 16p)$ where $t \equiv 3(mod\ 16)$ is a prime and so $g \equiv (2av + tp + 6ap)(mod\ p) \equiv 2av(mod\ p) \equiv a(1 + p)(mod\ p) \equiv a(mod\ p)$. Thus g is primitive root modulo p. Further, $g \equiv (2av + tp + 6ap)(mod\ 16) \equiv (a(1 + p) + tp + 6ap)(mod\ 16)$, again $g \equiv \{a(1 + 8k + 1) + (16k_1 + 3)(8k + 1) + 6a(8k + 1)\}(mod\ 16)$. Clearly, $g \equiv 3(mod\ 2) \equiv 1(mod\ 2)$. Hence, order of g modulo 2 is 1, again $g \equiv (a(1 + p) + tp + 6ap)(mod\ 4) \equiv \{a(1 + 8k + 1) + (16k_1 + 3)(8k + 1) + 6a(8k + 1)\}(mod\ 4) \equiv \{8(ak + a + 3k) + 3\}(mod\ 4) \equiv 3(mod\ 4)$ so $g^2 \equiv 1(mod\ 4)$.

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Therefore, order of g modulo 4 is 2. In the similar pattern, we can obtain that order of g modulo 8 and 16 are 2 and 4 respectively. Also, when $p \equiv r(mod\ 8)$ where $r = 3$ or 5 or 7, now for $r = 3$ or 5, $g \equiv (2av + tp + (r+1)ap)(mod\ 16p)$ and for $r = 7$, $g \equiv (4av + tp + (r+1)ap)(mod\ 16p)$, where $t \equiv (r+2)(mod\ p)$ is a prime will have the required properties. Further, due to order of g and order of q modulo p as $\frac{\phi(p)}{2}$ and $\phi(p)$ respectively, $g \neq q^i$ for some $1 \leq i \leq \frac{\phi(p)}{2} - 1$. □

Lemma 4. *For $0 \leq i \leq n-1$ and $0 \leq k \leq \frac{\phi(p^{n-i})}{2} - 1$, $t \not\equiv q^k(mod\ 16p^{n-i})$, where $t = \lambda, 2\lambda, 4\lambda, \mu, 2\mu, \nu, 2\nu, \eta, \xi, \rho, \chi$ and $\lambda = 1+2p^n$, $\mu = 1+4p^n$, $\nu = 1+6p^n$, $\eta = 1+8p^n$, $\xi = 1+10p^n$, $\rho = 1+12p^n$, $\chi = 1+14p^n$.*

Proof. Proof can be obtained by using lemma 2.1 and lemma 2.2. □

Theorem 1. *For $q \equiv 3(mod16)$ the q-cyclotomic cosets modulo $16p^n$ are given by,*
$\Omega_{k_1p^n} = \{k_1p^n\}$, *for* $k_1 \in \{0, 8\}$, $\Omega_{K_2p^n} = \{k_2p^n, k_2p^nq\}$, *for* $k_2 \in \{2, 4, 10\}$, $\Omega_{k_3p^n} = \{k_3p^n, k_3p^nq, k_3p^nq^2, k_3p^nq^3\}$ *for* $k_3 \in \{1, 5\}$ *and for* $0 \leq i \leq n-1$, $\Omega_{tg^kp^i} = \{tg^kp^i, tg^kp^iq, tg^kp^iq^2, ..., tg^kp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$,
$\Omega_{t_1g^kp^i} = \{t_1g^kp^i, t_1g^kp^iq, ..., t_1g^kp^iq^{\phi(p^{n-i})-1}\}$ *and* $\Omega_{t_2g^kp^i} = \{t_2g^kp^i, t_2g^kp^iq, ..., t_2g^kp^iq^{2\phi(p^{n-i})-1}\}$,
where $k = 0, 1$ and g is as defined in lemma 2.3,
(i) when $p \equiv 1(mod8)$ these are $(32n+7)$, for $t \in A$.
(ii) when $p \equiv 5(mod8)$ these are $(24n+7)$, for $t \in A_2$ and $t_1 \in B_1$.
(iii) when $p \equiv 3(mod4)$ these are $(14n+7)$, for $t \in \{8, 16\}$, $t_1 \in \{2, 4, 2\mu\}$ and $t_2 \in \{1, \mu\}$.

Proof. (i)By definition, it is obvious that $\Omega_0 = \{0\}$ is trivial. Since q is of the form $16k+3$, So $q \equiv 3(mod16)$, so $k_1p^nq \equiv k_1p^n(mod16p^n)$ and hence $\Omega_{k_1p^n} = \{k_1p^n\}$ also $q^2 \equiv 9(mod16)$, so $k_2p^nq^2 \equiv k_2p^n(mod16p^n)$ so $\Omega_{k_2p^n} = \{k_2p^n, k_2p^nq\}$ and $q^4 \equiv 1(mod16)$, so $k_3p^nq^4 \equiv k_3p^n(mod16p^n)$ so $\Omega_{k_3p^n} = \{k_3p^n, k_3p^nq, k_3p^nq^2, k_3p^nq^3\}$. By lemma 2.2; $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod16p^{n-i})$ equivalently, $tg^kp^iq^{\frac{\phi(p^{n-i})}{2}} \equiv tg^kp^i(mod16p^n)$, so $\Omega_{tg^kp^i} = \{tg^kp^i, tg^kp^iq, tg^kp^iq^2, ..., tg^kp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$. Equivalently, $t_1g^kp^iq^{\phi(p^{n-i})} \equiv t_1g^kp^i(mod16p^n)$, so $\Omega_{t_1g^kp^i} = \{t_1g^kp^i, t_1g^kp^iq, t_1g^kp^iq^2, ..., t_1g^kp^iq^{\phi(p^{n-i})-1}\}$, again $q^{2\phi(p^{n-i})} \equiv 1(mod16p^{n-i})$ equivalently, $t_2g^kp^iq^{2\phi(p^{n-i})} \equiv t_2g^kp^i(mod16p^n)$, so $\Omega_{t_2g^kp^i} = \{t_2g^kp^i, t_2g^kp^iq, t_2g^kp^iq^2, ..., t_2g^kp^iq^{2\phi(p^{n-i})-1}\}$.
Obviously, $|\Omega_{k_1p^n}| = 1$, $|\Omega_{k_2p^n}| = 2$, $|\Omega_{k_3p^n}| = 4$ for every k_1, k_2, k_3 and $|\Omega_{tp^i}| = |\Omega_{tgp^i}| = \frac{\phi(p^{n-i})}{2}$, So $\sum_{t\in A}|\Omega_{tp^i}| = \sum_{t\in A}|\Omega_{tgp^i}| = \frac{\phi(p^n)}{2} + \frac{\phi(p^{n-1})}{2} + ... + \frac{\phi(p)}{2} = \frac{p^n-p^{n-1}}{2} + \frac{p^{n-1}-p^{n-2}}{2} + ... + \frac{p-1}{2} = \frac{p^n-1}{2}$.
Hence $\sum_{k_1}|\Omega_{k_1p^n}| + \sum_{k_2}|\Omega_{k_2p^n}| + \sum_{k_3}|\Omega_{k_3p^n}| + \sum_{i=0}^{n-1}[\sum_{t\in A}|\Omega_{tp^i}| + |\Omega_{tgp^i}|] = 16 + \frac{32(p^n-1)}{2} = 16 + 16(p^n - 1) = 16p^n$ it follows that these are the only distinct q-cyclotomic cosets modulo $16p^n$ in this case. Similarly other parts can be proved on similar lines. □

3. Primitive Idempotents Corresponding to Ω_{tp^n}, $t \in B'$

Throughout this paper, we consider α to be $16p^n$th root of unity in some extension field of F. Let M_s be the minimal ideal in $R_{16p^n} = \frac{F[x]}{<x^{16p^n}-1>} \equiv FC_{16p^n}$, generated by $\frac{(x^{16p^n}-1)}{m_s(x)}$, where $m_s(x)$ is the minimal polynomial for α^s, $s \in \Omega_s$. We denote $P_s(x)$, the primitive idempotent in R_{16p^n}, corresponding to the minimal ideal M_s, given by $P_s(x) = \frac{1}{16p^n}\sum_{i=0}^{16p^n-1}\epsilon_i^s x^s$ where $\epsilon_i^s = \sum_{s\in\Omega_s}\alpha^{-is}$ and $\overline{C}_s = \sum_{s\in\Omega_s}x^s$.
Then,

$$P_s(x) = \frac{1}{16p^n}[\sum_{t\in B'}\epsilon_{tp^n}^s\overline{C}_{tp^n} + \sum_{i=0}^{n-1}\{\sum_{a\in\mathbb{A}}\{\epsilon_{ap^i}^s\overline{C}_{ap^i} + \epsilon_{agp^i}^s\overline{C}_{agp^i}\}\}] \ [4] \quad (4)$$

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Lemma 5. *For cyclotomic cosets Ω_{tp^n}, where $t \in B'$*
(i) $\Omega_{ap^n} = -\Omega_{5ap^n}$ *where* $a = \{1,2\}$.
(ii) $\Omega_{bp^n} = -\Omega_{bp^n}$ *where* $b = \{0,4,8\}$.

Proof. (i) Since $\Omega_{ap^n} = \{ap^n, ap^n q, ap^n q^2, ap^n q^3\}$. Thus $5ap^n q + ap^n = p^n(5aq + a) = p^n\{5a(16k + 3) + a\} = p^n(80ak + 16a) \equiv 0(mod16p^n)$, therefore $-\Omega_{ap^n} = \Omega_{5ap^n}$ or $\Omega_{ap^n} = -\Omega_{5ap^n}$.
Further (ii) is trivial. □

Lemma 6. *For cyclotomic cosets $\Omega_{p^i}, 0 \le i \le n-1$*
(i) If $p \equiv 1(mod4)$
$\eta^2\Omega_{p^i} = \eta\Omega_{\eta p^i} = \Omega_{p^i} = \nu^2\Omega_{p^i} = \chi^2\Omega_{p^i} = -\lambda^2\Omega_{p^i} = -\mu^2\Omega_{p^i} = -\xi^2\Omega_{p^i} = -\rho^2\Omega_{p^i}$
(ii) If $p \equiv 3(mod4)$
$\mu^2\Omega_{p^i} = \Omega_{\eta p^i} = \eta\Omega_{p^i} = \rho^2\Omega_{p^i} = -\lambda^2\Omega_{p^i} = \mu^2\Omega_{p^i} = \nu^2\Omega_{p^i} = -\xi^2\Omega_{p^i} = \rho^2\Omega_{p^i} = \chi^2\Omega_{p^i}$
(iii) $(2\lambda)^2\Omega_{p^i} = 2\Omega_{2p^i} = 4\Omega_{p^i} = \Omega_{4p^i} = (2\mu)^2\Omega_{p^i} = (2\nu)^2\Omega_{p^i}$
(iv) $(4\lambda)^2\Omega_{p^i} = \Omega_{16p^i} = 2\Omega_{8p^i} = 4\Omega_{4p^i} = 16\Omega_{p^i}$.

Proof. (i) Since $\eta = 1 + 8p^n$. So $\eta^2 = (1+8p^n)^2 = 1 + 64p^{2n} + 16p^n \equiv 1(mod16p^n)$
Hence $\eta^2 \equiv 1(mod16p^n) \Rightarrow \eta^2\Omega_p i = \Omega p^i$ Now $\eta^2\Omega_p i = \{\eta^2 p^i, \eta^2 p^i q, \eta^2 p^i q^2, ..., \eta^2 p^i q^{\frac{\phi(p^{n-i})}{2}-1}\}$
$= \eta\{\eta p^i, \eta p^i q, \eta p^i q^2, ..., \eta p^i q^{\frac{\phi(p^{n-i})}{2}-1}\} = \eta\Omega_{\eta p^i}$.
Now $\nu^2 = (1+6p^n)^2 = 1 + 36p^{2n} + 12p^n = 1(mod16p^n)$ and $\chi^2 = (1+14p^n)^2 = 1 + 196p^{2n} + 28p^n = 1(mod16p^n)$. Hence $\nu^2\Omega_{p^i} = \chi^2\Omega_{p^i} = \Omega_{p^i}$.
Similarly, result holds for remaining □

Theorem 2. *For $p \equiv 1(mod8)$, the explicit expressions for the primitive idempotents $P_{tp^n}, t \in B'$ in R_{16p^n} are given by:*

$$P_{tp^n}(x) = \frac{1}{16p^n}[\{\sum_{w\in B'} \alpha^{8twp^{2n}}\overline{C}_{wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in A}\sum_{k=0,1} \alpha^{8tag^k p^{n+i}}\overline{C}_{ag^k p^i}\}], \textit{for } t = 0,8.$$

$$P_{tp^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,4,5\}}\{(-1)^w 2\{\alpha^{2twp^{2n}} + \alpha^{6twp^{2n}}\}\overline{C}_{2wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in B_2}\sum_{k=0,1}(-1)^a \{\alpha^{2atg^k p^{n+i}} + \alpha^{6atg^k p^{n+i}}\}\overline{C}_{2ag^k p^i}\}], \textit{for } t = 1,5.$$

$$P_{2tp^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,4,5,8\}}\{(-1)^w\{\alpha^{2twp^{2n}} + \alpha^{6twp^{2n}}\}\overline{C}_{wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in A}\sum_{k=0,1}(-1)^a \{\alpha^{2atg^k p^{n+i}} + \alpha^{6atg^k p^{n+i}}\}\overline{C}_{ag^k p^i}\}], \textit{for } t = 1,5.$$

$$P_{4p^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,2,4,5\}}\{(-1)^w 2\overline{C}_{2wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in B_2}\sum_{k=0,1}(-1)^a 2\overline{C}_{2ag^k p^i}\}].$$

Proof. By definition, we have

$$P_s(x) = \frac{1}{16p^n}[\sum_{t\in B'}\epsilon^s_{tp^n}\overline{C}_{tp^n} + \sum_{i=0}^{n-1}\{\sum_{a\in A}\sum_{k=0,1}\epsilon^s_{ag^k p^i}\overline{C}_{ag^k p^i}\}]. \tag{5}$$

Since $\epsilon^s_k = \sum_{s\epsilon\Omega_s}\alpha^{-ks}$ and $\overline{C}_s = \sum_{s\epsilon\Omega_s} x^s$, where α is $16p^n$th root of unity in $GF(q)$. Taking $s = 0$, $\epsilon^0_k = 1$ for all $0 \le k \le 16p^n - 1$, so $P_0(x) = \frac{1}{16p^n}[\sum_{w\in B'}\overline{C}_{wp^n} + \sum_{i=0}^{n-1}\{\sum_{a\in A}\sum_{k=0,1}\overline{C}_{ag^k p^i}\}]$.

Evaluation of $P_{p^n}(x)$.
Since $\Omega_{p^n} = -\Omega_{5p^n}$, $\epsilon^{p^n}_k = \sum_{s\epsilon\Omega_{p^n}}\alpha^{-ks} = \sum_{s\epsilon\Omega_{5p^n}}\alpha^{sk} = \alpha^{5p^n k} + \alpha^{5p^n kq} + \alpha^{5p^n kq^2} + \alpha^{5p^n kq^3} = \alpha^{5p^n k} + \alpha^{7p^n k} + \alpha^{13p^n k} + \alpha^{15p^n k}$.

$\epsilon^{p^n}_0 = -\epsilon^{p^n}_{8p^n} = 4,$ $\qquad$ $\epsilon^{p^n}_{p^n} = \epsilon^{p^n}_{4p^n} = \epsilon^{p^n}_{5p^n} = 0,$

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$\epsilon^{p^n}_{2p^n} = -\epsilon^{p^n}_{10p^n} = -2(\alpha^{2p^{2n}} + \alpha^{6p^{2n}})$, $\epsilon^{p^n}_{g^k p^i} = \epsilon^{p^n}_{\lambda g^k p^i} = \epsilon^{p^n}_{\mu g^k p^i} = \epsilon^{p^n}_{\nu g^k p^i} = 0$,
$\epsilon^{p^n}_{2g^k p^i} = -\epsilon^{p^n}_{2\mu g^k p^i} = -2(\alpha^{2p^{n+i}} + \alpha^{6p^{n+i}})$, $\epsilon^{p^n}_{4g^k p^i} = -\epsilon^{p^n}_{4\lambda g^k p^i} = 0$,
$\epsilon^{p^n}_{8g^k p^i} = -\epsilon^{p^n}_{16g^k p^i} = -4$, $\epsilon^{p^n}_{2\lambda g^k p^i} = -\epsilon^{p^n}_{2\nu g^k p^i} = -2(\alpha^{2\lambda g^k p^{n+i}} + \alpha^{6\lambda g^k p^{n+i}})$,
$\epsilon^{p^n}_{\eta g^k p^i} = \epsilon^{p^n}_{\xi g^k p^i} = \epsilon^{p^n}_{\rho g^k p^i} = \epsilon^{p^n}_{\chi g^k p^i} = 0$. Using these in equation 5 the expressions for $P_{p^n}(x)$ can be obtained. Similarly, $P_{tp^n}(x)$ for $t \in B' - \{0,1\}$ can be obtained. □

Theorem 3. *For $p \equiv 3(mod4)$, the explicit expressions for the primitive idempotents P_{tp^n}, $t \in B'$ in R_{16p^n} are given by*

$$P_{tp^n}(x) = \frac{1}{16p^n}[\{\sum_{w\in B'} \alpha^{8twp^{2n}}\overline{C}_{wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in D}\sum_{k=0,1} \alpha^{8tag^k p^{n+i}}\overline{C}_{ag^k p^i}\}], \textit{for } t \in \{0,8\},$$

$$P_{tp^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,4,5\}} \{(-1)^t 2\{\alpha^{2twp^{2n}} + \alpha^{6twp^{2n}}\}\overline{C}_{2wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in\{1,4,8,\mu\}}\sum_{k=0,1}(-1)^a$$
$$\{\alpha^{2atg^k p^{n+i}} + \alpha^{6atg^k p^{n+i}}\}\overline{C}_{2ag^k p^i}\}], \textit{for } t = 1,5,$$

$$P_{2tp^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,4,5,8\}} \{(-1)^t\{\alpha^{2twp^{2n}} + \alpha^{6twp^{2n}}\}\overline{C}_{wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in D-\{2,2\mu\}}\sum_{k=0,1}(-1)^a$$
$$\{\alpha^{2atg^k p^{n+i}} + \alpha^{6atg^k p^{n+i}}\}\overline{C}_{ag^k p^i}\}] \textit{ for } t = 1,5,$$

$$P_{4p^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,2,4,5\}} \{(-1)^w 2\overline{C}_{2wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in\{1,2,4,8,\mu\}}\sum_{k=0,1}(-1)^a 2\overline{C}_{2ag^k p^i}\}],$$

where $D = \{1,2,4,8,16,\mu,2\mu\}$.

Proof. Proof is similar to that of theorem 2. □

Theorem 4. *For $p \equiv 5(mod8)$, the explicit expressions for the primitive idempotents P_{tp^n}, $t \in B'$ in R_{16p^n} are given by:*

$$P_{tp^n}(x) = \frac{1}{16p^n}[\{\sum_{w\in B'} \alpha^{8twp^{2n}}\overline{C}_{wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in B}\sum_{k=0,1} \alpha^{8tag^k p^{n+i}}\overline{C}_{ag^k p^i}\}], \textit{for } t = 0,8.$$

$$P_{tp^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,4,5\}} \{(-1)^w 2\{\alpha^{2twp^{2n}} + \alpha^{6twp^{2n}}\}\overline{C}_{2wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in\{1,4,8,\lambda,\mu,\nu\}}\sum_{k=0,1}(-1)^a$$
$$\{\alpha^{2atg^k p^{n+i}} + \alpha^{6atg^k p^{n+i}}\}\overline{C}_{2ag^k p^i}\}], \textit{for } t = 1,5.$$

$$P_{2tp^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,4,5,8\}} \{(-1)^w\{\alpha^{2twp^{2n}} + \alpha^{6twp^{2n}}\}\overline{C}_{wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in B-2B_1}\sum_{k=0,1}(-1)^a$$
$$\{\alpha^{2atg^k p^{n+i}} + \alpha^{6atg^k p^{n+i}}\}\overline{C}_{ag^k p^i}\}], \textit{for } t = 1,5.$$

$$P_{4p^n}(x) = \frac{1}{16p^n}[\sum_{w\in\{0,1,2,4,5\}} \{(-1)^w 2\overline{C}_{2wp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in B-B_1}\sum_{k=0,1}(-1)^a 2\overline{C}_{2ag^k p^i}\}],$$

where $B = \{1,2,4,8,16,\lambda,2\lambda,4\lambda,\mu,2\mu,\nu,2\nu\}$.

Proof. Proof can be derived on same pattern to theorem 2. □

4. Primitive Idempotents Corresponding to $\Omega_{tg^k p^i}$, where $t = 8,16$

For $0 \leq j \leq n-1$, we define: $H_j = p^j \sum_{s\epsilon\Omega_{8gp^j}} \alpha^s$; $I_j = p^j \sum_{s\epsilon\Omega_{16gp^j}} \alpha^s$; $Q_j = p^j \sum_{s\epsilon\Omega_{8p^j}} \alpha^s$;$R_j = p^j \sum_{s\epsilon\Omega_{16p^j}} \alpha^s$.

Since $Q_j^q = (p^j \sum_{s\epsilon\Omega_{8p^j}} \alpha^s)^q = (p^j)^q(\sum_{s\epsilon\Omega_{8p^j}} \alpha^s)^q = (p^j)^q \sum_{s\epsilon\Omega_{8p^j}} \alpha^{qs} = (p^j)^q \sum_{s\epsilon q\Omega_{8p^j}} \alpha^s$.

However, $(p^j)^q = p^j$, therefore $Q_j^q = p^j \sum_{s\epsilon q\Omega_{8p^j}} \alpha^s$. Moreover, Ω_{8p^j} is a cyclotomic coset, therefore $q\Omega_{8p^j} = \Omega_{8p^j}$. Hence $Q_j^q = p^j \sum_{s\epsilon\Omega_{8p^j}} \alpha^s = Q_j$ and so $Q_j \in GF(q)$. Similarly, $H_j, I_j, R_j \in GF(q)$.

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Lemma 7. $0 \le j \le n-1$, $I_j + R_j = \begin{cases} -p^{n-1}, \text{if } j = n-1 \\ 0, \quad otherwise. \end{cases}$

Proof. By definition, $I_j + R_j = p^j \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} (\delta^{gq^t} + \delta^{q^t})$, where $\delta = \alpha^{16p^j}$.

As the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ is a reduced residue system $mod(p^{n-j})$, therefore $I_j + R_j = p^j[\sum_{t=0}^{p^{n-j}} \delta^t - \sum_{t=1,p/t}^{p^{n-j}} \delta^t] = p^j[\sum_{t=0}^{p^{n-j}} \delta^t - \sum_{t=1}^{p^{n-j-1}} \delta^{pt}] = \begin{cases} -p^{n-1}, \text{if } j = n-1 \\ 0, \quad otherwise. \end{cases}$ □

Lemma 8. $0 \le j \le n-1$, $H_j + Q_j = \begin{cases} p^{n-1}, \text{if } j = n-1 \\ 0, \quad otherwise. \end{cases}$

Proof. This result can be obtained on similar pattern as that of lemma 4.1 and using the fact that $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ is a reduced residue system $mod(2p^{n-j})$. □

Lemma 9. *For* $0 \le i \le n-1$ *and* $p \equiv 1(mod8)$ *then*
(i) $\Omega_{tp^i} = -\Omega_{tp^i}$ *and hence* $\Omega_{tgp^i} = -\Omega_{tgp^i}$, *for* $t = 8, 16$,
(ii) $\Omega_{4p^i} = -\Omega_{4\lambda p^i}$ *and hence* $\Omega_{4gp^i} = -\Omega_{4\lambda gp^i}$,
(iii) $\Omega_{2(1+tp^n)p^i} = -\Omega_{2\{1+(6-t)p^n\}p^i}$ *and hence* $\Omega_{2(1+tp^n)gp^i} = -\Omega_{2\{1+(6-t)p^n\}gp^i}$, *for* $t = 0, 4$.
(iv) *For* $p = 8k+1$, k *is even then*
$\Omega_{(1+tp^n)p^i} = -\Omega_{\{1+(14-t)p^n\}p^i}$ *and hence* $\Omega_{(1+tp^n)gp^i} = -\Omega_{\{1+(14-t)p^n\}gp^i}$, *for* $t = 0, 2, 4, 6$.
(v) *For* $p = 8k+1$, k *is odd then*
$\Omega_{(1+tp^n)p^i} = -\Omega_{\{1+(22-t)p^n\}p^i}$ *and hence* $\Omega_{(1+tp^n)gp^i} = -\Omega_{\{1+(22-t)p^n\}gp^i}$, *for* $t = 0, 2, 8, 10$.

Proof. For $t = 0$ in (iv), Since $\chi = 1 + 14p^n \equiv -1(mod16)$ and $q^{\frac{\phi(p^n)}{2}} \equiv 1(mod16)$. Further, $q^{\frac{\phi(p^n)}{4}} \equiv 1(mod16)$, as $q \equiv 3(mod16)$, therefore $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(mod16)$. Also $q^{\frac{\phi(p^n)}{2}} \equiv 1(modp^n)$, so $q^{\frac{\phi(p^n)}{4}} \equiv -1(modp^n)$ and $\chi \equiv 1(modp^n)$, thus $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(modp^n)$. However $(16, p^n) = 1$, thus $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(mod16p^n)$ and, therefore $-\Omega_{\chi p^i} = \Omega_{p^i}$. Hence $-\Omega_{\chi gp^i} = \Omega_{gp^i}$.
Proof of remaining can be obtained on similar pattern. □

Lemma 10. *For* $0 \le i \le n-1$, $k = 0, 1$ *following holds:*
If $p \equiv 3(mod8)$, $\quad \Omega_{tp^i} = -\Omega_{tgp^i}$ *and hence* $\Omega_{tgp^i} = -\Omega_{tp^i}$, *for* $t \in D$.

Proof. Proof can be derived on same pattern to lemma 9. □

Lemma 11. *For* $p \equiv 1(mod8)$; $0 \le i \le n$; $0 \le j \le n-1$,

$$\sum_{s \epsilon \Omega_{t_1 p^j}} \alpha^{8gp^i s} = \sum_{s \epsilon \Omega_{2tp^j}} \alpha^{4gp^i s} = \sum_{s \epsilon \Omega_{4\lambda p^j}} \alpha^{2tgp^i s} = \sum_{s \epsilon \Omega_{8p^j}} \alpha^{t_1 gp^i s} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, & if\ i+j \ge n, \\ \frac{1}{p^j} H_{i+j}, & if\ i+j \le n-1, \ g \ne 1, \\ \frac{1}{p^j} Q_{i+j}, & if\ i+j \le n-1, \ g = 1, \end{cases}$$

where $t_1 \in A_1$ *and* $t \in B_1$.

Proof. As α is $16p^n$th root of unity in some extension field of $GF(q)$, so

$$\sum_{s \epsilon \Omega_{4\lambda p^j}} \alpha^{2\nu gp^i s} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{8(1+2p^n)(1+6p^n)gp^{i+j}q^t} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{8gp^{i+j}q^t} = \sum_{s \epsilon \Omega_{p^j}} \alpha^{8gp^i s}$$

Similarly, remaining can be obtained on simple multiplication.

If $\beta = \alpha^{8gp^{i+j}}$, then $\sum_{s \epsilon \Omega_{p^j}} \alpha^{8gp^i s} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{8gp^{i+j}q^t} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \beta^{q^t}$.

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For $i+j \geq n$, $\sum_{s \epsilon \Omega_{p^j}} \alpha^{8gp^i s} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{8gp^{i+j}q^t} = -\frac{\phi(p^{n-j})}{2}$.

For $i+j \leq n-1$, β is $2p^{n-i-j}$th root of unity. Then $\beta^{q^l} = \beta^{q^r}$, which is possible when $l \equiv r(mod\frac{\phi(p^{n-i-j})}{2})$, due to lemma 2.2. So $\sum_{s \epsilon \Omega_{p^j}} \alpha^{8gp^i s} = \frac{\phi(p^{n-j})}{\phi(p^{n-i-j})} \sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1} \beta^{q^t} = \frac{p^{i+j}}{p^j} \sum_{s \epsilon \Omega_{8gp^{i+j}}} \alpha^s = \begin{cases} \frac{1}{p^j} H_{i+j}, \ g \neq 1, \\ \frac{1}{p^j} Q_{i+j}, \ g = 1. \end{cases}$
□

Proof of lemmas 4.6 - 4.7 can be obtained on similar reasoning as that of lemma 4.5, using definition of I_j and R_j.

Lemma 12. *For $p \equiv 1(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1$,*

$$\sum_{s \epsilon \Omega_{t_1 p^j}} \alpha^{16gp^i s} = \sum_{s \epsilon \Omega_{2tp^j}} \alpha^{8gp^i s} = \sum_{s \epsilon \Omega_{4t_2 p^j}} \alpha^{4t_2 gp^i s} = \sum_{s \epsilon \Omega_{16p^j}} \alpha^{t_1 gp^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2}, & if \ i+j \geq n, \\ \frac{1}{p^j} I_{i+j}, \ if \ i+j \leq n-1, & g \neq 1, \\ \frac{1}{p^j} R_{i+j}, \ if \ i+j \leq n-1, & g = 1, \end{cases}$$

where $t_1 \in A_1$, $t_2 = 1, 4, \lambda$ and $t \in B_1$.

Lemma 13. *For $p \equiv 1(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1$,*

$$\sum_{s \epsilon \Omega_{2tp^j}} \alpha^{16g^k p^i s} = \sum_{s \epsilon \Omega_{4t_2 p^j}} \alpha^{8g^k p^i s} = \sum_{s \epsilon \Omega_{16p^j}} \alpha^{2tg^k p^i s} = \sum_{s \epsilon \Omega_{8p^j}} \alpha^{4t_2 g^k p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2}, & if \ i+j \geq n, \\ \frac{1}{p^j} I_{i+j}, & if \ i+j \leq n-1, \ k=1, \\ \frac{1}{p^j} R_{i+j}, & if \ i+j \leq n-1, \ k=0, \end{cases}$$

where $t = 1, 4, \lambda, \mu, \nu$ and $t_2 = 1, \lambda$

Lemma 14. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1$,*

$$\sum_{s \epsilon \Omega_{p^j}} \alpha^{8g^k p^i s} = \sum_{s \epsilon \Omega_{\mu p^j}} \alpha^{8g^k p^i s} = \begin{cases} -2\phi(p^{n-j}), & if \ i+j \geq n, \\ \frac{4}{p^j} H_{i+j}, & if \ i+j \leq n-1, \ k=1, \\ \frac{4}{p^j} Q_{i+j}, & if \ i+j \leq n-1, \ k=0. \end{cases}$$

Proof. Assume $\beta = \alpha^{8g^k p^{i+j}}$, so $\sum_{s \epsilon \Omega_{p^j}} \alpha^{8g^k p^i s} = \sum_{t=0}^{2\phi(p^{n-j})-1} \alpha^{8g^k p^{i+j} q^t} = \sum_{t=0}^{2\phi(p^{n-j})-1} \beta^{q^t}$.

For $i+j \geq n$, $\sum_{s \epsilon \Omega_{p^j}} \alpha^{8g^k p^i . s} = \sum_{t=0}^{2\phi(p^{n-j})-1} \alpha^{8g^k p^{i+j} q^t} = \{\alpha^{8g^k p^{i+j}} + \alpha^{8g^k p^{i+j} q} + ... + \alpha^{8g^k p^{i+j} q^{2\phi(p^{n-j})-1}}\}$
$= -2\phi(p^{n-j})$.

For $i+j \leq n-1$, then β is $2p^{n-i-j}$th root of unity. Then $\beta^{q^l} = \beta^{q^r}$ which is possible when $l \equiv r(mod\frac{\phi(p^{n-i-j})}{2})$, due to lemma 2.2. So,

$$\sum_{s \epsilon \Omega_{p^j}} \alpha^{8g^k p^i s} = \sum_{t=0}^{2\phi(p^{n-j})-1} \alpha^{8g^k p^{i+j} q^t} = \sum_{t=0}^{2\phi(p^{n-j})-1} \beta^{q^t} = \frac{2\phi(p^{n-j})}{\frac{\phi(p^{n-i-j})}{2}} \sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1} \beta^{q^t}$$

$$= \frac{4p^{i+j}}{p^j} \sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1} \beta^{q^t} = \frac{4p^{i+j}}{p^j} \sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1} \alpha^{8g^k p^{i+j} q^t} = \frac{4p^{i+j}}{p^j} \sum_{s \epsilon \Omega_{8g^k p^j}} \alpha^{p^i s} = \begin{cases} \frac{4}{p^j} H_{i+j}, \ k=1, \\ \frac{4}{p^j} Q_{i+j}, \ k=0. \end{cases}$$ □

Proof of lemmas 4.9 - 4.15 can be obtained on similar reasoning as that of lemma 4.5, 4.8, using definition of I_j and R_j.

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Lemma 15. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1,$*

$$\sum_{s\epsilon\Omega_{2p^j}} \alpha^{4g^k p^i s} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{2\mu g^k p^i s} = \begin{cases} -\phi(p^{n-j}), & if\ i+j \geq n, \\ \frac{2}{p^j} H_{i+j}, & if\ i+j \leq n-1,\ k=1, \\ \frac{2}{p^j} Q_{i+j}, & if\ i+j \leq n-1,\ k=0. \end{cases}$$

Lemma 16. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1,$*

$$\sum_{s\epsilon\Omega_{8p^j}} \alpha^{g^k p^i s} = \sum_{s\epsilon\Omega_{8p^j}} \alpha^{\mu g^k p^i s} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, & if\ i+j \geq n, \\ \frac{1}{p^j} H_{i+j}, & if\ \ i+j \leq n-1,\ k=1 \\ \frac{1}{p^j} Q_{i+j}, & if\ \ i+j \leq n-1,\ k=0. \end{cases}$$

Lemma 17. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1,$*

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{16g^k p^i s} = \sum_{s\epsilon\Omega_{\mu p^j}} \alpha^{16g^k p^i s} = \begin{cases} 2\phi(p^{n-j}), & if\ i+j \geq n, \\ \frac{4}{p^j} I_{i+j}, & if\ i+j \leq n-1,\ k=1, \\ \frac{4}{p^j} R_{i+j}, & if\ i+j \leq n-1,\ k=0. \end{cases}$$

Lemma 18. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1,$*

$$\sum_{s\epsilon\Omega_{2p^j}} \alpha^{8g^k p^i s} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{8g^k p^i s} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{4g^k p^i s} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{16g^k p^i s} = \begin{cases} \phi(p^{n-j}), & if\ i+j \geq n, \\ \frac{2}{p^j} I_{i+j}, & if\ i+j \leq n-1,\ k=1, \\ \frac{2}{p^j} R_{i+j}, & if\ i+j \leq n-1,\ k=0. \end{cases}$$

Lemma 19. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1,$*

$$\sum_{s\epsilon\Omega_{2p^j}} \alpha^{16g^k p^i s} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{16g^k p^i s} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{8g^k p^i s} = \begin{cases} \phi(p^{n-j}), & if\ i+j \geq n, \\ \frac{2}{p^j} R_{i+j}, & if\ i+j \leq n-1,\ k=1, \\ \frac{2}{p^j} I_{i+j}, & if\ i+j \leq n-1,\ k=0, \end{cases}$$

Lemma 20. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1,$*

$$\sum_{s\epsilon\Omega_{8p^j}} \alpha^{2t_1 g^k p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{t_2 g^k p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2}, & if\ i+j \geq n, \\ \frac{1}{p^j} I_{i+j}, & if\ i+j \leq n-1,\ k=1, \\ \frac{1}{p^j} R_{i+j}, & if\ i+j \leq n-1,\ k=0, \end{cases} \quad where\ t_1 = 1,4,\mu\ and\ t_2 = t_1, 16.$$

Lemma 21. *For $p \equiv 3(mod8); 0 \leq i \leq n; 0 \leq j \leq n-1,$*

$$\sum_{s\epsilon\Omega_{8p^j}} \alpha^{4t_1 g^k p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{2t_2 g^k p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2}, & if\ i+j \geq n, \\ \frac{1}{p^j} R_{i+j}, & if\ \ i+j \leq n-1,\ k=1, \\ \frac{1}{p^j} I_{i+j}, & if\ \ i+j \leq n-1,\ k=0, \end{cases} \quad where\ t_1 = 1,4\ and\ t_2 = 1,\mu.$$

Theorem 5. *If $p \equiv 1(mod8)$ the expressions for the primitive idempotents corresponding to $\Omega_{8g^k p^i}$, $\Omega_{16g^k p^i}$ in R_{16p^n} are as follows:*

$$P_{8p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\{\sum_{t\in B'}(-1)^t \overline{C}_{tp^n}\} + \sum_{i=n-j}^{n-1}\{\sum_{a\in A}\sum_{k=0,1}(-1)^a \overline{C}_{ag^k p^i}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\sum_{a\in A_1}\{Q_{i+j}\overline{C}_{ap^i} + H_{i+j}\overline{C}_{agp^i}\} + \sum_{a\in A_2}\{R_{i+j}\overline{C}_{ap^i} + I_{i+j}\overline{C}_{agp^i}\}\}],$$

$$P_{16p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t\in B'}\overline{C}_{tp^n} + \sum_{i=n-j}^{n-1}\{\sum_{a\in A}\sum_{k=0,1}\overline{C}_{ag^k p^i}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\sum_{a\in A}\{R_{i+j}\overline{C}_{ap^i} + I_{i+j}\overline{C}_{agp^i}\}],$$

where $Q_{n-1} = \frac{p^{n-1}}{2}(1+\sqrt{p}),\ H_{n-1} = \frac{p^{n-1}}{2}(1-\sqrt{p}),\ I_{n-1} = \frac{p^{n-1}}{2}(\sqrt{p}-1)$
and $R_{n-1} = \frac{p^{n-1}}{2}(-\sqrt{p}-1)$ *and for all* $j \leq n-2,\ Q_j = H_j = I_j = R_j = 0.$

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Proof. Since $\Omega_{8p^j} = -\Omega_{8p^j}$, as obtained in lemma 4.4, so $\epsilon_k^{8p^j} = \sum_{s\epsilon\Omega_{8p^j}} \alpha^{-ks} = \sum_{s\epsilon\Omega_{p^j}} \alpha^{8ks}$.

Thus $\epsilon_{tp^n}^{8p^j} = (-1)^t \frac{\phi(p^{n-j})}{2}$ for $t \in B'$ and using lemma 4.3, 4.5 − 4.7, wherever required we obtain,

$$\epsilon_{tg^kp^i}^{8p^j} = \begin{cases} -\frac{\phi(p^{n-j})}{2} \text{ if } i+j \geq n, \\ \frac{1}{p^j}Q_{i+j} \text{ if } i+j \leq n-1,\ k=0, \\ \frac{1}{p^j}H_{i+j} \text{ if } i+j \leq n-1,\ k=1, \end{cases} \text{ for } t \in A_1, \quad \epsilon_{2tg^kp^i}^{8p^j} = \begin{cases} \frac{\phi(p^{n-j})}{2} \text{ if } i+j \geq n, \\ \frac{1}{p^j}R_{i+j} \text{ if } i+j \leq n-1,\ k=0, \\ \frac{1}{p^j}I_{i+j} \text{ if } i+j \leq n-1,\ k=1, \end{cases}$$

for $t \in B_2 = \{1,2,4,8,\lambda,2\lambda,\mu,\nu\}$,
Using all these in (3.1), the expression for P_{8p^j} is obtained.
Using lemma 4.3, 4.5 − 4.7, the expressions for P_{16p^j}, P_{8gp^j} and P_{16gp^j} can be derived. However, the expressions for P_{8gp^j} and P_{16gp^j} can be obtained by interchanging Q_{i+j}, R_{i+j} by H_{i+j}, I_{i+j} respectively and vice versa in the expressions of $P_{8p^j}(x)$, $P_{16p^j}(x)$,
Since $\overline{C}_k = \sum_{s\epsilon\Omega_k} x^s$ and $(\overline{C}_k)_{\alpha^{16p^j}} = \overline{C}_k(\alpha^{16p^j}) = \sum_{s\epsilon\Omega_k} (\alpha^{16p^j})^s$. Therefore $\overline{C}_{tp^n}(\alpha^{16p^j}) = 1$ for $t = 0,8$; $\overline{C}_{tp^n}(\alpha^{16p^j}) = 4$ for $t = 1,5$; $\overline{C}_{tp^n}(\alpha^{16p^j}) = 2$ for $t = 2,4,10$;.

$$\overline{C}_{ap^i}(\alpha^{16p^j}) = \begin{cases} \frac{\phi(p^{n-j})}{2},\ if\ i+j \geq n \\ \frac{1}{p^j}R_{i+j},\ if\ i+j \leq n-1. \end{cases} \text{ and } \overline{C}_{agp^i}(\alpha^{16p^j}) = \begin{cases} \frac{\phi(p^{n-j})}{2}, if\ i+j \geq n, \\ \frac{1}{p^j}I_{i+j}, if\ i+j \leq n-1, \end{cases}$$

where $a \in A$. Using all these in $P_{16p^j}(\alpha^{16p^j}) = 1$, to obtain

$$16p^n = \phi(p^{n-j})[8 + 8\sum_{i=n-j}^{n-1} \phi(p^{n-i})] + \frac{16}{p^j}\sum_{i=0}^{n-j-1} \frac{1}{p^i}(R^2{}_{i+j} + I^2{}_{i+j})$$

which in turn implies $\frac{1}{p^j}\sum_{i=0}^{n-j-1} \frac{1}{p^i}(R^2{}_{i+j} + I^2{}_{i+j}) = \frac{p^{n-1}}{2}(p+1)$

In particular, for $j = n-1$, $\frac{1}{p^{n-1}}(R^2{}_{n-1} + I^2{}_{n-1}) = \frac{p^{n-1}}{2}(p+1)$. Using lemma 4.1, to obtain $I_{n-1} = \frac{p^{n-1}}{2}(\sqrt{p}-1)$ and $R_{n-1} = \frac{p^{n-1}}{2}(-\sqrt{p}-1)$ and so $I_{n-2} = R_{n-2} = I_{n-3} = R_{n-3} = ... = 0$.
Relations for Q_{i+j} and H_{i+j} can be derived using lemma 4.2 and the fact that $P_{8p^j}(\alpha^{8p^j}) = 1$. □

Theorem 6. *For $p \equiv 3(mod8)$, the expressions for the primitive idempotents corresponding to $\Omega_{8g^kp^i}$, $\Omega_{16g^kp^i}$ in R_{16p^n} are as follows:*

$$P_{8p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\{\sum_{t\in B'}(-1)^t\overline{C}_{tp^n}\} + \sum_{i=n-j}^{n-1}\{\sum_{a\in D}\sum_{k=0,1}(-1)^a\overline{C}_{ag^kp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{H_{i+j}\{\overline{C}_{p^i} + \overline{C}_{\mu p^i}\}$$
$$+ I_{i+j}\{\overline{C}_{2p^i} + \overline{C}_{4gp^i} + \overline{C}_{8p^i} + \overline{C}_{16gp^i} + \overline{C}_{2\mu p^i}\} + Q_{i+j}\{\overline{C}_{gp^i} + \overline{C}_{\mu gp^i}\} + R_{i+j}\{\overline{C}_{2gp^i} + \overline{C}_{4p^i} + \overline{C}_{8gp^i} + \overline{C}_{16p^i} + \overline{C}_{2\mu gp^i}\}\}].$$

$$P_{16p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t\in B'}\overline{C}_{tp^n} + \sum_{i=n-j}^{n-1}\{\sum_{a\in D}\sum_{k=0,1}\overline{C}_{ag^kp^i}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{I_{i+j}\{\overline{C}_{p^i} + \overline{C}_{2gp^i} + \overline{C}_{4p^i} + \overline{C}_{8gp^i} +$$
$$\overline{C}_{16p^i} + \overline{C}_{\mu p^i} + \overline{C}_{2\mu gp^i}\} + R_{i+j}\{\overline{C}_{gp^i} + \overline{C}_{2p^i} + \overline{C}_{4gp^i} + \overline{C}_{8p^i} + \overline{C}_{16gp^i} + \overline{C}_{\mu gp^i} + \overline{C}_{2\mu p^i}\}\}].$$

The expressions for P_{8gp^j} and P_{16gp^j} can be obtained by interchanging Q_{i+j}, R_{i+j} by H_{i+j}, I_{i+j} respectively in the expressions of $P_{8p^j}(x)$, $P_{16p^j}(x)$,
where $Q_{n-1} = \frac{p^{n-1}}{2}(1-\sqrt{-p})$, $H_{n-1} = \frac{p^{n-1}}{2}(1+\sqrt{-p})$, $I_{n-1} = \frac{p^{n-1}}{2}(\sqrt{-p}-1)$ and
$R_{n-1} = \frac{p^{n-1}}{2}(-\sqrt{-p}-1)$ and for all $j \leq n-2$, $Q_j = H_j = I_j = R_j = 0$.

Proof. These expressions can be obtained using lemmas 4.9 - 4.15 and on similar reasoning as that of theorem 4.16. Also relations can be derived using lemma 4.1 - 4.2 and $P_{8p^j}(\alpha^{8p^j}) = 1$, $P_{16p^j}(\alpha^{16p^j}) = 1$. □

5. Primitive Idempotents Corresponding to $\Omega_{4tg^kp^i}$, where $t = 1, \lambda$

For $0 \leq j \leq n-1$, define
$G_j = p^j \sum_{s\epsilon\Omega_{4gp^j}} \alpha^s$, $P_j = p^j \sum_{s\epsilon\Omega_{4p^j}} \alpha^s$. Due to similar procedure as in section 4, $G_j, P_j \in GF(q)$.

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Proof of lemma 5.1 $-$ 5.3 can be obtained on similar lines as that of lemma 4.5, 4..8 and represent G_{i+j}, P_{i+j}.

Lemma 22. *For $p \equiv 1(mod8)$; $0 \le i \le n$; $0 \le j \le n-1$,*

$$\sum_{s\epsilon\Omega_{t_1p^j}} \alpha^{4g^kp^is} = \sum_{s\epsilon\Omega_{t_2p^j}} \alpha^{4\lambda g^kp^is} = -\sum_{s\epsilon\Omega_{t_2p^j}} \alpha^{4g^kp^is} = -\sum_{s\epsilon\Omega_{t_1p^j}} \alpha^{4\lambda g^kp^is} = \sum_{s\epsilon\Omega_{2tp^j}} \alpha^{2tg^kp^is} = \sum_{s\epsilon\Omega_{2p^j}} \alpha^{2\mu g^kp^is}$$
$$= \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{2\nu g^kp^is} = -\sum_{s\epsilon\Omega_{2p^j}} \alpha^{2t_3g^kp^is} = -\sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{2t_3g^kp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{t_1g^kp^is} = \sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{t_2g^kp^is} = -\sum_{s\epsilon\Omega_{4p^j}}$$
$$\alpha^{t_2g^kp^is} = -\sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{t_1g^kp^is} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{1}{p^j}G_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j}P_{i+j}, & if\ i+j \le n-1,\ k=0, \end{cases}$$

where $t_1 = 1, \mu, \eta, \rho,\ \ t_2 = \lambda, \nu, \xi, \chi,\ \ t_3 = \lambda, \nu,\ \ t \in B_1$

Lemma 23. *For $p \equiv 3(mod8)$; $0 \le i \le n$; $0 \le j \le n-1$,*

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{4g^kp^is} = \sum_{s\epsilon\Omega_{\mu p^j}} \alpha^{4g^kp^is} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{2}{p^j}G_{i+j}, & if\ i+j \le n-1,\ k=1 \\ \frac{2}{p^j}P_{i+j}, & if\ i+j \le n-1,\ k=0. \end{cases}$$

Lemma 24. *For $p \equiv 3(mod8)$; $0 \le i \le n$; $0 \le j \le n-1$,*

$$\sum_{s\epsilon\Omega_{2p^j}} \alpha^{2tg^kp^is} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{2\mu g^kp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{tg^kp^is} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{1}{p^j}G_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j}P_{i+j}, & if\ i+j \le n-1,\ k=0. \end{cases} \quad where\ t = 1, \mu$$

Theorem 7. *For $p \equiv 1(mod8)$, the expressions for the primitive idempotents corresponding to $\Omega_{4g^kp^i}$, $\Omega_{4\lambda g^kp^i}$ are given by:*

$$P_{4p^j}(x) = \tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{2}\{\{\sum_{t\in\{0,1,2,4,5\}}(-1)^t\overline{C}_{2tp^n}\} + \sum_{i=n-j}^{n-1}\{\sum_{a\in A_2}\sum_{k=0,1}\alpha^{4ag^kp^{i+j}}\overline{C}_{ag^kp^i}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}$$
$$\{P_{i+j}\{-\overline{C}_{p^i}+\overline{C}_{\lambda p^i}-\overline{C}_{\mu p^i}+\overline{C}_{\nu p^i}-\overline{C}_{\eta p^i}+\overline{C}_{\xi p^i}-\overline{C}_{\rho p^i}+\overline{C}_{\chi p^i}\}+Q_{i+j}\{\overline{C}_{2p^i}+\overline{C}_{2\lambda p^i}+\overline{C}_{2\mu p^i}+\overline{C}_{2\nu p^i}\}+$$
$$R_{i+j}\{\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}+\overline{C}_{4\lambda p^i}\}+G_{i+j}\{-\overline{C}_{gp^i}+\overline{C}_{\lambda gp^i}-\overline{C}_{\mu gp^i}+\overline{C}_{\nu gp^i}-\overline{C}_{\eta gp^i}+\overline{C}_{\xi gp^i}-\overline{C}_{\rho gp^i}+$$
$$\overline{C}_{\chi gp^i}\}+H_{i+j}\{\overline{C}_{2gp^i}+\overline{C}_{2\lambda gp^i}+\overline{C}_{2\mu gp^i}+\overline{C}_{2\nu gp^i}\}+I_{i+j}\{\overline{C}_{4gp^i}+\overline{C}_{8gp^i}+\overline{C}_{16gp^i}+\overline{C}_{4\lambda gp^i}\}\}],$$

$$P_{4\lambda p^j}(x) = \tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{2}\{\{\sum_{t\in\{0,1,2,4,5\}}(-1)^t\overline{C}_{2tp^n}\} + \sum_{i=n-j}^{n-1}\{\sum_{a\in A_2}\sum_{k=0,1}\alpha^{4ag^kp^{i+j}}\overline{C}_{ag^kp^i}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}$$
$$\{P_{i+j}\{\overline{C}_{p^i}-\overline{C}_{\lambda p^i}+\overline{C}_{\mu p^i}-\overline{C}_{\nu p^i}+\overline{C}_{\eta p^i}-\overline{C}_{\xi p^i}+\overline{C}_{\rho p^i}-\overline{C}_{\chi p^i}\}+Q_{i+j}\{\overline{C}_{2p^i}+\overline{C}_{2\lambda p^i}+\overline{C}_{2\mu p^i}+\overline{C}_{2\nu p^i}\}+$$
$$R_{i+j}\{\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}+\overline{C}_{4\lambda p^i}\}+G_{i+j}\{\overline{C}_{gp^i}-\overline{C}_{\lambda gp^i}+\overline{C}_{\mu gp^i}-\overline{C}_{\nu gp^i}+\overline{C}_{\eta gp^i}-\overline{C}_{\xi gp^i}+\overline{C}_{\rho gp^i}-$$
$$\overline{C}_{\chi gp^i}\}+H_{i+j}\{\overline{C}_{2gp^i}+\overline{C}_{2\lambda gp^i}+\overline{C}_{2\mu gp^i}+\overline{C}_{2\nu gp^i}\}+I_{i+j}\{\overline{C}_{4gp^i}+\overline{C}_{8gp^i}+\overline{C}_{16gp^i}+\overline{C}_{4\lambda gp^i}\}\}].$$

The expressions for P_{4gp^j} and $P_{4\lambda gp^j}$ can be obtained by replacing $\alpha^{up^{i+j}}$, P_{i+j}, Q_{i+j}, R_{i+j} by $\alpha^{ugp^{i+j}}$, G_{i+j}, H_{i+j}, I_{i+j} respectively and vice versa in the expressions of $P_{4p^j}(x)$, $P_{4\lambda p^j}(x)$,

where $P_{n-1} = \frac{p^{n-1}}{2}(\sqrt{-p})$, $G_{n-1} = 0$ and Q_{n-1}, R_{n-1}, H_{n-1}, I_{n-1} are defined in theorem 5 and for $j \le n-2$, $P_j = G_j = 0$.

Proof. These expressions can be derived using lemmas 4.3, 4.5 $-$ 4.7, 5.1 with similar procedure as in theorem 4.16. Also the relations can be derived using $P_{4p^j}(\alpha^{4p^j}) = 1$ and $P_{4p^j}(\alpha^{4gp^j}) = 0$. $\square$

Theorem 8. *For $p \equiv 3(mod8)$, the expressions for the primitive idempotents analogous to $\Omega_{4g^kp^i}$ are specified as:*

$$P_{4p^j}(x) = \tfrac{1}{16p^n}[\phi(p^{n-j})\{\sum_{t\in\{0,1,2,4,5\}}(-1)^t\overline{C}_{2tp^n}\} + \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\sum_{a\in D-\{1,\mu\}}\{\alpha^{4agp^{i+j}}\overline{C}_{ap^i} + \alpha^{4ap^{i+j}}\overline{C}_{agp^i}\}\} +$$

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$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{G_{i+j}\{\overline{C}_{p^i}+\overline{C}_{\mu p^i}\}+2H_{i+j}\{\overline{C}_{2p^i}+\overline{C}_{2\mu p^i}\}+2I_{i+j}\{\overline{C}_{4p^i}+\overline{C}_{16p^i}+\overline{C}_{8gp^i}\}+2R_{i+j}\{\overline{C}_{8p^i}+\overline{C}_{4gp^i}+\overline{C}_{16gp^i}\}+P_{i+j}\{\overline{C}_{gp^i}+\overline{C}_{\mu gp^i}\}+2Q_{i+j}\{\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}\}],$

$P_{4gp^j}(x)=\frac{1}{16p^n}[\phi(p^{n-j})\{\sum_{t\in\{0,1,2,4,5\}}(-1)^t\overline{C}_{2tp^n}\}+\phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\sum_{a\in D-\{1,\mu\}}\sum_{k=0,1}\{\alpha^{4ag^kp^{i+j}}\overline{C}_{ag^kp^i}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{P_{i+j}\{\overline{C}_{p^i}+\overline{C}_{\mu p^i}\}+2Q_{i+j}\{\overline{C}_{2p^i}+\overline{C}_{2\mu p^i}\}+2R_{i+j}\{\overline{C}_{4p^i}+\overline{C}_{16p^i}+\overline{C}_{8gp^i}\}+2I_{i+j}\{\overline{C}_{8p^i}+\overline{C}_{4gp^i}+\overline{C}_{16gp^i}\}+G_{i+j}\{\overline{C}_{gp^i}+\overline{C}_{\mu gp^i}\}+2H_{i+j}\{\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}\}],$

where $P_{n-1}=0$, $G_{n-1}=0$ *and* $Q_{n-1}, R_{n-1}, H_{n-1}, I_{n-1}$ *are defined in theorem 4.17 and for* $j\le n-2$, $P_j=G_j=0$.

Proof. Due to lemma 4.4, $\Omega_{4p^j}=-\Omega_{4gp^j}$, so $\epsilon^{4p^j}_{k'}=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{-k's}=\sum_{s\epsilon\Omega_{4gp^j}}\alpha^{k's}$.

$\epsilon^{4p^j}_{2tp^n}=(-1)^t\phi(p^{n-j})$, for $t\in\{0,1,2,4,5\}$ $\qquad \epsilon^{4p^j}_{tp^n}=0$ for $t\in\{1,5\}$. Due to lemma 5.2 - 5.3, 4.8-4.15, to obtain,

$$\epsilon^{4p^j}_{g^kp^i}=\epsilon^{4p^j}_{\mu g^kp^i}=\begin{cases}0, & if\ i+j\ge n,\\ \frac{1}{p^j}G_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{1}{p^j}P_{i+j}, & if\ i+j\le n-1,\ k=1,\end{cases}$$

$$\epsilon^{4p^j}_{2g^kp^i}=\epsilon^{4p^j}_{2\mu g^kp^i}=\begin{cases}-\phi(p^{n-j}), & if\ i+j\ge n,\\ \frac{2}{p^j}H_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{2}{p^j}Q_{i+j}, & if\ i+j\le n-1,\ k=1,\end{cases}$$

$$\epsilon^{4p^j}_{4g^{k+1}p^i}=\epsilon^{4p^j}_{16g^{k+1}p^i}=\begin{cases}\phi(p^{n-j}), & if\ i+j\ge n,\\ \frac{2}{p^j}I_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{2}{p^j}R_{i+j}, & if\ i+j\le n-1,\ k=1,\end{cases}\quad \epsilon^{4p^j}_{8g^kp^i}=\begin{cases}\phi(p^{n-j}), & if\ i+j\ge n,\\ \frac{2}{p^j}I_{i+j}, & if\ i+j\le n-1,\ k=1,\\ \frac{2}{p^j}R_{i+j}, & if\ i+j\le n-1,\ k=0,\end{cases}$$

Using all these in equation (3.1), to get the expressions of P_{4p^j}. Using similar reason the expressions for P_{4gp^j} can be derived.

Evaluation of P_{i+j} and G_{i+j},

By definition of primitive idempotents, $P_{4p^j}(\alpha^{4gp^j})=0$.

$\overline{C}_{tp^n}(\alpha^{4gp^j})=1$ for $t\in\{0,8\}$, $\quad \overline{C}_{(2t)p^n}(\alpha^{4gp^j})=2(-1)^t$, for $t\in\{1,2,5\}$, $\quad \overline{C}_{tp^n}(\alpha^{4gp^j})=0$, for $t\in\{1,5\}$.

Due to lemma 5.2 - 5.3, 4.8- 4.15, to obtain,

$$\overline{C}_{g^kp^i}(\alpha^{4gp^j})=\overline{C}_{\mu g^kp^i}(\alpha^{4gp^j})=\begin{cases}0, & if\ i+j\ge n,\\ \frac{2}{p^i}G_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{2}{p^i}P_{i+j}, & if\ i+j\le n-1,\ k=1.\end{cases}$$

$$\overline{C}_{2g^kp^i}(\alpha^{4gp^j})=\overline{C}_{2\mu g^kp^i}(\alpha^{4gp^j})=\begin{cases}-\phi(p^{n-i}), & if\ i+j\ge n,\\ \frac{2}{p^j}H_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{2}{p^j}Q_{i+j}, & if\ i+j\le n-1,\ k=1.\end{cases}$$

$$\overline{C}_{4g^kp^i}(\alpha^{4gp^j})=\begin{cases}\phi(p^{n-i}), & if\ i+j\ge n,\\ \frac{2}{p^j}I_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{2}{p^j}R_{i+j}, & if\ i+j\le n-1,\ k=1.\end{cases}$$

$$\overline{C}_{16g^kp^i}(\alpha^{4gp^j})=\begin{cases}\frac{\phi(p^{n-i})}{2}, & if\ i+j\ge n,\\ \frac{1}{p^j}I_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{1}{p^j}R_{i+j}, & if\ i+j\le n-1,\ k=1.\end{cases}\quad \overline{C}_{8g^kp^i}(\alpha^{4p^j})=\begin{cases}\frac{\phi(p^{n-i})}{2}, & if\ i+j\ge n,\\ \frac{1}{p^j}I_{i+j}, & if\ i+j\le n-1,\ k=0,\\ \frac{1}{p^j}R_{i+j}, & if\ i+j\le n-1,\ k=1.\end{cases}$$

Using all these in $P_{4p^j}(\alpha^{4gp^j})=0$, which gives

$$0=\phi(p^{n-j})[1+2+2+1+2+8\sum_{i=n-j}^{n-1}\phi(p^{n-i})]+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{(4P^2{}_{i+j}+4G^2{}_{i+j})}{p^i}+$$

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$\frac{(8Q^2{}_{i+j} + 8H^2{}_{i+j})}{p^i} + \frac{(8R^2{}_{i+j} + 8I^2{}_{i+j})}{p^i}\}.$

Using the value of Q_{i+j}, H_{i+j}, I_{i+j}, R_{i+j} from theorem 4.17, which in turn implies

$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\frac{1}{p^i}(P^2{}_{i+j} + G^2{}_{i+j}) = 0$. In particular, for $j = n-1$, $G^2{}_{n-1} + P^2{}_{n-1} = 0$.

Again using $P_{4p^j}(\alpha^{4p^j}) = 1$, to get $\frac{1}{p^{n-1}}(G_{n-1}P_{n-1}) = 0$.

Thus $P_{n-1} = 0$ and $G_{n-1} = 0$, and for all $j < n-1$, $P_{n-2} = G_{n-2} = P_{n-3} = G_{n-3} = \ldots = 0$. $\square$

6. Primitive Idempotents Corresponding to $\Omega_{2tg^kp^i}$, $t \in B_1$

For $0 \le j \le n-1$, define

$C_j = p^j \sum_{s\epsilon\Omega_{2\lambda gp^j}} \alpha^s$, $F_j = p^j \sum_{s\epsilon\Omega_{2gp^j}} \alpha^s$, $K_j = p^j \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^s$, $O_j = p^j \sum_{s\epsilon\Omega_{2p^j}} \alpha^s$, Due to similar procedure as in section 4, $C_j, F_j, K_j, O_j, \in GF(q)$.

Proof of lemma $6.1 - 6.4$ can be obtained on similar lines as that of lemma 4.5, 4.8 and represent C_{i+j}, F_{i+j}, G_{i+j}, K_{i+j}, O_{i+j}, P_{i+j}.

Lemma 25. *For $p \equiv 1(mod8)$ and $0 \le i \le n; 0 \le j \le n-1$,*

$$\sum_{s\epsilon\Omega_{t_1p^j}} \alpha^{2g^kp^is} = \sum_{s\epsilon\Omega_{t_2p^j}} \alpha^{2\lambda g^kp^is} = \sum_{s\epsilon\Omega_{t_3p^j}} \alpha^{2\mu g^kp^is} = \sum_{s\epsilon\Omega_{t_4p^j}} \alpha^{2\nu g^kp^is} = -\sum_{s\epsilon\Omega_{t_1p^j}} \alpha^{2\mu g^kp^is} = -\sum_{s\epsilon\Omega_{t_3p^j}} \alpha^{2g^kp^is}$$
$$= -\sum_{s\epsilon\Omega_{t_2p^j}} \alpha^{2\nu g^kp^is} = -\sum_{s\epsilon\Omega_{t_4p^j}} \alpha^{2\lambda g^kp^is} = \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{t_2g^kp^is} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{t_3g^kp^is} = \sum_{s\epsilon\Omega_{2p^j}} \alpha^{t_1g^kp^is} = \sum_{s\epsilon\Omega_{2\nu p^j}}$$
$$\alpha^{t_4g^kp^is} = -\sum_{s\epsilon\Omega_{2p^j}} \alpha^{t_3g^kp^is} = -\sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{t_4g^kp^is} = -\sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{t_1g^kp^is} = -\sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{t_2g^kp^is}$$
$$= \begin{cases} \frac{\phi(p^{n-j})}{4}\{\alpha^{2g^kp^{i+j}} + \alpha^{6g^kp^{i+j}}\}, & if\ i+j \ge n, \\ \frac{1}{p^j}F_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j}O_{i+j}, & if\ i+j \le n-1,\ k=0, \end{cases} \quad where\ t_1 = 1,\eta,\quad t_2 = \lambda,\xi,\quad t_3 = \mu,\rho,\quad t_4 = \nu,\chi.$$

Lemma 26. *For $p \equiv 1(mod8)$ and $0 \le i \le n, 0 \le j \le n-1$,*

$$\sum_{s\epsilon\Omega_{t_1p^j}} \alpha^{2\lambda g^kp^is} = \sum_{s\epsilon\Omega_{t_2p^j}} \alpha^{2g^kp^is} = \sum_{s\epsilon\Omega_{t_3p^j}} \alpha^{2\nu g^kp^is} = \sum_{s\epsilon\Omega_{t_4p^j}} \alpha^{2\mu g^kp^is} = -\sum_{s\epsilon\Omega_{t_1p^j}} \alpha^{2\nu g^kp^is} = -\sum_{s\epsilon\Omega_{t_2p^j}}$$
$$\alpha^{2\mu g^kp^is} = -\sum_{s\epsilon\Omega_{t_3p^j}} \alpha^{2\lambda g^kp^is} = -\sum_{s\epsilon\Omega_{t_4p^j}} \alpha^{2g^kp^is} = \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{t_1g^kp^is} = \sum_{s\epsilon\Omega_{2p^j}} \alpha^{t_2g^kp^is} = \sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{t_3g^kp^is}$$
$$= \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{t_4g^kp^is} = -\sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{t_1g^kp^is} = -\sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{t_2g^kp^is} = -\sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{t_3g^kp^is} = -\sum_{s\epsilon\Omega_{2p^j}} \alpha^{t_4g^kp^is} =$$
$$\begin{cases} \frac{\phi(p^{n-j})}{4}\{\alpha^{2\lambda g^kp^{i+j}} + \alpha^{6\lambda g^kp^{i+j}}\}, & if\ i+j \ge n, \\ \frac{1}{p^j}C_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j}K_{i+j}, & if\ i+j \le n-1,\ k=0, \end{cases} \quad where\ t_1 = 1,\eta,\ t_2 = \lambda,\xi,\ t_3 = \mu,\rho,\ t_4 = \nu,\chi.$$

Lemma 27. *For $p \equiv 3(mod8)$ and $0 \le i \le n, 0 \le j \le n-1$,*

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{2g^kp^is} = \sum_{s\epsilon\Omega_{\mu p^j}} \alpha^{2\mu g^kp^is} = -\sum_{s\epsilon\Omega_{p^j}} \alpha^{2\mu g^kp^is} = \begin{cases} \phi(p^{n-j})\{\alpha^{2g^kp^{i+j}} + \alpha^{6g^kp^{i+j}}\}, & if\ i+j \ge n, \\ \frac{2}{p^j}F_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{2}{p^j}O_{i+j}, & if\ i+j \le n-1,\ k=0. \end{cases}$$

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Lemma 28. *For $p \equiv 3(mod 8)$ and $0 \le i \le n, 0 \le j \le n-1$,*
$\sum_{s\epsilon\Omega_{2p^j}} \alpha^{g^k p^i s} = -\sum_{s\epsilon\Omega_{2p^j}} \alpha^{\mu g^k p^i s} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{\mu g^k p^i s} = -\sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{g^k p^i s} =$
$$\begin{cases} \frac{\phi(p^{n-j})}{2}\{\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}}\}, \text{ if } i+j \ge n, \\ \frac{1}{p^j}F_{i+j}, \quad \text{if } i+j \le n-1,\ k=1, \\ \frac{1}{p^j}O_{i+j}, \quad \text{if } i+j \le n-1,\ k=0. \end{cases}$$

Theorem 9. *For $p \equiv 1(mod 8)$, the expressions for the primitive idempotents corresponding to $\Omega_{2tg^k p^i}$, where $t \in B_1$ are given by:*

$P_{2p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{4}\{\{\sum_{t\in B'}(-1)^t\{\alpha^{2\lambda t p^{n+j}} + \alpha^{6\lambda t p^{n+j}}\}\overline{C}_{tp^n}\} + \sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})\{-\overline{C}_{g^k p^i} + \overline{C}_{\mu g^k p^i} - \overline{C}_{\eta g^k p^i} + \overline{C}_{\rho g^k p^i}\} + 2\{-\overline{C}_{4g^k p^i} + \overline{C}_{8g^k p^i} + \overline{C}_{16g^k p^i} - \overline{C}_{4\lambda g^k p^i}\} + (\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})\{-\overline{C}_{\lambda g^k p^i} + \overline{C}_{\nu g^k p^i} - \overline{C}_{\xi g^k p^i} + \overline{C}_{\chi g^k p^i}\}\}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{K_{i+j}\{-\overline{C}_{p^i} + \overline{C}_{\mu p^i} - \overline{C}_{\eta p^i} + \overline{C}_{\rho p^i}\} + P_{i+j}\{-\overline{C}_{2p^i} + \overline{C}_{2\lambda p^i} - \overline{C}_{2\mu p^i} + \overline{C}_{2\nu p^i}\} + Q_{i+j}\{\overline{C}_{4p^i} + \overline{C}_{4\lambda p^i}\} + R_{i+j}\{\overline{C}_{8p^i} + \overline{C}_{16p^i}\} + O_{i+j}\{-\overline{C}_{\lambda p^i} + \overline{C}_{\nu p^i} - \overline{C}_{\xi p^i} + \overline{C}_{\chi p^i}\} + C_{i+j}\{-\overline{C}_{gp^i} + \overline{C}_{\mu gp^i} - \overline{C}_{\eta gp^i} + \overline{C}_{\rho gp^i}\} + G_{i+j}\{-\overline{C}_{2gp^i} + \overline{C}_{2\lambda gp^i} - \overline{C}_{2\mu gp^i} + \overline{C}_{2\nu gp^i}\} + H_{i+j}\{\overline{C}_{4gp^i} + \overline{C}_{4\lambda gp^i}\} + I_{i+j}\{\overline{C}_{8gp^i} + \overline{C}_{16gp^i}\} + F_{i+j}\{-\overline{C}_{\lambda gp^i} + \overline{C}_{\nu gp^i} - \overline{C}_{\xi gp^i} + \overline{C}_{\chi gp^i}\}\}]$.

$P_{2\lambda p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{4}\{\sum_{t\in B'}(-1)^t\{\alpha^{2tp^{n+j}} + \alpha^{6tp^{n+j}}\}\overline{C}_{tp^n} + \sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})\{-\overline{C}_{g^k p^i} + \overline{C}_{\mu g^k p^i} - \overline{C}_{\eta g^k p^i} + \overline{C}_{\rho g^k p^i}\} + 2\{-\overline{C}_{4g^k p^i} + \overline{C}_{8g^k p^i} + \overline{C}_{16g^k p^i} - \overline{C}_{4\lambda g^k p^i}\} + (\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})\{-\overline{C}_{\lambda g^k p^i} + \overline{C}_{\nu g^k p^i} - \overline{C}_{\xi g^k p^i} + \overline{C}_{\chi g^k p^i}\}\}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{O_{i+j}\{-\overline{C}_{p^i} + \overline{C}_{\mu p^i} - \overline{C}_{\eta p^i} + \overline{C}_{\rho p^i}\} + P_{i+j}\{\overline{C}_{2p^i} - \overline{C}_{2\lambda p^i} + \overline{C}_{2\mu p^i} - \overline{C}_{2\nu p^i}\} + Q_{i+j}\{\overline{C}_{4p^i} + \overline{C}_{4\lambda p^i}\} + R_{i+j}\{\overline{C}_{8p^i} + \overline{C}_{16p^i}\} + K_{i+j}\{-\overline{C}_{\lambda p^i} + \overline{C}_{\nu p^i} - \overline{C}_{\xi p^i} + \overline{C}_{\chi p^i}\} + F_{i+j}\{-\overline{C}_{gp^i} + \overline{C}_{\mu gp^i} - \overline{C}_{\eta gp^i} + \overline{C}_{\rho gp^i}\} + G_{i+j}\{\overline{C}_{2gp^i} - \overline{C}_{2\lambda gp^i} + \overline{C}_{2\mu gp^i} - \overline{C}_{2\nu gp^i}\} + H_{i+j}\{\overline{C}_{4gp^i} + \overline{C}_{4\lambda gp^i}\} + I_{i+j}\{\overline{C}_{8gp^i} + \overline{C}_{16gp^i}\} + C_{i+j}\{-\overline{C}_{\lambda gp^i} + \overline{C}_{\nu gp^i} - \overline{C}_{\xi gp^i} + \overline{C}_{\chi gp^i}\}\}]$.

$P_{2\mu p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{4}\{\sum_{t\in B'}\{\alpha^{2tp^{n+j}} + \alpha^{6tp^{n+j}}\}\overline{C}_{tp^n} + \sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})\{\overline{C}_{g^k p^i} - \overline{C}_{\mu g^k p^i} + \overline{C}_{\eta g^k p^i} - \overline{C}_{\rho g^k p^i}\} + 2\{-\overline{C}_{4g^k p^i} + \overline{C}_{8g^k p^i} + \overline{C}_{16g^k p^i} - \overline{C}_{4\lambda g^k p^i}\} + (\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})\{\overline{C}_{\lambda g^k p^i} - \overline{C}_{\nu g^k p^i} + \overline{C}_{\xi g^k p^i} - \overline{C}_{\chi g^k p^i}\}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{K_{i+j}\{\overline{C}_{p^i} - \overline{C}_{\mu p^i} + \overline{C}_{\eta p^i} - \overline{C}_{\rho p^i}\} + P_{i+j}\{-\overline{C}_{2p^i} + \overline{C}_{2\lambda p^i} - \overline{C}_{2\mu p^i} + \overline{C}_{2\nu p^i}\} + Q_{i+j}\{\overline{C}_{4p^i} + \overline{C}_{4\lambda p^i}\} + R_{i+j}\{\overline{C}_{8p^i} + \overline{C}_{16p^i}\} + O_{i+j}\{\overline{C}_{\lambda p^i} - \overline{C}_{\nu p^i} + \overline{C}_{\xi p^i} - \overline{C}_{\chi p^i}\} + C_{i+j}\{\overline{C}_{gp^i} - \overline{C}_{\mu gp^i} + \overline{C}_{\eta gp^i} - \overline{C}_{\rho gp^i}\} + G_{i+j}\{-\overline{C}_{2gp^i} + \overline{C}_{2\lambda gp^i} - \overline{C}_{2\mu gp^i} + \overline{C}_{2\nu gp^i}\} + H_{i+j}\{\overline{C}_{4gp^i} + \overline{C}_{4\lambda gp^i}\} + I_{i+j}\{\overline{C}_{8gp^i} + \overline{C}_{16gp^i}\} + F_{i+j}\{\overline{C}_{\lambda gp^i} - \overline{C}_{\nu gp^i} + \overline{C}_{\xi gp^i} - \overline{C}_{\chi gp^i}\}\}]$.

$P_{2\nu p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{4}\{\sum_{t\in B'}\{\alpha^{2tp^{n+j}} + \alpha^{6tp^{n+j}}\}\overline{C}_{tp^n} + \sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})\{\overline{C}_{g^k p^i} - \overline{C}_{\mu g^k p^i} + \overline{C}_{\eta g^k p^i} - \overline{C}_{\rho g^k p^i}\} + 2\{-\overline{C}_{4g^k p^i} + \overline{C}_{8g^k p^i} + \overline{C}_{16g^k p^i} - \overline{C}_{4\lambda g^k p^i}\} + (\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})\{\overline{C}_{\lambda g^k p^i} - \overline{C}_{\nu g^k p^i} + \overline{C}_{\xi g^k p^i} - \overline{C}_{\chi g^k p^i}\}\}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{O_{i+j}\{\overline{C}_{p^i} - \overline{C}_{\mu p^i} + \overline{C}_{\eta p^i} - \overline{C}_{\rho p^i}\} + P_{i+j}\{\overline{C}_{2p^i} - \overline{C}_{2\lambda p^i} + \overline{C}_{2\mu p^i} - \overline{C}_{2\nu p^i}\} + Q_{i+j}\{\overline{C}_{4p^i} + \overline{C}_{4\lambda p^i}\} + R_{i+j}\{\overline{C}_{8p^i} + \overline{C}_{16p^i}\} + K_{i+j}\{\overline{C}_{\lambda p^i} - \overline{C}_{\nu p^i} + \overline{C}_{\xi p^i} - \overline{C}_{\chi p^i}\} + F_{i+j}\{\overline{C}_{gp^i} - \overline{C}_{\mu gp^i} + \overline{C}_{\eta gp^i} - \overline{C}_{\rho gp^i}\} + G_{i+j}\{\overline{C}_{2gp^i} - \overline{C}_{2\lambda gp^i} + \overline{C}_{2\mu gp^i} - \overline{C}_{2\nu gp^i}\} + H_{i+j}\{\overline{C}_{4gp^i} + \overline{C}_{4\lambda gp^i}\} + I_{i+j}\{\overline{C}_{8gp^i} + \overline{C}_{16gp^i}\} + C_{i+j}\{\overline{C}_{\lambda gp^i} - \overline{C}_{\nu gp^i} + \overline{C}_{\xi gp^i} - \overline{C}_{\chi gp^i}\}\}]$.

Where C_{i+j}, F_{i+j}, K_{i+j}, O_{i+j} can be obtained from the following relations,
$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{K_{i+j}C_{i+j} + O_{i+j}F_{i+j}}{p^i}\} = -\frac{p^{n-1}}{4}(1-p), \quad \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{K_{i+j}O_{i+j} + C_{i+j}F_{i+j}}{p^i}\} = -\frac{p^{n-1}}{4}(3p+1).$$
$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{K^2{}_{i+j} + C^2{}_{i+j} + F^2{}_{i+j} + O^2{}_{i+j}}{p^i}\} = -\frac{p^{n-1}}{2}(1-p).$$

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$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{K_{i+j}F_{i+j}+O_{i+j}C_{i+j}}{p^i}\} = -\frac{p^{n-1}}{4}(1-p).$$

The expressions for P_{2tgp^j}, where $t \in B_1$ can be obtained by interchanging P_{i+j}, Q_{i+j}, R_{i+j}, K_{i+j}, O_{i+j} by G_{i+j}, H_{i+j}, I_{i+j}, C_{i+j}, F_{i+j} and $\alpha^{up^{i+j}}$ by $\alpha^{ugp^{i+j}}$ respectively and vice versa in the expressions of P_{2tp^j}, where $t \in B_1$.

Proof. These expressions can be obtained using lemmas 4.3, 4.5 − 4.7, 5.1, 25 − 26 with same procedure to the theorem 5. Also the relations can be derived using $P_{2p^j}(\alpha^{2p^j}) = 1$, $P_{2p^j}(\alpha^{2gp^j}) = 0$, $P_{2p^j}(\alpha^{2\lambda p^j}) = 0$ and $P_{2p^j}(\alpha^{2\nu gp^j}) = 0$. □

Theorem 10. *For $p \equiv 3(mod 8)$, the expressions for the primitive idempotents corresponding to $\Omega_{2g^kp^i}$, $\Omega_{2\mu g^kp^i}$ are given by*

$P_{2p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\{\sum_{t\in B'}(\alpha^{2gtp^{n+j}}+\alpha^{6gtp^{n+j}})\overline{C}_{tp^n}\}+\sum_{i=n-j}^{n-1}\{(\alpha^{2gp^{i+j}}+\alpha^{6gp^{i+j}})\{\overline{C}_{p^i}-\overline{C}_{\mu p^i}\}+2\{-\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}\}+(\alpha^{2p^{i+j}}+\alpha^{6p^{i+j}})\{\overline{C}_{gp^i}-\overline{C}_{\mu gp^i}\}+2\{-\overline{C}_{4gp^i}+\overline{C}_{8gp^i}+\overline{C}_{16gp^i}\}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{F_{i+j}\{\overline{C}_{p^i}-\overline{C}_{\mu p^i}\}+G_{i+j}\{\overline{C}_{2p^i}+\overline{C}_{2\mu p^i}\}+2H_{i+j}\overline{C}_{4p^i}+2I_{i+j}\{\overline{C}_{8p^i}+\overline{C}_{16gp^i}\}+2R_{i+j}\{\overline{C}_{16p^i}+\overline{C}_{8gp^i}\}+O_{i+j}\{\overline{C}_{gp^i}-\overline{C}_{\mu gp^i}\}+P_{i+j}\{\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}+2Q_{i+j}\overline{C}_{4gp^i}\}]$.

$P_{2\mu p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\{\sum_{t\in B'}(-1)^t(\alpha^{2gtp^{n+j}}+\alpha^{6gtp^{n+j}})\overline{C}_{tp^n}\}+\sum_{i=n-j}^{n-1}\{(\alpha^{2gp^{i+j}}+\alpha^{6gp^{i+j}})\{-\overline{C}_{p^i}+\overline{C}_{\mu p^i}\}+2\{-\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}\}+(\alpha^{2p^{i+j}}+\alpha^{6p^{i+j}})\{-\overline{C}_{gp^i}+\overline{C}_{\mu gp^i}\}+2\{-\overline{C}_{4gp^i}+\overline{C}_{8gp^i}+\overline{C}_{16gp^i}\}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{F_{i+j}\{-\overline{C}_{p^i}+\overline{C}_{\mu p^i}\}+G_{i+j}\{\overline{C}_{2p^i}+\overline{C}_{2\mu p^i}\}+2H_{i+j}\overline{C}_{4p^i}+2I_{i+j}\{\overline{C}_{8p^i}+\overline{C}_{16gp^i}\}+2R_{i+j}\{\overline{C}_{16p^i}+\overline{C}_{8gp^i}\}+O_{i+j}\{-\overline{C}_{gp^i}+\overline{C}_{\mu gp^i}\}+P_{i+j}\{\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}+2Q_{i+j}\overline{C}_{4gp^i}\}]$,

the expressions for P_{2gp^j} and $P_{2\mu gp^j}$ can be obtained by interchanging P_{i+j}, Q_{i+j}, R_{i+j}, O_{i+j} by G_{i+j}, H_{i+j}, I_{i+j}, F_{i+j} and $\alpha^{up^{i+j}}$ by $\alpha^{ugp^{i+j}}$ respectively and vice versa in the expression of P_{2p^j} and $P_{2\mu p^j}$. where F_{i+j}, O_{i+j} can be obtained from the following relations,

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\frac{F_{i+j}O_{i+j}}{p^i} = \frac{p^{n-1}(1+p)}{2}, \qquad \frac{1}{p^j}\sum_{i=0}^{n-j-1}\frac{F^2{}_{i+j}+O^2{}_{i+j}}{p^i} = p^{n-1}(p-1).$$

Proof. These expressions can be derived using lemmas 4.4, 14 − 21, 23 − 24 , 27 − 28 with same procedure to the theorem 5. Also relations can be derived using $P_{2p^j}(\alpha^{2p^j}) = 1$ and $P_{2p^j}(\alpha^{2gp^j}) = 0$. □

7. Primitive Idempotents Corresponding to $\Omega_{tg^kp^i}$, $t \in A_1$

For $0 \le j \le n-1$, define $A_j = p^j\sum_{s\epsilon\Omega_{gp^j}}\alpha^s$; $B_j = p^j\sum_{s\epsilon\Omega_{\lambda gp^j}}\alpha^s$; $D_j = p^j\sum_{s\epsilon\Omega_{\mu gp^j}}\alpha^s$; $E_j = p^j\sum_{s\epsilon\Omega_{\nu gp^j}}\alpha^s$; $J_j = p^j\sum_{s\epsilon\Omega_{\lambda p^j}}\alpha^s$; $L_j = p^j\sum_{s\epsilon\Omega_{\mu p^j}}\alpha^s$; $M_j = p^j\sum_{s\epsilon\Omega_{\nu p^j}}\alpha^s$; $N_j = p^j\sum_{s\epsilon\Omega_{p^j}}\alpha^s$.

Using similar procedure as in section 4 to obtain $A_j, B_j, D_j, E_j, J_j, L_j, M_j, N_j \in GF(q)$.

Lemma 29. *For $0 \le j \le n-1$, $L_j - \beta^{p^j}N_j = 0$; $M_j - \beta^{p^j}J_j = 0$; $D_j - \beta^{gp^j}A_j = 0$; $E_j - \beta^{gp^j}B_j = 0$; where $\beta = \alpha^{4p^n}$.*

Proof. Since $\alpha^{\mu p^j} = \alpha^{(1+4p^n)p^j} = \alpha^{p^j}\beta^{p^j}$. So $L_j = p^j\sum_{s\epsilon\Omega_{\mu p^i}}\alpha^s = p^j[\alpha^{\mu p^i}+\alpha^{q\mu p^i}+\alpha^{q^2\mu p^i}+\ldots+\alpha^{q^{\frac{\phi(p^{n-j})}{2}-1}\mu p^i}] = p^j[\alpha^{p^i}\beta^{p^i}+\alpha^{qp^i}\beta^{p^i}+\alpha^{q^2p^i}\beta^{p^i}+\ldots+\alpha^{p^iq^{\frac{\phi(p^{n-j})}{2}-1}}\beta^{p^i}] = \beta^{p^i}[p^j\sum_{s\epsilon\Omega_{p^j}}\alpha^s] = \beta^{p^i}N_j$.

Remaining can be obtained in similar lines. □

Proof of lemma 7.2 − 7.7 is similar to that of lemma 4.5 and 4.8.

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Lemma 30. *For* $p \equiv 1(mod4)$, $0 \le i \le n; 0 \le j \le n-1$, $\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\xi g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\mu p^j}} \alpha^{\rho g^k p^i s} =$
$\sum\limits_{s\epsilon\Omega_{\nu p^j}} \alpha^{\nu g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\eta p^j}} \alpha^{\eta g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\chi p^j}} \alpha^{\chi g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\nu p^j}} \alpha^{\chi g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\lambda g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\mu p^j}} \alpha^{\mu g^k p^i s} =$
$-\sum\limits_{s\epsilon\Omega_{\xi p^j}} \alpha^{\xi g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\eta p^j}} \alpha^{g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\rho p^j}} \alpha^{\rho g^k p^i s} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{1}{p^j} A_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j} N_{i+j}, & if\ i+j \le n-1,\ k=0. \end{cases}$

Lemma 31. *For* $p \equiv 1(mod4)$, $0 \le i \le n; 0 \le j \le n-1$,
$\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{\lambda g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\eta p^j}} \alpha^{\xi g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\mu p^j}} \alpha^{\nu g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\rho p^j}} \alpha^{\chi g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{\xi g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\nu p^j}} \alpha^{\rho g^k p^i s}$
$= -\sum\limits_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\eta g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\mu p^j}} \alpha^{\chi g^k p^i s} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{1}{p^j} B_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j} J_{i+j}, & if\ i+j \le n-1,\ k=0. \end{cases}$

Lemma 32. *For* $p \equiv 1(mod4)$, $0 \le i \le n; 0 \le j \le n-1$,
$\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{\mu g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\nu g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\eta p^j}} \alpha^{\rho g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\xi p^j}} \alpha^{\chi g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{\rho g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\chi g^k p^i s}$
$= -\sum\limits_{s\epsilon\Omega_{\mu p^j}} \alpha^{\eta g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\nu p^j}} \alpha^{\xi g^k p^i s} = \begin{cases} 0, & if\ \ i+j \ge n, \\ \frac{1}{p^j} D_{i+j}, & if\ i+j \le n-1,\ \ k=1, \\ \frac{1}{p^j} L_{i+j}, & if\ i+j \le n-1,\ \ k=0. \end{cases}$

Lemma 33. *For* $p \equiv 1(mod4)$, $0 \le i \le n; 0 \le j \le n-1$,
$\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{\nu g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\rho g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\mu p^j}} \alpha^{\xi g^k p^i s} = \sum\limits_{s\epsilon\Omega_{\eta p^j}} \alpha^{\chi g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{\chi g^k p^i s} - \sum\limits_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\mu g^k p^i s}$
$= -\sum\limits_{s\epsilon\Omega_{\nu p^j}} \alpha^{\eta g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\xi p^j}} \alpha^{\rho g^k p^i s} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{1}{p^j} E_{i+j}, & if\ i+j \le n-1,\ \ k=1, \\ \frac{1}{p^j} M_{i+j}, & if\ i+j \le n-1,\ \ k=0. \end{cases}$

Lemma 34. *For* $p \equiv 3(mod4)$, $0 \le i \le n; 0 \le j \le n-1$,
$$\sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{g^k p^i s} = -\sum\limits_{s\epsilon\Omega_{\mu p^j}} \alpha^{\mu g^k p^i s} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{1}{p^j} A_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j} N_{i+j}, & if\ i+j \le n-1,\ k=0. \end{cases}$$

Lemma 35. *For* $p \equiv 3(mod4)$, $0 \le i \le n; 0 \le j \le n-1$,
$$\sum\limits_{s\epsilon\Omega_{\mu g^k p^j}} \alpha^{p^i s} = \sum\limits_{s\epsilon\Omega_{p^j}} \alpha^{\mu g^k p^i s} = \begin{cases} 0, & if\ i+j \ge n, \\ \frac{1}{p^j} D_{i+j}, & if\ i+j \le n-1,\ k=1, \\ \frac{1}{p^j} L_{i+j}, & if\ i+j \le n-1,\ k=0. \end{cases}$$

Theorem 11. *For* $p \equiv 1(mod8)$, *the expressions for the primitive idempotents corresponding to* $\Omega_{tg^k p^i}$, *where* $t \in A_1$ *are given by:*
If $\Omega_{p^i} = -\Omega_{\chi p^i}$,

$P_{p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{4}\{\{\sum\limits_{t\in\{0,1,4,5\}}(-1)^t(\alpha^{2t\lambda p^{n+j}} + \alpha^{6t\lambda p^{n+j}})\overline{C}_{2tp^n}\} + \sum\limits_{i=n-j}^{n-1}\{\sum\limits_{k=0,1}\{(\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})$
$\{-\overline{C}_{2g^k p^i} + \overline{C}_{2\mu g^k p^i}\} - 2\overline{C}_{8g^k p^i} + 2\overline{C}_{16g^k p^i} + (\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})\{-\overline{C}_{2\lambda g^k p^i} + \overline{C}_{2\nu g^k p^i}\}\}\} + \frac{1}{p^j}\sum\limits_{i=0}^{n-j-1}\{M_{i+j}$
$\{-\overline{C}_{p^i} + \overline{C}_{\eta p^i}\} + K_{i+j}\{-\overline{C}_{2p^i} + \overline{C}_{2\mu p^i}\} + P_{i+j}\{-\overline{C}_{4p^i} + \overline{C}_{4\lambda p^i}\} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + L_{i+j}\{-\overline{C}_{\lambda p^i} +$
$\overline{C}_{\xi p^i}\} + O_{i+j}\{-\overline{C}_{2\lambda p^i} + \overline{C}_{2\nu p^i}\} + J_{i+j}\{-\overline{C}_{\mu p^i} + \overline{C}_{\rho p^i}\} + N_{i+j}\{-\overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\} + E_{i+j}\{-\overline{C}_{gp^i} + \overline{C}_{\eta gp^i}\} +$

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$C_{i+j}\{-\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}+G_{i+j}\{-\overline{C}_{4gp^i}+\overline{C}_{4\lambda gp^i}\}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}+D_{i+j}\{-\overline{C}_{\lambda gp^i}+\overline{C}_{\xi gp^i}\}+F_{i+j}\{-\overline{C}_{2\lambda gp^i}+\overline{C}_{2\nu gp^i}\}+B_{i+j}\{-\overline{C}_{\mu gp^i}+\overline{C}_{\rho gp^i}\}+A_{i+j}\{-\overline{C}_{\nu gp^i}+\overline{C}_{\chi gp^i}\}\}].$

If $\Omega_{p^i}=-\Omega_{\nu p^i}$ *then* $P_{p^j}(x)$ *can be obtained by replacing the sign* $+$ *to* $-$ *and* $-$ *to* $+$ *of the terms when* $\overline{C}_{tp^i}$ *has odd suffix.*

If $\Omega_{\lambda p^i}=-\Omega_{\rho p^i}$,

$$P_{\lambda p^j}(x)=\tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{4}\{\{\sum_{t\in\{0,1,4,5\}}(-1)^t(\alpha^{2tp^{n+j}}+\alpha^{6tp^{n+j}})\overline{C}_{2tp^n}\}+\sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2g^kp^{i+j}}+\alpha^{6g^kp^{i+j}})\{-\overline{C}_{2g^kp^i}+\overline{C}_{2\mu g^kp^i}\}-2\overline{C}_{8g^kp^i}+2\overline{C}_{16g^kp^i}+(\alpha^{2\lambda g^kp^{i+j}}+\alpha^{6\lambda g^kp^{i+j}})\{-\overline{C}_{2\lambda g^kp^i}+\overline{C}_{2\nu g^kp^i}\}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{L_{i+j}\{-\overline{C}_{p^i}+\overline{C}_{\eta p^i}\}+O_{i+j}\{-\overline{C}_{2p^i}+\overline{C}_{2\mu p^i}\}+P_{i+j}\{\overline{C}_{4p^i}-\overline{C}_{4\lambda p^i}\}+Q_{i+j}\overline{C}_{8p^i}+R_{i+j}\overline{C}_{16p^i}+M_{i+j}\{\overline{C}_{\lambda p^i}-\overline{C}_{\xi p^i}\}+K_{i+j}\{-\overline{C}_{2\lambda p^i}+\overline{C}_{2\nu p^i}\}+N_{i+j}\{\overline{C}_{\mu p^i}-\overline{C}_{\rho p^i}\}+J_{i+j}\{-\overline{C}_{\nu p^i}+\overline{C}_{\chi p^i}\}+D_{i+j}\{-\overline{C}_{gp^i}+\overline{C}_{\eta gp^i}\}+F_{i+j}\{-\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}+G_{i+j}\{\overline{C}_{4gp^i}-\overline{C}_{4\lambda gp^i}\}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}+E_{i+j}\{\overline{C}_{\lambda gp^i}-\overline{C}_{\xi gp^i}\}+C_{i+j}\{-\overline{C}_{2\lambda gp^i}+\overline{C}_{2\nu gp^i}\}+A_{i+j}\{\overline{C}_{\mu gp^i}-\overline{C}_{\rho gp^i}\}+B_{i+j}\{-\overline{C}_{\nu gp^i}+\overline{C}_{\chi gp^i}\}\}\}].$$

If $\Omega_{\lambda p^i}=-\Omega_{\mu p^i}$, *then* $P_{\lambda p^j}(x)$ *can be obtained by replacing the sign of the terms* $+$ *to* $-$ *and* $-$ *to* $+$ *when* $\overline{C}_{tp^i}$ *has odd suffix.*

If $\Omega_{\mu p^i}=-\Omega_{\xi p^i}$,

$$P_{\mu p^j}(x)=\tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{4}\{\{\sum_{t\in\{0,1,4,5\}}(\alpha^{2t\lambda p^{n+j}}+\alpha^{6t\lambda p^{n+j}})\overline{C}_{2tp^n}\}+\sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2\lambda g^kp^{i+j}}+\alpha^{6\lambda g^kp^{i+j}})\{\overline{C}_{2g^kp^i}-\overline{C}_{2\mu g^kp^i}\}-2\overline{C}_{8g^kp^i}+2\overline{C}_{16g^kp^i}+(\alpha^{2g^kp^{i+j}}+\alpha^{6g^kp^{i+j}})\{\overline{C}_{2\lambda g^kp^i}-\overline{C}_{2\nu g^kp^i}\}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{J_{i+j}\{-\overline{C}_{p^i}+\overline{C}_{\eta p^i}\}+K_{i+j}\{\overline{C}_{2p^i}-\overline{C}_{2\mu p^i}\}+P_{i+j}\{-\overline{C}_{4p^i}+\overline{C}_{4\lambda p^i}\}+Q_{i+j}\overline{C}_{8p^i}+R_{i+j}\overline{C}_{16p^i}+N_{i+j}\{\overline{C}_{\lambda p^i}-\overline{C}_{\xi p^i}\}+O_{i+j}\{\overline{C}_{2\lambda p^i}-\overline{C}_{2\nu p^i}\}+M_{i+j}\{\overline{C}_{\mu p^i}-\overline{C}_{\rho p^i}\}+L_{i+j}\{-\overline{C}_{\nu p^i}+\overline{C}_{\chi p^i}\}+B_{i+j}\{-\overline{C}_{gp^i}+\overline{C}_{\eta gp^i}\}+C_{i+j}\{\overline{C}_{2gp^i}-\overline{C}_{2\mu gp^i}\}+G_{i+j}\{-\overline{C}_{4gp^i}+\overline{C}_{4\lambda gp^i}\}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}+A_{i+j}\{\overline{C}_{\lambda gp^i}-\overline{C}_{\xi gp^i}\}+F_{i+j}\{\overline{C}_{2\lambda gp^i}-\overline{C}_{2\nu gp^i}\}+E_{i+j}\{\overline{C}_{\mu gp^i}-\overline{C}_{\rho gp^i}\}+D_{i+j}\{-\overline{C}_{\nu gp^i}+\overline{C}_{\chi gp^i}\}\}\}].$$

If $\Omega_{\mu p^i}=-\Omega_{\lambda p^i}$, *then* $P_{\mu p^j}(x)$ *can be obtained by replacing the sign of the term* $+$ *to* $-$ *and* $-$ *to* $+$ *when* $\overline{C}_{tp^i}$ *has odd suffix.*

If $\Omega_{\nu p^i}=-\Omega_{\eta p^i}$,

$$P_{\nu p^j}(x)=\tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{4}\{\{\sum_{t\in\{0,1,4,5\}}(\alpha^{2tp^{n+j}}+\alpha^{6tp^{n+j}})\overline{C}_{2tp^n}\}+\sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2g^kp^{i+j}}+\alpha^{6g^kp^{i+j}})\{\overline{C}_{2g^kp^i}-\overline{C}_{2\mu g^kp^i}\}-2\overline{C}_{8g^kp^i}+2\overline{C}_{16g^kp^i}+(\alpha^{2\lambda g^kp^{i+j}}+\alpha^{6\lambda g^kp^{i+j}})\{\overline{C}_{2\lambda g^kp^i}-\overline{C}_{2\nu g^kp^i}\}\}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{N_{i+j}\{-\overline{C}_{p^i}+\overline{C}_{\eta p^i}\}+O_{i+j}\{\overline{C}_{2p^i}-\overline{C}_{2\mu p^i}\}+P_{i+j}\{\overline{C}_{4p^i}-\overline{C}_{4\lambda p^i}\}+Q_{i+j}\overline{C}_{8p^i}+R_{i+j}\overline{C}_{16p^i}+J_{i+j}\{-\overline{C}_{\lambda p^i}+\overline{C}_{\xi p^i}\}+K_{i+j}\{\overline{C}_{2\lambda p^i}-\overline{C}_{2\nu p^i}\}+L_{i+j}\{-\overline{C}_{\mu p^i}+\overline{C}_{\rho p^i}\}+M_{i+j}\{-\overline{C}_{\nu p^i}+\overline{C}_{\chi p^i}\}+A_{i+j}\{-\overline{C}_{gp^i}+\overline{C}_{\eta gp^i}\}+F_{i+j}\{\overline{C}_{2gp^i}-\overline{C}_{2\mu gp^i}\}+G_{i+j}\{\overline{C}_{4gp^i}-\overline{C}_{4\lambda gp^i}\}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}+B_{i+j}\{-\overline{C}_{\lambda gp^i}+\overline{C}_{\xi gp^i}\}+C_{i+j}\{\overline{C}_{2\lambda gp^i}-\overline{C}_{2\nu gp^i}\}+D_{i+j}\{-\overline{C}_{\mu gp^i}+\overline{C}_{\rho gp^i}\}+E_{i+j}\{-\overline{C}_{\nu gp^i}+\overline{C}_{\chi gp^i}\}\}].$$

If $\Omega_{\nu p^i}=-\Omega_{p^i}$, *then* $P_{\nu p^j}(x)$ *can be obtained by replacing the sign of the term* $+$ *to* $-$ *and* $-$ *to* $+$ *when* $\overline{C}_{tp^i}$ *has odd suffix.*

If $\Omega_{\eta p^i}=-\Omega_{\nu p^i}$,

$$P_{\eta p^j}(x)=\tfrac{1}{16p^n}{}_{i+j}[\tfrac{\phi(p^{n-j})}{4}\{\{\sum_{t\in\{0,1,4,5\}}(-1)^t(\alpha^{2tp^{n+j}}+\alpha^{6tp^{n+j}})\overline{C}_{2tp^n}\}+\sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2\lambda g^kp^{i+j}}+\alpha^{6\lambda g^kp^{i+j}})\{-\overline{C}_{2g^kp^i}+\overline{C}_{2\mu g^kp^i}\}-2\overline{C}_{8g^kp^i}+2\overline{C}_{16g^kp^i}+(\alpha^{2g^kp^{i+j}}+\alpha^{6g^kp^{i+j}})\{-\overline{C}_{2\lambda g^kp^i}+\overline{C}_{2\nu g^kp^i}\}\}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{M_{i+j}\{\overline{C}_{p^i}-\overline{C}_{\eta p^i}\}+K_{i+j}\{-\overline{C}_{2p^i}+\overline{C}_{2\mu p^i}\}+P_{i+j}\{-\overline{C}_{4p^i}+\overline{C}_{4\lambda p^i}\}+Q_{i+j}\overline{C}_{8p^i}+R_{i+j}\overline{C}_{16p^i}+L_{i+j}\{\overline{C}_{\lambda p^i}-\overline{C}_{\xi p^i}\}+O_{i+j}\{-\overline{C}_{2\lambda p^i}+\overline{C}_{2\nu p^i}\}+J_{i+j}\{\overline{C}_{\mu p^i}-\overline{C}_{\rho p^i}\}+N_{i+j}\{\overline{C}_{\nu p^i}-\overline{C}_{\chi p^i}\}+E_{i+j}\{\overline{C}_{gp^i}-\overline{C}_{\eta gp^i}\}+C_{i+j}\{-\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}+G_{i+j}\{-\overline{C}_{4gp^i}+\overline{C}_{4\lambda gp^i}\}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}+D_{i+j}\{\overline{C}_{\lambda gp^i}-\overline{C}_{\xi gp^i}\}+F_{i+j}\{-\overline{C}_{2\lambda gp^i}+\overline{C}_{2\nu gp^i}\}+B_{i+j}\{\overline{C}_{\mu gp^i}-\overline{C}_{\rho gp^i}\}+A_{i+j}\{\overline{C}_{\nu gp^i}-\overline{C}_{\chi gp^i}\}\}].$$

If $\Omega_{\eta p^i}=-\Omega_{\chi p^i}$ *then* $P_{\eta p^j}(x)$ *can be obtained by replacing the sign of the term* $+$ *to* $-$ *and* $-$ *to* $+$ *when* $\overline{C}_{tp^i}$ *has odd suffix.*

If $\Omega_{\xi p^i}=-\Omega_{\mu p^i}$,

$$P_{\xi p^j}(x)=\tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{4}\{\{\sum_{t\in\{0,1,4,5\}}(-1)^t(\alpha^{2tp^{n+j}}+\alpha^{6tp^{n+j}})\overline{C}_{2tp^n}\}+\sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2g^kp^{i+j}}+\alpha^{6g^kp^{i+j}})\{-\overline{C}_{2g^kp^i}+\overline{C}_{2\mu g^kp^i}\}-2\overline{C}_{8g^kp^i}+2\overline{C}_{16g^kp^i}+(\alpha^{2\lambda g^kp^{i+j}}+\alpha^{6\lambda g^kp^{i+j}})\{-\overline{C}_{2\lambda g^kp^i}+\overline{C}_{2\nu g^kp^i}\}\}\}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{L_{i+j}\{\overline{C}_{p^i}-\overline{C}_{\eta p^i}\}+O_{i+j}\{-\overline{C}_{2p^i}+\overline{C}_{2\mu p^i}\}+P_{i+j}\{\overline{C}_{4p^i}-\overline{C}_{4\lambda p^i}\}+Q_{i+j}\overline{C}_{8p^i}+R_{i+j}\overline{C}_{16p^i}+M_{i+j}\{-\overline{C}_{\lambda p^i}+\overline{C}_{\xi p^i}\}+K_{i+j}\{-\overline{C}_{2\lambda p^i}+\overline{C}_{2\nu p^i}\}+N_{i+j}\{-\overline{C}_{\mu p^i}+\overline{C}_{\rho p^i}\}+J_{i+j}\{\overline{C}_{\nu p^i}-\overline{C}_{\chi p^i}\}+D_{i+j}\{\overline{C}_{gp^i}-\overline{C}_{\eta gp^i}\}+F_{i+j}\{-\overline{C}_{2gp^i}+\overline{C}_{2\mu gp^i}\}+G_{i+j}\{\overline{C}_{4gp^i}-\overline{C}_{4\lambda gp^i}\}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}+E_{i+j}\{-\overline{C}_{\lambda gp^i}+\overline{C}_{\xi gp^i}\}+C_{i+j}\{-\overline{C}_{2\lambda gp^i}+\overline{C}_{2\nu gp^i}\}+A_{i+j}\{-\overline{C}_{\mu gp^i}+\overline{C}_{\rho gp^i}\}+B_{i+j}\{\overline{C}_{\nu gp^i}-\overline{C}_{\chi gp^i}\}\}\}].$$

If $\Omega_{\xi p^i}=-\Omega_{\rho p^i}$, *then* $P_{\xi p^j}(x)$ *can be obtained by replacing the sign of the term* $+$ *to* $-$ *and* $-$ *to* $+$ *when* $\overline{C}_{tp^i}$ *has odd suffix.*

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If $\Omega_{\rho p^i} = -\Omega_{\lambda p^i}$,

$P_{\rho p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{4}\{\{\sum_{t\in\{0,1,4,5\}}(\alpha^{2t\lambda p^{n+j}} + \alpha^{6t\lambda p^{n+j}})\overline{C}_{2tp^n}\} + \sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})$ $\{\overline{C}_{2g^k p^i} - \overline{C}_{2\mu g^k p^i}\} - 2\overline{C}_{8g^k p^i} + 2\overline{C}_{16g^k p^i} + (\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})\{\overline{C}_{2\lambda g^k p^i} - \overline{C}_{2\nu g^k p^i}\}\}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{J_{i+j}\{\overline{C}_{p^i} - \overline{C}_{\eta p^i}\} + K_{i+j}\{\overline{C}_{2p^i} - \overline{C}_{2\mu p^i}\} + P_{i+j}\{-\overline{C}_{4p^i} + \overline{C}_{4\lambda p^i}\} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + N_{i+j}\{-\overline{C}_{\lambda p^i} + \overline{C}_{\xi p^i}\} + O_{i+j}\{\overline{C}_{2\lambda p^i} - \overline{C}_{2\nu p^i}\} + M_{i+j}\{-\overline{C}_{\mu p^i} + \overline{C}_{\rho p^i}\} + L_{i+j}\{\overline{C}_{\nu p^i} - \overline{C}_{\chi p^i}\} + B_{i+j}\{\overline{C}_{gp^i} - \overline{C}_{\eta gp^i}\} + C_{i+j}\{\overline{C}_{2gp^i} - \overline{C}_{2\mu gp^i}\} + G_{i+j}\{-\overline{C}_{4gp^i} + \overline{C}_{4\lambda gp^i}\} + H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + A_{i+j}\{-\overline{C}_{\lambda gp^i} + \overline{C}_{\xi gp^i}\} + F_{i+j}\{\overline{C}_{2\lambda gp^i} - \overline{C}_{2\nu gp^i}\} + E_{i+j}\{-\overline{C}_{\mu gp^i} + \overline{C}_{\rho gp^i}\} + D_{i+j}\{\overline{C}_{\nu gp^i} - \overline{C}_{\chi gp^i}\}\}\}]$.

If $\Omega_{\rho p^i} = -\Omega_{\xi p^i}$, *then* $P_{\rho p^j}(x)$ *can be obtained by replacing the sign of the term* $+$ *to* $-$ *and* $-$ *to* $+$ *when* $\overline{C}_{tp^i}$ *has odd suffix.*

If $\Omega_{\chi p^i} = -\Omega_{p^i}$,

$P_{\chi p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{4}\{\{\sum_{t\in\{0,1,4,5\}}(\alpha^{2tp^{n+j}} + \alpha^{6tp^{n+j}})\overline{C}_{2tp^n}\} + \sum_{i=n-j}^{n-1}\{\sum_{k=0,1}\{(\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})$ $\overline{C}_{2g^k p^i} - 2\overline{C}_{8g^k p^i} + 2\overline{C}_{16g^k p^i} + (\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})\overline{C}_{2\lambda g^k p^i} - (\alpha^{2g^k p^{i+j}} + \alpha^{6g^k p^{i+j}})\overline{C}_{2\mu g^k p^i} - (\alpha^{2\lambda g^k p^{i+j}} + \alpha^{6\lambda g^k p^{i+j}})\overline{C}_{2\nu g^k p^i}\}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{N_{i+j}\{\overline{C}_{p^i} - \overline{C}_{\eta p^i}\} + O_{i+j}\{\overline{C}_{2p^i} - \overline{C}_{2\mu p^i}\} + P_{i+j}\{\overline{C}_{4p^i} - \overline{C}_{4\lambda p^i}\} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + J_{i+j}\{\overline{C}_{\lambda p^i} - \overline{C}_{\xi p^i}\} + K_{i+j}\{\overline{C}_{2\lambda p^i} - \overline{C}_{2\nu p^i}\} + L_{i+j}\{\overline{C}_{\mu p^i} - \overline{C}_{\rho p^i}\} + M_{i+j}\{\overline{C}_{\nu p^i} - \overline{C}_{\chi p^i}\} + A_{i+j}\{\overline{C}_{gp^i} - \overline{C}_{\eta gp^i}\} + F_{i+j}\{\overline{C}_{2gp^i} - \overline{C}_{2\mu gp^i}\} + G_{i+j}\{\overline{C}_{4gp^i} - \overline{C}_{4\lambda gp^i}\} + H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + B_{i+j}\{\overline{C}_{\lambda gp^i} - \overline{C}_{\xi gp^i}\} + C_{i+j}\{\overline{C}_{2\lambda gp^i} - \overline{C}_{2\nu gp^i}\} + D_{i+j}\{\overline{C}_{\mu gp^i} - \overline{C}_{\rho gp^i}\} + E_{i+j}\{\overline{C}_{\nu gp^i} - \overline{C}_{\chi gp^i}\}\}]$.

If $\Omega_{\chi p^i} = -\Omega_{\eta p^i}$, *then* $P_{\chi p^j}(x)$ *can be obtained by replacing the sign of the term* $+$ *to* $-$ *and* $-$ *to* $+$ *when* $\overline{C}_{tp^n}$ *has odd suffix.*

The expressions for P_{tgp^j}, *where* $t \in A_1$ *can be obtained by interchanging* $\alpha^{up^{i+j}}$, P_{i+j}, Q_{i+j}, R_{i+j}, K_{i+j}, O_{i+j}, L_{i+j}, M_{i+j}, N_{i+j}, J_{i+j} *by* $\alpha^{ugp^{i+j}}$, G_{i+j}, H_{i+j}, I_{i+j}, C_{i+j}, F_{i+j}, D_{i+j}, E_{i+j}, A_{i+j}, B_{i+j} *respectively and vice versa, also change the sign* $+$ *to* $-$ *and vice versa of* $\overline{C}_{tgp^i}$ *when* $\overline{C}_{tgp^i}$ *has odd suffix only,*
where A_{i+j}, B_{i+j}, D_{i+j}, E_{i+j}, J_{i+j}, L_{i+j}, M_{i+j}, N_{i+j} *can be obtained, using lemma* 29 *and the following relations,*

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{M_{i+j}N_{i+j} + A_{i+j}E_{i+j} + J_{i+j}L_{i+j} + B_{i+j}D_{i+j}}{p^i}\} = -2p^n,$$

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{M^2{}_{i+j} + E^2{}_{i+j} + L^2{}_{i+j} + D^2{}_{i+j} + J^2{}_{i+j} + B^2{}_{i+j} + N^2{}_{i+j} + A^2{}_{i+j}}{p^i}\} = 0,$$

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{M_{i+j}B_{i+j} + N_{i+j}D_{i+j} - L_{i+j}A_{i+j} - J_{i+j}E_{i+j}}{p^i}\} = 0,$$

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{M_{i+j}D_{i+j} + B_{i+j}N_{i+j} - E_{i+j}L_{i+j} - J_{i+j}A_{i+j}}{p^i}\} = 0.$$

Proof. These expressions can be obtained using lemmas 4.3, $11-13$, 22, $25-26$ and $30-33$ with same procedure like to theorem 5. Also the relations can be derived using $P_{p^j}(\alpha^{p^j}) = 1$, $P_{p^j}(\alpha^{\nu p^j}) = 0$, $P_{p^j}(\alpha^{\lambda gp^j}) = 0$ and $P_{p^j}(\alpha^{\mu gp^j}) = 0$. □

Theorem 12. *For* $p \equiv 3(mod8)$, *the expressions for the primitive idempotents corresponding to* $\Omega_{g^k p^i}$, $\Omega_{\mu g^k p^i}$ *are given by:*

$P_{p^j}(x) = \frac{1}{16p^n}[\phi(p^{n-j})\{\sum_{t\in\{0,1,4,5\}}(\alpha^{2tgp^{n+j}} + \alpha^{6tgp^{n+j}})\overline{C}_{2tp^n} + \sum_{i=n-j}^{n-1}\{(\alpha^{2gp^{i+j}} + \alpha^{6gp^{i+j}})\{\overline{C}_{2p^i} - \overline{C}_{2\mu p^i}\} - 2\overline{C}_{8p^i} + 2\overline{C}_{16p^i} + (\alpha^{2p^{i+j}} + \alpha^{6p^{i+j}})\{\overline{C}_{2gp^i} - \overline{C}_{2\mu gp^i}\} - 2\overline{C}_{8gp^i} + 2\overline{C}_{16gp^i}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{C}_{p^i} + 2F_{i+j}\overline{C}_{2p^i} + 2G_{i+j}\overline{C}_{4p^i} + 4H_{i+j}\overline{C}_{8p^i} + 4I_{i+j}\overline{C}_{16p^i} + D_{i+j}\overline{C}_{\mu p^i} - 2F_{i+j}\overline{C}_{2\mu p^i} + N_{i+j}\overline{C}_{gp^i} + 2O_{i+j}\overline{C}_{2gp^i} + 2P_{i+j}\overline{C}_{4gp^i} + 4Q_{i+j}\overline{C}_{8gp^i} + 4R_{i+j}\overline{C}_{16gp^i} + L_{i+j}\overline{C}_{\mu gp^i} - 2O_{i+j}\overline{C}_{2\mu gp^i}\}]$

$P_{\mu p^j}(x) = \frac{1}{16p^n}[\phi(p^{n-j})\{\sum_{t\in\{0,1,4,5\}}(-1)^t(\alpha^{2tgp^{n+j}} + \alpha^{6tgp^{n+j}})\overline{C}_{2tp^n} + \sum_{i=n-j}^{n-1}\{(\alpha^{2gp^{i+j}} + \alpha^{6gp^{i+j}})\{-\overline{C}_{2p^i} + \overline{C}_{2\mu p^i}\} - 2\overline{C}_{8p^i} + 2\overline{C}_{16p^i} + (\alpha^{2p^{i+j}} + \alpha^{6p^{i+j}})\{-\overline{C}_{2gp^i} + \overline{C}_{2\mu gp^i}\} - 2\overline{C}_{8gp^i} + 2\overline{C}_{16gp^i}\}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{D_{i+j}\overline{C}_{p^i} - 2F_{i+j}\overline{C}_{2p^i} + 2G_{i+j}\overline{C}_{4p^i} + 4H_{i+j}\overline{C}_{8p^i} + 4I_{i+j}\overline{C}_{16p^i} - A_{i+j}\overline{C}_{\mu p^i} + 2F_{i+j}\overline{C}_{2\mu p^i} + L_{i+j}\overline{C}_{gp^i} - 2O_{i+j}\overline{C}_{2gp^i} + 2P_{i+j}\overline{C}_{4gp^i} + 4Q_{i+j}\overline{C}_{8gp^i} + 4R_{i+j}\overline{C}_{16gp^i} - N_{i+j}\overline{C}_{\mu gp^i} + 2O_{i+j}\overline{C}_{2\mu gp^i}\}]$.

The expressions for P_{gp^j} *and* $P_{\mu gp^j}$ *can be obtained by replacing* P_{i+j}, Q_{i+j}, R_{i+j}, O_{i+j}, L_{i+j}, N_{i+j} *by* G_{i+j}, H_{i+j}, I_{i+j}, F_{i+j}, D_{i+j}, A_{i+j} *and* $\alpha^{up^{i+j}}$ *by* $\alpha^{ugp^{i+j}}$ *respectively and vice versa, where* A_{i+j}, D_{i+j}, N_{i+j}, L_{i+j} *can be obtained using lemma* 29 *and the following relations,*

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{A^2{}_{i+j} + N^2{}_{i+j} + L^2{}_{i+j} + D^2{}_{i+j}}{p^i}\} = 0, \quad \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{N_{i+j}A_{i+j} + D_{i+j}L_{i+j}}{p^i}\} = 0.$$

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Proof. These expressions can be obtained using lemmas 4.4, 14 − 21, 23 − 24, 27 − 28 and 34 − 35 with same procedure like to theorem 5. Also the relations can be derived using $P_{pj}(\alpha^{p^j}) = 1$ and $P_{pj}(\alpha^{gp^j}) = 0$. □

8. Dimension and Generating Polynomials

The polynomial $m_s(x) = \prod_{s\epsilon\Omega_s}(x - \alpha^s)$ denote the minimal polynomial for α^s and so the generating polynomial for cyclic code M_s of length $16p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{16p^n}-1}{m_s(x)}$ and the dimension of M_s is equal to the cardinality of the class Ω_s [5].

Theorem 13. *(i) The generating polynomial for the codes M_{tp^n}, for $t = \{0,1,2,4,5,8,10\}$ are $(1 + x + x^2 + ... + x^{(16p^n-1)}$, $(x^8 - 1)(x^2 + \beta^6)(x^2 + \beta^2)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x + \beta^2)(x + \beta^6)(x^4 - 1)(x^8 + 1)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x^2 - 1)(x^4 + 1)(x^8 + 1)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x^8 - 1)(x^2 - \beta^6)(x^2 - \beta^2)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x^8 + 1)(x^4 + 1)(x^2 + 1)(x - 1)(1 + x^{16} + ... + x^{16(p^n-1)})$, and $(x^8 + 1)(x^4 - 1)(x - \beta^2)(x - \beta^6)(1 + x^{16} + ... + x^{16(p^n-1)})$ respectively, where β is 16th root of unity.*

(ii) The generating polynomial for $M_{8p^i} \oplus M_{8gp^i}$, $M_{16p^i} \oplus M_{16gp^i}$ and $M_{p^i} \oplus M_{2p^i} \oplus M_{4p^i} \oplus M_{\lambda p^i} \oplus M_{2\lambda p^i} \oplus M_{4\lambda p^i} \oplus M_{\mu p^i} \oplus M_{2\mu p^i} \oplus M_{\nu p^i} \oplus M_{2\nu p^i} \oplus M_{\eta p^i} \oplus M_{\xi p^i} \oplus M_{\rho p^i} \oplus M_{\chi p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{4gp^i} \oplus M_{\lambda gp^i} \oplus M_{2\lambda gp^i} \oplus M_{4\lambda gp^i} \oplus M_{\mu gp^i} \oplus M_{2\mu gp^i} \oplus M_{\nu gp^i} \oplus M_{2\nu gp^i} \oplus M_{\eta gp^i} \oplus M_{\xi gp^i} \oplus M_{\rho gp^i} \oplus M_{\chi gp^i}$ are $(x^{p^{n-i-1}} + 1)(x^{p^{n-i}} - 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1)(1 + x^{16p^{n-i}} + ... + x^{16p^{n-i}(p^i-1)})$, $(x^{p^{n-i-1}} - 1)(x^{p^{n-i}} + 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1)(1 + x^{16p^{n-i}} + ... + x^{16p^{n-i}(p^i-1)})$ and $(x^{2p^{n-i-1}} + 1)(x^{4p^{n-i-1}} + 1)(x^{8p^{n-i-1}} + 1)(x^{2p^{n-i}} - 1)(1 + x^{16p^{n-i}} + ... + x^{16p^{n-i}(p^i-1)})$ respectively.

Proof. (i) The minimal polynomial for α^{tp^n}, for $t = \{0,1,2,4,5,8,10\}$ are $(x - 1)$, $(x - \alpha^{p^n})(x - \alpha^{p^n q})(x - \alpha^{p^n q^2})(x - \alpha^{p^n q^3}) = (x - \beta)(x - \beta^q)(x - \beta^{q^2})(x - \beta^{q^3}) = (x - \beta)(x - \beta^3)(x + \beta)(x + \beta^3) = (x^2 - \beta^2)(x^2 - \beta^6)$, $(x - \beta^2)(x - \beta^6)$, $(x^2 + 1)$, $(x^2 + \beta^2)(x^2 + \beta^6)$, $(x + 1)$ and $(x + \beta^2)(x + \beta^6)$ respectively. The corresponding generating polynomials are $(1 + x + x^2 + ... + x^(16p^n - 1))$, $(x^8 - 1)(x^2 + \beta^6)(x^2 + \beta^2)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x + \beta^2)(x + \beta^6)(x^4 - 1)(x^8 + 1)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x^2 - 1)(x^4 + 1)(x^8 + 1)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x^8 - 1)(x^2 - \beta^6)(x^2 - \beta^2)(1 + x^{16} + ... + x^{16(p^n-1)})$, $(x^8 + 1)(x^4 + 1)(x^2 + 1)(x - 1)(1 + x^{16} + ... + x^{16(p^n-1)})$ and $(x^8 + 1)(x^4 - 1)(x - \beta^2)(x - \beta^6)(1 + x^{16} + ... + x^{16(p^n-1)})$.

(ii) The product of minimal polynomial satisfied by α^{8p^i} and α^{8gp^i} is $(\frac{x^{p^{n-i}}+1}{x^{p^{n-i-1}}+1})$. Therefore, the generating polynomial for $M_{8p^i} \oplus M_{8gp^i}$ is $(x^{p^{n-i-1}} + 1)(x^{p^{n-i}} - 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1)(1 + x^{16p^{n-i}} + ... + x^{16p^{n-i}(p^i-1)})$. The product of minimal polynomial satisfied by α^{16p^i} and α^{16gp^i} is $(\frac{x^{p^{n-i}}-1}{x^{p^{n-i-1}}-1})$.

Therefore, the generating polynomial for $M_{16p^i} \oplus M_{16gp^i}$ is $(x^{p^{n-i-1}} - 1)(x^{p^{n-i}} + 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1)(1 + x^{16p^{n-i}} + ... + x^{16p^{n-i}(p^i-1)})$. Also the product of minimal polynomial satisfied by $\alpha^{p^i}, \alpha^{2p^i}, \alpha^{4p^i}, ...\alpha^{\rho gp^i}, \alpha^{\chi gp^i}$ is $\frac{(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)}{(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)}$. Therefore, the generating polynomial for $M_{p^i} \oplus M_{2p^i} \oplus M_{4p^i} \oplus M_{\lambda p^i} \oplus M_{2\lambda p^i} \oplus M_{4\lambda p^i} \oplus M_{\mu p^i} \oplus M_{2\mu p^i} \oplus M_{\nu p^i} \oplus M_{2\nu p^i} \oplus M_{\eta p^i} \oplus M_{\xi p^i} \oplus M_{\rho p^i} \oplus M_{\chi p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{4gp^i} \oplus M_{\lambda gp^i} \oplus M_{2\lambda gp^i} \oplus M_{4\lambda gp^i} \oplus M_{\mu gp^i} \oplus M_{2\mu gp^i} \oplus M_{\nu gp^i} \oplus M_{2\nu gp^i} \oplus M_{\eta gp^i} \oplus M_{\xi gp^i} \oplus M_{\rho gp^i} \oplus M_{\chi gp^i}$ is $(x^{2p^{n-i-1}} + 1)(x^{4p^{n-i-1}} + 1)(x^{8p^{n-i-1}} + 1)(x^{2p^{n-i}} - 1)(1 + x^{16p^{n-i}} + ... + x^{16p^{n-i}(p^i-1)})$. □

9. Minimum Distance

If l is a cyclic code of length m generated by $g(x)$ and its minimum distance is d, then the code $\bar{l}$ of length mk generated by $g(x)(1 + x^m + x^{2m} + ... + x^{(k-1)m})$ is a repetition code of l repeated k times and its minimum distance is dk [3]. Here, we find the minimum distance of the minimal cyclic code M_s of length $16p^n$, generated by the primitive idempotent P_s.

Theorem 14. *Each of the codes M_{tp^n}, (i) for $t = 0,2,8,10$, are of minimum distance $16p^n$*
(ii) for $t = 1,4,5$, are of minimum distance $8p^n$

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(iii) For $0 \le i \le n-1$, the minimum distance of the cyclic codes $M_{ag^k p^i}$ for $a = 8, 16$, are greater than or equal $32p^i$ and for $a \in A - \{8, 16\}$ are greater than or equal to $16p^i$.

Proof. (i) Since the generating polynomial for the code M_0 is $(1 + x + x^2 + ... + x^{16p^n-1})$ which is itself a polynomial of length $16p^n$, hence its minimum distance is $16p^n$.
The generating polynomial for the code M_{2p^n} is $(x+\beta^2)(x+\beta^6)(x^4-1)(x^8+1)(1+x^{16}+...+x^{16(p^n-1)})$. If we take a cyclic code of length 16 generated by the polynomial $(x+\beta^2)(x+\beta^6)(x^4-1)(x^8+1)$ then the minimum distance of this code is 16. Since the cyclic code of length $16p^n$ with generating polynomial $(x+\beta^2)(x+\beta^6)(x^4-1)(x^8+1)(1+x^{16}+...+x^{16(p^n-1)})$ is a repetition of the cyclic code of length 16 with generating polynomial $(x+\beta^2)(x+\beta^6)(x^4-1)(x^8+1)$ repeated p^n times, therefore its minimum distance is $16p^n$.
Similarly, the minimum distance of each of the cyclic codes M_{tp^n}, where $t = 8, 10$, can be obtained.
(ii) The generating polynomial for the cyclic code M_{p^n} is $(x^8-1)(x^2+\beta^6)(x^2+\beta^2)(1+x^{16}+...+x^{16(p^n-1)})$, which is a repetition code of the cyclic code of length 8 with generating polynomial $(x^8-1)(x^2+\beta^6)(x^2+\beta^2)$, repeated p^n times. Therefore its minimum distance is $8p^n$. Similarly, the minimum distance of each of the cyclic codes M_{tp^n}, where $t = 4, 5$, can be obtained.
(iii) Since the product of generating polynomials of the cyclic codes M_{8p^i} and M_{8gp^i} is $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$. Therefore, if we take a cyclic code C of length $16p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)$ then the minimum distance of this code is 32.Since the cyclic code C_1 of length $16p^n$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$ is a repitition of the code C repeated p^n times. Hence its minimum distance is $32p^i$.
The codes corresponding to M_{8p^i} and M_{8gp^i} are the sub codes of the above code, so their minimum distance is greater than or equal to $32p^i$.
Similarly, the minimum distance of the cyclic code M_{16p^i} and M_{16gp^i} of length $16p^n$ are also greater than or equal to $32p^i$.
The product of generating polynomial for the cyclic codes $M_{ap^i}, M_{agp^i}, a \in A - \{8, 16\}$ is $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)(x^{2p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$. Therefore if we take a code C of length $16p^{n-i}$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)(x^{2p^{n-i}}+1)$, then the minimum distance of this code is 16. Since the cyclic code of length $16p^n$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)(x^{2p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$ is a repitition of the code C repeated p^i times. Hence its minimum distance is $16p^i$.
Since the codes corresponding to the cyclic codes $M_{ap^i}, M_{agp^i}, a \in A - \{8, 16\}$ are the sub codes of the above code, so this minimum distance is greater than or equal to $16p^i$. □

10. Example

Example 10.1. Cyclic codes of length 48.
Take $p = 3$, $n = 1$,$16p^n = 48$, $q = 67$. Then the q-cyclotomic cosets are
$\Omega_0 = \{0\}$, $\Omega_1 = \{1, 19, 25, 43\}$, $\Omega_2 = \{2, 38\}$, $\Omega_3 = \{3, 9, 27, 33\}$, $\Omega_4 = \{4, 28\}$, $\Omega_5 = \{5, 23, 29, 47\}$, $\Omega_6 = \{6, 18\}$, $\Omega_7 = \{7, 13, 31, 37\}$, $\Omega_8 = \{8\}$, $\Omega_{10} = \{10, 46\}$, $\Omega_{11} = \{11, 17, 35, 41\}$, $\Omega_{12} = \{12, 36\}$, $\Omega_{14} = \{14, 26\}$, $\Omega_{15} = \{15, 21, 39, 45\}$, $\Omega_{16} = \{16\}$, $\Omega_{20} = \{20, 44\}$, $\Omega_{22} = \{22, 34\}$, $\Omega_{24} = \{24\}$, $\Omega_{30} = \{30, 42\}$, $\Omega_{32} = \{32\}$, $\Omega_{40} = \{40\}$.
Minimal polynomials for α^t, where $t \in C = \{0, ..., 8, 10, 11, 12, 14, 15, 16, 20, 22, 24, 30, 32, 40\}$ are $x-1$, x^4-3x^2-29, $x^2-3x-29$, x^4-20x^2-1, x^2-30, x^4-23x^2+30, $x^2-20x-1$, x^4+3x^2-29, $x-30$, $x^2-23x+30$, x^4+23x^2+30, x^2+1, $x^2+3x-29$, x^4+20x^2-1, $x-29$, x^2+29, $x^2+23x+30$, $x+1$, $x^2+20x-1$, $x+30$ and $x+29$ respectively.
The minimal codes M_t, where $t \in C$ of length 48 are as follows:

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Code	Dim.	Min. Bound	Distance	Generating Polynomial
M_{8a}	1	48		$\sum_{t=0}^{47}\{38^{a(t+1)}x^t\}(mod67)$, where $a=0,\ 3$
M_{8b}	1	$32 \leq d \leq 48$		$\sum_{t=0}^{47}\{38^{b(t+1)}x^t\}(mod67)$, where $b=1,2,4,5$
M_1	4	$16 \leq d \leq 48$		$37+47x^2+38x^4+66x^8+44x^{10}+37x^{12}+29x^{16}+64x^{18}+66x^{20}+30x^{24}+20x^{26}+29x^{28}+x^{32}+23x^{34}+30x^{36}+38x^{40}+3x^{42}+x^{44}$
M_2	2	$16 \leq d \leq 48$		$37+47x+38x^2+66x^4+44x^5+37x^6+29x^8+64x^9+66x^{10}+30x^{12}+20x^{13}+29x^{14}+x^{16}+23x^{17}+30x^{18}+38x^{20}+3x^{21}+x^{22}+37x^{24}+47x^{25}+38x^{26}+66x^{28}+44x^{29}+37x^{30}+29x^{32}+64x^{33}+66x^{34}+30x^{36}+20x^{37}+29x^{38}+x^{40}+23x^{41}+30x^{42}+38x^{44}+3x^{45}+x^{46}$
M_3	4	24		$1+47x^2+66x^4+66x^8+20x^{10}+x^{12}+x^{16}+47x^{18}+66x^{20}+66x^{24}+20x^{26}+x^{28}+x^{32}+47x^{34}+66x^{36}+66x^{40}+20x^{42}+x^{44}$
M_4	2	$16 \leq d \leq 48$		$\sum_{t=0}^{23}\{66^{t+1}x^{2t}\}(mod67)$
M_5	4	$16 \leq d \leq 48$		$29+20x^2+30x^4+66x^8+64x^{10}+29x^{12}+37x^{16}+44x^{18}+66x^{20}+38x^{24}+47x^{26}+37x^{28}+x^{32}+3x^{34}+38x^{36}+30x^{40}+23x^{42}+x^{44}$
M_6	2	48		$1+47x+66x^2+66x^4+20x^5+x^6+x^8+47x^9+66x^{10}+66x^{12}+20x^{13}+x^{14}+x^{16}+47x^{17}+66x^{18}+66x^{20}+20x^{21}+x^{22}+x^{24}+47x^{25}+66x^{26}+66x^{28}+20x^{29}+x^{30}+x^{32}+47x^{33}+66x^{34}+66x^{36}+20x^{37}+x^{38}+x^{40}+47x^{41}+66x^{42}+66x^{44}+20x^{45}+x^{46}$
M_7	4	$16 \leq d \leq 48$		$37+20x^2+38x^4+66x^8+23x^{10}+37x^{12}+29x^{16}+3x^{18}+66x^{20}+30x^{24}+47x^{26}+29x^{28}+x^{32}+44x^{34}+30x^{36}+38x^{40}+64x^{42}+x^{44}$

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Code	Dim.	Min. Distance Bound	Generating Polynomial
M_{10}	2	$16 \leq d \leq 48$	$29+20x+30x^2+66x^4+64x^5+29x^6+37x^8+44x^9+66x^{10}+38x^{12}+47x^{13}+37x^{14}+x^{16}+3x^{17}+38x^{18}+30x^{20}+23x^{21}+x^{22}+29x^{24}+20x^{25}+30x^{26}+66x^{28}+64x^{29}+29x^{30}+37x^{32}+44x^{33}+66x^{34}+38x^{36}+47x^{37}+37x^{38}+x^{40}+3x^{41}+38x^{42}+30x^{44}+23x^{45}+x^{46}$
M_{11}	4	$16 \leq d \leq 48$	$29+47x^2+30x^4+66x^8+3x^{10}+29x^{12}+37x^{16}+23x^{18}+66x^{20}+38x^{24}+20x^{26}+37x^{28}+x^{32}+64x^{34}+38x^{36}+30x^{40}+44x^{42}+x^{44}$
M_{12}	2	24	$\sum_{t=0}^{23}\{66^{t+1}x^{2t}\}(mod 67)$
M_{14}	2	$16 \leq d \leq 48$	$37+20x+38x^2+66x^4+23x^5+37x^6+29x^8+3x^9+66x^{10}+30x^{12}+47x^{13}+29x^{14}+x^{16}+44x^{17}+30x^{18}+38x^{20}+64x^{21}+x^{22}+37x^{24}+20x^{25}+38x^{26}+66x^{28}+23x^{29}+37x^{30}+29x^{32}+3x^{33}+66x^{34}+30x^{36}+47x^{37}+29x^{38}+x^{40}+44x^{41}+30x^{42}+38x^{44}+64x^{45}+x^{46}$
M_{15}	4	24	$1+20x^2+66x^4+66x^8+47x^{10}+x^{12}+x^{16}+20x^{18}+66x^{20}+66x^{24}+47x^{26}+x^{28}+x^{32}+20x^{34}+66x^{36}+66x^{40}+47x^{42}+x^{44}$
M_{20}	2	$16 \leq d \leq 48$	$\sum_{t=0}^{23}\{30^{t+1}x^{2t}\}(mod 67)$
M_{22}	2	$16 \leq d \leq 48$	$29+47x+30x^2+66x^4+3x^5+29x^6+37x^8+23x^9+66x^{10}+38x^{12}+20x^{13}+37x^{14}+x^{16}+64x^{17}+38x^{18}+30x^{20}+44x^{21}+x^{22}+29x^{24}+47x^{25}+30x^{26}+66x^{28}+3x^{29}+29x^{30}+37x^{32}+23x^{33}+66x^{34}+38x^{36}+20x^{37}+37x^{38}+x^{40}+64x^{41}+38x^{42}+30x^{44}+44x^{45}+x^{46}$
M_{30}	2	48	$1+20x+66x^2+66x^4+47x^5+x^6+x^8+20x^9+66x^{10}+66x^{12}+47x^{13}+x^{14}+x^{16}+20x^{17}+66x^{18}+66x^{20}+47x^{21}+x^{22}+x^{24}+20x^{25}+66x^{26}+66x^{28}+47x^{29}+x^{30}+x^{32}+20x^{33}+66x^{34}+66x^{36}+47x^{37}+x^{38}+x^{40}+20x^{41}+66x^{42}+66x^{44}+47x^{45}+x^{46}$

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