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An Inductive Proof of Stafford's Theorem

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Abstract: Let K be an algebraically closed field of characteristic zero. The generalized Weyl algebra $A_{n,f}$ is defined by generators $x_1, x_2, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$ subject to certain commutation relations and additional structure determined by a collection of functions $f = (f_1, \ldots, f_n)$. We focus on the structure of left and right ideals in $A_{n,f}(K)$, particularly proving that every left or right ideal can be generated by two elements. The proof is based on showing that if a left ideal can be generated by three elements, it can be reduced to two elements by applying the Noetherian property of the ring and an iterative reduction process. This result complements the simplicity of $A_{n,f}(K)$, as established in prior work.

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1. Introduction

In [2], Stafford proved that every left or right ideal of the Weyl algebra $A_n(K) = K[x_1, x_2, \ldots, x_n]\langle\partial_1, \partial_2, \ldots, \partial_n\rangle$, where K is a field of characteristic zero, is generated by two elements. This result was extended in [3] by Caro and Levcovitz, who demonstrated the same property for the ring of differential operators $D_n = K[[x_1, x_2, \ldots, x_n]]\langle\partial_1, \partial_2, \ldots, \partial_n\rangle$, which is defined over the ring of formal power series $K[[x_1, \ldots, x_n]]$.

The concept of generalized Weyl algebras was introduced by Bavula in [5] and has since become a central topic in the study of noncommutative algebra. Generalized Weyl algebras have been extensively studied by various authors, both prior to and following Bavula's definition. Noteworthy contributions to the theory include the works of Hodges [6,10], Jordan [11], Joseph [4], Rosenberg [12], Smith [7], and Stafford [13]. Several important families of algebras, including the Weyl algebra and primitive quotients of $U(sl_2)$, can be constructed as generalized Weyl algebras.

In this paper, we prove the result that every left or right ideal of a generalized Weyl algebra is generated by two elements. This result aligns with the conjecture that in any (noncommutative) Noetherian simple ring, every left or right ideal is generated by two elements. To prove this, we follow the method of Stafford's theorem as presented in Bjork's book [1], making the necessary modifications to address the specific structure of generalized Weyl algebras.

Let K be an algebraically closed field with characteristic zero. The generalized Weyl algebra $A_{n,f}$ of degree n is an algebra generated by $x_1, x_2, ..., x_n, y_1, ..., y_n, z_1, ..., z_n$ with identity subject to the following relations

$$[x_i, x_j] = [y_i, y_j] = [z_i, z_j] = [x_i, y_j] = [x_i, z_j] = [y_i, z_j] = 0$$

for all $i \neq j$,

$$[x_i, y_i] = y_i, \quad [x_i, z_i] = -z_i, \quad y_i z_i = f_i(x_i), \quad z_i y_i = f_i(x_i + 1)$$

for all $i = 1, 2, ..., n$ and $f = (f_1, f_2, ..., f_n)$.

A polynomial f is said to be separating if for any pair of roots α, β of f, $\alpha - \beta \notin \mathbb{Z}$.

If each f_i is separating then, $A_{n,f}(K)$ is simple.

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2. Statements and Structure of Proof

Theorem 1. *Every left(right) ideal in $A_{n,f}$ can be generated by two elements.*

Our first key observation is that it is sufficient to prove that if a left ideal L is generated by three elements, then it can be generated by two elements. To demonstrate this, suppose the claim holds and let L be a left ideal. Since $A_{n,f}$ is a left Noetherian ring, L has a finite set of generators $a_1, a_2, \ldots, a_s$. If $s \geq 3$, we can find elements b_1 and b_2 such that

$$A_{n,f}b_1 + A_{n,f}b_2 = A_{n,f}a_1 + A_{n,f}a_2 + A_{n,f}a_3,$$

and thus L is generated by the set $\{b_1, b_2, a_4, \ldots, a_s\}$. By repeatedly applying this reasoning, we can reduce the number of generators of L to two.

Theorem 2. *Let a,b,c be three elements in $A_{n,f}$. Then there exist d and e in $A_{n,f}$ such that $c \in A_{n,f}\,(a + dc) + A_{n,f}\,(b + ec)$.*

Theorem 2 implies that

$$A_{n,f}(a + dc) + A_{n,f}(b + ec) = A_{n,f}a + A_{n,f}b + A_{n,f}c.$$

We only need to demonstrate that the right-hand side is contained in the left-hand side, as the reverse inclusion is straightforward. Specifically, we need to show that a and b belong to the sum $A_{n,f}(a + dc) + A_{n,f}(b + ec)$. We have

$$a = (a + dc) - dc \in A_{n,f}(a + dc) + A_{n,f}(b + ec),$$

and similarly, we can show that $b \in A_{n,f}(a + dc) + A_{n,f}(b + ec)$. Therefore, the result follows.

If $1 \leq r \leq n - 1$, we obtain the subring

$$A_{r,f} = K\langle x_1, \ldots, x_r, y_1, \ldots, y_r, z_1, \ldots, z_r\rangle$$

of $A_{n,f}$.

Now, since $y_{r+1}, \ldots, y_n$ commute with the elements of $A_{r,f}$, we get the polynomial ring

$$A_{r,f}[y_{r+1}, \ldots, y_n].$$

If $r = 0$, we define $A_{0,f} = K$ and the polynomial subring $K[y_1, \ldots, y_n]$.

Proposition 1. *(r) Given $0 \leq r \leq n - 1$, there exists some $q_r \neq 0 \in A_{r,f}\,[y_{r+1}, ..., y_n]$ and d_r and e_r in $A_{n,f}$ such that $q_r c \in A_{n,f}\,(a + d_r c) + A_{n,f}\,(b + e_r c)$.*

In particular, when $r = 0$ we get,

Proposition 2. *There exists some $q_0 \neq 0 \in K[y_1, ..., y_n]$ and d_0 and e_0 in $A_{n,f}$ such that $q_0 c \in A_{n,f}\,(a + d_0 c) + A_{n,f}\,(b + e_0 c)$.*

Proposition 4 is essentially one part of the proof of Theorem 2. Starting from Proposition 4, we can begin reducing the number of y-variables in the polynomial q_0.

From this point onward, we focus our attention on the induction process as r moves from $n - 1$ to 0, which will ultimately prove Proposition 4.

The strategy in the following proofs is that each step is motivated by and derived from the subsequent one. Thus, we employ a backward argument until we reach Section 4.

Proposition 3. *Let $0 \leq r \leq n-1$ and let $q_{r+1} \in A_{r+1}[y_{r+2}, ..., y_n]$ where $q_{r+1} \neq 0$. If u and v are two elements in $A_{n,f}$ with $v \neq 0$, then there exists $g \in A_{n,f}$ and $q_r \in A_{r,f}[y_{r+1}, ..., y_n]$ such that $q_r \in A_{n,f}q_{r+1} + A_{n,f}(u + vg)$.*

Proof. We will use induction. If $r = n$, choose $d_n = e_n = 0$. Since $A_{n,f}$ is an Ore domain, we know that $A_{n,f}c \cap (A_{n,f}a + A_{n,f}b) \neq 0$. Therefore, there exists $t \neq 0 \in A_{n,f}c \cap (A_{n,f}a + A_{n,f}b)$. Thus, we have $t = q_n c \in A_{n,f}a + A_{n,f}b$.

We will apply Proposition 5 to prove that Proposition 3(r+1) implies Proposition 3(r). By assumption, there exists $q_{r+1} \in A_{r+1,f}[y_{r+2}, \ldots, y_n]$ such that $q_{r+1}c \in A_{n,f}(a + d_{r+1}c) + A_{n,f}(b + e_{r+1}c)$. Let $a_1 = a + d_{r+1}c$ and $b_1 = b + e_{r+1}c$.

If we can find $q_r \in A_{r,f}[y_{r+1}, \ldots, y_n]$ such that $q_r c \in A_{n,f}(a_1 + d_r c) + A_{n,f}(b_1 + e_r c)$, then Proposition 3(r) follows with d_r replaced by $d_r + d_{r+1}$ and e_r replaced by $e_r + e_{r+1}$. Therefore, without loss of generality, we assume that $a_1 = a$ and $b_1 = b$ from the start. Thus, we can write $q_{r+1}c = h_1 a + h_2 b$, and we assume both h_1 and h_2 are nonzero, using the fact that $A_{n,f}a \cap A_{n,f}b \neq 0$.

Similarly, since $A_{n,f}$ is a right Ore domain, we can find nonzero elements s and t such that $sq_{r+1}c = tb$.

Now, apply Proposition 5 to q_{r+1} with $u = sq_{r+1}$ and $v = tg_2$. This gives $g \in A_{n,f}$ and $q_r \neq 0 \in A_{r,f}[y_{r+1}, \ldots, y_n]$ such that

$$q_r \in A_{n,f}q_{r+1} + A_{n,f}(sq_{r+1} + tg_2 g),$$

which implies that

$$q_r c \in A_{n,f}q_{r+1}c + A_{n,f}(sq_{r+1}c + tg_2 gc).$$

Let us now consider the left ideal

$$L = A_{n,f}(a + g_1 gc) + A_{n,f}(b + g_2 gc).$$

If we can prove that $q_{r+1}c$ and $sq_{r+1}c + tg_2 gc$ belong to L, then it follows that $q_r c \in L$, which gives Proposition 3(r) with $d_r = g_1 g$ and $e_r = g_2 f$.

First, observe that

$$q_{r+1}c = h_1 a + h_2 b = h_1 a + h_2 b + (h_1 g_1 + h_2 g_2)gc,$$

where the additional term is zero because $h_1 g_1 + h_2 g_2 = 0$. Rewriting the expression, we get

$$q_{r+1}c = h_1(a + g_1 gc) + h_2(b + g_2 gc) \in L.$$

Second, we have

$$sq_{r+1}c + tg_2 gc = tb + tg_2 gc = t(b + g_2 gc) \in L.$$

Thus, the proof is complete.. □

3. Partial Quotient Rings of $A_{n,f}$

If $0 \leq r \leq n$, we define $\mathcal{D}_r$ as the quotient ring of $A_{r,f}$. Since the elements $y_{r+1}, \ldots, y_n$ commute with the elements in $\mathcal{D}_r$, we also obtain the division ring $\mathcal{D}_r(y_{r+1}, \ldots, y_n)$, which, by definition, is the quotient ring of $A_{r,f}[y_{r+1}, \ldots, y_n]$.

Next, we define $\mathcal{R} = \mathcal{D}_r(y_{r+1}, \ldots, y_n)\langle x_{r+1}, \ldots, x_n\rangle$. If $r = n$, then $\mathcal{R} = \mathcal{D}_n$. If $0 \leq r < n$, then $\mathcal{R}$ is referred to as a partial quotient ring of $A_{n,f}$. Furthermore, if $r \in \mathcal{R}$, we can express r as $r = q^{-1}a$, where $a \in A_{n,f}$ and $q \neq 0 \in A_{r,f}[y_{r+1}, \ldots, y_n]$.

Lemma 1. *Let $0 \neq q_{r+1} \in A_{r+1,f}[y_{r+2}, ..., y_n]$ and let u and v belong to $A_{n,f}$ with $v \neq 0$. Then there exists some $g \in A_{n,f}$ such that $\mathcal{R} = \mathcal{R}q_{r+1} + \mathcal{R}(u + vg)$.*

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It is straightforward to observe that Proposition 5 and Lemma 6 imply each other. Lemma 6 asserts that $1 = r_1 q_{r+1} + r_2(u + vg)$, where r_1 and r_2 are elements of $\mathcal{R}$. In the partial quotient ring $\mathcal{R}$, we can find some $q_r \neq 0 \in A_{r,f}[y_{r+1}, \dots, y_n]$ such that $r_j = q_r^{-1} a_j$, where a_1 and a_2 are elements of $A_{n,f}$. By the properties of an Ore domain, it follows that $q_r = a_1 q_{r+1} + a_2(u + vg) \in A_{n,f} q_{r+1} + A_{n,f}(u + vg)$.

On the other hand, since $q_r \in A_{n,f} q_{r+1} + A_{n,f}(u + vg)$, we have the following containment:

$$\mathcal{R} = \mathcal{R} q_r \subset \mathcal{R} q_{r+1} + \mathcal{R}(u + vg).$$

Thus, we obtain:

$$\mathcal{R} = \mathcal{R} q_{r+1} + \mathcal{R}(u + vg).$$

To prove Lemma 6, we will use the ring $S = \mathcal{D}_r(y_{r+1}, \dots, y_n)\langle x_{r+1}\rangle$, which appears as a subring of $\mathcal{R}$. First, we note that $A_{r+1,f}[y_{r+2}, \dots, y_n]$ is a subring of S, and that S contains $K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$.

Proposition 4. *Let $\delta_1, ..., \delta_m$ be some finite set of K-linearly independent elements in $K < x_{r+1}, y_{r+1}, z_{r+1} >$ and let $\rho \neq 0 \in A_{r+1,f}[y_{r+2}, ..., y_n]$. Let $S^{(m+1)} = S\varepsilon_0 + ... + S\varepsilon_m$ be a free S-module of rank $m + 1$ and let $S^{(m+1)}\rho$ denote the submodule generated by $\rho\varepsilon_0, ..., \rho\varepsilon_m$. Then there exists some $g \in K < x_{r+1}, y_{r+1}, z_{r+1} >$ such that $S^{(m+1)} = S^{(m+1)}\rho + S(\varepsilon_0 + \delta_1 g\varepsilon_1 + ... + \delta_m g\varepsilon_m)$.*

We'll postpone the proof of Proposition 7 for now.

Proposition 5. *Let $0 \leq r \leq n$ and $q_r \in A_{r,f}[y_{r+1}, ..., y_n]$ and let $\{a_1, ..., a_t\}$ be a finite set in $A_{n,f}$. Then there exists some $\rho \neq 0 \in A_{r,f}[y_{r+1}, ..., y_n]$ such that $\rho a_j \in A_{n,f} q_r$.*

Proof. For $r = n$, the result follows immediately since $A_{n,f}$ is an Ore domain. Suppose $0 \leq r \leq n - 1$ and $a \in A_{n,f}$. Then $a q_r^{-1} \in \mathcal{R}$ can be expressed in the form:

$$a q_r^{-1} = \sum_{\alpha,\beta} q_{\alpha\beta}^{-1} x^\alpha z^\beta$$

where $q_{\alpha\beta} \neq 0 \in A_{r,f}[y_{r+1}, \dots, y_n]$, $x^\alpha = x_{r+1}^{\alpha_1} \cdots x_n^{\alpha_{n-r}}$, and $z^\beta = z_{r+1}^{\beta_1} \cdots z_n^{\beta_{n-r}}$. Since $A_{r,f}[y_{r+1}, \dots, y_n]$ is an Ore domain, we can find a common nonzero multiple ρ' of $q_{\alpha\beta}$'s. Thus, we have:

$$\rho' a q_r^{-1} = \sum_{\alpha,\beta} \rho' q_{\alpha\beta}^{-1} x^\alpha z^\beta \in A_{n,f}.$$

Applying this for each a_j, we obtain a certain $\rho'_j \in A_{r,f}[y_{r+1}, \dots, y_n] \setminus \{0\}$ such that $\rho'_j \in A_{n,f}\, q_r$. Therefore, if ρ is a nontrivial left common multiple of $\rho'_1, \dots, \rho'_t$, we get $\rho a_j \in A_{n,f} q_r$.
□

Proof. Proof of Lemma 6. Let us remove $x_{r+1}, y_{r+1}, z_{r+1}$ from $A_{n,f}$ and consider the subring

$$K\langle x_1, \dots, x_r, x_{r+2}, \dots, x_n, y_1, \dots, y_r, y_{r+2}, \dots, y_n, z_2, \dots, z_r, z_{r+2}, \dots, z_n\rangle,$$

which we denote by $A(r+1)$. Now, $A_{n,f} \cong A(r+1) \otimes_K K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$.

Let q_{r+1}, u, v be the elements that occur in Lemma 6. We can express v as

$$v = \delta_1 G_1 + \cdots + \delta_m G_m,$$

where $\delta_1, \dots, \delta_m$ are K-linearly independent elements in $K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$, and $G_1, \dots, G_m$ belong to $A(r+1)$.

Now, since $A(r+1) \cong A_{n-1,f}$ is a simple ring, the two-sided ideal generated by $G_1, \dots, G_m$ is the entire ring $A(r+1)$. This gives finite sets $H_1, \dots, H_t$ and $Y_1, \dots, Y_t$ in $A(r+1)$ such that

$$1 = \sum_{v=1}^{t} \sum_{j=1}^{m} Y_v G_j H_v,$$

and thus,

$$A(r+1) = \sum_{v=1}^{t} \sum_{j=1}^{m} A(r+1) G_j H_v.$$

Identifying $A(r+1)$ with a subring of $\mathcal{R}$, we conclude that

$$\mathcal{R} = \sum_{v=1}^{t} \sum_{j=1}^{m} \mathcal{R} G_j H_v.$$

Sublemma 1: To each m-tuple $B_1, \dots, B_m$ in $A(r+1)$, there exists some $g \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \mathcal{R}B_1 + \dots + \mathcal{R}B_m = \mathcal{R}q_{r+1} + \mathcal{R}(u + \delta_1 g B_1 + \dots + \delta_m g B_m).$$

Proof: Since $q_{r+1} \neq 0 \in A_{r+1,f}[y_{r+2}, \dots, y_n]$, it follows from Proposition 8 that there exists some $\rho \neq 0 \in A_{r+1,f}[y_{r+2}, \dots, y_n]$ such that $\rho B_j \in A_{n,f} q_{r+1}$ for each $j = 1, 2, \dots, m$, and in addition $\rho u \in A_{n,f} q_{r+1}$.

Using Proposition 7, we get some $g \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$S^{(m+1)} = S^{(m+1)}\rho + S(\varepsilon_0 + \delta_1 g \varepsilon_1 + \dots + \delta_m g \varepsilon_m).$$

Since S is a subring of $\mathcal{R}$, it follows that

$$\mathcal{R}^{(m+1)} = \mathcal{R}^{(m+1)}\rho + \mathcal{R}(\varepsilon_0 + \delta_1 g \varepsilon_1 + \dots + \delta_m g \varepsilon_m).$$

Consider the $\mathcal{R}$-linear map π from $\mathcal{R}^{(m+1)}$ to $\mathcal{R}$ defined by $\pi(\varepsilon_0) = u$ and $\pi(\varepsilon_j) = B_j$ for each $1 \le j \le m$. Then, the image of π becomes $\mathcal{R}u + \mathcal{R}B_1 + \dots + \mathcal{R}B_m$.

Since ρu and ρB_j belong to $A_{n,f} q_{r+1}$, we see that $\pi(\rho \varepsilon_v) = \rho B_v$ for $0 \le v \le m$, so that $\pi(\mathcal{R}^{(m+1)}\rho) \subset \mathcal{R}q_{r+1}$. We conclude, using the above equality, that

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}B_j = \mathcal{R}q_{r+1} + \pi(\mathcal{R}^{(m+1)}) \subset \mathcal{R}q_{r+1} + \mathcal{R}\pi(\varepsilon_0 + \delta_1 g \varepsilon_1 + \dots + \delta_m g \varepsilon_m).$$

Thus, we obtain

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}B_j = \mathcal{R}q_{r+1} + \mathcal{R}(u + \delta_1 g B_1 + \dots + \delta_m g B_m).$$

The other direction is obvious, and hence

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}B_j = \mathcal{R}q_{r+1} + \mathcal{R}(u + \delta_1 g B_1 + \dots + \delta_m g B_m).$$

Proof continued: Let us apply Sublemma 1 with $B_j = G_j H_1$. This gives some $g_1 \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}G_j H_1 = \mathcal{R}q_{r+1} + \mathcal{R}\left(u + \sum_{j=1}^{m} \delta_j g_1 G_j H_1\right).$$

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Recall $v = \sum_{j=1}^{m} \delta_j G_j$, and since g_1 commutes with G_j in $A_{n,f}$ for all j, we get

$$\sum_{j=1}^{m} \delta_j g_1 G_j H_1 = \sum_{j=1}^{m} \delta_j G_j g_1 H_1 = v g_1 H_1,$$

and hence

$$\mathcal{R} q_{r+1} + \mathcal{R} u + \sum_{j=1}^{m} \mathcal{R} G_j H_1 = \mathcal{R} q_{r+1} + \mathcal{R}(u + v g_1 H_1).$$

Let us now apply Sublemma 1 with u replaced by $u + vg_1H_1$ and with $B_j = G_jH_2$. This gives $g_2 \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$\mathcal{R} q_{r+1} + \mathcal{R}(u + v g_1 H_1) + \sum_{j=1}^{m} \mathcal{R} G_j H_2 = \mathcal{R} q_{r+1} + \mathcal{R}(u + v g_1 H_1 + v g_2 H_2).$$

Using the previous equation, we can rewrite the left-hand side and conclude that

$$\mathcal{R} q_{r+1} + \mathcal{R} u + \sum_{j=1}^{m} \mathcal{R} G_j H_1 + \sum_{j=1}^{m} \mathcal{R} G_j H_2 = \mathcal{R} q_{r+1} + \mathcal{R}(u + v g_1 H_1 + v g_2 H_2).$$

In the next step, we apply Sublemma 1 with $B_j = G_jH_3$ and u replaced by $u + vg_1H_1 + vg_2H_2$, and so on. After t steps, we can conclude that

$$\mathcal{R} q_{r+1} + \mathcal{R}(u + v g_1 H_1 + \cdots + v g_t H_t) = \mathcal{R} q_{r+1} + \mathcal{R} u + \sum_{v=1}^{t} \sum_{j=1}^{m} \mathcal{R} G_j H_v = \mathcal{R},$$

where the last equality follows because $\mathcal{R} = \sum_{v=1}^{t} \sum_{j=1}^{m} \mathcal{R} G_j H_v$. Hence, Lemma 6 follows with $g = g_1H_1 + \cdots + g_tH_t$, that is,

$$\mathcal{R} = \mathcal{R} q_{r+1} + \mathcal{R}(u + vg).$$

□

Some results : $z^k x^m = (x+k)^m z^k$, $x^m z^k = z^k (x-k)^m$, $y^k x^m = (x-k)^m y^k$, $x^m y^k = y^k (x+k)^m$. $[x,y] = y$, $[x,z] = -z$, $yz = f(x)$, $zy = f(x+1)$.

4. The Ring S

$S = D(y) < x >$ where D is division ring . Each element in S can be written as $p_0 + p_1(y)x + ... + p_t(y)x^t$. The largest integer t such that $p_t \neq 0$ in $D(y)$ is called degree of the element.

Proposition 6. *The ring S is simple ring i.e. it has no non-trivial two-sided ideals.*

Proposition 7. *(Division Algorithm): Let α and $\beta \in S$ with $\alpha \neq 0$. Then there exists γ and $\theta \in S$ such that $\beta = \gamma\alpha + \theta$ where $\theta = 0$ or $deg(\theta) < deg(\alpha)$.*

Proof. It is straightforward. □

Proposition 8. *Let α be a non-zero element in S. Then the left S-module $S/S\alpha$ has finite length.*

Proof. Suppose that $\deg(\alpha) = t$. Given an element β in S, by Proposition 10 we can find elements γ in S and $p_0, \ldots, p_{t-1}$ in $D(y)$ such that

$$\beta = \gamma\alpha + p_0 + p_1 x + \cdots + p_{t-1} x^{t-1}.$$

Thus, the $D(y)$-vector space $S/S\alpha$ has no dimension greater than t. On the other hand, each S-submodule of $S/S\alpha$ induces a $D(y)$-subspace of $S/S\alpha$. Hence, length$(S/S\alpha) \leq t$. □

Let K be a subfield in S then $K < x, y, z >$ ia a subring in S.

Proposition 9. *Let $\delta_1, ..., \delta_m$ be K- linearly independent elements in $K < x, y, z >$ and $\alpha \neq 0 \in S$ and let $S^{(m)} = S\varepsilon_1 + ... + S\varepsilon_m$ be a free S-module of rank m. Let M be the S - submodule of $S^{(m)}$ which is generated by $\{\alpha\delta_1 g\varepsilon_1 + ... + \alpha\delta_m g\varepsilon_m : g \in K < x, y, z >\}$ then $M = S^{(m)}$.*

Proof. Let us first observe that both the assumption and the conclusion are unchanged if the m-tuple $\delta_1, \ldots, \delta_m$ is replaced by an m-tuple $\beta_1, \ldots, \beta_m$ where $\beta_j = \sum a_{jv}\delta_v$ and (a_{jv}) is an invertible matrix of size (m, m) with $a_{jv} \in K$. While we replace $\delta_1, \ldots, \delta_m$ by $\beta_1, \ldots, \beta_m$ under K-linear transformation (a_{jv}), we also replace the free generators $\varepsilon_1, \ldots, \varepsilon_m$ of $S^{(m)}$ by $\zeta_1, \ldots, \zeta_m$ where $\zeta_j = \sum b_{jv}\varepsilon_v$ and $(b_{jv}) = (a_{jv})^{-1}$.

By reordering the δ_j if required, we may assume that $\deg(\delta_1) \geq \deg(\delta_2) \geq \cdots \geq \deg(\delta_m)$. Hence, there exists a positive integer w and some $1 \leq q \leq m$ such that $w = \deg(\delta_1) = \cdots = \deg(\delta_q)$ while $\deg(\delta_j) < w$ if $j > q$. If $1 \leq j \leq q$, we can write $\delta_j = r_j + p_j(y)x^w$ where $\deg(r_j) < w$ and $p_j(y) \in K[y]$.

We can assume that the y-polynomials $p_1, \ldots, p_q$ have decreasing degrees. If $\deg(p_1) = \deg(p_2) = \mu$, we can choose $k \in K$ such that $p_2 - kp_1$ has degree $< \mu$ and replace δ_2 by $\delta_2 - k\delta_1$ while $\delta_1, \delta_3, \ldots, \delta_m$ are unchanged.

Armed with these normalizations, we begin to prove $\varepsilon_1 \in M$. First, let $\deg(\alpha) = t$. We can assume that $\alpha = \alpha_0 + x^t$ where $\deg(\alpha_0) < t$. If $1 \leq j \leq m$, $\alpha\delta_j = \psi_j + p_j(y)x^{w+t}$ where $\deg(\psi_j) < t + w$.

Using the relations $[x, y] = y$, $[x, z] = -z$, $yz = f(x)$, $zy = f(x+1)$, $x^2y - yx^2 = y(2x+1)$, $y(2x+1)y - y^2(2x+1) = 2!y^2$, $[z, y] = f(x+1) - f(x)$

$$= x - (x-1) = 1, \;\; zy^2 - y^2z = 2y, \;\; z(2y) - (2y)z = 2!,$$

and so on.

The $(t+w)$-fold commutator of $\alpha\delta_j$ and y will be $(t+w)!p_j(y)y^{t+w}$ for $1 \leq j \leq q$, while it is 0 for $q < j \leq m$. Since $\alpha\delta_1\varepsilon_1 + \cdots + \alpha\delta_m\varepsilon_m \in M$ and $[\alpha\delta_j, y] = \alpha\delta_j y - y(\alpha\delta_j)$, we have

$$[\alpha\delta_1, y]\varepsilon_1 + \cdots + [\alpha\delta_m, y]\varepsilon_m \in M.$$

Proceeding in this way, we get

$$(t+w)!p_1(y)y^{t+w}\varepsilon_1 + \cdots + (t+w)!p_q(y)y^{t+w}\varepsilon_q \in M.$$

After dividing by $(t+w)!$, we get

$$v_1 = p_1(y)y^{t+w}\varepsilon_1 + \cdots + p_q(y)y^{t+w}\varepsilon_q \in M.$$

Since $[\alpha\delta_j g, z] = \alpha\delta_j(gz) - z(\alpha\delta_j g)$ for all $g \in M$ and $v_1 \in M$, we have

$$[p_1 y^{(t+w)}, z]\varepsilon_1 + \cdots + [p_q y^{(t+w)}, z]\varepsilon_q \in M.$$

Suppose $\deg(p_1) = r$ and take the $(r+t+w)$-fold commutator with z; then we'll get

$$(-1)^{t+w+r}(r+t+w)!\mu^{(r+t+w)}\varepsilon_1 \in M,$$

where μ is the leading coefficient of p_1, and finally, we can divide by the coefficient of ε_1 to get $\varepsilon_1 \in M$.

Restricting attention to the $(m-1)$-tuple $\delta_1, \ldots, \delta_{m-1}$ and the S-module $S^{(m-1)} = S\varepsilon_2 + \cdots + S\varepsilon_m$, the Proposition follows by induction over m. □

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Proposition 10. *Let $\delta_1, ..., \delta_m$ be K - linearly independent elements in $K < x, y, z >$ and let M be submodule of $S^{(m)} = S\varepsilon_1 + ... + S\varepsilon_m$ such that the S - module $S^{(m)}/M$ has finite length. If $\alpha \neq 0 \in S$ we can find $g \in K < x, y, z >$ such that $S^{(m)} = M + S(\alpha\delta_1 g\varepsilon_1 + ... + \alpha\delta_m g\varepsilon_m)$.*

Proof. We proceed by induction on the length of the S-module $S^{(m)}/M$. If $M = S^{(m)}$, there is nothing to prove. Therefore, assume that the Proposition holds for any submodule M' with length$(S^{(m)}/M') <$ length$(S^{(m)}/M)$. By Proposition 12, there exists some $g_0 \in K\langle x, y, z\rangle$ such that

$$\alpha\delta_1 g_0\varepsilon_1 + \cdots + \alpha\delta_m g_0\varepsilon_m \notin M.$$

Consider an S-submodule M' such that $M \subset M' \subset M + S\left(\sum_{j=1}^m \alpha\delta_j g_0\varepsilon_j\right)$ with M'/M irreducible. The existence of M' is guaranteed by the fact that $S^{(m)}/M$ has finite length.

Additionally, since $S^{(m)}/M$ has finite length and the ring S is not left Artinian, we conclude that there exists some non-zero $\beta \in S$ such that $\beta\varepsilon_j \in M$ for each $1 \leq j \leq m$. In other words, the S-submodule $S^{(m)}\beta \subset M$. Since S is an Ore domain, we can choose $t \neq 0 \in S$ such that $t\alpha\delta_j g_0 \in S\beta$ for each $1 \leq j \leq m$.

Applying the induction hypothesis to M' and $\alpha' = t\alpha$, we obtain the existence of $g_1 \in K\langle x, y, z\rangle$ such that

$$S^{(m)} = M' + S\left(\sum_{j=1}^{m} t\alpha\delta_j g_1\varepsilon_j\right).$$

Now, define $N = M + S(\alpha\delta_j g_1\varepsilon_j)$. If $N = S^{(m)}$, the proof is complete, as g_1 provides the desired element. Otherwise, consider the submodule $N_0 = M + S\left(\sum_{j=1}^m t\alpha\delta_j g_1\varepsilon_j\right)$. We claim that $N_0 = N$. Clearly, $N_0 \subset N$ because

$$\sum t\alpha\delta_j g_1\varepsilon_j = t\sum \alpha\delta_j g_1\varepsilon_j \in N.$$

Since $S^{(m)} = M' + P$, where $P = S\left(\sum_{j=1}^m t\alpha\delta_j g_1\varepsilon_j\right)$, it follows that

$$S^{(m)}/N_0 = (M' + P)/(M + P) \cong M'/(M + M' \cap P),$$

and the module $S^{(m)}/N_0$ is either zero or irreducible because M'/M is irreducible. However, $N_0 \neq S^{(m)}$. Therefore, $S^{(m)}/N_0$ is irreducible. Hence, $N = N_0$.

Next, define $N_1 = M + S\left(\sum_{j=1}^m \alpha\delta_j(g_0 + g_1)\varepsilon_j\right)$. Since $t\alpha\delta_j g_0 \in S\beta$ and $\beta\varepsilon_j \in M$ for all $1 \leq j \leq m$, we have

$$\sum_{j=1}^{m} t\alpha\delta_j g_0\varepsilon_j \in M \subset N_1.$$

Thus, $N = N_0 \subset N_1$. Since

$$\sum_{j=1}^{m} \alpha\delta_j g_1\varepsilon_j \in N \subset N_1,$$

we conclude that

$$\sum_{j=1}^{m} \alpha\delta_j g_0\varepsilon_j \in N_1.$$

On the other hand, we have

$$M' \subset M + S\left(\sum_{j=1}^{m} \alpha\delta_j g_0\varepsilon_j\right) \subset N_1.$$

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Consequently,

$$S^{(m)} = M' + S\left(\sum_{j=1}^{m} t\alpha\delta_j g_1\right) \subset N_1.$$

Therefore, $g = g_0 + g_1 \in K\langle x, y, z\rangle$ satisfies the required condition. □

Corollary 1. *Let $\rho \neq 0 \in S$ and $\delta_1, ..., \delta_m$ be K - linearly independent elements in $K < x, y, z >$. Then there exists some $g \in K < x, y, z >$ such that $S^{(m)} = S^{(m)}\rho + S(\sum_{j=1}^{m} \rho\delta_j g\varepsilon_j)$.*

Proof. Proposition 11 applied to $\alpha = \rho$ and used on the direct sum of copies of S, implies that the S-module $S^{(m)}/S^{(m)}\rho$ has a finite length. Now the result follows from Proposition 13 with $\alpha = \rho$ and $M = S^{(m)}\rho$. □

Recall that we consider the ring $S = \mathcal{D}_r(y_{r+1}, \ldots, y_n)\langle x_{r+1}\rangle$. If $r < n$, we first consider the division ring $D = \mathcal{D}_r(y_{r+2}, \ldots, y_n)$, and then we observe that $S = D(y)\langle x\rangle$, where x is the variable x_{r+1} and y represents y_{r+1}. If we allow the element ρ in Proposition 2 to be an arbitrary non-zero element in S, we obtain a more general statement. In conclusion, Proposition 2 holds if we can establish the necessary conditions.

Proposition 11. *Let $\delta_1, ..., \delta_m$ be K - linearly independent elements in $K < x, y, z >$ and let $\rho \neq 0 \in S$ and consider the free S - module $S^{(m+1)} = S\varepsilon_0 + ... + S\varepsilon_m$ then there exists some $g \in K < x, y, z >$ such that $S^{(m+1)} = S^{(m+1)}\rho + S(\varepsilon_0 + \delta_1 g\varepsilon_1 + ... + \delta_m g\varepsilon_m)$.*

Proof. We can certainly identify $S^{(m)}$ with the S-submodule of $S^{(m+1)}$ generated by $\varepsilon_1, \ldots, \varepsilon_m$. By Corollary 14, we can choose $g \in K\langle x, y, z\rangle$ such that

$$S^{(m)} = S^{(m)}\rho + S\left(\sum_{j=1}^{m} \rho\delta_j g\varepsilon_j\right) \tag{4.1}$$

Consider an element $v \in S^{(m+1)}$. Then there exists $\alpha_0 \in S$ and $w \in S^{(m)}$ such that $v = \alpha_0\varepsilon_0 + w$. By Proposition 10, we can write $\alpha_0 = \beta_0\rho + \theta$ for some elements β_0 and θ in S. According to Equation (3.1), there exist $\beta_1, \ldots, \beta_m, \lambda \in S$ such that

$$w = \beta_1\rho\varepsilon_1 + \cdots + \beta_m\rho\varepsilon_m + \lambda(\rho\delta_1 g\varepsilon_1 + \cdots + \rho\delta_m g\varepsilon_m).$$

Therefore,

$$v = (\beta_0\rho\varepsilon_0 + \beta_1\rho\varepsilon_1 + \cdots + \beta_m\rho\varepsilon_m) + \lambda\rho(\varepsilon_0 + \delta_1 g\varepsilon_1 + \cdots + \delta_m g\varepsilon_m) + (\theta - \lambda\rho)\varepsilon_0.$$

Hence,

$$S^{(m+1)} = S^{(m+1)}\rho + S\rho(\varepsilon_0 + \delta_1 g\varepsilon_1 + \cdots + \delta_m g\varepsilon_m) + S\varepsilon_0. \tag{4.2}$$

Now set $N = S^{(m+1)}\rho + S(\varepsilon_0 + \delta_1 g\varepsilon_1 + \cdots + \delta_m g\varepsilon_m)$. Our goal is to show that $N = S^{(m+1)}$. First, note that

$$N \supset S^{(m+1)}\rho + S\rho\left(\varepsilon_0 + \sum_{j=1}^{m} \delta_j g\varepsilon_j\right) \supset S^{(m+1)}\rho + S\rho\left(\sum_{j=1}^{m} \delta_j g\varepsilon_j\right)$$

Second, we claim that

$$S^{(m+1)}\rho + S\rho\left(\sum_{j=1}^{m} \delta_j g\varepsilon_j\right) \supset S\rho\varepsilon_0 + S^{(m)}.$$

Indeed, consider $v \in S\rho\varepsilon_0 + S^{(m)}$. Then there exist $\lambda_0 \in S$ and $w \in S^{(m)}$ such that $v = \lambda_0\rho\varepsilon_0 + w$. So, using Equation (3.1), we can write

$$v = (\lambda_0\rho\varepsilon_0 + \lambda_1\rho\varepsilon_0 + \cdots + \lambda_m\rho\varepsilon_m) + \beta\rho(\delta_1 g\varepsilon_1 + \cdots + \delta_m g\varepsilon_m),$$

for some elements $\lambda_1, \ldots, \lambda_m, \beta \in S$. Therefore, $v \in S^{(m+1)}\rho + S\rho\left(\sum_{j=1}^{m} \delta_j g\varepsilon_j\right)$ as we claimed.

Finally, since

$$(\delta_1 g\varepsilon_1 + \cdots + \delta_m g\varepsilon_m) \in S\rho\varepsilon_0 + S^{(m)} \subset N,$$

we have $\varepsilon_0 \in N$. By Equation 10, we conclude that $N = S^{(m+1)}$. The proof is complete. □

5. The Final Part of the Proof of Theorem 2

We have established Proposition 4, which provides an element $q_0 \neq 0 \in K[y_1, \ldots, y_n]$ and elements d and e in $A_{n,f}$ such that $q_0 c \in A_{n,f}(a + dc) + A_{n,f}(b + ec)$. This result serves as the basis for an induction, which allows us to prove the following counterparts to Proposition 3(r).

Proposition 12. *(r) There exists some $q_r \neq 0 \in K[y_1, ..., y_r]$ and d_r and e_r in $A_{n,f}$ such that $q_r c \in A_{n,f}\,(a + d_r c) + A_{n,f}\,(b + e_r c)$.*

When $r = 0$, q_0 is a non-zero scalar in the field K, and Theorem 2 follows. Proposition 4 corresponds to the case $r = n$. It remains to prove the induction step: Proposition 16($r+1$) $\Longrightarrow$ Proposition 16(r) for all $0 \leq r \leq n-1$. This can be demonstrated in the same manner as the implications Proposition 3($r+1$) $\Rightarrow$ Proposition 3(r), by replacing Proposition 5 with...

Proposition 13. *Let $0 \leq r \leq n-1$ and $q_{r+1} \neq 0 \in K[y_1, ..., y_{r+1}]$ and let u and v in $A_{n,f}$ with $v \neq 0$. Then there exists some $q_r \in K[y_1, ..., y_r]$ and $g \in A_{n,f}$ such that $q_r \in A_{n,f}q_{r+1} + A_{n,f}(u + vg)$.*

We introduce the ring $\mathcal{R} = K(y_1, ..., y_r) < x_1, ..., x_n, y_{r+1}, ..., y_n, z_{r+1}, ..., z_n >$ which is partial quotient ring of $A_{n,f}$. If $t \in \mathcal{R}$ then $t = q_r^{-1}a$ for some $a \in A_{n,f}$ and some $q_r \in K[y_1, ..., y_r]$.

Lemma 2. *Let $q_{r+1} \neq 0 \in K[y_1, ..., y_{r+1}]$ and let u and v in $A_{n,f}$ with $v \neq 0$, then there exists some $g \in A_{n,f}$ such that $\mathcal{R} = \mathcal{R}q_{r+1} + \mathcal{R}(u + vg)$.*

It is straightforward to verify that Lemma 18 and Proposition 17 imply each other. To prove Lemma 18, we will use the ring $S_1 = K(y_1, \ldots, y_r)\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$, which appears as a subring of $\mathcal{R}$. We observe that $K[y_1, \ldots, y_{r+1}]$ is a subring of S_1, and it also contains the generalized Weyl algebra $K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ of degree one.

Proposition 14. *Let $\delta_1, ..., \delta_m$ be K- linearly independent elements in $K < x_{r+1}, y_{r+1}, z_{r+1} >$ and $\rho \neq 0 \in K[y_1, ..., y_{r+1}]$. Let $S_1^{(m+1)} = S_1\varepsilon_0 + S_1\varepsilon_1 + ... + S_1\varepsilon_m$ be free S_1 -module of rank $m+1$ and let $S_1^{(m+1)}\rho$ denote the submodule generated by $\rho\varepsilon_0, ..., \rho\varepsilon_m$. Then there exists some $g \in K < x_{r+1}, y_{r+1}, z_{r+1} >$ such that $S_1^{(m+1)} = S_1^{(m+1)}\rho + S_1(\varepsilon_0 + \delta_1 g\varepsilon_1 + ... + \delta_m g\varepsilon_m)$.*

We'll postpone the proof Proposition 19 for now

Proposition 15. *Let $q \neq 0 \in K[y_1, ..., y_n]$. If $a_1, ..., a_t \in A_{n,f}$ then there exists $\rho \neq 0 \in K[y_1, ..., y_n]$ such that $\rho a_j \in A_{n,f}q$ for all j.*

Proof. Let $a \in A_{n,f}$. Then $aq^{-1} \in \mathcal{R}$ can be written as a finite sum

$$\sum_{\alpha,\beta} q_{\alpha\beta}^{-1} p_{\alpha\beta} x^{\alpha} z^{\beta},$$

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where $q_{\alpha\beta} \in K[y_1, \ldots, y_n] \setminus \{0\}$ and $p_{\alpha\beta} \in K[y_1, \ldots, y_n]$. Choose $\rho' \in K[y_1, \ldots, y_n] \setminus \{0\}$ to be a common multiple of the $q_{\alpha\beta}$'s. Then, we have

$$\rho' a q^{-1} = \sum_{\alpha,\beta} \rho' q_{\alpha\beta}^{-1} p_{\alpha\beta} x^{\alpha} z^{\beta} \in A_{n,f}.$$

Applying this result to each a_j, we obtain a certain $\rho'_j \neq 0 \in K[y_1, \ldots, y_n]$ such that $\rho'_j a_j \in A_{n,f}$. Therefore, if ρ is a non-zero left common multiple of $\rho_1, \ldots, \rho_t$, we get $\rho a_j \in A_{n,f} q$ for every j. □

Proof. Proof of Lemma 18. Suppose

$$A(r+1) = K\langle x_1, \ldots, x_r, x_{r+2}, \ldots, x_n, y_1, \ldots, y_r, y_{r+2}, \ldots, y_n, z_1, \ldots, z_r, z_{r+2}, \ldots, z_n \rangle.$$

Then, we have the isomorphism $A_{n,f} \cong A(r+1) \otimes_K K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$. We can express $v = \delta_1 G_1 + \cdots + \delta_m G_m$, where $\delta_1, \ldots, \delta_m$ are K-linearly independent elements in $K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ and $G_1, \ldots, G_m$ are elements of $A(r+1)$. Since $A(r+1) \cong A_{n-1,f}$ is simple, it follows that the two-sided ideal generated by $G_1, \ldots, G_m$ is the entire ring $A(r+1)$.

This implies the existence of finite sets $\{H_1, \ldots, H_t\}$ and $\{Y_1, \ldots, Y_t\}$ such that

$$1 = \sum_{v=1}^{t} \sum_{j=1}^{m} Y_v G_j H_v,$$

and hence

$$A(r+1) = \sum_{j=1}^{m} \sum_{v=1}^{t} A(r+1) G_j H_v.$$

Identifying $A(r+1)$ as a subring of $\mathcal{R}$, we conclude that

$$\mathcal{R} = \sum_{j=1}^{m} \sum_{v=1}^{t} \mathcal{R} G_j H_v.$$

At this stage, we need to...

Sublemma 1: To each m-tuple $B_1, \ldots, B_m$ in $A(r+1)$, there exists some $g \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$\mathcal{R} q_{r+1} + \mathcal{R} u + \mathcal{R} B_1 + \cdots + \mathcal{R} B_m = \mathcal{R} q_{r+1} + \mathcal{R}(u + \delta_1 g B_1 + \cdots + \delta_m g B_m).$$

Proof: Since $q_{r+1} \neq 0 \in K[y_1, \ldots, y_r, y_{r+1}]$, it follows from Proposition 20 that there exists some $\rho \neq 0 \in K[y_1, \ldots, y_{r+1}]$ such that $\rho B_j \in A_{n,f}$ for each $j = 1, 2, \ldots, m$, and in addition, $\rho u \in A_{n,f}$.

Using Proposition 19, we obtain some $g \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$S_1^{(m+1)} = S_1^{(m+1)} \rho + S_1(\varepsilon_0 + \delta_1 g \varepsilon_1 + \cdots + \delta_m g \varepsilon_m).$$

Since S_1 is a subring of $\mathcal{R}$, it follows that

$$\mathcal{R}^{(m+1)} = \mathcal{R}\rho + \mathcal{R}(\varepsilon_0 + \delta_1 g \varepsilon_1 + \cdots + \delta_m g \varepsilon_m).$$

Consider an $\mathcal{R}$-linear map π from $\mathcal{R}^{(m+1)}$ to $\mathcal{R}$ defined by $\pi(\varepsilon_0) = u$ and $\pi(\varepsilon_j) = B_j$ for each $1 \leq j \leq m$.

Since $\rho u, \rho B_1, \ldots, \rho B_m \in A_{n,f}$, it follows that

$$\pi(\mathcal{R}^{(m+1)}) = \pi(\mathcal{R}\rho\varepsilon_0 + \cdots + \mathcal{R}\rho\varepsilon_m) = \mathcal{R}\rho u + \mathcal{R}\rho B_1 + \cdots + \mathcal{R} B_m \subset \mathcal{R} q_{r+1}.$$

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Thus, we have

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}B_j = \mathcal{R}q_{r+1} + \pi(\mathcal{R}^{(m+1)}) \subset \mathcal{R}q_{r+1} + \mathcal{R}\pi(\varepsilon_0 + \delta_1 g \varepsilon_1 + \cdots + \delta_m g \varepsilon_m).$$

Finally, this simplifies to

$$\mathcal{R}q_{r+1} + \mathcal{R}(u + \delta_1 g B_1 + \cdots + \delta_m g B_m),$$

and the Sublemma follows since the opposite direction is obvious.

Proof continued. Applying the Sublemma with $B_j = G_j H_1$, we obtain some $g_1 \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}G_j H_1 = \mathcal{R}q_{r+1} + \mathcal{R}\left(u + \sum_{j=1}^{m} \delta_j g_1 G_j H_1\right).$$

Recall that $v = \sum_{j=1}^{m} \delta_j G_j$, and since g_1 and G_j commute in $A_{n,f}$ for all j, we have

$$\sum_{j=1}^{m} \delta_j g_1 G_j H_1 = v g_1 H_1,$$

so that

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}G_j H_1 = \mathcal{R}q_{r+1} + \mathcal{R}(u + v g_1 H_1).$$

Now, let us apply Sublemma 1 with u replaced by $u + v g_1 H_1$ and with $B_j = G_j H_2$. This gives $g_2 \in K\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$ such that

$$\mathcal{R}q_{r+1} + \mathcal{R}(u + v g_1 H_1) + \sum_{j=1}^{m} \mathcal{R}G_j H_2 = \mathcal{R}q_{r+1} + \mathcal{R}(u + v g_1 H_1 + v g_2 H_2).$$

Using the previous equation, we can rewrite the left-hand side and conclude that

$$\mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \mathcal{R}G_j H_1 + \sum_{j=1}^{m} \mathcal{R}G_j H_2 = \mathcal{R}q_{r+1} + \mathcal{R}(u + v g_1 H_1 + v g_2 H_2).$$

Similarly, for $B_j = G_j H_3$ and with u replaced by $u + v g_1 H_1 + v g_2 H_2$, and continuing in this way, after t steps we conclude that

$$\mathcal{R}q_{r+1} + \mathcal{R}(u + v g_1 H_1 + \cdots + v g_t H_t) = \mathcal{R}q_{r+1} + \mathcal{R}u + \sum_{j=1}^{m} \sum_{v=1}^{t} \mathcal{R}G_j H_v = \mathcal{R},$$

where the last equality follows because $\mathcal{R} = \sum_{j=1}^{m} \sum_{v=1}^{t} \mathcal{R}G_j H_v$. Hence, Lemma 18 follows with $g = g_1 H_1 + \cdots + g_t H_t$. $\square$

Let $S = K(y_1) < x, y, z >$.

Proposition 16. *S is a simple ring.*

Proposition 17. *Let $\alpha \neq 0 \in S$. Then the left S - module $S/S\alpha$ has a finite length.*

Proposition 18. *Let $\delta_0, ..., \delta_m$ be a finite set of K - linearly independent elements in $K < x, y, z >$ and let $\alpha \neq 0 \in S$ and let $S^{(m)} = S\varepsilon_1 + ... + S\varepsilon_m$ be a free S - module of rank m. Let M be the S - submodule of $S^{(m)}$ which is generated by $\{\alpha\delta_1 g \varepsilon_1 + ... + \alpha\delta_m g \varepsilon_m : g \in K < x, y, z >\}$. Then $M = S^{(m)}$.*

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Proof. Let us first observe that both the assumption and the conclusion remain unchanged if the m-tuple $\delta_1, \ldots, \delta_m$ is replaced by an m-tuple $\beta_1, \ldots, \beta_m$ where $\beta_j = \sum_{v=1}^{m} a_{jv}\delta_v$ and (a_{jv}) is an invertible matrix of size (m, m) with $a_{jv} \in K$. While we replace $\delta_1, \ldots, \delta_m$ by $\beta_1, \ldots, \beta_m$ under the K-linear transformation (a_{jv}), we also replace the free generators $\varepsilon_1, \ldots, \varepsilon_m$ of $S^{(m)}$ by $\zeta_1, \ldots, \zeta_m$ where $\zeta_j = \sum_{v=1}^{m} b_{jv}\varepsilon_v$ and $(b_{jv}) = (a_{jv})^{-1}$.

By reordering the δ_j if necessary, we may assume that

$$\deg(\delta_1) \geq \deg(\delta_2) \geq \cdots \geq \deg(\delta_m).$$

Hence, there exists a positive integer w and some $1 \leq q \leq m$ such that $\deg(\delta_1) = \cdots = \deg(\delta_q) = w$ and $\deg(\delta_j) < w$ for $j > q$. If $1 \leq j \leq q$, we can write

$$\delta_j = r_j + p_j(y)x^w$$

where $\deg(r_j) < w$ and $p_j(y) \in K[y]$.

We can also assume that the polynomials $p_1, \ldots, p_q$ have decreasing degrees. If $\deg(p_1) = \deg(p_2)$, we can choose $k \in K$ such that $\deg(p_2 - kp_1) < \mu$, and replace δ_2 by $\delta_2 - k\delta_1$ while leaving $\delta_1, \delta_3, \ldots, \delta_m$ unchanged.

Armed with these normalizations, we begin to prove that $\varepsilon_1 \in M$. First, let $\deg(\alpha) = t$. We can assume that

$$\alpha = \alpha_0 + h(y_1)k(y)x^t$$

where $\deg(\alpha_0) < t$ and $h(y_1) \in K[y_1]$.

Using the relations

$$[x, y] = y, \quad [x, z] = -z, \quad yz = f(x), \quad zy = f(x+1), \quad x^2y - yx^2 = y(2x+1),$$

$$y(2x+1)y - y^2(2x+1) = 2!y^2, \quad [z, y] = f(x+1) - f(x) = x - (x-1) = 1, \quad zy^2 - y^2z = 2y,$$

$$z(2y) - (2y)z = 2!$$

and so on, we proceed as follows.

For $1 \leq j \leq q$, we get

$$\alpha\delta_j = \varphi_j + h(y_1)\Psi_j x^{t+w}$$

where $\deg(\varphi_j) < t + w$; and for $q < j \leq m$, we have $\deg(\alpha\delta_j) < t + w$.

The $(t+w)$-fold commutator of $\alpha\delta_j$ and y will be $(t+w)!h(y_1)\Psi_j y^{t+w}$ for $1 \leq j \leq q$ and 0 for $q < j \leq m$. Since

$$\alpha\delta_1\varepsilon_1 + \cdots + \alpha\delta_m\varepsilon_m \in M$$

and

$$[\alpha\delta_j, y] = \alpha\delta_j y - y(\alpha\delta_j),$$

we have

$$[\alpha\delta_1, y]\varepsilon_1 + \cdots + [\alpha\delta_m, y]\varepsilon_m \in M.$$

Proceeding in this way, we obtain

$$(t+w)!h(y_1)\Psi_1 y^{t+w}\varepsilon_1 + \cdots + (t+w)!h(y_1)\Psi_q y^{t+w}\varepsilon_q \in M.$$

After dividing by $(t+w)!h(y_1)$, we get

$$v_1 = \Psi_1 y^{t+w}\varepsilon_1 + \cdots + \Psi_q y^{t+w}\varepsilon_q \in M.$$

Since $[\alpha\delta_j g, z] = \alpha\delta_j(gz) - z(\alpha\delta_j g)$ for all $g \in M$ and $v_1 \in M$, we have

$$[\Psi_1 y^{t+w}, z]\varepsilon_1 + \cdots + [\Psi_q y^{t+w}, z]\varepsilon_q \in M.$$

Suppose $\deg(p_1) = r$. Taking the $(r+t+w)$-fold commutator with z, we obtain

$$(-1)^{t+t+w}(r+t+w)!\mu^{r+t+w}\varepsilon_1 \in M,$$

where μ is the leading coefficient of Ψ_1. Finally, we can divide by the coefficient of ε_1 to get

$$\varepsilon_1 \in M.$$

Restricting our attention to the $(m-1)$-tuple $\delta_1, \ldots, \delta_{m-1}$ and the S-module $S^{(m-1)} = S\varepsilon_2 + \cdots + S\varepsilon_m$, the proposition follows by induction on m. □

Proposition 19. *Let $\delta_1, ..., \delta_m$ be K - linearly independent elements in $K < x, y, z >$ and let M be a submodule of $S^{(m)} = S\varepsilon_1 + ... + S\varepsilon_m$ such that the S- module $S^{(m)}/M$ has a finite length. If $\alpha \neq 0 \in S$ we can find $g \in K < x, y, z >$ such that $S^{(m)} = M + S(\alpha\delta_1 g\varepsilon_1 + ... + \alpha\delta_1 g\varepsilon_m)$.*

Proof. Analogous to Proposition 13. □

Corollary 2. *Let $\rho \neq 0 \in S$ and let $\delta_1, ..., \delta_m$ be K-linearly independent elements in $K < x, y, z >$. Then there exists some $g \in K < x, y, z >$ such that $S^{(m)} = S^{(m)}\rho + S(\sum_{j=1}^{m} \rho\delta_j g\varepsilon_j)$.*

Proof. Analogous to Corollary 14. □

Recall that $S_1 = K(y_1, \ldots, y_r)$. If $r < n$, consider $S = K(y_1, \ldots, y_r)\langle x_{r+1}, y_{r+1}, z_{r+1}\rangle$. If we allow the element ρ in Proposition 4.7 to be an arbitrary non-zero element in S, we obtain a more general statement. Summing up, Proposition 19 follows once we establish...

Proposition 20. *Let $\delta_1, ..., \delta_m$ be K - linearly independent elements in $K < x_{r+1}, y_{r+1}, z_{r+1} >$ and let $\rho \neq 0 \in S$ and consider the free S - module $S^{(m+1)} = S\varepsilon_0 + S\varepsilon_1 + ... + S\varepsilon_m$. Then there exists some $g \in K < x_{r+1}, y_{r+1}, z_{r+1} >$ such that $S^{(m+1)} = S^{(m+1)} + S(\varepsilon_0 + \delta_1 g\varepsilon_1 + ... + \delta_m g\varepsilon_m)$.*

Proof. Analogous to Proposition 15. □

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