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Article

Vehicle Directional Cosine Calculation Method

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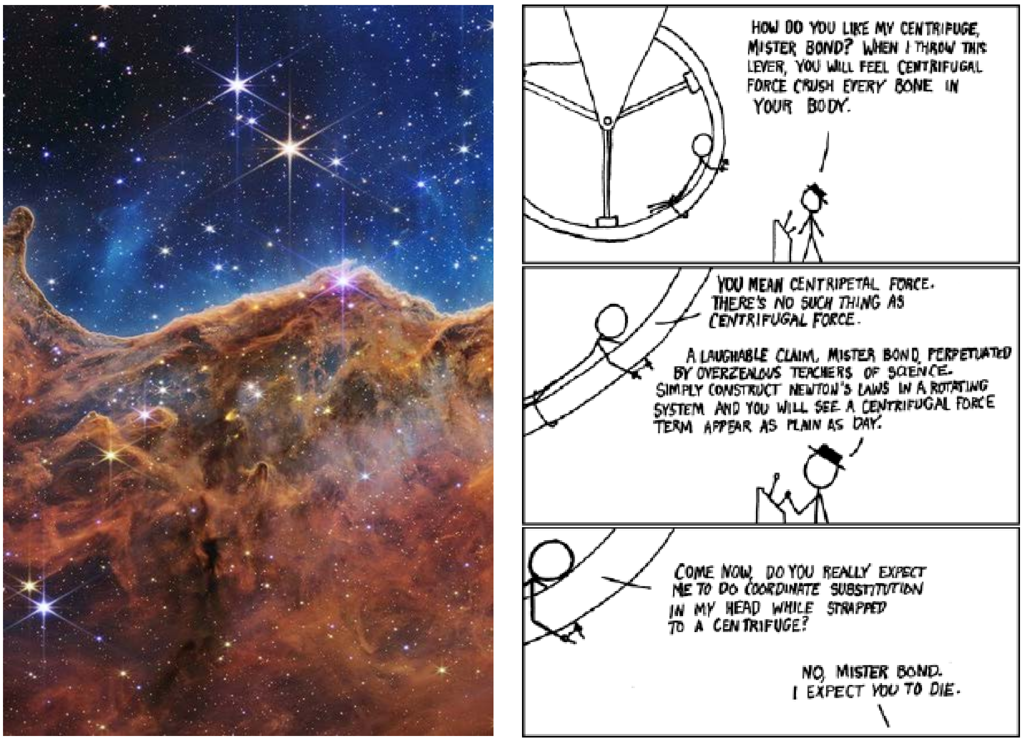
Abstract: Teaching kinematic rotations is a daunting task for even some of the most advanced mathematical minds. However, envisioning and explaining the three-dimensional rotations can be highly simplified by changing the paradigm. This paradigm change allows for a high school student with a understanding of geometry to be able to not only develop the matrix but also explain the rotations at a collegiate level. The proposed method includes the assumption of a point (P) within the initial three-dimensional frame with axes ($\hat{x}_i$, $\hat{y}_i$, $\hat{z}_i$). It then utilizes a two-dimensional rotation view (2DRV) to measure how the coordinates of point P translate after the initial axis is rotated instead of using the established Euler's formula. The equations are used in matrix notation to develop a direction cosine matrix (DCM) for future equations. This method provides a high school student with an elementary and repeatable process to compose and explain kinematic rotations.

Keywords: kinematics; direction cosine matrixes; education



(**a**) (**b**)

Figure 1. (**a**) First images from the James Webb Space Telescope peering into deep space where the "inertial", non-moving reference frame is often mathematically placed. Credits: NASA, ESA, CSA, and STSc. [1] (**b**) NASA humor over the ubiquity of difficulty learning kinematics. [2] Image usage is consistent with NASA policy, "NASA content (images, videos, audio, etc.) are generally not copyrighted and may be used for educational or informational purposes without needing explicit permissions." [3].

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1. Introduction

When you can measure what you are speaking about and express it in numbers, you know something about it, and when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind. It may be the beginning of knowledge, but you have scarcely in your thought advanced to the stage of a science. Lord Baron Kelvin of Largs [4]

This manuscript describes the proposed method as a step-by-step process to simply develop and understand a set of equations to translate positioning and rotations from one perspective to another. The direction cosine equations are essential to correctly understand automation of precision motion expressions. Teaching the concept of an initial axis ($\hat{x}_i$, $\hat{y}_i$, $\hat{z}_i$) rotating to another axis ($\widehat{x'}$, $\widehat{y'}$, $\widehat{z'}$) becomes much easier if the initial axis is viewed as a two-dimensional rotation view (2DRV). The new viewpoint of the axis and the rotation makes the mathematics and visual comprehension more palatable.

1.1. Educational Importance

Recently, Manurung [5] recommended new methods for teaching kinematics in a journal on education (perhaps not even read by scientific and mathematical fields' researchers). In 2017, Ramma [6] published a public, scathing rebuke of kinematics' pedagogy in a popular online blog. The rebuke reported the results of a study of twenty-six physics teachers from twenty-six of a country's secondary schools who had been teaching for an average of five years to students between 16 and 17 years old. Kinematics typically represent three-dimensional motion with two-dimensional representations, while one kinematic representation (the quaternion) involves depicting *four*-dimensional representations of *three*-dimensional motion in *two*-dimensional graphics. Potentially confounding confusion is immediately self-evident. The Ramma study confirmed the well-known fact that kinematics is difficult to learn, but additionally assessed a key issue laid at the feet of pedagogy and limitations of the topic's teaching.

Ayop [7] evaluated the assessments themselves and resulting teaching strategies for kinematics. Shodiqin [8] very recently elaborated student difficulties in understanding kinematics, while Núñez [9] shortly afterwards elaborated some difficulties themselves hinting at the relative importance of re-thinking the topic's pedagogical methods. After these recommendations, Moyo [10] performed a focused case study for high schools in Botswana validating the omnipresent sources of difficulties and potential impacts.

To advance our species' understanding of the world and our ability to shape future scientists, technicians, and engineers, we must develop them earlier, faster, and more efficiently. With that in mind, finding and developing more effective ways to deliver academics to students is critical to future development. Due to the mathematical requirements, rotational kinematics is often a topic that is broached after calculus and linear algebra, if not even later in a student's academic career. However, with the right tools and perspective, students can be taught the basics of axes rotations at a much earlier developmental timeframe.

Laying this foundation at a younger age allows for a more significant expansion of knowledge later. The purpose of this manuscript's proposal is to outline how to efficiently develop this foundational understanding, significantly accelerating the academic growth of students.

1.2 History of Kinematic and Directional Cosine Matrices

The theorems used to translate rigid bodies in Euclidean space date back to 1775 when Euler wrote: "General formulas for the translation of arbitrary rigid bodies" [11]. Over the decades and centuries, the translation of rigid bodies mathematics has been refined and expanded, e.g. by Chasles [12] in the early 1800's, by Lord Kelvin in the late 1800's [13], by Whittaker [14] in the early 1900's. Several authors [15–19] from the USA's national aeronautics and space administration (NASA) published updates.

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In the late 1960's Meyer [15] proposed a method for expanding a matrix of direction cosines, while Jordan [16–17] evaluated computation errors using direction cosines and proposed a novel direction cosine algorithm (as done in this present manuscript). Kane [18] articulated many challenging aspects of kinematics in the early 1970's. Meanwhile, Haley [19] described manifestations of affects due to kinematics of integrated robotics. In the 1990's King [20] illustrated the deleterious effects on the accuracy of global position systems, while Dunn [21] extended the conclusions to include satellite laser ranging. Following the turn of the century, in 2001 Xing [22] proposed alternate forms of attitude kinematics de-emphasizing matrices of nine direction cosines in favor of three-parameter kinematics (e.g., Euler angles) or four-parameter kinematics (e.g., quaternions, Rodrigues-Gibbs vector, etc.).

Most recently, the significance of direction cosine matrices and their derivation has been resurged in the 2018 publication by Smeresky [23] seeking to develop more accurate and efficient calculations revisited by Cole [24] and Sandberg [25] just this year, trying to discern errors and resulting deleterious effects in applications. Regardless of which derivation is used, each requires the development of at least two rotational matrices, and ultimately understanding the development of the direction cosine matrix is critical to understanding kinematics.

1.3 Present-day purpose and innovations presented

While the usual math equations and examples in Appendix A are still as useful today as they were when they were developed, these examples are not immune to adaption or improvement. This work aims to provide tools to students and teachers to learn and or instruct three-dimensional kinematic rotations without the need for higher-level mathematics. The conclusion is that linear algebra and calculus are not required for direction cosine matrix development or understanding. Simple equations can be used to develop the rotation matrix.

1.3.1. Innovations proposed

1. Two-dimensional rotation view (2DRV) about a third dimension.
2. Equations of direction cosine matrix (DCM).

Section two describes the overall method of direction cosine matrix creation. Section three outlines how direction cosine matrices fit into dynamics, 2DRV, and the proposed method of teaching direction cosine matrix development. Section four and five provide a conclusion and review other discussion points, respectively.

2. Methods and Results

The approach utilized in this manuscript is based on the rotation of a coordinate system about a fixed point. Any point's position in the rotated coordinate system can be calculated based on three separate rotations about an axis. Breaking down a single rotation about an axis is equitable to a two-dimensional plane of points rotating about a point. Each rotation can be described and depicted in a 2DRV, so long as the axis and rotation are defined correctly. In a 2DRV, any point in the initial plane can be described in the new plane using the similarity of right triangles and equations of motion functions. The generic rotation conversion equations can be transferred into a matrix format to develop a direction cosine matrix. This method generates the same rotation matrices as the Euler equations but is more straightforward.

2.1. Dynamics

Dynamics is a division of mechanics that deals with how rigid bodies propagate with time in relation to force, mass, momentum, and energy. Dynamics can be dissected into two components kinetic (linear & rotational forces acting on bodies) and kinematic (motion of bodies regardless of forces). This manuscript only represents a new calculation method of kinematic equations (i.e., direction cosine matrix). However, without

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understanding some of the basic principles of kinetics, the purpose of the direction cosine matrix can be missed.

2.1.1 Kinetics

The Kinetic equations describe linear and rotation forces (disturbance and control) on the rigid body. Equations (1) and (2) consider all the linear and nonlinear motion in a fictional non-rotating reference frame utilizing Michel Chasles and Giulio Mozzi's theorem to combine Newton and Euler's theorems.

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = [M] \begin{bmatrix} \ddot{x} - \dot{y}\omega_z + \dot{z}\omega_y \\ \ddot{y} - \dot{z}\omega_x + \dot{x}\omega_z \\ \ddot{z} - \dot{x}\omega_y + \dot{y}\omega_x \end{bmatrix} \tag{1}$$

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{bmatrix} J_{xx}\dot{\omega}_x + J_{xy}\dot{\omega}_y + J_{xz}\dot{\omega}_z - J_{xy}\omega_x\omega_z - J_{yy}\omega_y\omega_z - J_{yz}{\omega_z}^2 + J_{xz}\omega_x\omega_y + J_{zz}\omega_z\omega_y + J_{yz}{\omega_y}^2 \\ J_{yx}\dot{\omega}_x + J_{yy}\dot{\omega}_y + J_{yz}\dot{\omega}_z - J_{yz}\omega_x\omega_y - J_{zz}\omega_x\omega_z - J_{xz}{\omega_x}^2 + J_{xx}\omega_x\omega_z + J_{xy}\omega_z\omega_y + J_{xz}{\omega_z}^2 \\ J_{zx}\dot{\omega}_x + J_{zy}\dot{\omega}_y + J_{zz}\dot{\omega}_z - J_{xx}\omega_x\omega_y - J_{xz}\omega_y\omega_z - J_{xy}{\omega_y}^2 + J_{yy}\omega_x\omega_y + J_{yz}\omega_z\omega_x + J_{xy}{\omega_x}^2 \end{bmatrix} \tag{2}$$

Table 1. Proximal variable and nomenclature definitions.

Term	Definition	Term	Definition	Term	Definition	Term	Definition
F_x	External resultant force in *inertial* x-direction	ω_x	Angular velocity about x-axis	$\dot{\omega}_x$	Angular acceleration about x-axis	J_{xx}	Moment of inertia along x with respect to x
F_y	External resultant force in *inertial* y-direction	ω_y	Angular velocity about y-axis	$\dot{\omega}_y$	Angular acceleration about y-axis	$J_{xy} = J_{yx}$	Product of inertia along x with respect to y
F_z	External resultant force in *inertial* z-direction	ω_z	Angular velocity about z-axis	$\dot{\omega}_z$	Angular acceleration about z-axis	$J_{xz} = J_{zx}$	Product of inertia along x with respect to z
$\ddot{x}$	Acceleration in *inertial* x-direction	$\dot{x}$	Velocity in x-direction	M	External resultant moment	J_{yy}	Moment of inertia along y with respect to y
$\ddot{y}$	Acceleration in *inertial* y-direction	$\dot{y}$	Velocity in y-direction			$J_{yz} = J_{zy}$	Product of inertia along y with respect to z
$\ddot{z}$	Acceleration in *inertial* z-direction	$\dot{z}$	Velocity in z-direction			J_{zz}	Moment of inertia along z with respect to z

The non-rotating reference frame or the inertial frame is a fictional construct and is used to describe a position that is not accelerating or subject to a gravitational field. As no known position in the universe is not accelerating or subject to a gravitational field, one is developed. If Newton and Euler's equations of motion were calculated not using an inertial frame of reference, Newtonian laws of motion would not necessarily hold to be true. Once the equations of motion are evaluated in the inertial frame, the equations of motion can then be expressed in the body frame and then translated into other frames of reference. The kinematics described in section 2.1.2 can translate position and motions into different reference frames [26].

2.1.2. Kinematics

Kinematics is used to translate equations of motion such as Equations (1 & 2) that are calculated the inertial frame (expressed in the body frame) into other reference systems [27]. The non-rotating perspective is required to calculate the disturbance and maneuvering forces correctly, as described in Section 2.1.1 Kinetics for Newtonian and Euler equations. Once the equations of motion are set, translation equations like direction cosine matrices and/or quaternions translate the motion into other frames of reference [28].

Quaternions have many beneficial features over classic direction cosine matrices [28]. However, the four-dimensional variables that make up quaternions are not intuitive to the typical three-dimensional human perspective. Ultimately converting the four-dimension variables to Euler angles makes direction cosine matrix development critical for human-machine interfaces (HMI). Understanding the development of the direction cosine matrix is foundational to understanding how the kinematic rotation is calculated.

2.2. Direction cosine matrix (DCM)

Teaching three-dimensional rotations is challenging to present utilizing two-dimensional resources that most teachers have available; typical examples include those in Appendix A. The DCM equations are based on three separate one-dimensional rotations matrices multiplied in sequence to equate to one three-dimensional rotation equation. This manuscript uses the 2DRV to help depict and write the translation equations. The 2DRV takes advantage of the fact that a reference frame can be viewed from any aspect. Specifically, 2DRV depicts the positive axis being rotated about will face directly out of the page, and the perpendicular positive axes will point up and to the right, as seen in Figure 2.

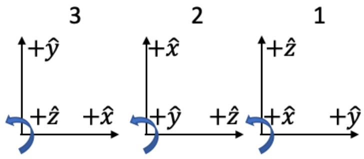
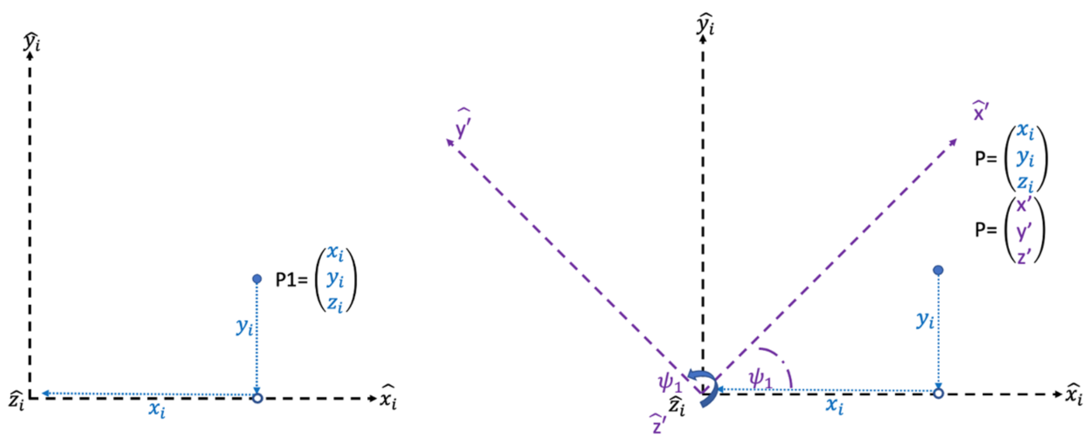


Figure 2. Two-dimensional viewpoint (2DRV) of 3-Rotation, 2-Rotation & 1-Rotation. All rotations are utilizing right hand revolutions. The rotation about the zˆ-axis, yˆ-axis, and xˆ-axis is known as the "3", "2", and "1" rotation respectively

This depiction is helpful because the initial and final axis is colinear, and the only changes happen to points and vectors, not on that axis. This tactic is referred to as 2DRV, which simplifies the dimensionality and makes the development of the equations simpler to see and understand. Developing each rotation creates a matrix and must be done one revolution at a time. Each rotation is identified by the axis it is rotated about. After each matrix is created, they can be combined to create an overarching direction cosine matrix.

The following direction cosine matrix creation uses the proposed method. Using the 3, 2, and 1 order of rotations, or a 321-DCM, will take the inertial frame to a body frame. In this description, the order of the rotation also follows the number order. However, it is essential to note that any combination of rotations or translation types can be done similarly. While the order does not matter, the 3-rotation is the easiest to conceptualize as a horizontal $\hat{x}$-axis and a vertical $\hat{y}$-axis is the most common representation and makes for a solid foundation. First, any point P with coordinates (x_i,y_i, z_i) is placed in the inertial space with axes ($\hat{x}_i$, $\hat{y}_i$, $\hat{z}_i$), as seen in Figure 3a.



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(a) (b)

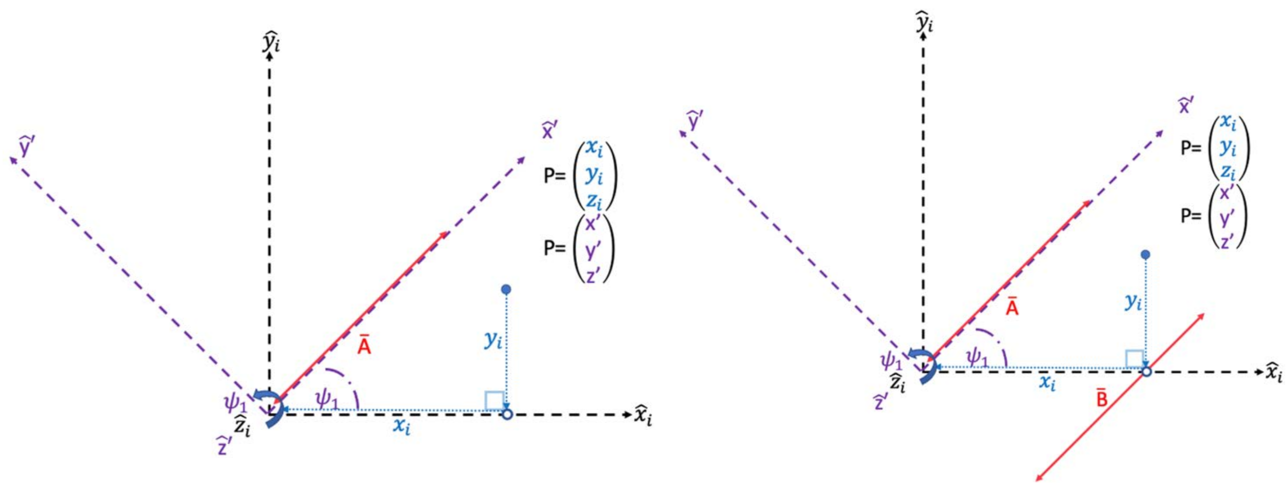
Figure 3. (a) Point P1 is dropped within the inertial orientation to develop the direction cosine matrix. (b) 3-Rotation about the $\hat{z}_i$ & $\widehat{z'}$ axes to develop P's coordinates in the new reference frame.

Table 2. Proximal variable and nomenclature definitions for Figure 3 to Figure 8.

Term	Definition	Term	Definition	Term	Definition	Term	Definition
$\hat{x}_i$	x-axis in the inertial non-rotating reference frame	x_i	point P's location on x-axis in the inertial non-rotating reference frame	$\widehat{x'}$	x-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\widehat{z'}$ axis	x'	point P's location on the $\widehat{x'}$ -axis after the rotation about the $\hat{z}_i$ axis to the $\widehat{z'}$ axis
$\hat{y}_i$	y-axis in the inertial non-rotating reference frame	y_i	point P's location on y-axis in the inertial non-rotating reference frame	$\widehat{y'}$	y-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\widehat{z'}$ axis	y'	point P's location on the $\widehat{y'}$ -axis after the rotation about the $\hat{z}_i$ axis to the $\widehat{z'}$ axis
$\hat{z}_i$	z-axis in the inertial non-rotating reference frame	z_i	point P's location on z-axis in the inertial non-rotating reference frame	$\widehat{z'}$	z-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\widehat{z'}$ axis	z'	point P's location on the $\widehat{z'}$ -axis after the rotation about the $\hat{z}_i$ axis to the $\widehat{z'}$ axis
x'_{xi}	line B segment determined by angle ψ_1 and the hypotonus x_i	x'_{yi}.	line B segment determined by the angle ψ_1 and hypotonus y_i	ψ_1	angle of rotation about the $\hat{z}_i$ axis to the $\widehat{z'}$ axis	ψ_2	complementary angle to ψ_1
$\bar{A}$	line equivalent to Point P's measurement along the $\widehat{x'}$ axis	$\bar{B}$	line parallel and equidistant to $\bar{A}$ crossing through (x_i,0,0)	$\bar{C}$	line perpendicular to $\bar{A}$ and $\bar{B}$ equidistant to crossing through point P	$\bar{D}$	line perpendicular to $\bar{A}$ and $\bar{B}$ equidistant to crossing through the origin/rotation point
						$\overline{ABCD}$	rectangle developed by lines $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$

[1] Tables may have a footer.

Next, a rotation about the body frame $\hat{z}_i$ is rotated by ψ_1 to $\widehat{z'}$, as seen in Figure 3b. In Figure 4, P1 has different coordinates after the rotation to the new frame of reference. The difference between the coordinates can be calculated using simple equations.



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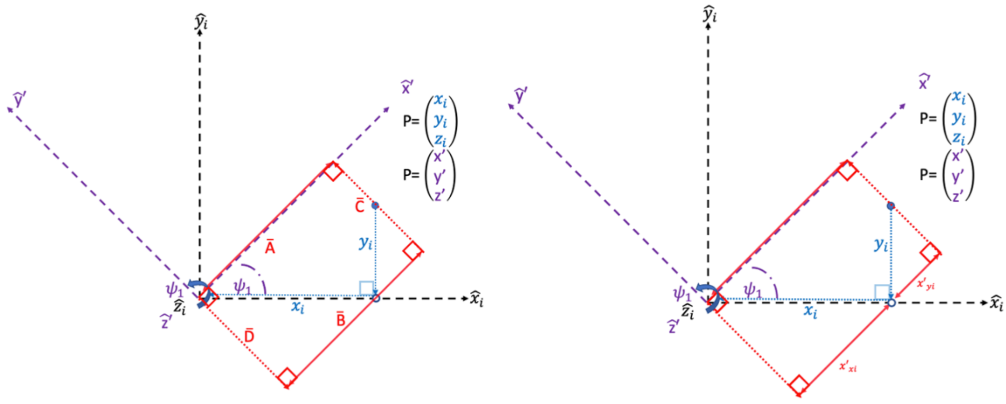
(a) (b)

Figure 4. (a) 3-Rotation about the $\hat{z_i}$ & $\widehat{z'}$ axes with the line $\bar{A}$ equivalent to x'. (b) 3-Rotation about the $\hat{z_i}$ & $\widehat{z'}$ axes with parallel lines $\bar{A}$ & $\bar{B}$ equivalent to x'.

Then, we identify a line $\bar{A}$ exists along the rotated axis equates to x' as seen in Figure 4a and another similar parallel line $\bar{B}$ with the same length that runs through the (x_i,0,0) as seen in Figure 4b.



(a) (b)

Figure 5. (a) 3-Rotation about the $\hat{z_i}$ & $\widehat{z'}$ axes adding parallel lines $\bar{C}$ & $\bar{D}$ to create rectangle $\overline{ABCD}$. (b) 3-Rotation about the $\hat{z_i}$ & $\widehat{z'}$ axes & dividing $\bar{B}$ into measurable sections x'_{xi} & x'_{yi}. A determination was made to calculate the equations for x' first. Both equations of x' and y' will be calculated eventually and the order does not matter x' is a bit easier to visualize in this case.

To measure the length of parallel lines $\bar{A}$ *and* $\bar{B}$, a rectangle can be made using two other perpendicular similar parallel lines $\bar{C}$ *and* $\bar{D}$ to develop rectangle $\overline{ABCD}$ that starts at the origin, aligns with the new rotation frame, and bisects the point P and (x_i,0,0), as seen in Figure 5a. Therefore, lengths of $\bar{A}, \bar{B},$ *and* x' are all equal.

By measuring line $\bar{B}$ in segments, we can take the summation to calculate line $\bar{A}$. Segments can be identified by which dimension ($\hat{x_i}$ or $\hat{y_i}$) of the axis in the initial orientation results in the x'_{xi} and x'_{yi} as seen in Figure 5b. Therefore x' can be calculated by the sum of x'_{xi} and x'_{yi} giving equation (3).

$$\mathrm{x}' = x'_{xi} + x'_{yi} \tag{3}$$

To calculate x'_{xi} and x'_{yi}, identify ψ_2 is a complementary angle to ψ_1. We can identify corresponding angles of the transversal of x_i across $\bar{A}$ *and* $\bar{B}$. Further, an additional complementary angle (made where y_i and $\bar{C}$ intersect at point P) can be identified as associated to the corresponding angles of where x_i crosses $\bar{B}$ as seen in Figure 6.

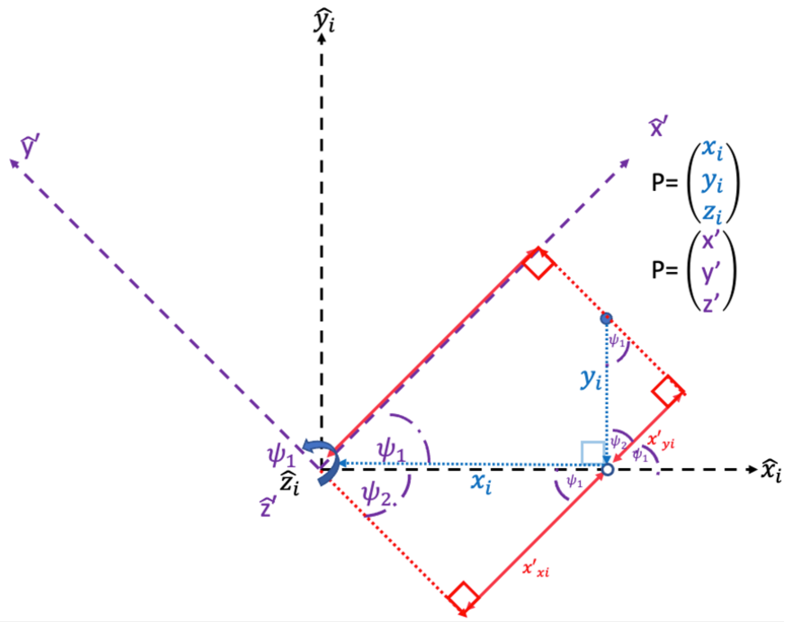


Figure 6. 3-Rotation about the $\hat{z}_i$ & $\hat{z}'$ axes and identifying angles ψ_1 & ψ_2 & pertinent corresponding angles.

Now two triangles can be identified, seen in Figure 7 with two defined angles identified as ψ_2 and ψ_1 each with a defined side x_i or y_i.

It can be seen that point P's location on the x' axis is positive and because x_i and y_i is positive, y' is a summation of each segment x'_{xi} and x'_{yi}. Therefore both x'_{xi} and x'_{yi} are a positive contribution. These conditions define x'_{xi} and x'_{yi} by ψ_1 and x_i or y_i in equations (4) and (5) using the definitions of sine and cosine as seen in Figure 8. Substituting equations (4) and (5) into equation (3) provides equation (6), the first equation of the 3-DCM.

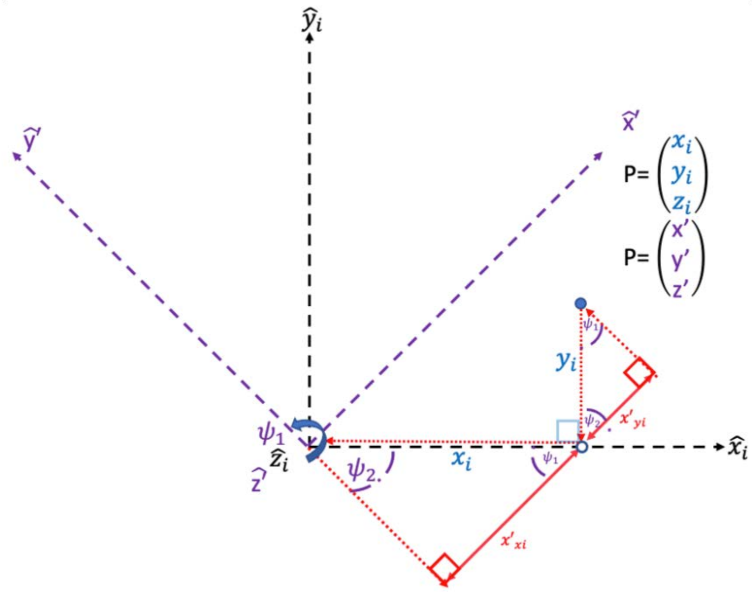


Figure 7. 3-Rotation about the $\hat{z}_i$ & $\hat{z}'$ axes highlighting the pertinent similar right triangles.

$$x'_{xi} = x_i cos(\psi_1) \tag{4}$$

$$x'_{yi} = y_i \sin(\psi_1) \tag{5}$$

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$$x' = x_i cos(\psi_1) + y_i \sin(\psi_1) + z_i 0 \tag{6}$$

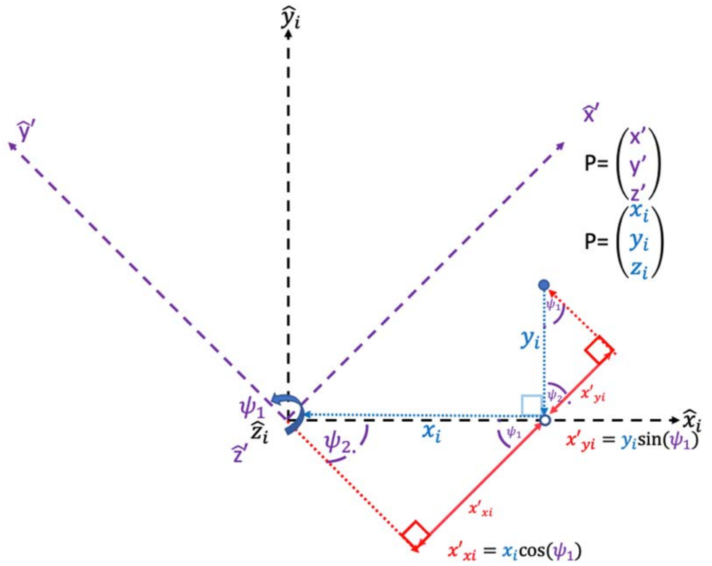

Figure 8. 3-Rotation about the $\hat{z}_i$ & $\hat{z}'$ axes highlighting segments x'_{xi} & x'_{yi} equations.

The subsequent equation of the 3-DCM is for y'. To calculate y' the same rectangle $\overline{ABCD}$, the two triangles and the defined angles from previous Figures (3-8) will be used except focusing on lines $\bar{C}$ and $\bar{D}$. Figure 9 shows that P is in the negative axis of y' and is only a portion of line $\bar{C}$. Because the rotation is about $z'$and z_i no additional length is provided to x' by z_i so can be multiplied by 0.

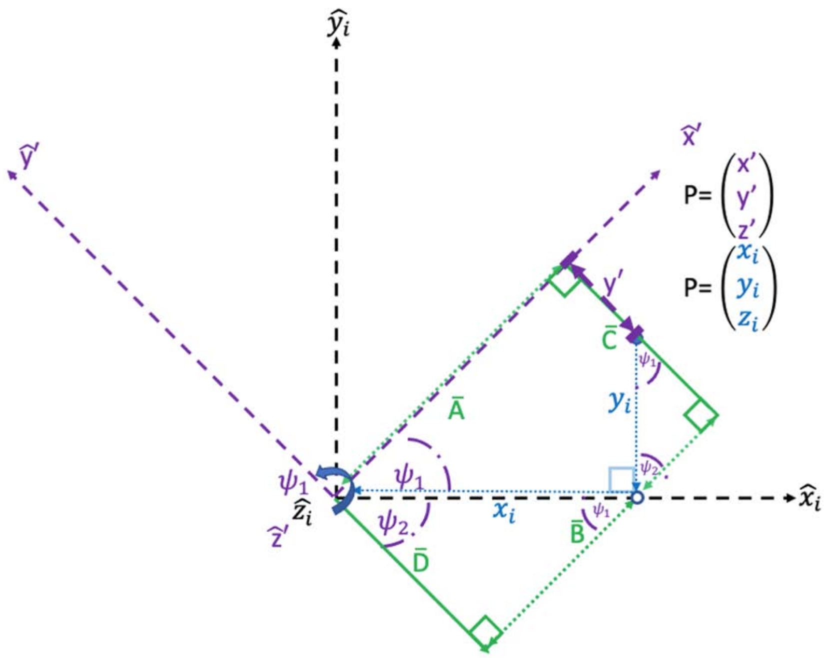

Figure 9. 3-Rotation about the $\hat{z}_i$ & $\hat{z}'$ axes using the rectangle $\overline{ABCD}$, the corresponding angles ψ_1 & ψ_2, & the same similar triangles built-in Figures 3-8 except to measure y' instead of x'.

Table 3. Proximal variable and nomenclature definitions for Figure 9 to Figure 11.

Term	Definition	Term	Definition	Term	Definition	Term	Definition
$\hat{x}_i$	x-axis in the inertial non-rotating reference frame	x_i	point P's location on x-axis in the inertial non-rotating reference frame	$\hat{x}'$	x-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	x'	point P's location on the $\hat{x}'$-axis after the rotation about the $\hat{z}_i$ axis to the $\hat{z}'$ axis
$\hat{y}_i$	y-axis in the inertial non-rotating reference frame	y_i	point P's location on y-axis in the inertial non-rotating reference frame	$\hat{y}'$	y-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	y'	point P's location on the $\hat{y}'$-axis after the rotation about the $\hat{z}_i$ axis to the $\hat{z}'$ axis
$\hat{z}_i$	z-axis in the inertial non-rotating reference frame	z_i	point P's location on z-axis in the inertial non-rotating reference frame	$\hat{z}'$	z-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	z'	point P's location on the $\hat{z}'$-axis after the rotation about the $\hat{z}_i$ axis to the $\hat{z}'$ axis
y'_{xi}	line $\bar{D}$ segment determined by angle ψ_1 and the hypotonus x_i	y'_{yi}.	line $\bar{C}$ segment determined by the angle ψ_1 and hypotonus y_i	ψ_1	angle of rotation about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	ψ_2	complementary angle to ψ_1
$\bar{A}$	line equivalent to point P's measurement along the $\hat{x}'$ axis	$\bar{B}$	line parallel and equidistant to $\bar{A}$ crossing through (x_i,0,0)	$\bar{C}$	line perpendicular to $\bar{A}$ and $\bar{B}$ equidistant to crossing through point P	$\bar{D}$	line perpendicular to $\bar{A}$ and $\bar{B}$ equidistant to crossing through the origin/rotation point
						$\overline{ABCD}$	rectangle developed by lines $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$

By segmenting line $\bar{C}$ into y' and y'_{yi}, we can take the difference of line $\bar{D}$ or y'_{xi} and a segment of y'_{yi} to calculate y'. The segments can be identified by which dimension (x or y) of the axis in the initial orientation results in the y'_{xi} and y'_{yi} as seen in Figure 10. Labeling the pertinent sides of two triangles y_{xi} and y_{yi} in Figure 11, we can use a similar equation to equation (3) to calculate y', and we get equation (7).

$$y' = y'_{xi} + y'_{yi} \quad (7)$$

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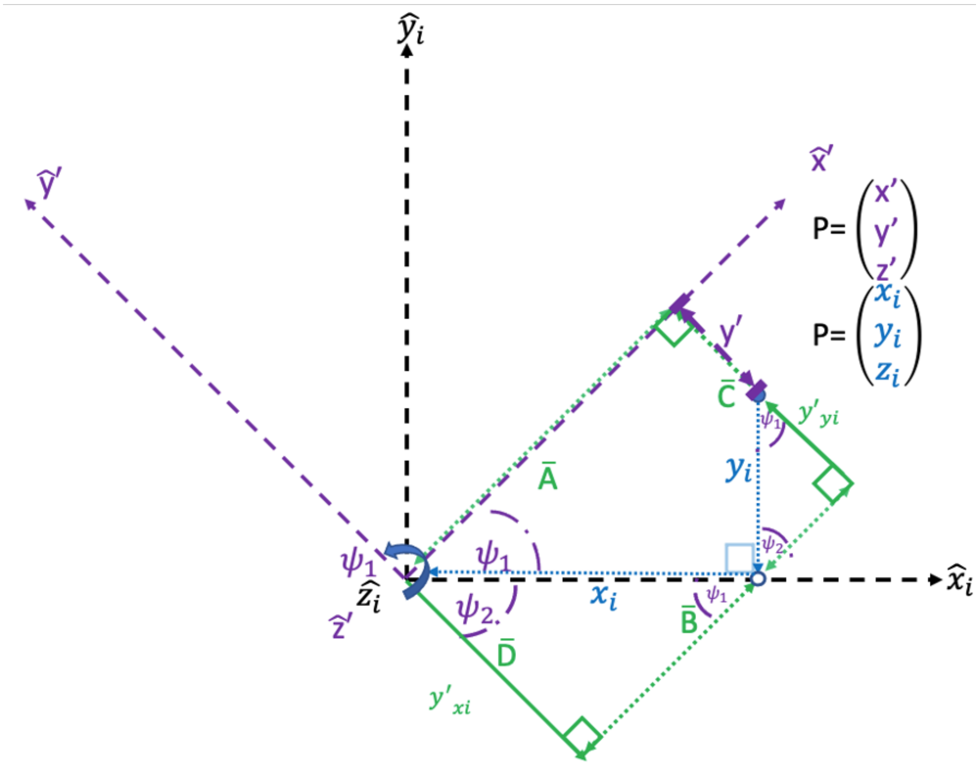

Figure 10. 3-Rotation about the $\hat{z}_i$ & $\hat{z}'$ axes highlighting y'_{xi} & y'_{yi} as part of rectangle $\overline{ABCD}$.

y' must be the negative difference between y'_{xi} and y'_{yi} for three reasons: P lies on the negative $\hat{y}'$ axis, x_i and y_i are positive, and because y' is less than the summation of y'_{xi} *and* y'_{yi}. Therefore y' will be a difference between the segments, and the longer segment y'_{xi}, must be negative in this case. These conditions define y'_{xi} and y'_{yi} by ψ_1 and x_i or y_i in equations (8) and (9) using the definitions of sine and cosine as seen in Figure 11. Substituting equations (8) and (9) into equation (7) provides equation (10), the second equation of the 3-DCM. [1] Because the rotation is about $\hat{z}_i$ and $\hat{z}'$ no additional length is provided to y' by $\hat{z}_i$ so can be multiplied by 0.

$$-y'_{xi} = x_i \sin(\psi_1) \tag{8}$$

$$y'_{yi} = y_i \cos(\psi_1) \tag{9}$$

$$y' = -x_i \sin(\psi_1) + y_i \cos(\psi_1) + z_i 0 \tag{10}$$

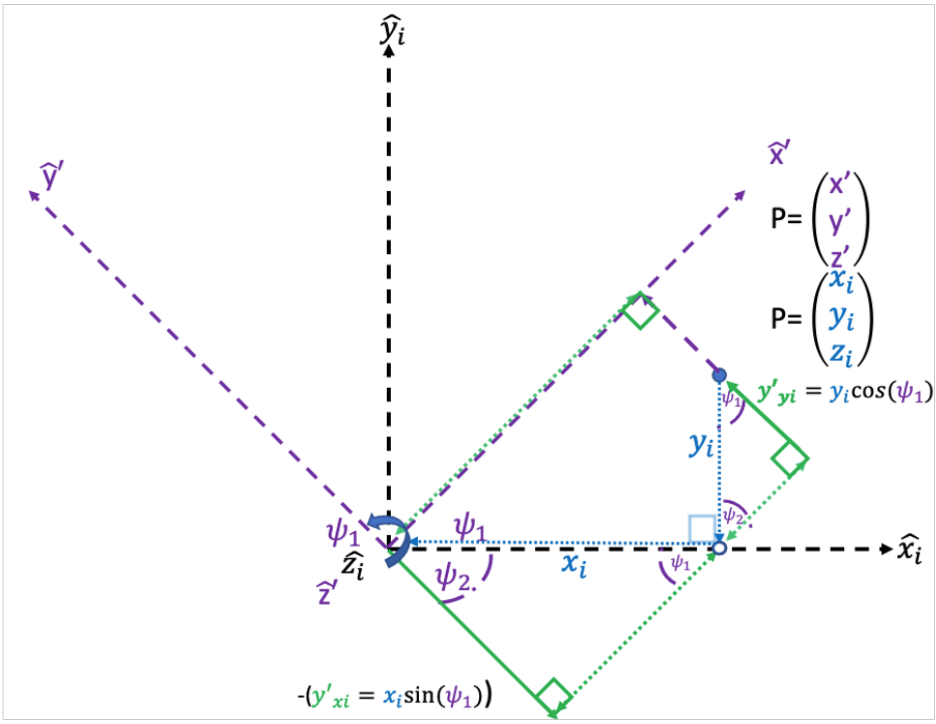

Figure 11. 3-Rotation about the $\hat{z}_i$ & $\hat{z}'$ axes & highlighting equations for segments y'_{xi} & y'_{yi}.

Equation (11) for z' is the last for this rotation. It is straightforward because the revolution is about $\hat{z}_i$ to $\hat{z}'$, hence z'equates to z_i. Because the rotation is about $\hat{z}_i$ to $\hat{z}'$ no additional length is provided to z' by x_i or y_i so both can be multiplied by 0.

$$y' = x_i 0 + y_i 0 + z_i \tag{11}$$

Finally, bringing the equations (6), (10), and (11) into the format that a student can change into a matrix format seen in equation (24), the 3-rotation matrix can be seen in equation (12).

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\psi_1) & \sin(\psi_1) & 0 \\ -\sin(\psi_1) & \cos(\psi_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{12}$$

Continuing to the 2-rotation in Figure 12, the $\hat{y}'$ axis is rotated by θ_1 to $\hat{y}''$. Figure 1 shows the view of the rotation is changed so that the positive axis of the rotating axis ($\hat{y}'$ and $\hat{y}''$) is viewed as coming directly out of the page. Point P is now taken from (x',y',z') to (x'',y'',z''), similar to Figure 8 and Figure 11. Then, similar equations can be developed using the same mathematics equations (13-22).

$$z''_{z'} = z'\cos(\theta_1) \tag{13}$$
$$z''_{x'} = x'\sin(\theta_1) \tag{14}$$

$$x''_{z'} = -z'\sin(\theta_1) \tag{15}$$
$$x''_{x'} = x'\cos(\theta_1) \tag{16}$$

$$z'' = z''_{z'} + z''_{x'} \tag{17}$$
$$x'' = x''_{z'} + x''_{x'} \tag{18}$$

$$x'' = x'\cos(\theta_1) + y'0 - z'\sin(\theta_1) \tag{19}$$

$$y'' = x'0 + y' + z'0 \tag{20}$$

$$z'' = x'\sin(\theta_1) + y'0 + z'\cos(\theta_1) \tag{21}$$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & 0 & -\sin(\theta_1) \\ 0 & 1 & 0 \\ \sin(\theta_1) & 0 & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{22}$$

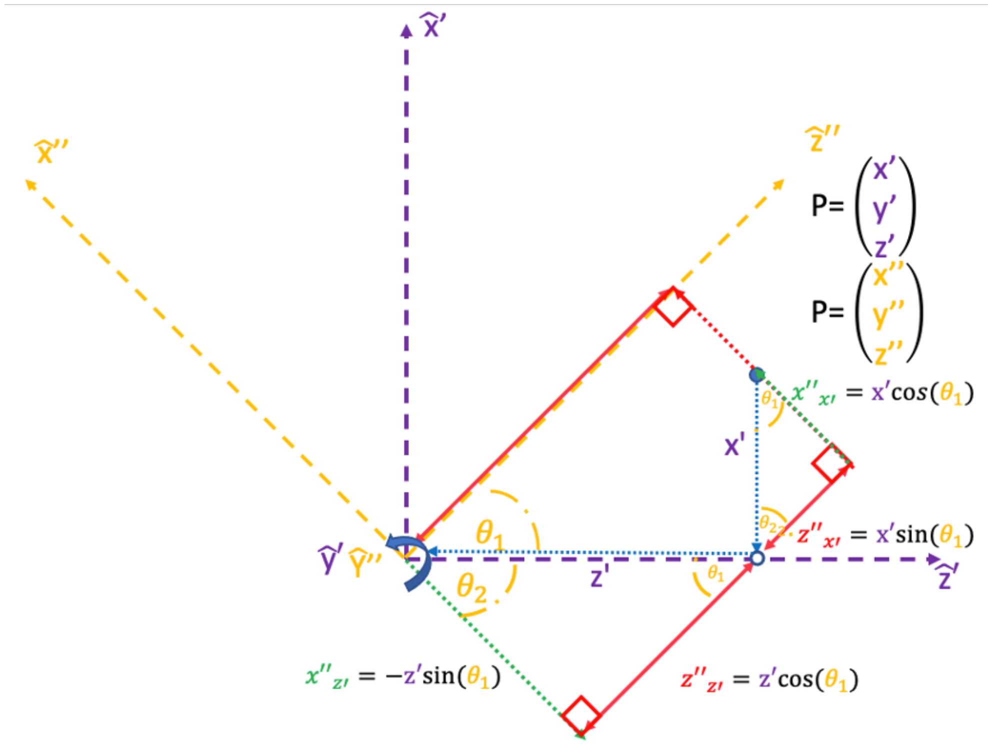

Figure 12. 2-Rotation about the $\hat{y}'$ & $\hat{y}''$ axes to develop P1's coordinates in the new reference frame with all similar angles, shapes & equations established in Figures 3-11.

Table 4. Proximal variable and nomenclature definitions for Figure 12.

Term	Definition	Term	Definition	Term	Definition	Term	Definition
$\hat{x}'$	x-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	x'	point P's location on the $\hat{x}'$-axis after the rotation about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	$\hat{x}''$	x-axis in the rotated reference frame about the $\hat{y}'$ axis to the $\hat{y}''$ axis	x''	point P's location on the $\hat{x}''$-axis after the rotation about the $\hat{y}'$ axis to the $\hat{y}''$ axis
$\hat{y}'$	y-axis in the new rotated reference frame about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	y'	point P's location on the $\hat{y}'$-axis after the rotation about the $\hat{z}_i$ axis to the $\hat{z}'$ axis	$\hat{y}''$	y-axis in the rotated reference frame about the $\hat{y}'$ axis to the $\hat{y}''$ axis	y''	point P's location on the $\hat{y}''$-axis after the rotation about the $\hat{y}'$ axis to the $\hat{y}''$ axis
$\hat{z}'$	z-axis in the new rotated reference frame about the $\hat{z}_i$	z'	point P's location on the $\hat{z}'$-axis after the rotation	$\hat{z}''$	z-axis in the rotated reference frame about	z''	point P's location on the $\hat{z}''$-axis after the rotation

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	axis to the $\hat{z}'$ axis		about the $\hat{z}_i$ axis to the $\hat{z}'$ axis		the $\hat{y}'$ axis to the $\widehat{y''}$ axis		about the $\hat{y}'$ axis to the $\widehat{y''}$ axis
$x''_{z'}$	Determined using the same techniques as y'_{xi}	$x''_{x'}$.	Determined using the same techniques as y'_{yi}.	θ_1	angle of rotation about the $\hat{y}'$ axis to the $\widehat{y''}$ axis	θ_2	complementary angle to θ_1
$z''_{z'}$	Determined using the same techniques as x'_{xi}	$z''_{x'}$.	Determined using the same techniques as x'_{yi}.				

Finally, the 1-rotation in Figure 13 is developed in the same manner as the 3-Rotation and 2-Rotation. Where the x'' axis is rotated by ϕ_1 to x_b thereby providing the final conversion from (x'',y'',z'') to (x_b,y_b,z_b) coordinates; this provides equations (23-32).

$$y_{by''} = y''\cos(\phi_1) \tag{23}$$

$$y_{bz''} = z''\sin(\phi_1) \tag{24}$$

$$z_{by''} = -y''\sin(\phi_1) \tag{25}$$

$$z_{bz''} = z''\cos(\phi_1) \tag{26}$$

$$y_b = y_{by''} + y_{bz''} \tag{27}$$

$$z_b = z_{by''} + z_{bz''} \tag{28}$$

$$x_b = x'' + y''0 + z''0 \tag{29}$$

$$y_b = x''0 + y''\cos(\phi_1) + z''\sin(\phi_1) \tag{30}$$

$$z_b = x''0 + -y''\sin(\phi_1) + z''\cos(\phi_1) \tag{31}$$

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_1) & \sin(\phi_1) \\ 0 & -\sin(\phi_1) & \cos(\phi_1) \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \tag{32}$$

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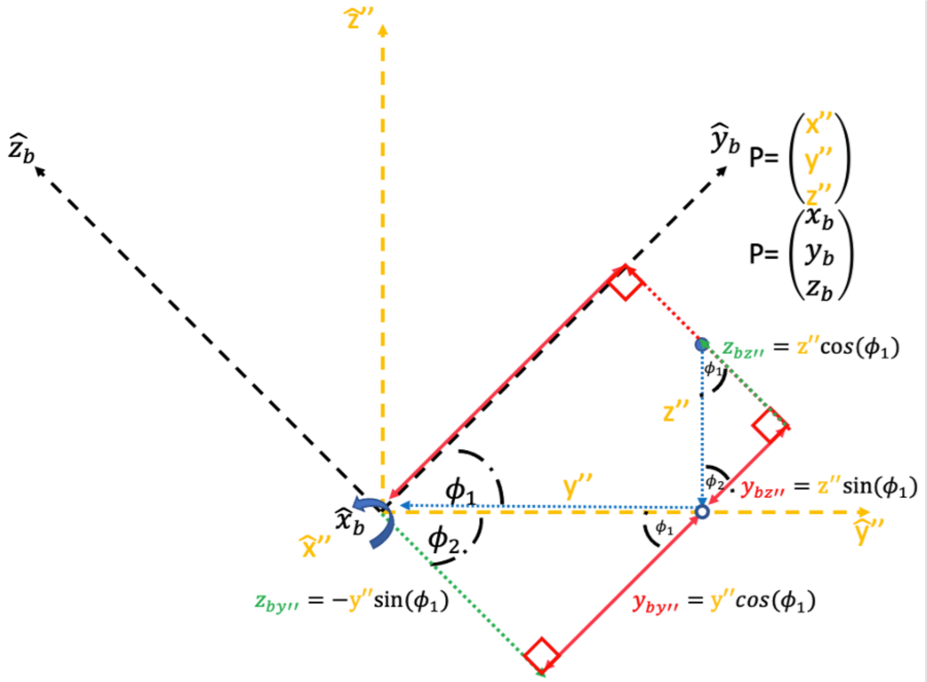

Figure 13. 1-Rotation about the $\widehat{x''}$ & $\widehat{x_b}$ axes to develop P1's coordinates in the new reference frame with all similar angles, shapes & equations established in Figures 3-11.

Table 5. Proximal variable and nomenclature definitions for Figure 13.

Term	Definition	Term	Definition	Term	Definition	Term	Definition
$\widehat{x''}$	x-axis in the rotated reference frame about the $\widehat{y'}$ axis to the $\widehat{y''}$ axis	x''	point P's location on the $\widehat{x''}$-axis after the rotation about the $\widehat{y'}$ axis to the $\widehat{y''}$ axis	$\widehat{x_b}$	x-axis in the rotated reference frame about the $\widehat{y''}$ axis to the $\widehat{x_b}$ axis	x_b	point P's location on the $\widehat{x_b}$-axis after the rotation about the $\widehat{y''}$ axis to the $\widehat{x_b}$ axis
$\widehat{y''}$	y-axis in the rotated reference frame about the $\widehat{y'}$ axis to the $\widehat{y''}$ axis	y''	point P's location on the $\widehat{y''}$-axis after the rotation about the $\widehat{y'}$ axis to the $\widehat{y''}$ axis	$\widehat{y_b}$	y-axis in the rotated reference frame about the $\widehat{y''}$ axis to the $\widehat{x_b}$ axis	y_b	point P's location on the $\widehat{y_b}$-axis after the rotation about the $\widehat{y''}$ axis to the $\widehat{x_b}$ axis
$\widehat{z''}$	z-axis in the rotated reference frame about the $\widehat{y'}$ axis to the $\widehat{y''}$ axis	z''	point P's location on the $\widehat{z''}$-axis after the rotation about the $\widehat{y'}$ axis to the $\widehat{y''}$ axis	$\widehat{z_b}$	z-axis in the rotated reference frame about the $\widehat{y''}$ axis to the $\widehat{x_b}$ axis	z_b	point P's location on the $\widehat{z_b}$-axis after the rotation about the $\widehat{y''}$ axis to the $\widehat{x_b}$ axis
$z_{by''}$	Determined using the same techniques as y'_{xi}	$z_{bz''}$.	Determined using the same techniques as y'_{yi}.	ϕ_1	angle of rotation about the $\widehat{y''}$ axis to the $\widehat{x_b}$ axis	ϕ_2	complementary angle to ϕ_1
$y_{by''}$	Determined using the same techniques as	$y_{bz''}$.	Determined using the same techniques as x'_{yi}.				

x'_{xi}

Finally, taking each rotation matrix from equations (12), (22), and (32), the matrices can be multiplied to move from the inertial to the body frame seen in equation (33). However, the three matrices can be multiplied to develop one overarching direction cosine matrix in a simplified equation (34), the 321-DCM.

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi_1 & sin\phi_1 \\ 0 & -sin\phi & cos\phi_1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ 0 & 1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} cos\psi_1 & sin\psi_1 & 0 \\ -sin\psi_1 & cos\psi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos\theta cos\psi & \cos\theta sin\psi & -\sin\theta \\ sin\phi sin\theta cos\psi - cos\phi sin\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi\cos\theta \\ cos\phi sin\theta cos\psi + sin\phi sin\psi & cos\phi sin\theta sin\psi - sin\phi cos\psi & cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{34}$$

3. Discussion

Using the two-dimensional rotation view coupled with the proposed method of creating direction cosine matrix provides a simplified and repeatable process of development. However, there are a couple of recommendations to keep the process simple. Simple decisions on the order of axis rotation, point P's location, and the rotation angle can have significant difficulty if not adequately thought through.

Starting with the 3-rotation is the easiest to conceptualize as a horizontal $\hat{x}$-axis and a vertical $\hat{y}$-axis is the most common representation and makes for a solid foundation where another rotation may be more challenging to visualize. Deciding which axis rotation to start with is not highly consequential to the overall difficulty but beginning with the 3-rotation may help a new student grasp the concept.

Using a point P that lies in the positive $\hat{x}$, $\hat{y}$, or $\hat{z}$ axis is recommended. A negative point P, while not insurmountable, can add unrequired difficulty to the problem. If point P lies in either the negative $\hat{x}$, $\hat{y}$, or $\hat{z}$ axis prior to the rotation, the calculations become more confusing to maintain perspective because it is easy to overlook which variable is negative and how it affects the equations. Ultimately, if x_i, y_i, or z_i of point P (x_i,y_i,z_i) is negative, that negative factor must be maintained through the equations to get the proper matrix.

It is recommended to use an acute rotation angle. Using an obtuse rotation angle can further complicate the process; however, similar rectangles, triangles, and their corresponding angles can be found. The question is which angle to utilize for the triangle. The most straightforward method found was to take the cosine and sin of the obtuse angle and find the corresponding acute angles that equate to it but may be its opposite sign. Then use those equations, but you must carry the sign of the obtuse angle throughout the rotation. It is recommended for ease of calculation to start with the 3-rotation, maintain point P in the positive $\hat{x}$, $\hat{y}$, and $\hat{z}$ axis, and use an acute angle.

Direction cosine matrix (DCM) rotations, while helpful, struggle with computations to express specific movements. Specifically, DCM struggle when kinematic singularities come into play. A kinematic singularity is when the cosine of the rotational angles used for the DCM equals zero. It is a regular practice to divide by the cosine of the rotation angle to calculate the Euler-angles rate, so when the cosine of rotation is zero, it is an issue. This division by the cosine of the rotation angle when the cosine of the rotation angle is zero causes a calculation error when the systems try to calculate infinity. For these reasons, the quaternions do much of the heavy lifting of the rotations.

4. Conclusions

The method provided is generalized such that it should work for all rotations and any non-zero-point P. However, some may find the process lengthy and confusing compared to other methods. The development of a DCM, while a foundational portion of kinematics, is but a small step. Once a DCM is created, it does not need to be recalculated to be utilized.

The 2DRV does not allow a member to see all three rotations simultaneously. 2DRV simplifies a single rotation, but does not always help visualize all three rotations. It is recommended that this method be used in tandem with a three-dimensional view to help link the equations to the movement.

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Appendix A

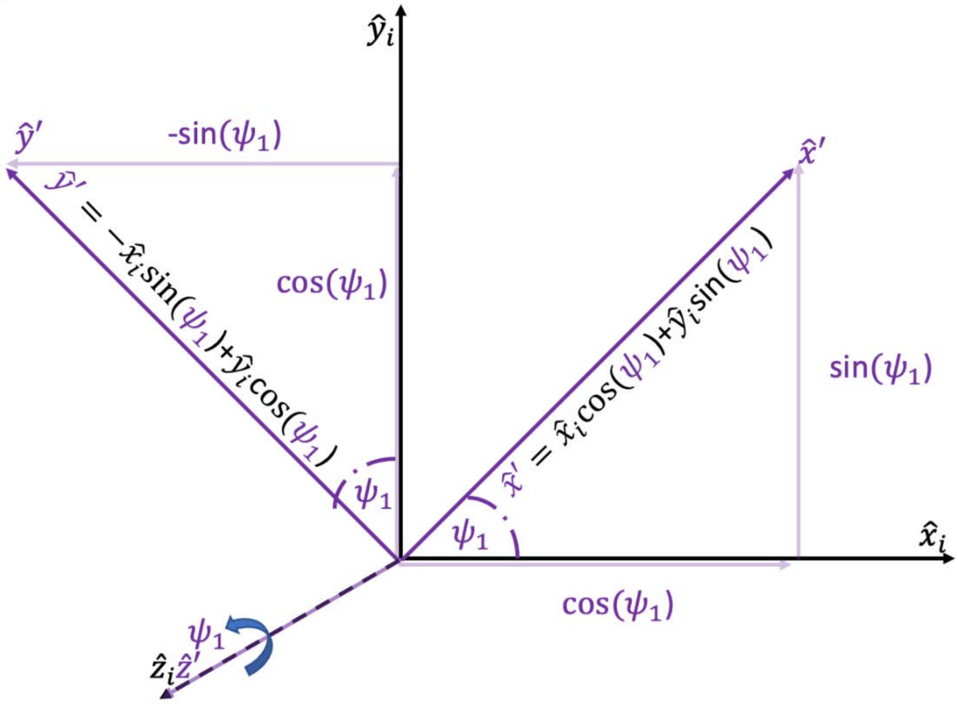

Figure A1. 3-Rotation about the $\hat{z}_i$ & $\hat{z}'$ axes [23].

$$\begin{Bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{Bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{Bmatrix} \quad \textbf{(1a)}$$

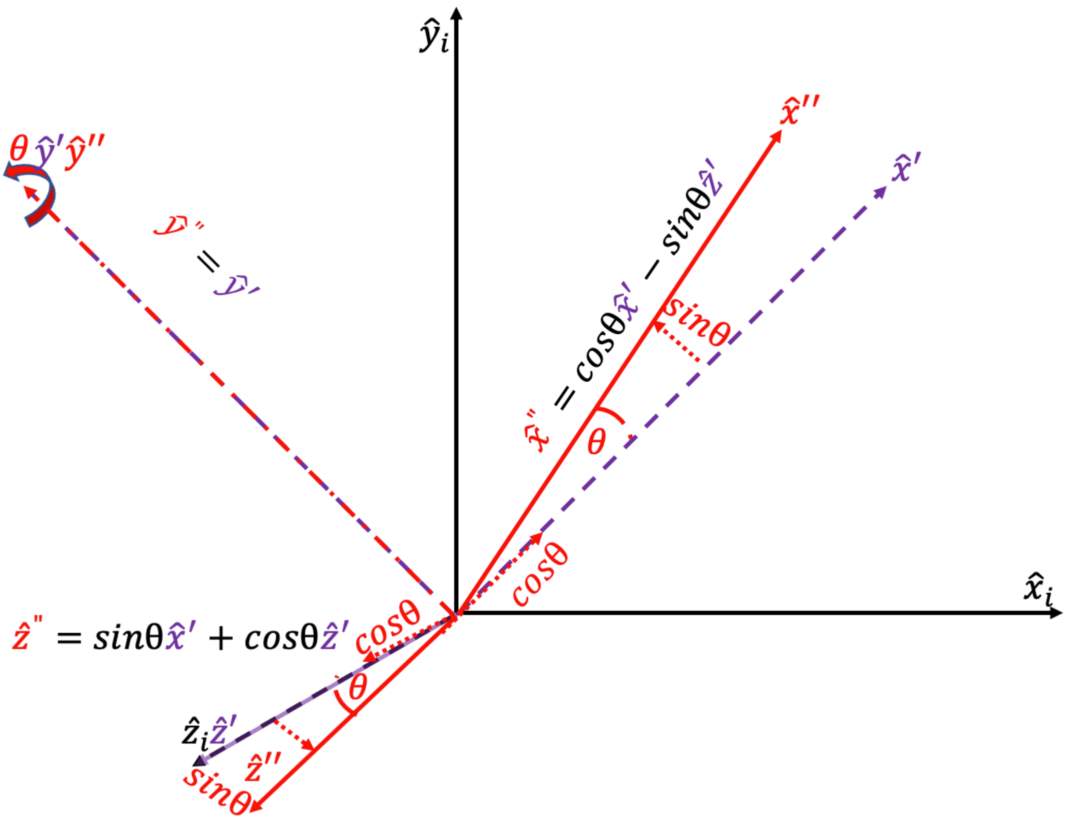

Figure A2. 2-Rotation about the $\hat{y}'$ & $\hat{y}''$ axes [23].

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$$\begin{Bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{Bmatrix} \quad (2a)$$

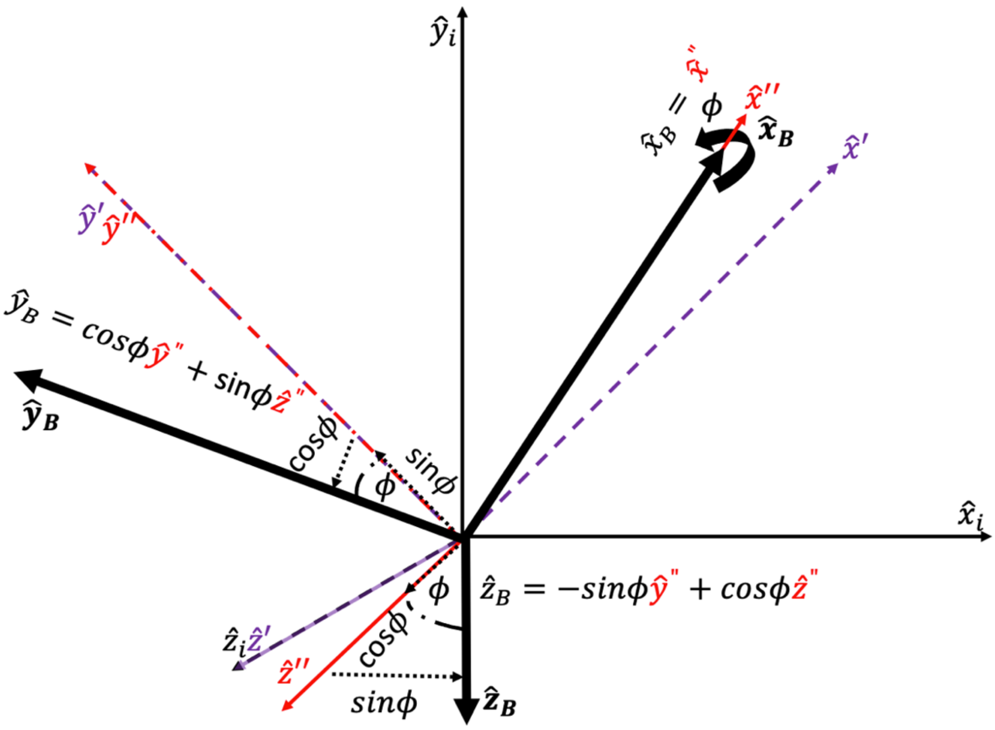

Figure A3. 3-Rotation about the $\widehat{x''}$ & $\widehat{x_b}$ axes [23].

$$\begin{Bmatrix} \hat{x}_B \\ \hat{y}_B \\ \hat{z}_B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{Bmatrix} \quad (3a)$$

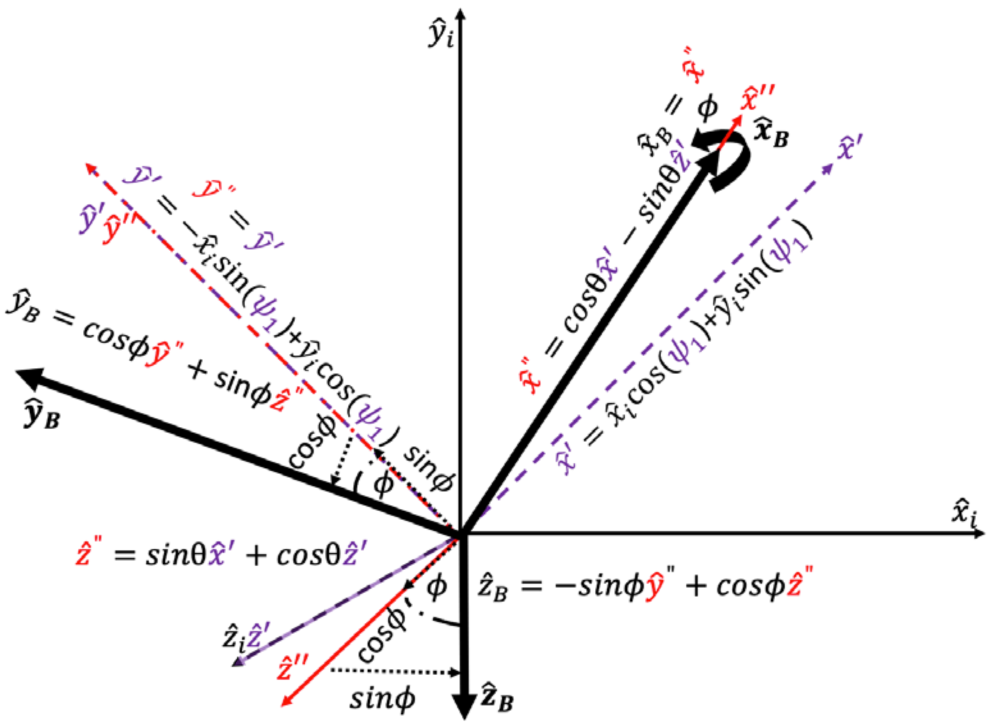

Figure A4. Compiling the 321-Rotations in one view with all equations [23].

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$$\begin{Bmatrix} \hat{x}_B \\ \hat{y}_B \\ \hat{z}_B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & sin\phi \\ 0 & -sin\phi & cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{Bmatrix} \quad \text{(4a)}$$

$$\begin{Bmatrix} \hat{x}_B \\ \hat{y}_B \\ \hat{z}_B \end{Bmatrix} = \begin{bmatrix} \cos\theta cos\psi & \cos\theta sin\psi & -\sin\theta \\ sin\phi sin\theta cos\psi - cos\phi sin\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & sin\phi\cos\theta \\ cos\phi sin\theta cos\psi + sin\phi sin\psi & cos\phi sin\theta sin\psi - sin\phi cos\psi & cos\phi\cos\theta \end{bmatrix} \begin{Bmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \end{Bmatrix} \quad \text{(5a)}$$