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Not peer-reviewed version

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Posted Date: 15 July 2026

doi: 10.20944/preprints202607.1148.v1

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Quantum Entanglement from Theory of Classical Gaussian Processes

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Abstract

We propose a novel approach to the problem of interconnecting the probabilistic formalisms of classical and quantum physics, focusing on its most challenging aspect: the classical probabilistic generation of entangled states. We show that the statistics encoded in the density operators of composite quantum systems correspond to fourth-order classical statistics. Specifically, to generate a density operator, one must consider the covariance of a random covariance operator. We term this framework the Double Covariance Model (DCM). This double covariance possesses a non-trivial internal structure that arises from the interplay between two distinct time scales, combining temporal and statistical covariances. In this article, we exploit a well-known property of Gaussian processes: the second-order moment determines the moments of higher orders, specifically the fourth-order moment. This Gaussian reduction simplifies the DCM by reducing it to second-order statistics. Utilizing (circular) Gaussian processes simplifies and generalizes the DCM construction for entangled states, rendering it mathematically rigorous. Furthermore, it clarifies the classical probabilistic meaning of concurrence, a foundational quantitative measure of entanglement.

Keywords: classical vs. quantum probability; double covariance model; quantum entangled state; classical Gaussian processes; fourth-order vs. second order statistics; micro- and macro-time scales

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1. Introduction

Understanding the interrelation between the probabilistic formalisms of classical and quantum physics remains a complex foundational problem. This topic has been analyzed and explored from various perspectives[1] One of the most challenging issues is the possibility of classical probabilistic modeling of the states of composite quantum systems, with a specific emphasis on the generation of entangled states. A new approach to this problem was proposed in [24] within the framework of the *Double Covariance Model* (DCM), wherein density operators are classically generated as covariances of covariances.

In practice, this classical-to-quantum transition is quite subtle. The DCM is based on the interplay between two distinct time scales:

- a fine time scale (subquantum),
- a rough time scale (quantum).

The quantum scale corresponds to the scale of physical measurements, whereas the subquantum time scale represents the ontic time scale—the scale of reality as it is; see [25] for the coupling with the hydrodynamic limit model; cf. [4]).

The first-order covariance operator $\hat{C}_\Delta$ is obtained via temporal correlation over the interval Δ, which determines the measurement time scale. This temporal covariance is a random operator

[1] For foundational overviews, see the monographs [22,23]. Key mathematical works reviewing the structural interplay between classical and quantum probability include [2,3,15,16,29,31]; for epistemological analyses, see [27,34,35,37,40]. For physical frameworks exploring classical probabilistic structures beyond standard quantum theory, see, for example, [4,7–14,17,18,21,28,30,32,36,39,42].

$\hat{C}_\Delta = \hat{C}_\Delta(\omega)$, where the randomness represents statistical randomness described by the Kolmogorov probability model. Ultimately, the DCM operates with the statistical covariance operator of this random operator, thereby employing fourth-order statistics. This construction leads to the "double covariance operator" $\widehat{C} : H_A \otimes H_B \to H_A \otimes H_B$. The classical-to-quantum transition from the double covariance operator $\widehat{C}$ to the corresponding density operator $\hat{\rho}_C$ is achieved via trace normalization:

$$\widehat{C} \to \hat{\rho}_C = \widehat{C}/\mathrm{Tr}\widehat{C}. \quad (1)$$

All density operators, including those of entangled states, can be generated in this manner. Further development of the DCM [25] led to the derivation of quantum Markovian dynamics by exploring the framework of the hydrodynamic limit.

In light of the above considerations, the DCM shapes entangled quantum states on the basis of fourth-order classical statistics—specifically, the covariance of covariances. In this paper, we exploit a well-known property of Gaussian processes: the second-order covariance completely determines the covariances of higher orders, including the fourth-order covariance used in the DCM. Our primary aim is the Gaussian reduction of the DCM to second-order covariances, which enables the generation of entangled states using classical Gaussian processes. Utilizing these processes simplifies and generalizes the foundational construction proposed in [24], rendering it mathematically rigorous. Furthermore, this Gaussian formalism provides a subquantum interpretation of concurrence—one of the primary quantitative measures of entanglement—in terms of energy redistribution.

The classical (subquantum) states of the subsystems S_A and S_B of a composite quantum system S_{AB} are mathematically described by circular Gaussian stochastic processes X_t and Y_t taking values in their corresponding Hilbert spaces. Surprisingly, the Gaussian DCM analysis demonstrates that entangled quantum states can be generated by statistically independent Gaussian processes. Within the DCM, the fundamental origin of entanglement is not statistical dependence, but rather temporal synchronization at the subquantum time scale. This synchronization can be realized via various orthogonal decompositions of the space $L_2(\Delta)$; such decompositions determine the synchronized energy redistribution of the subquantum signals X_t and Y_t.

We emphasize that DCM is in its early stages of development and does not aim to provide a comprehensive classical probabilistic framework for all of quantum mechanics. At present, DCM addresses a specific foundational challenge: constructing a viable classical probabilistic model that accounts for quantum entanglement. Beyond this foundational role, DCM offers immediate pragmatic utility for the classical probabilistic generation of entangled states, with applications ranging from quantum-inspired computing to quantum-like modeling in cognition and decision-making [26]. Finally, we highlight a key foundational characteristic of DCM: broad classes of distinct classical states (stochastic processes) can map to the exact same quantum state, a property that shares conceptual ground with the work of 't Hooft [17,18] (see also [39]).

2. Mathematical and Physical Preliminaries

2.1. Physics

The Double Covariance Model (DCM) is intended as a classical probabilistic framework for the generation of quantum states of composite systems. We consider a bipartite quantum system

$$S_{AB} = S_A + S_B,$$

with subsystem state spaces represented by complex Hilbert spaces

$$H_A, \qquad H_B.$$

The corresponding quantum state space of the composite system is the tensor product

$$H_A \otimes H_B.$$

In the DCM, the quantum state of the composite system is not taken as fundamental. Instead, it is generated from a deeper classical probabilistic description at a finer time scale. The basic subquantum variables are stochastic processes

$$X = \{X_t\}_{t\in\Delta}, \qquad Y = \{Y_t\}_{t\in\Delta},$$

taking values in the subsystem Hilbert spaces H_A and H_B, respectively. These processes represent the classical states of the subsystems S_A and S_B at the subquantum time scale.

The DCM is based on the interplay of two distinct levels of description.

(i) At the *subquantum* time scale, the processes X_t and Y_t describe the fine temporal dynamics of the subsystems.

(ii) At the *quantum* or observational time scale, one does not observe the instantaneous values of these processes. Instead, one forms a temporal covariance over a time window Δ, and then takes the statistical covariance of this random temporal covariance.

Thus the DCM is built from two successive averaging procedures: first a temporal averaging over the observation window Δ, and then a statistical averaging over the probability space of random parameters. This is why the resulting object is a *double covariance*: it is a fourth-order classical statistical object constructed as the covariance of a random covariance operator.

The physical significance of the model is that it provides a classical probabilistic mechanism for the generation of quantum states, including entangled states. In particular, the DCM suggests that quantum entanglement need not originate from ordinary second-order statistical dependence between the subquantum processes X and Y. Instead, it may arise from a more subtle temporal organization of the subquantum signals within the observation window Δ. In the Gaussian setting studied in this paper, this temporal organization is encoded in the geometry of the Karhunen–Loève modes and in the integrated tensors generated by them.

As was emphasized, the DCM is based on the interplay between two time scales: a subquantum (microscopic) time scale and a quantum (macroscopic) time scale. We denote the corresponding time variables by t and τ, respectively. The length $|\Delta|$ of the observation window plays the role of a unit of macroscopic time. In the present paper we work only with the subquantum time variable t. We do not study the evolution of the quantum state $\hat{\rho} = \hat{\rho}(\tau)$; for the derivation of quantum Markov dynamics within the DCM by means of the hydrodynamic limit formalism, see [25]. Our focus here is on the generation of density operators at a fixed macroscopic time scale.

The quantum time scale is interpreted as the time scale of observational dynamics. In this respect, we follow Bohr's view of quantum mechanics as a theory of measurement outcomes and their probabilistic structure, rather than a direct description of underlying microscopic processes [34,35].

The DCM is related only indirectly to the traditional foundational programme of hidden-variable theories [5,6]. The subquantum stochastic processes that represent classical states in the DCM should not be identified with hidden variables in the conventional sense. Rather, the DCM can be viewed as a natural development of Prequantum Classical Statistical Field Theory (PCSFT) [21], a theory of classical random fields underlying quantum phenomena.

At the same time, the DCM differs essentially from PCSFT. PCSFT is formulated in terms of second-order moments of random fields, whereas the DCM is based on fourth-order moments - a double-covariance construction and, crucially, on the interplay between microscopic and macroscopic time scales.

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2.2. Mathematics

All Hilbert spaces in this paper are complex. For Hilbert spaces H_1 and H_2, we denote by $L(H_1, H_2)$ the space of continuous linear operators from H_1 to H_2. In particular, $L(H) := L(H, H)$. (In the finite dimensional case these are simply spaces of linear operators.)

Throughout the paper, H_A and H_B denote finite-dimensional complex Hilbert spaces associated with the two subsystems. We shall repeatedly use the canonical identification of the tensor product space $H_A \otimes H_B$ with the space of linear operators $L(H_B, H_A)$, given on simple tensors by

$$a \otimes b \longleftrightarrow |a\rangle\langle b|, \qquad a \in H_A,\ b \in H_B.$$

Accordingly, we will freely pass between vectors in $H_A \otimes H_B$ and the corresponding operators in $L(H_B, H_A)$.

Let $(\Omega, \mathcal{F}, P)$ be a Kolmogorov probability space. Here Ω is a set of random parameters ("elementary events"), $\mathcal{F}$ is a σ-algebra of its subsets representing events, and P is a probability measure on Ω. Let $H_A,\ H_B$ be (finite-dimensional) complex Hilbert spaces associated with the subsystems S_A and S_B, respectively. We consider H_A- and H_B-valued stochastic processes $X = \{X_t\}_{t\in\Delta}$, $Y = \{Y_t\}_{t\in\Delta}$, defined on $(\Omega, \mathcal{F}, P)$, where $\Delta \subset \mathbb{R}$ is a finite observation interval.

Centered circular Gaussian processes

We shall assume throughout that the processes under consideration are *centered circular Gaussian*.

Definition 1. *Let H be a complex separable Hilbert space and let*

$$X = \{X_t\}_{t\in\Delta}$$

be an H-valued stochastic process. We say that X is a centered circular Gaussian process *if for every $n \geq 1$, every choice of times*

$$t_1, \ldots, t_n \in \Delta,$$

and every choice of vectors

$$h_1, \ldots, h_n \in H,$$

the complex random vector

$$(\langle h_1, X_{t_1}\rangle, \ldots, \langle h_n, X_{t_n}\rangle) \in \mathbb{C}^n$$

is jointly complex Gaussian, centered,

$$\mathbb{E}\langle h_j, X_{t_j}\rangle = 0, \qquad j = 1, \ldots, n,$$

and circularly symmetric in the sense that

$$e^{i\theta}(\langle h_1, X_{t_1}\rangle, \ldots, \langle h_n, X_{t_n}\rangle) \stackrel{d}{=} (\langle h_1, X_{t_1}\rangle, \ldots, \langle h_n, X_{t_n}\rangle)$$

for every $\theta \in [0, 2\pi)$.

For centered complex Gaussian processes, circularity is equivalent to the vanishing of all pseudo-covariances:

$$\mathbb{E}\big[\langle h, X_t\rangle\, \langle g, X_s\rangle\big] = 0, \qquad h, g \in H, \quad t, s \in \Delta.$$

Hence the law of a centered circular Gaussian process is completely determined by its covariance kernel.

doi:10.20944/preprints202607.1148.v1

Covariance kernels of the pair (X, Y)

For the pair of processes X and Y, we define the operator-valued second-order covariance kernels

$$\hat{R}_{XX}(t,s) = \mathbb{E}\big[\,|X_t\rangle\langle X_s|\,\big] \in L(H_A),$$

$$\hat{R}_{YY}(t,s) = \mathbb{E}\big[\,|Y_t\rangle\langle Y_s|\,\big] \in L(H_B),$$
$$\hat{R}_{XY}(t,s) = \mathbb{E}\big[\,|X_t\rangle\langle Y_s|\,\big] \in L(H_B, H_A),$$
$$\hat{R}_{YX}(t,s) = \mathbb{E}\big[\,|Y_t\rangle\langle X_s|\,\big] \in L(H_A, H_B).$$

These kernels form the block covariance kernel of the joint process

$$(X_t, Y_t) : \Omega \to H_A \oplus H_B,$$

namely

$$\hat{\Gamma}(t,s) = \begin{pmatrix} \hat{R}_{XX}(t,s) & \hat{R}_{XY}(t,s) \\ \hat{R}_{YX}(t,s) & \hat{R}_{YY}(t,s) \end{pmatrix}.$$

If X and Y are jointly Gaussian and

$$\hat{R}_{XY}(t,s) \equiv 0 \qquad (t,s \in \Delta),$$

then the processes X and Y are independent. This fact will be used repeatedly below.

Temporal Hilbert spaces and covariance operators

To formulate the Karhunen–Loève decomposition, we introduce the temporal Hilbert spaces

$$\mathcal{H}_A := L^2(\Delta; H_A), \qquad \mathcal{H}_B := L^2(\Delta; H_B),$$

with inner products

$$\langle u, v\rangle_{\mathcal{H}_A} = \int_\Delta \langle u(t), v(t)\rangle_{H_A}\, dt,$$

$$\langle u, v\rangle_{\mathcal{H}_B} = \int_\Delta \langle u(t), v(t)\rangle_{H_B}\, dt.$$

A process $X = \{X_t\}_{t\in\Delta}$ with

$$\mathbb{E}\int_\Delta \|X_t\|^2_{H_A}\, dt < \infty$$

may be viewed as a random element of $\mathcal{H}_A$. Its covariance operator is the positive trace-class operator

$$\hat{T}_X : \mathcal{H}_A \to \mathcal{H}_A$$

defined by

$$\langle \hat{T}_X u, v\rangle_{\mathcal{H}_A} = \mathbb{E}\big[\langle X, u\rangle_{\mathcal{H}_A}\, \overline{\langle X, v\rangle_{\mathcal{H}_A}}\big], \qquad u, v \in \mathcal{H}_A.$$

Equivalently, $\hat{T}_X$ is the integral operator with kernel $\hat{R}_{XX}$:

$$(\hat{T}_X u)(t) = \int_\Delta \hat{R}_{XX}(t,s) u(s)\, ds.$$

Similarly, the covariance operator of Y is the positive trace-class operator

$$\hat{T}_Y : \mathcal{H}_B \to \mathcal{H}_B$$

defined by

$$\langle \hat{T}_Y u, v\rangle_{\mathcal{H}_B} = \mathbb{E}\big[\langle Y, u\rangle_{\mathcal{H}_B}\, \overline{\langle Y, v\rangle_{\mathcal{H}_B}}\big], \qquad u, v \in \mathcal{H}_B,$$

 doi:10.20944/preprints202607.1148.v1

or equivalently,

$$(\hat{T}_Y u)(t) = \int_\Delta \hat{R}_{YY}(t,s)u(s)\,ds.$$

Karhunen–Loève modes

Since $\hat{T}_X$ and $\hat{T}_Y$ are positive trace-class operators, they admit spectral decompositions

$$\hat{T}_X = \sum_{k\geq 1} \lambda_k\,|u_k\rangle\langle u_k|, \qquad \lambda_k \geq 0,$$

$$\hat{T}_Y = \sum_{\ell\geq 1} \mu_\ell\,|v_\ell\rangle\langle v_\ell|, \qquad \mu_\ell \geq 0,$$

where

$$\{u_k\} \subset \mathcal{H}_A, \qquad \{v_\ell\} \subset \mathcal{H}_B$$

are orthonormal families. These functions are the *Karhunen–Loève modes* of the processes X and Y.

Because X and Y are centered circular Gaussian processes, their laws are completely determined by $\hat{T}_X$ and $\hat{T}_Y$, and the corresponding Karhunen–Loève expansions take the form

$$X_t(\omega) = \sum_{k\geq 1} \sqrt{\lambda_k}\,\xi_k(\omega)\,u_k(t),$$

$$Y_t(\omega) = \sum_{\ell\geq 1} \sqrt{\mu_\ell}\,\eta_\ell(\omega)\,v_\ell(t),$$

where $\{\xi_k\}$ and $\{\eta_\ell\}$ are standard circular complex Gaussian random variables satisfying

$$\mathbb{E}[\xi_k] = 0, \qquad \mathbb{E}[\xi_k\overline{\xi_m}] = \delta_{km}, \qquad \mathbb{E}[\xi_k\xi_m] = 0,$$

$$\mathbb{E}[\eta_\ell] = 0, \qquad \mathbb{E}[\eta_\ell\overline{\eta_m}] = \delta_{\ell m}, \qquad \mathbb{E}[\eta_\ell\eta_m] = 0.$$

If X and Y are independent, then the two families $\{\xi_k\}$ and $\{\eta_\ell\}$ are independent as well.

The KL modes will play a central role in the Gaussian reduction of the DCM and in the constructive realization of entangled states.

The double covariance construction

We now recall the basic DCM construction. Define the tensor-valued process

$$Z_t = X_t \otimes Y_t \in H_A \otimes H_B.$$

For a time window Δ, define the time-averaged tensor amplitude

$$C_\Delta = \frac{1}{|\Delta|}\int_\Delta Z_t\,dt = \frac{1}{|\Delta|}\int_\Delta X_t \otimes Y_t\,dt \in H_A \otimes H_B.$$

This is a random vector $C_\Delta = C_\Delta(\omega)$ in $H_A \otimes H_B$.

The *double covariance operator* is defined as the statistical covariance of this random tensor:

$$\widehat{C} = \mathbb{E}\big[\,|C_\Delta\rangle\langle C_\Delta|\,\big] \in L(H_A \otimes H_B).$$

The corresponding quantum state is obtained by trace normalization:

$$\rho_C = \frac{\widehat{C}}{\operatorname{Tr}\widehat{C}}.$$

 doi:10.20944/preprints202607.1148.v1

Using the identification $H_A \otimes H_B \cong L(H_B, H_A)$, the random vector C_Δ may equivalently be viewed as the random operator

$$\widehat{C}_\Delta = \frac{1}{|\Delta|}\int_\Delta |X_t\rangle\langle Y_t|\,dt \in L(H_B, H_A).$$

Thus the DCM may be interpreted either as the covariance of the random tensor $C_\Delta \in H_A \otimes H_B$ or as the covariance of the random operator $\widehat{C}_\Delta \in L(H_B, H_A)$.

Expanding the definition of $\widehat{C}$, we obtain

$$\widehat{C} = \frac{1}{|\Delta|^2}\int_\Delta\int_\Delta \hat{K}(t,s)\,dt\,ds,$$

where

$$\hat{K}(t,s) = \mathbb{E}\Big[|X_t \otimes Y_t\rangle\langle X_s \otimes Y_s|\Big] \in L(H_A \otimes H_B).$$

The kernel $\hat{K}(t,s)$ is a fourth-order object. The Gaussian reduction theorem proved in the next section shows that for jointly Gaussian processes it is completely determined by the second-order covariance kernels

$$\hat{R}_{XX}, \quad \hat{R}_{YY}, \quad \hat{R}_{XY}, \quad \hat{R}_{YX}.$$

This reduction is the basic mathematical mechanism that allows one to describe DCM states in terms of Gaussian covariance data alone.

2.3. *Second-Order Covariance Kernels*

Define

$$\hat{R}_{XX}(t,s) = \mathbb{E}\big[|X_t\rangle\langle X_s|\big] \in L(H_A),\ \hat{R}_{YY}(t,s) = \mathbb{E}\big[|Y_t\rangle\langle Y_s|\big] \in L(H_B), \tag{2}$$

$$\hat{R}_{XY}(t,s) = \mathbb{E}\big[|X_t\rangle\langle Y_s|\big] \in L(H_B, H_A),\ \hat{R}_{YX}(t,s) = \mathbb{E}\big[|Y_t\rangle\langle X_s|\big] \in L(H_A, H_B). \tag{3}$$

These covariances form the covariance operator of the joint Gaussian process

$$\hat{\Gamma}(t,s) = \begin{pmatrix} \hat{R}_{XX}(t,s) & \hat{R}_{XY}(t,s) \\ \hat{R}_{YX}(t,s) & \hat{R}_{YY}(t,s) \end{pmatrix}. \tag{4}$$

3. Wick Reduction

Let $u, u' \in H_A$ and $v, v' \in H_B$. Then

$$\langle u \otimes v, \hat{K}(t,s)(u' \otimes v')\rangle = \mathbb{E}\Big[\langle u, X_t\rangle\langle v, Y_t\rangle\overline{\langle u', X_s\rangle\langle v', Y_s\rangle}\Big].$$

Since the underlying variables are jointly Gaussian and circular, Isserlis–Wick theorem yields (see appendix for details):

$$\mathbb{E}\Big[\langle u, X_t\rangle\langle v, Y_t\rangle\overline{\langle u', X_s\rangle\langle v', Y_s\rangle}\Big] = \langle u, \hat{R}_{XX}(t,s)u'\rangle\langle v, \hat{R}_{YY}(t,s)v'\rangle + \langle u, \hat{R}_{XY}(t,s)v'\rangle\langle v, \hat{R}_{YX}(t,s)u'\rangle.$$

Define the exchange operator $\hat{E}(t,s)$ by

$$\langle u \otimes v, \hat{E}(t,s)(u' \otimes v')\rangle = \langle u, \hat{R}_{XY}(t,s)v'\rangle\langle v, \hat{R}_{YX}(t,s)u'\rangle. \tag{5}$$

Then

$$\hat{K}(t,s) = \hat{R}_{XX}(t,s) \otimes \hat{R}_{YY}(t,s) + \hat{E}(t,s). \tag{6}$$

Consequently,

$$\hat{C} = \frac{1}{\Delta^2}\int_\Delta\int_\Delta \Big(\hat{R}_{XX}(t,s) \otimes \hat{R}_{YY}(t,s) + \hat{E}(t,s)\Big)dt\,ds. \tag{7}$$

3.1. Gaussian Reduction Theorem

Theorem 1 (Gaussian Reduction of DCM)**.** *Let X_t and Y_t be jointly Gaussian Hilbert-space-valued stochastic processes with finite second moments.*

Then the DCM covariance operator $\widehat{C} = \mathbb{E}[|C_\Delta\rangle\langle C_\Delta|]$ and therefore the normalized density operator $\rho = \frac{\widehat{C}}{\mathrm{Tr}\,\widehat{C}}$ are completely determined by the second-order covariance operator

$$\hat{\Gamma}(t,s) = \begin{pmatrix} \hat{R}_{XX}(t,s) & \hat{R}_{XY}(t,s) \\ \hat{R}_{YX}(t,s) & \hat{R}_{YY}(t,s) \end{pmatrix}$$

of the joint Gaussian process (X_t, Y_t). Equivalently, all fourth-order moments entering the DCM construction admit a Wick reduction to quadratic expressions in the covariance kernels, and no independent fourth-order information contributes to the double-covariance operator.

Proof. The operator $\widehat{C}$ depends on the kernel

$$\hat{K}(t,s) = \mathbb{E}[|X_t \otimes Y_t\rangle\langle X_s \otimes Y_s|].$$

For arbitrary vectors $u, u' \in H_A$ and $v, v' \in H_B$, matrix elements of $\hat{K}(t,s)$ are fourth-order moments of jointly Gaussian scalar variables. Applying Isserlis–Wick theorem expresses each such fourth-order moment uniquely in terms of pair covariances. Hence all matrix elements of $\hat{K}(t,s)$ are determined by $\hat{R}_{XX}, \hat{R}_{YY}, \hat{R}_{XY}, \hat{R}_{YX}$. Therefore $\hat{K}(t,s)$ itself is determined by $\hat{\Gamma}(t,s)$. Integrating over the observation window yields $\widehat{C}$, and normalization yields ρ. □

Remark 1. *For non-Gaussian processes the theorem generally fails. Higher-order cumulants contribute to $K(t,s)$ and the DCM density operator may then contain genuinely fourth-order statistical information.*

4. Gaussian Realization of Arbitrary Pure States of Composite Quantum Systems

We investigate the possibility of generating arbitrary finite-dimensional pure quantum states of composite quantum systems within the DCM framework by means of Gaussian Hilbert-space-valued stochastic processes. The construction is based on a special organization of Karhunen–Loève modes into *orthogonal temporal sectors* associated with the Schmidt decomposition of the target state.

The resulting realization reveals an unexpected feature of DCM. The generated quantum state is determined by the geometry of time-averaged tensor modes rather than by ordinary second-order cross-covariances. In particular, the construction remains valid even when the underlying Gaussian processes are independent.

Previous DCM constructions [24] demonstrated that arbitrary pure quantum states of composite systems (e.g., the Bell states) can be generated from suitably organized stochastic processes. Those constructions relied on explicitly synchronized temporal structures and were generally non-Gaussian. The purpose of the present work is to investigate whether arbitrary pure states can also be generated within the class of Gaussian Hilbert-space-valued stochastic processes.

4.1. The DCM State Construction

The following elementary lemma will be used repeatedly.

Lemma 1. *Assume that $C_\Delta(\omega) = \alpha(\omega)|\Psi\rangle$ (a.e.), where α is a complex valued random variable with finite second moment and vector $\Psi \neq 0$ is fixed. Then $\hat{\rho}_C = \frac{|\Psi\rangle\langle\Psi|}{\|\Psi\|^2}$. In particular, if $\|\Psi\| = 1$, then $\rho_C = |\Psi\rangle\langle\Psi|$.*

Proof. Since $|C_\Delta\rangle\langle C_\Delta| = |\alpha|^2|\Psi\rangle\langle\Psi|$, one obtains $E[|C_\Delta\rangle\langle C_\Delta|] = E[|\alpha|^2]\,|\Psi\rangle\langle\Psi|$. Furthermore, $\|C_\Delta\|^2 = |\alpha|^2\|\Psi\|^2$, hence $E\big[\|C_\Delta\|^2\big] = E\big[|\alpha|^2\big]\,\|\Psi\|^2$. Substitution into the definition of ρ_C gives the result. □

doi:10.20944/preprints202607.1148.v1

4.2. Construction of Gaussian Processes

Let

$$\{u_k\}_{k\geq 1} \subset \mathcal{H}_A \equiv L^2(\Delta; H_A), \ \{v_\ell\}_{\ell\geq 1} \subset \mathcal{H}_B \equiv L^2(\Delta; H_B)$$

be orthonormal families, and let $\{\lambda_k\}_{k\geq 1}$, $\{\mu_\ell\}_{\ell\geq 1}$ be positive summable sequences;

$$\sum_{k=1}^{\infty} \lambda_k < \infty, \ \sum_{\ell=1}^{\infty} \mu_\ell < \infty.$$

Let $\{\xi_k\}_{k\geq 1}$ and $\{\eta_\ell\}_{\ell\geq 1}$ be jointly Gaussian standard random variables with a prescribed covariance structure.

Define the Hilbert-space-valued stochastic processes

$$X_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k\, u_k(t), \ Y_t = \sum_{\ell=1}^{\infty} \sqrt{\mu_\ell}\, \eta_\ell\, v_\ell(t).$$

These series converge in $L^2(\Omega; \mathcal{H}_A)$ and $L^2(\Omega; \mathcal{H}_B)$, respectively. Consequently, X_t and Y_t are centered Gaussian Hilbert-space-valued stochastic processes.

Their covariance kernels are the operator-valued kernels

$$\hat{R}_{XX}(t,s) = E[|X_t\rangle\langle X_s|], \ \hat{R}_{YY}(t,s) = E[|Y_t\rangle\langle Y_s|].$$

A direct computation yields

$$\hat{R}_{XX}(t,s) = \sum_{k=1}^{\infty} \lambda_k\, |u_k(t)\rangle\langle u_k(s)|, \ \hat{R}_{YY}(t,s) = \sum_{\ell=1}^{\infty} \mu_\ell\, |v_\ell(t)\rangle\langle v_\ell(s)|.$$

The corresponding covariance operators on $\mathcal{H}_A$ $\mathcal{H}_B$ are positive and trace class. Their eigenfunctions are $\{u_k\}$ and $\{v_\ell\}$, with corresponding eigenvalues $\{\lambda_k\}$ and $\{\mu_\ell\}$.

Substituting the above expansions into DCM gives

$$C_\Delta = \sum_{k,\ell} \sqrt{\lambda_k \mu_\ell}\, \xi_k \eta_\ell\, W_{k\ell}$$

where

$$W_{k\ell} = \frac{1}{\Delta} \int_\Delta u_k(t) \otimes v_\ell(t)\, dt.$$

Thus the DCM temporal covariance, and therefore the resulting DCM state, is completely determined by the deterministic tensor family $\{W_{k\ell}\}$.

4.3. Temporal Schmidt-Sector Construction

Let

$$|\Psi\rangle = \sum_{r=1}^{m} s_r\, e_r \otimes f_r$$

be a normalized Schmidt decomposition of a pure state, $s_r > 0, \sum_{r=1}^{m} s_r^2 = 1$.

Define

$$S = \sum_{r=1}^{m} s_r.$$

Denote by $L^2(\Delta)$ space of square integrable complex-valued functions defined on the segment Δ. Consider its decomposition into direct sum of subspaces:

$$L^2(\Delta) = F_1 \oplus F_2 \oplus \cdots \oplus F_m,$$

where each subspace F_r is infinite-dimensional.

 doi:10.20944/preprints202607.1148.v1

For every r, choose an orthonormal basis $\{f_r^{(k)}\}_{k\geq 1}$ in the subspace F_r. Thus

$$\int_\Delta f_r^{(k)}(t) f_q^{(\ell)}(t)\,dt = \delta_{rq}\delta_{k\ell}.$$

Define

$$u_k(t) = \frac{1}{\sqrt{S}} \sum_{r=1}^{m} \sqrt{s_r}\, f_r^{(k)}(t)\, e_r, \; v_k(t) = \frac{1}{\sqrt{S}} \sum_{r=1}^{m} \sqrt{s_r}\, f_r^{(k)}(t)\, f_r.$$

Lemma 2. *The families*

$$\{u_k\}_{k\geq 1} \subset \mathcal{H}_A, \quad \{v_k\}_{k\geq 1} \subset \mathcal{H}_B$$

are orthonormal.

Proof. Using the orthogonality relations,

$$\langle u_k, u_\ell\rangle = \frac{1}{S} \sum_{r,q} \sqrt{s_r s_q}\, \langle f_r^{(k)}, f_q^{(\ell)}\rangle\, \langle e_r, e_q\rangle = \frac{1}{S} \sum_{r} s_r \delta_{k\ell} = \delta_{k\ell}.$$

The proof for the family $\{v_k\}$ is identical. □

Coordinate representation of the Gaussian processes.

Let

$$\{\lambda_k\}_{k\geq 1}, \; \{\mu_k\}_{k\geq 1},$$

be positive summable sequences and let

$$\{\xi_k\}_{k\geq 1}, \; \{\eta_k\}_{k\geq 1},$$

be independent standard circular Gaussian random variables.

The Gaussian processes generated by the above orthonormal systems are

$$X_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k\, u_k(t),$$

and

$$Y_t = \sum_{k=1}^{\infty} \sqrt{\mu_k}\, \eta_k\, v_k(t).$$

Substituting the definitions of u_k and v_k gives

$$X_t = \frac{1}{\sqrt{S}} \sum_{r=1}^{m} \sqrt{s_r} \left(\sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k\, f_r^{(k)}(t) \right) e_r,$$

and

$$Y_t = \frac{1}{\sqrt{S}} \sum_{r=1}^{m} \sqrt{s_r} \left(\sum_{k=1}^{\infty} \sqrt{\mu_k}\, \eta_k\, f_r^{(k)}(t) \right) f_r.$$

Thus each Hilbert-space component of the processes occupies its own temporal sector F_r. The Schmidt coefficients s_r determine the relative weights of the temporal sectors, while the random Fourier coefficients are generated by the Gaussian variables $\{\xi_k\}$ and $\{\eta_k\}$

 doi:10.20944/preprints202607.1148.v1

4.4. Expected Energies of the Gaussian Processes

The coordinate representation of the processes immediately allows one to compute their instantaneous and expected energies.

Since

$$X_t = \frac{1}{\sqrt{S}} \sum_{r=1}^{m} \sqrt{s_r} \left(\sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k f_r^{(k)}(t) \right) e_r,$$

and the vectors $\{e_r\}_{r=1}^{m}$ form an orthonormal basis of H_A,

$$\begin{aligned} \|X_t\|^2 &= \langle X_t, X_t \rangle \\ &= \frac{1}{S} \sum_{r=1}^{m} s_r \left| \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k f_r^{(k)}(t) \right|^2. \end{aligned}$$

Similarly,

$$\|Y_t\|^2 = \frac{1}{S} \sum_{r=1}^{m} s_r \left| \sum_{k=1}^{\infty} \sqrt{\mu_k}\, \eta_k f_r^{(k)}(t) \right|^2.$$

These quantities are random and describe the instantaneous energies of the two Gaussian processes.

Taking expectations and using

$$E(\xi_k \overline{\xi_\ell}) = \delta_{k\ell},$$

we obtain

$$\begin{aligned} E\|X_t\|^2 &= \frac{1}{S} \sum_{r=1}^{m} s_r E \left| \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k f_r^{(k)}(t) \right|^2 \\ &= \frac{1}{S} \sum_{r=1}^{m} s_r \sum_{k=1}^{\infty} \lambda_k |f_r^{(k)}(t)|^2. \end{aligned}$$

Analogously,

$$E\|Y_t\|^2 = \frac{1}{S} \sum_{r=1}^{m} s_r \sum_{k=1}^{\infty} \mu_k |f_r^{(k)}(t)|^2.$$

Integrating over the interval Δ and using the orthonormality of the systems $\{f_r^{(k)}\}_{k \geq 1}$,

$$\int_\Delta |f_r^{(k)}(t)|^2\, dt = 1,$$

yields

$$\begin{aligned} \int_\Delta E\|X_t\|^2\, dt &= \frac{1}{S} \sum_{r=1}^{m} s_r \sum_{k=1}^{\infty} \lambda_k \\ &= \sum_{k=1}^{\infty} \lambda_k, \end{aligned}$$

since

$$\sum_{r=1}^{m} s_r = S.$$

Similarly,

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$$\int_\Delta E\|Y_t\|^2\,dt = \sum_{k=1}^{\infty} \mu_k.$$

Thus the total expected energies of the two Gaussian processes coincide with the traces of their covariance operators,

$$\int_\Delta E\|X_t\|^2\,dt = \mathrm{Tr}(Q_X) = \sum_{k=1}^{\infty} \lambda_k,$$

and

$$\int_\Delta E\|Y_t\|^2\,dt = \mathrm{Tr}(Q_Y) = \sum_{k=1}^{\infty} \mu_k.$$

Moreover, the contribution of the r-th temporal sector to the total expected energy equals

$$E_r(X) = \frac{s_r}{S}\sum_{k=1}^{\infty} \lambda_k,$$

and

$$E_r(Y) = \frac{s_r}{S}\sum_{k=1}^{\infty} \mu_k.$$

Hence the relative expected energy carried by the r-th temporal sector is

$$\frac{E_r(X)}{\sum_{q=1}^m E_q(X)} = \frac{s_r}{S}, \qquad \frac{E_r(Y)}{\sum_{q=1}^m E_q(Y)} = \frac{s_r}{S}.$$

Therefore the Schmidt decomposition coefficients s_r admit a direct stochastic interpretation: they determine how the expected energy of each Gaussian process is distributed among the mutually orthogonal temporal sectors F_r.

Remark 2. *One may ask whether it is necessary to use the same temporal basis*

$$\{f_r^{(k)}\}_{k\geq 1} \subset F_r$$

in the constructions of both families $\{u_k\}$ and $\{v_k\}$. Suppose instead that the second family is constructed from another collection of functions

$$\{g_r^{(k)}\}_{k\geq 1} \subset F_r.$$

Repeating the computation of the tensors $W_{k\ell}$ yields

$$W_{k\ell} = \frac{1}{S|\Delta|}\sum_{r,q=1}^{m} \sqrt{s_r s_q}\left(\int_\Delta f_r^{(k)}(t)\,g_q^{(\ell)}(t)\,dt\right) e_r \otimes f_q.$$

Hence the identity

$$W_{k\ell} = \frac{\delta_{k\ell}}{S|\Delta|}\sum_{r=1}^{m} s_r\, e_r \otimes f_r$$

holds provided the two systems satisfy the biorthogonality relations

$$\int_\Delta f_r^{(k)}(t)\,g_q^{(\ell)}(t)\,dt = \delta_{rq}\delta_{k\ell}.$$

Thus, from the viewpoint of the tensor realization alone, biorthogonality is sufficient.

However, the present construction requires the families $\{u_k\}$ and $\{v_k\}$ to be Karhunen–Loève modes of Gaussian processes. Consequently, both $\{f_r^{(k)}\}$ and $\{g_r^{(k)}\}$ must be complete orthonormal systems of the

 doi:10.20944/preprints202607.1148.v1

corresponding temporal sectors F_r. A standard result from Hilbert space theory states that the biorthogonal basis of a complete orthonormal basis coincides with the basis itself, up to unimodular phase factors. Therefore,

$$g_r^{(k)}(t) = e^{i\theta_r^{(k)}} f_r^{(k)}(t),$$

for suitable real numbers $\theta_r^{(k)}$.

Consequently, apart from inessential phase factors (which may be absorbed into the Schmidt vectors), the use of the same temporal basis in both Gaussian processes is essentially forced by the Karhunen–Loève structure. The apparent symmetry of the construction is therefore not an additional assumption but a consequence of simultaneously requiring tensor realization and orthonormal Karhunen–Loève expansions.

5. The Fundamental Tensor Identity

The key observation is that the integrated tensor modes $W_{k\ell}$ are all collinear with the target state.

Proposition 1. *For the above construction,*

$$W_{k\ell} = \frac{\delta_{k\ell}}{S|\Delta|}|\Psi\rangle.$$

Proof. Using the definitions of u_k and v_ℓ,

$$u_k(t) \otimes v_\ell(t) = \frac{1}{S}\sum_{r,q} \sqrt{s_r s_q}\, f_r^{(k)}(t)\, f_q^{(\ell)}(t)\, e_r \otimes f_q.$$

Therefore

$$W_{k\ell} = \frac{1}{S|\Delta|}\sum_{r,q} \sqrt{s_r s_q}\left(\int_\Delta f_r^{(k)}(t) f_q^{(\ell)}(t)\, dt\right) e_r \otimes f_q.$$

Using

$$\int_\Delta f_r^{(k)}(t) f_q^{(\ell)}(t)\, dt = \delta_{rq}\delta_{k\ell},$$

we obtain

$$W_{k\ell} = \frac{\delta_{k\ell}}{S|\Delta|}\sum_r s_r\, e_r \otimes f_r.$$

Since

$$|\Psi\rangle = \sum_r s_r\, e_r \otimes f_r,$$

the result follows.

□

6. Constructive Gaussian Realization

In Section 8 we constructed centered circular Gaussian processes

$$X_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \xi_k\, u_k(t), \;\; Y_t = \sum_{\ell=1}^{\infty} \sqrt{\mu_\ell}\, \eta_\ell\, v_\ell(t),$$

 doi:10.20944/preprints202607.1148.v1

where

$$\{u_k\}_{k\geq 1} \subset \mathcal{H}_A, \ \{v_\ell\}_{\ell\geq 1} \subset \mathcal{H}_B$$

are orthonormal systems,

$$\{\lambda_k\}, \ \{\mu_\ell\}$$

are positive summable sequences, and $\{\xi_k\}$, $\{\eta_\ell\}$ are independent standard circular Gaussian random variables.

We first calculate the covariance operators of these processes. For a centered Hilbert-space-valued Gaussian process $X = \{X_t\}_{t\in\Delta}$, its covariance operator $\hat{T}_X : \mathcal{H}_A \to \mathcal{H}_A$ is defined by

$$\hat{T}_X = E\big[\,|X\rangle\langle X|\,\big],$$

that is,

$$\langle \hat{T}_X f, g\rangle = E\Big[\langle X, f\rangle\overline{\langle X, g\rangle}\Big], \qquad f, g \in \mathcal{H}_A.$$

Equivalently,

$$(\hat{T}_X f)(t) = \int_\Delta \hat{K}_{XX}(t,s) f(s)\, ds,$$

where

$$\hat{K}_{XX}(t,s) = E[X_t \otimes X_s^*]$$

is the covariance kernel.

Similarly one defines the covariance operator

$$\hat{T}_Y = E\big[\,|Y\rangle\langle Y|\,\big]$$

of the process Y.

Using

$$E(\xi_k\overline{\xi_\ell}) = \delta_{k\ell},$$

one immediately obtains

$$\hat{T}_X = \sum_{k=1}^{\infty} \lambda_k\, |u_k\rangle\langle u_k|,$$

that is,

$$(\hat{T}_X f)(t) = \sum_{k=1}^{\infty} \lambda_k \langle f, u_k\rangle u_k(t).$$

Consequently,

$$\hat{K}_{XX}(t,s) = \sum_{k=1}^{\infty} \lambda_k\, u_k(t) u_k(s)^*.$$

Likewise,

$$\hat{T}_Y = \sum_{\ell=1}^{\infty} \mu_\ell\, |v_\ell\rangle\langle v_\ell|,$$

with covariance kernel

$$\hat{K}_{YY}(t,s) = \sum_{\ell=1}^{\infty} \mu_\ell\, v_\ell(t) v_\ell(s)^*.$$

Thus the covariance operators are diagonal in the orthonormal systems $\{u_k\}$ and $\{v_\ell\}$, with eigenvalues $\{\lambda_k\}$ and $\{\mu_\ell\}$, respectively.

In the previous sections, we constructed particular orthonormal systems associated with the Schmidt decomposition of a prescribed target state $|\Psi\rangle = \sum_{r=1}^{m} s_r\, e_r \otimes f_r$, and proved that the corresponding deterministic tensors satisfy

$$W_{k\ell} = \frac{1}{|\Delta|} \int_\Delta u_k(t) \otimes v_\ell(t)\, dt = \frac{\delta_{k\ell}}{S|\Delta|} |\Psi\rangle.$$

Substituting these tensors into the Gaussian reduction formula

$$C_\Delta = \sum_{k,\ell} \sqrt{\lambda_k \mu_\ell}\, \xi_k \eta_\ell\, W_{k\ell},$$

yields

$$C_\Delta = \frac{|\Psi\rangle}{S|\Delta|} \sum_{k=1}^{\infty} \sqrt{\lambda_k \mu_k}\, \xi_k \eta_k.$$

Define

$$\alpha(\omega) = \frac{1}{S|\Delta|} \sum_{k=1}^{\infty} \sqrt{\lambda_k \mu_k}\, \xi_k(\omega) \eta_k(\omega).$$

Then

$$C_\Delta(\omega) = \alpha(\omega) |\Psi\rangle.$$

To apply Lemma 1 it remains to verify that

$$E|\alpha|^2 < \infty.$$

Since the Gaussian processes are independent,

$$E[\xi_k \eta_k \overline{\xi_\ell \eta_\ell}] = E[\xi_k \overline{\xi_\ell}]\, E[\eta_k \overline{\eta_\ell}] = \delta_{k\ell},$$

and therefore

$$\begin{aligned} E|\alpha|^2 &= \frac{1}{S^2|\Delta|^2} \sum_{k,\ell} \sqrt{\lambda_k \mu_k \lambda_\ell \mu_\ell}\, E[\xi_k \eta_k \overline{\xi_\ell \eta_\ell}] \\ &= \frac{1}{S^2|\Delta|^2} \sum_{k=1}^{\infty} \lambda_k \mu_k. \end{aligned}$$

Since

$$\sum_{k=1}^{\infty} \lambda_k < \infty, \qquad \sum_{k=1}^{\infty} \mu_k < \infty,$$

both sequences are bounded. Hence

$$\sum_{k=1}^{\infty} \lambda_k \mu_k \le \left(\sup_k \mu_k \right) \sum_{k=1}^{\infty} \lambda_k < \infty,$$

and consequently

$$E|\alpha|^2 < \infty.$$

We have therefore established the following realization theorem.

Theorem 2 (Constructive Gaussian Realization)**.** *For every finite-dimensional pure state*

$$|\Psi\rangle = \sum_{r=1}^{m} s_r\, e_r \otimes f_r,$$

 doi:10.20944/preprints202607.1148.v1

there exist centered circular Gaussian Hilbert-space-valued stochastic processes

$$X_t : \Omega \to H_A, \qquad Y_t : \Omega \to H_B,$$

such that

$$\rho_\Delta = |\Psi\rangle\langle\Psi|.$$

Proof. The preceding calculations show that

$$C_\Delta(\omega) = \alpha(\omega)\Psi,$$

where

$$E|\alpha|^2 < \infty.$$

The conclusion therefore follows directly from Lemma 1. □

Remark 3. *The Gaussian realization theorem reveals a structural feature of the DCM framework that differs substantially from the standard quantum-information intuition. In the present construction, one may choose the cross-covariance kernel to vanish identically,*

$$\hat{R}_{XY}(t,s) \equiv 0.$$

For jointly Gaussian processes this implies that X_t and Y_t are independent. Nevertheless, the resulting DCM state may be an arbitrary pure entangled state, including maximally entangled states such as Bell states. Thus the generation of entanglement in the DCM framework does not require nontrivial second-order cross-correlations between the underlying stochastic processes.

Instead, the entanglement is encoded in the temporal geometry of the deterministic tensors $W_{k\ell}$ which enter the time-averaged amplitude

$$C_\Delta = \frac{1}{|\Delta|}\int_\Delta X_t \otimes Y_t \, dt.$$

The DCM construction therefore demonstrates that strong entanglement may emerge entirely from fourth-order temporal coherence, even when all ordinary second-order cross-covariances vanish.

We can formulate this property of DCM as mathematical statement:

Corollary 1 (Entanglement without cross-covariance)**.** *For every finite-dimensional pure state*

$$|\Psi\langle \in H_A \otimes H_B,$$

there exist independent Gaussian processes X_t and Y_t such that the corresponding DCM state satisfies

$$\hat{\rho} = |\Psi\rangle\langle\Psi|.$$

In particular, there exist independent Gaussian processes whose DCM state is maximally entangled.

Proof. Choose

$$\hat{R}_{XY}(t,s) \equiv 0.$$

For jointly Gaussian processes this implies independence. By Theorem 2, the realization depends only on the deterministic tensor family

$$W_{k\ell} = \frac{1}{|\Delta|}\int_\Delta u_k(t) \otimes v_\ell(t)\, dt$$

doi:10.20944/preprints202607.1148.v1

and not on the cross-covariance kernel. Therefore the resulting DCM state remains

$$\hat{\rho} = |\Psi\rangle\langle\Psi|.$$

□

6.1. A Concrete Two-Dimensional Example

We illustrate the realization theorem by an explicit construction. Let $\Delta = [0, 2\pi]$, and let

$$H_A = \mathrm{span}\{e_1, e_2\}, \ H_B = \mathrm{span}\{f_1, f_2\},$$

where both bases are orthonormal. We decompose the temporal Hilbert space as $L^2([0, 2\pi]) = F_1 \oplus F_2$, where

$$F_1 = \mathrm{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(kt)}{\sqrt{\pi}} : k \geq 1\right\}, \ F_2 = \mathrm{span}\left\{\frac{\sin(kt)}{\sqrt{\pi}} : k \geq 1\right\}.$$

Let

$$|\Psi\rangle = s_1 e_1 \otimes f_1 + s_2 e_2 \otimes f_2, \ s_1, s_2 > 0, s_1^2 + s_2^2 = 1,$$

and define $S = s_1 + s_2$. The Karhunen–Loève modes are chosen as

$$u_k(t) = \frac{1}{\sqrt{S}}\left(\sqrt{s_1}\frac{\cos(kt)}{\sqrt{\pi}}e_1 + \sqrt{s_2}\frac{\sin(kt)}{\sqrt{\pi}}e_2\right), \ v_k(t) = \frac{1}{\sqrt{S}}\left(\sqrt{s_1}\frac{\cos(kt)}{\sqrt{\pi}}f_1 + \sqrt{s_2}\frac{\sin(kt)}{\sqrt{\pi}}f_2\right).$$

The orthogonality of the trigonometric system implies that both families are orthonormal.

Let $\{\xi_k\}_{k\geq 1}, \{\eta_k\}_{k\geq 1}$, be independent standard circular Gaussian random variables and let $\{\lambda_k\}$, $\{\mu_k\}$, be positive summable sequences. The corresponding Gaussian processes are presented in the coordinate form as

$$X_t = \frac{\sqrt{s_1}}{\sqrt{\pi S}}\left(\sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\xi_k \cos(kt)\right)e_1 + \frac{\sqrt{s_2}}{\sqrt{\pi S}}\left(\sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\xi_k \sin(kt)\right)e_2, \tag{8}$$

and

$$Y_t = \frac{\sqrt{s_1}}{\sqrt{\pi S}}\left(\sum_{k=1}^{\infty} \sqrt{\mu_k}\,\eta_k \cos(kt)\right)f_1 + \frac{\sqrt{s_2}}{\sqrt{\pi S}}\left(\sum_{k=1}^{\infty} \sqrt{\mu_k}\,\eta_k \sin(kt)\right)f_2. \tag{9}$$

Thus the first Hilbert-space component of each process occupies the cosine sector F_1, while the second occupies the sine sector F_2. The deterministic tensors are $W_{k\ell} = \frac{1}{2\pi}\int_0^{2\pi} u_k(t) \otimes v_\ell(t)\,dt$. Substituting the expressions for u_k and v_ℓ gives

$$W_{k\ell} = \frac{\delta_{k\ell}}{2\pi S}(s_1 e_1 \otimes f_1 + s_2 e_2 \otimes f_2) = \frac{\delta_{k\ell}}{2\pi S}|\Psi\rangle.$$

7. Concurrence and Temporal Energy Distribution

We start with the two qubit case considered in section 6.1. To be more concrete we continue the example presented in setcion 6.1.

The expected energies computed above show that, for the present realization, the cosine and sine temporal sectors carry relative energies

$$p_1 := \frac{E_1(X)}{E_1(X) + E_2(X)} = \frac{s_1}{s_1 + s_2}, \qquad p_2 := \frac{E_2(X)}{E_1(X) + E_2(X)} = \frac{s_2}{s_1 + s_2},$$

with $p_1 + p_2 = 1$. The same relations hold for the process Y.

 doi:10.20944/preprints202607.1148.v1

Thus the realization theorem associates to the Schmidt coefficients (s_1, s_2) a temporal energy distribution (p_1, p_2) given by

$$s_1 = \frac{p_1}{\sqrt{p_1^2 + p_2^2}}, \qquad s_2 = \frac{p_2}{\sqrt{p_1^2 + p_2^2}}.$$

Substituting into the concurrence formula for a pure two-qubit state,

$$\mathcal{C}(\Psi) = 2 s_1 s_2,$$

we obtain

$$\mathcal{C}(\Psi) = \frac{2 p_1 p_2}{p_1^2 + p_2^2}.$$

Hence, within the present DCM realization, concurrence is determined by the distribution of expected stochastic energy between the two temporal sectors. It vanishes when one sector carries all the energy, and it is maximal when the energy is equally distributed:

$$p_1 = p_2 = \frac{1}{2} \quad \Longrightarrow \quad \mathcal{C}(\Psi) = 1.$$

Therefore, in this construction, entanglement is controlled not directly by the quadratic weights s_1^2, s_2^2, but by the relative sector energies

$$p_r = \frac{E_r(X)}{E_1(X) + E_2(X)} = \frac{E_r(Y)}{E_1(Y) + E_2(Y)}, \qquad r = 1, 2.$$

The Schmidt coefficients are recovered from these energy fractions by the above normalization formulas.

Now we consider the general multidimensional case. Let

$$|\Psi\rangle = \sum_{r=1}^{m} s_r \, e_r \otimes f_r, \qquad s_r \geq 0, \qquad \sum_{r=1}^{m} s_r^2 = 1,$$

be the Schmidt decomposition of the target pure state. A standard concurrence-type entanglement measure for pure bipartite states is

$$\mathcal{C}(\Psi) = \sqrt{2(1 - \operatorname{Tr} \hat{\rho}_A^2)},$$

where

$$\hat{\rho}_A = \operatorname{Tr}_{H_B} |\Psi\rangle\langle\Psi| = \sum_{r=1}^{m} s_r^2 \, |e_r\rangle\langle e_r|.$$

Hence

$$\mathcal{C}(\Psi) = \sqrt{2\left(1 - \sum_{r=1}^{m} s_r^4\right)}.$$

For the Gaussian realization constructed above, the expected energies of the temporal sectors satisfy

$$E_r(X) \propto s_r, \qquad E_r(Y) \propto s_r, \qquad r = 1, \ldots, m.$$

Therefore the normalized sector energies are

$$p_r := \frac{E_r(X)}{\sum_{q=1}^{m} E_q(X)} = \frac{E_r(Y)}{\sum_{q=1}^{m} E_q(Y)} = \frac{s_r}{\sum_{q=1}^{m} s_q}.$$

 doi:10.20944/preprints202607.1148.v1

Writing

$$S := \sum_{q=1}^{m} s_q,$$

we have $s_r = Sp_r$. Since $\sum_{r=1}^{m} s_r^2 = 1$, it follows that

$$1 = \sum_{r=1}^{m} s_r^2 = S^2 \sum_{r=1}^{m} p_r^2,$$

and therefore

$$S = \frac{1}{\sqrt{\sum_{r=1}^{m} p_r^2}}.$$

Consequently,

$$s_r = \frac{p_r}{\sqrt{\sum_{q=1}^{m} p_q^2}}, \qquad r = 1, \ldots, m.$$

Substituting into the concurrence formula yields

$$\mathcal{C}(\Psi) = \sqrt{2\left(1 - \frac{\sum_{r=1}^{m} p_r^4}{\left(\sum_{q=1}^{m} p_q^2\right)^2}\right)}.$$

Thus, in the present DCM realization, the entanglement of the target pure state is completely determined by the distribution of expected stochastic energy among the temporal sectors. The Schmidt coefficients are recovered from the sector energy fractions $\{p_r\}_{r=1}^{m}$ by the normalization formula

$$s_r = \frac{p_r}{\sqrt{\sum_{q=1}^{m} p_q^2}},$$

and the concurrence is then given by the above expression.

In the two-dimensional case $m = 2$, this reduces to

$$\mathcal{C}(\Psi) = \frac{2p_1 p_2}{p_1^2 + p_2^2},$$

which coincides with the formula obtained above.

8. Recovery from the Gaussian Reduction Theorem

The realization theorem may also be derived directly from the Gaussian Reduction Theorem established in Section 1. Consider the special case $\hat{R}_{XY}(t,s) = 0$. Then the exchange contribution $\mathcal{E}(t,s)$ vanishes identically and the two-time kernel reduces to

$$\hat{K}(t,s) = \hat{R}_{XX}(t,s) \otimes \hat{R}_{YY}(t,s).$$

Using the expansions introduced above, $\hat{R}_{XX}(t,s) = \sum_k \lambda_k |u_k(t)\rangle\langle u_k(s)|$, and $\hat{R}_{YY}(t,s) = \sum_\ell \mu_\ell |v_\ell(t)\rangle\langle v_\ell(s)|$, we obtain

$$\hat{K}(t,s) = \sum_{k,\ell} \lambda_k \mu_\ell |u_k(t) \otimes v_\ell(t)\rangle\langle u_k(s) \otimes v_\ell(s)|.$$

Integrating over the observation interval and using the definition of the DCM density operator yields $C = \sum_{k,\ell} \lambda_k \mu_\ell |W_{k\ell}\rangle\langle W_{k\ell}|$, where $W_{k\ell} = \frac{1}{|\Delta|}\int_\Delta u_k(t) \otimes v_\ell(t)\, dt$. By Proposition 1, $W_{k\ell} = \frac{\delta_{k\ell}}{S|\Delta|}|\Psi\rangle$. Substitution gives $C = \gamma |\Psi\rangle\langle\Psi|$.

Thus the Gaussian realization theorem follows directly from the Gaussian Reduction Theorem.

 doi:10.20944/preprints202607.1148.v1

Surprisingly the exchange term $\mathcal{E}$ is not required for the generation of entangled states. In the present construction, entanglement is generated entirely by the temporal geometry encoded in the tensors $W_{k\ell}$.

9. Towards a General Theory of Gaussian DCM States

The Gaussian realization theorem established in the previous section corresponds to a highly special situation. The temporal modes are arranged so that the deterministic tensors

$$W_{k\ell} = \frac{1}{|\Delta|} \int_\Delta u_k(t) \otimes v_\ell(t)\, dt$$

satisfy

$$W_{k\ell} = \delta_{k\ell}\, c\, |\Psi\rangle.$$

Consequently,

$$C_\Delta(\omega) = \alpha(\omega)|\Psi\rangle,$$

and the resulting DCM state is pure.

The Gaussian Reduction Theorem suggests a much broader viewpoint. For independent Gaussian processes,

$$C = \sum_{k,\ell} \lambda_k \mu_\ell |W_{k\ell}\rangle\langle W_{k\ell}|,$$

so that the resulting DCM state is completely determined by the family

$$\mathcal{W} = \{W_{k\ell}\}.$$

Thus the fundamental object of Gaussian DCM theory is not the covariance kernel itself but rather the geometry of the temporal tensor family.

This naturally leads to the following realizability problem.

Realizability Problem. Characterize all tensor families

$$\{W_{k\ell}\} \subset H_A \otimes H_B$$

that admit a representation

$$W_{k\ell} = \frac{1}{|\Delta|} \int_\Delta u_k(t) \otimes v_\ell(t)\, dt,$$

where

$$\{u_k\} \subset \mathcal{H}_A, \qquad \{v_\ell\} \subset \mathcal{H}_B$$

are orthonormal systems.

The realization theorem provides one highly nontrivial example of such a family. Different temporal organizations produce different tensor geometries and therefore different Gaussian DCM states.

Several elementary structural observations follow immediately.

Pure states.

The operator

$$C = \sum_{k,\ell} \lambda_k \mu_\ell |W_{k\ell}\rangle\langle W_{k\ell}|$$

has rank one if and only if all nonzero tensors are collinear,

$$W_{k\ell} = c_{k\ell}|\Psi\rangle.$$

After normalization,
$$\hat{\rho} = \frac{|\Psi\rangle\langle\Psi|}{\|\Psi\|^2}.$$
The realization theorem established above is precisely of this type.

Separable states.

Suppose every tensor factors,
$$W_{k\ell} = a_{k\ell} \otimes b_{k\ell}.$$
Then
$$|W_{k\ell}\rangle\langle W_{k\ell}| = |a_{k\ell}\rangle\langle a_{k\ell}| \otimes |b_{k\ell}\rangle\langle b_{k\ell}|,$$
and therefore
$$\widehat{C} = \sum_{k,\ell} \lambda_k \mu_\ell (|a_{k\ell}\rangle\langle a_{k\ell}|) \otimes (|b_{k\ell}\rangle\langle b_{k\ell}|).$$
Hence the normalized DCM state is separable.

Outside these two extreme situations, the geometry of the family $\{W_{k\ell}\}$ becomes essential. The remainder of this section develops a general framework for analyzing entanglement through the structure of these temporal tensors.

9.1. Common Schmidt Decompositions

A particularly tractable situation occurs when all tensors $W_{k\ell}$ share a common Schmidt basis.

Definition 2. *The family $\{W_{k\ell}\}$ is said to admit a common Schmidt decomposition if there exist orthonormal systems*
$$\{e_r\}_{r=1}^R \subset H_A, \qquad \{f_r\}_{r=1}^R \subset H_B,$$
such that
$$W_{k\ell} = \sum_{r=1}^R a_r^{(k\ell)}\, e_r \otimes f_r$$
for all pairs (k, ℓ).

The common Schmidt decomposition condition for the family $\{W_{k\ell}\}$ may also be reformulated in operator terms by introducing the associated Hilbert–Schmidt operators
$$\hat{M}_{k\ell} := \frac{1}{|\Delta|} \int_\Delta |u_k(t)\rangle\langle v_\ell(t)|\, dt.$$
It has the form:
$$\hat{M}_{k\ell} = \sum_{r=1}^R a_r^{(k\ell)}\, |e_r\rangle\langle f_r|.$$
The coefficients are
$$a_r^{(k\ell)} = \langle e_r, \hat{M}_{k\ell} f_r\rangle = \frac{1}{|\Delta|} \int_\Delta \langle e_r, u_k(t)\rangle\langle v_\ell(t), f_r\rangle\, dt.$$
Thus the Schmidt coefficients are obtained directly from temporal overlaps of the mode functions.

Under the standard tensor–operator correspondence, the family $\{W_{k\ell}\} \subset H_A \otimes H_B$ admits a common Schmidt decomposition if and only if the operators $\{\hat{M}_{k\ell}\}$ admit a simultaneous singular-value decomposition. We shall not pursue this operator-theoretic characterization here, since for the Gaussian constructions considered below a more transparent sufficient condition is given by the temporal sector decomposition introduced in Proposition 3.

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9.2. The Schmidt-Coherence Matrix

Assume from now on that a common Schmidt decomposition exists.

Define

$$\Gamma_{rs} = \sum_{k,\ell} \lambda_k \mu_\ell \, a_r^{(k\ell)} \overline{a_s^{(k\ell)}}.$$

The matrix

$$\Gamma = (\Gamma_{rs})_{r,s}$$

will be called the Schmidt-coherence matrix.

It is positive semidefinite. Indeed,

$$\Gamma = AA^*,$$

where the columns of A are

$$\sqrt{\lambda_k \mu_\ell} \left(a_1^{(k\ell)}, \dots, a_R^{(k\ell)} \right)^T.$$

Substituting the Schmidt expansions into the Gaussian reduction formula yields

$$\widehat{C} = \sum_{r,s} \Gamma_{rs} \, |e_r \otimes f_r\rangle\langle e_s \otimes f_s|.$$

Thus the DCM covariance operator is completely determined by the Schmidt-coherence matrix.

9.3. Entanglement Criterion

We now derive a sufficient condition for entanglement.

Proposition 2 (Schmidt-coherence criterion). *Assume that the family $\{W_{k\ell}\}$ admits a common Schmidt decomposition.*

If

$$\Gamma_{rs} \neq 0$$

for some $r \neq s$, then the normalized DCM state

$$\hat{\rho}_C = \frac{\widehat{C}}{\mathrm{Tr}(\widehat{C})}$$

is entangled.

Proof. Taking the partial transpose on the second subsystem gives

$$\hat{C}^\Gamma = \sum_{r,s} \Gamma_{rs} \, |e_r \otimes f_s\rangle\langle e_s \otimes f_r|.$$

Fix $r \neq s$. The subspace

$$\mathcal{H}_{rs} = \mathrm{span}\{e_r \otimes f_s, e_s \otimes f_r\}$$

is invariant under C^Γ.

Restricted to this subspace, $\hat{C}^\Gamma$ is represented by

$$\begin{pmatrix} 0 & \Gamma_{rs} \\ \overline{\Gamma_{rs}} & 0 \end{pmatrix}.$$

Its eigenvalues are

$$\pm|\Gamma_{rs}|.$$

Hence, if

$$\Gamma_{rs} \neq 0,$$

 doi:10.20944/preprints202607.1148.v1

the operator C^Γ possesses a negative eigenvalue and therefore is not positive.

Since positivity of the partial transpose is necessary for separability, the state $\hat{\rho}_C = \frac{\widehat{C}}{\mathrm{Tr}(\widehat{C})}$ cannot be separable. $\square$

9.4. Relation to the Realization Theorem

The realization theorem obtained earlier corresponds to a particularly simple instance of the common Schmidt decomposition framework. Let

$$|\Psi\rangle = \sum_{r=1}^{m} s_r\, e_r \otimes f_r, \qquad s_r \geq 0, \qquad \sum_{r=1}^{m} s_r^2 = 1,$$

and let X, Y be the Gaussian processes constructed in the realization theorem. Then the associated temporal tensors satisfy

$$W_{k\ell} = \delta_{k\ell}\, \frac{1}{S|\Delta|}\, |\Psi\rangle, \qquad S := \sum_{r=1}^{m} s_r.$$

Equivalently,

$$W_{k\ell} = \delta_{k\ell}\, c \sum_{r=1}^{m} s_r\, e_r \otimes f_r, \qquad c := \frac{1}{S|\Delta|}.$$

Thus the family $\{W_{k\ell}\}$ has a common Schmidt decomposition with coefficients

$$a_r^{(k\ell)} = \delta_{k\ell}\, c\, s_r.$$

The corresponding Schmidt-coherence matrix is therefore

$$\Gamma_{rs} = \sum_{k,\ell} \lambda_k \mu_\ell\, a_r^{(k\ell)} \overline{a_s^{(k\ell)}} = |c|^2 \sum_k \lambda_k \mu_k\, s_r s_s.$$

Hence

$$\Gamma_{rs} = \beta\, s_r s_s, \qquad \beta := |c|^2 \sum_k \lambda_k \mu_k.$$

In particular, Γ has rank one. Substituting into the general formula for C, we obtain

$$C = \sum_{r,s} \Gamma_{rs}\, |e_r \otimes f_r\rangle\langle e_s \otimes f_s| = \beta \sum_{r,s} s_r s_s\, |e_r \otimes f_r\rangle\langle e_s \otimes f_s| = \beta\, |\Psi\rangle\langle\Psi|.$$

After normalization, this yields

$$\rho_C = |\Psi\rangle\langle\Psi|,$$

which is exactly the conclusion of the realization theorem.

9.5. Open Problems

The preceding discussion suggests several natural problems.

1. Characterize all tensor families
$$\{W_{k\ell}\}$$
that admit a common Schmidt decomposition.
2. Determine temporal conditions on the Gaussian modes
$$u_k(t), \qquad v_\ell(t),$$
that imply simultaneous singular-value diagonalizability of the operators $\hat{M}_{k\ell}$.
3. Develop entanglement criteria that do not require a common Schmidt decomposition.

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4. Characterize all tensor families realizable in the form

$$W_{k\ell} = \frac{1}{|\Delta|} \int_\Delta u_k(t) \otimes v_\ell(t)\, dt.$$

These problems point toward a general geometric theory of Gaussian DCM states beyond the pure-state realization theorem established in the present work.

9.6. A Sufficient Condition for a Common Schmidt Decomposition

The common Schmidt decomposition assumption admits a natural sufficient condition expressed directly in terms of the temporal mode functions. The basic idea is to assign different Schmidt directions to pairwise orthogonal temporal sectors. Then all mixed sector contributions vanish after time integration, and the resulting family $\{W_{k\ell}\}$ acquires a common Schmidt decomposition.

Proposition 3 (Temporal sector decomposition)**.** *Assume that the temporal Hilbert space admits an orthogonal decomposition*

$$L^2(\Delta) = \left(\bigoplus_{r=1}^{R} F_r \right) \oplus F_\perp,$$

where $F_1, \ldots, F_R \subset L^2(\Delta)$ are pairwise orthogonal closed subspaces and $F_\perp$ is their orthogonal complement.

Let

$$\{e_r\}_{r=1}^R \subset H_A, \qquad \{f_r\}_{r=1}^R \subset H_B$$

be orthonormal systems, and suppose that the Gaussian modes admit representations

$$u_k(t) = \sum_{r=1}^R \phi_{kr}(t) e_r, \qquad v_\ell(t) = \sum_{r=1}^R \psi_{\ell r}(t) f_r,$$

where

$$\phi_{kr} \in F_r, \qquad \psi_{\ell r} \in F_r \qquad (r = 1, \ldots, R).$$

Then the family

$$W_{k\ell} = \frac{1}{|\Delta|} \int_\Delta u_k(t) \otimes v_\ell(t)\, dt$$

admits a common Schmidt decomposition. More precisely,

$$W_{k\ell} = \sum_{r=1}^R a_r^{(k\ell)}\, e_r \otimes f_r,$$

where

$$a_r^{(k\ell)} = \frac{1}{|\Delta|} \int_\Delta \phi_{kr}(t) \overline{\psi_{\ell r}(t)}\, dt.$$

Proof. Substituting the mode expansions gives

$$u_k(t) \otimes v_\ell(t) = \sum_{r,s=1}^R \phi_{kr}(t) \overline{\psi_{\ell s}(t)}\, e_r \otimes f_s,$$

and therefore

$$W_{k\ell} = \sum_{r,s=1}^R \left(\frac{1}{|\Delta|} \int_\Delta \phi_{kr}(t) \overline{\psi_{\ell s}(t)}\, dt \right) e_r \otimes f_s.$$

Since $F_r \perp F_s$ for $r \neq s$, and

$$\phi_{kr} \in F_r, \qquad \psi_{\ell s} \in F_s,$$

we have

$$\int_{\Delta} \phi_{kr}(t)\overline{\psi_{\ell s}(t)}\,dt = 0 \qquad (r \neq s).$$

Hence all off-diagonal terms vanish and only the contributions with $r = s$ remain:

$$W_{k\ell} = \sum_{r=1}^{R} \left(\frac{1}{|\Delta|} \int_{\Delta} \phi_{kr}(t)\overline{\psi_{\ell r}(t)}\,dt \right) e_r \otimes f_r.$$

This is exactly the claimed common Schmidt decomposition. □

Proposition 3 shows that the temporal-sector decomposition reduces the construction of the family $\{W_{k\ell}\}$ to the construction of the scalar coefficients

$$a_r^{(k\ell)} = \frac{1}{|\Delta|} \int_{\Delta} \phi_{kr}(t)\overline{\psi_{\ell r}(t)}\,dt.$$

Once the Schmidt directions

$$e_1 \otimes f_1, \ldots, e_R \otimes f_R$$

are fixed, the resulting DCM state is determined by these coefficients and hence by the Schmidt–coherence matrix

$$\Gamma_{rs} = \sum_{k,\ell} \lambda_k \mu_\ell \, a_r^{(k\ell)} \overline{a_s^{(k\ell)}}.$$

A particularly transparent specialization is obtained when, inside each temporal sector F_r, all A-components are proportional to one common profile and all B-components are proportional to another common profile in the same sector.

Corollary 2 (Paired sector profiles). *Assume the hypotheses of Proposition 3. In addition, suppose that for each $r = 1, \ldots, R$ there exist functions*

$$\chi_r^A \in F_r, \qquad \chi_r^B \in F_r,$$

and complex coefficients $\alpha_{kr}, \beta_{\ell r} \in \mathbb{C}$ such that

$$\phi_{kr}(t) = \alpha_{kr}\chi_r^A(t), \qquad \psi_{\ell r}(t) = \beta_{\ell r}\chi_r^B(t).$$

Define the sector overlap constants

$$\eta_r := \frac{1}{|\Delta|} \int_{\Delta} \chi_r^A(t)\overline{\chi_r^B(t)}\,dt.$$

Then

$$W_{k\ell} = \sum_{r=1}^{R} \eta_r \, \alpha_{kr}\overline{\beta_{\ell r}} \, e_r \otimes f_r.$$

Equivalently,

$$a_r^{(k\ell)} = \eta_r \, \alpha_{kr}\overline{\beta_{\ell r}}.$$

Proof. By Proposition 3,

$$a_r^{(k\ell)} = \frac{1}{|\Delta|} \int_{\Delta} \phi_{kr}(t)\overline{\psi_{\ell r}(t)}\,dt.$$

Substituting

$$\phi_{kr}(t) = \alpha_{kr}\chi_r^A(t), \qquad \psi_{\ell r}(t) = \beta_{\ell r}\chi_r^B(t),$$

we obtain

$$a_r^{(k\ell)} = \frac{1}{|\Delta|} \int_{\Delta} \alpha_{kr}\chi_r^A(t)\,\overline{\beta_{\ell r}\chi_r^B(t)}\,dt = \alpha_{kr}\overline{\beta_{\ell r}} \frac{1}{|\Delta|} \int_{\Delta} \chi_r^A(t)\overline{\chi_r^B(t)}\,dt.$$

 doi:10.20944/preprints202607.1148.v1

This is exactly
$$a_r^{(k\ell)} = \eta_r\, \alpha_{kr} \overline{\beta_{\ell r}}.$$

Therefore
$$W_{k\ell} = \sum_{r=1}^{R} \eta_r\, \alpha_{kr} \overline{\beta_{\ell r}}\, e_r \otimes f_r.$$

□

Corollary 2 shows that the coefficients of the common Schmidt decomposition factor into three ingredients:

- the amplitudes α_{kr} of the A-modes in sector F_r,
- the amplitudes $\beta_{\ell r}$ of the B-modes in sector F_r,
- the temporal overlap constants

$$\eta_r = \frac{1}{|\Delta|} \int_\Delta \chi_r^A(t) \overline{\chi_r^B(t)}\, dt.$$

Accordingly,
$$a_r^{(k\ell)} = \eta_r\, \alpha_{kr} \overline{\beta_{\ell r}},$$

and the Schmidt–coherence matrix takes the form

$$\Gamma_{rs} = \eta_r \overline{\eta_s} \sum_{k,\ell} \lambda_k \mu_\ell\, \alpha_{kr} \overline{\alpha_{ks}}\, \overline{\beta_{\ell r}} \beta_{\ell s}.$$

Thus the temporal-sector approach provides a concrete constructive mechanism: the orthogonal sectors determine the common Schmidt directions, while the amplitudes α_{kr}, $\beta_{\ell r}$ and the sector overlaps η_r determine the coefficients $a_r^{(k\ell)}$ and hence the resulting DCM state.

10. Examples: Entangled Gaussian DCM States Beyond the Rank-One Realization Theorem

The realization theorem of Section 6 corresponds to the special rank-one situation in which all nonzero tensors W_{kl} are collinear with a single target vector, so that the resulting DCM state is pure. Proposition 2 shows that the common Schmidt decomposition framework is substantially more general: one may construct tensor families $\{W_{kl}\}$ that are not collinear, and nevertheless obtain entangled states. In this subsection we present two elementary examples in the two-qubit case.

Throughout this subsection let
$$H_A = H_B = \mathbb{C}^2$$

with fixed orthonormal bases

$$\{e_1, e_2\} \subset H_A, \qquad \{f_1, f_2\} \subset H_B.$$

We work in the common Schmidt basis

$$e_1 \otimes f_1, \qquad e_2 \otimes f_2.$$

Example 1 (Bell-type mixed entangled state generated by two tensors). *Assume that only two tensors are nonzero, namely*
$$W_{11} = e_1 \otimes f_1 + e_2 \otimes f_2, \qquad W_{22} = e_1 \otimes f_1 - e_2 \otimes f_2,$$

and that
$$W_{k\ell} = 0 \qquad \textit{for all other pairs } (k,\ell).$$

 doi:10.20944/preprints202607.1148.v1

Then the Gaussian DCM covariance operator is

$$\widehat{C} = \lambda_1\mu_1 \, |W_{11}\rangle\langle W_{11}| + \lambda_2\mu_2 \, |W_{22}\rangle\langle W_{22}|.$$

Expanding the two rank-one operators gives

$$|W_{11}\rangle\langle W_{11}| = |e_1 \otimes f_1\rangle\langle e_1 \otimes f_1| + |e_1 \otimes f_1\rangle\langle e_2 \otimes f_2| + |e_2 \otimes f_2\rangle\langle e_1 \otimes f_1| + |e_2 \otimes f_2\rangle\langle e_2 \otimes f_2|,$$

and

$$|W_{22}\rangle\langle W_{22}| = |e_1 \otimes f_1\rangle\langle e_1 \otimes f_1| - |e_1 \otimes f_1\rangle\langle e_2 \otimes f_2| - |e_2 \otimes f_2\rangle\langle e_1 \otimes f_1| + |e_2 \otimes f_2\rangle\langle e_2 \otimes f_2|.$$

Therefore

$$\widehat{C} = (\lambda_1\mu_1 + \lambda_2\mu_2)\Big(|e_1 \otimes f_1\rangle\langle e_1 \otimes f_1| + |e_2 \otimes f_2\rangle\langle e_2 \otimes f_2|\Big)$$
$$+(\lambda_1\mu_1 - \lambda_2\mu_2)\Big(|e_1 \otimes f_1\rangle\langle e_2 \otimes f_2| + |e_2 \otimes f_2\rangle\langle e_1 \otimes f_1|\Big).$$

Equivalently, the Schmidt–coherence matrix is

$$\Gamma = \begin{pmatrix} \lambda_1\mu_1 + \lambda_2\mu_2 & \lambda_1\mu_1 - \lambda_2\mu_2 \\ \lambda_1\mu_1 - \lambda_2\mu_2 & \lambda_1\mu_1 + \lambda_2\mu_2 \end{pmatrix}.$$

Hence

$$\Gamma_{12} = \lambda_1\mu_1 - \lambda_2\mu_2.$$

By Proposition 2, the normalized DCM state

$$\hat{\rho}_C = \frac{\widehat{C}}{\mathrm{Tr}\widehat{C}}$$

is entangled whenever

$$\lambda_1\mu_1 \neq \lambda_2\mu_2.$$

Thus unequal Gaussian weights produce a genuinely mixed entangled state. If

$$\lambda_1\mu_1 = \lambda_2\mu_2,$$

then the off-diagonal terms cancel and

$$\hat{\rho} = \frac{1}{2}\Big(|e_1 \otimes f_1\rangle\langle e_1 \otimes f_1| + |e_2 \otimes f_2\rangle\langle e_2 \otimes f_2|\Big),$$

which is separable.

Example 2 (Bell state plus a product contribution)**.** *We now consider a family consisting of one entangled tensor and one product tensor:*

$$W_{11} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2), \qquad W_{22} = e_1 \otimes f_1,$$

and

$$W_{k\ell} = 0 \qquad \textit{for all other pairs } (k, \ell).$$

Then

$$\widehat{C} = \lambda_1\mu_1 \, |W_{11}\rangle\langle W_{11}| + \lambda_2\mu_2 \, |W_{22}\rangle\langle W_{22}|.$$

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A direct computation yields

$$|W_{11}\rangle\langle W_{11}| = \frac{1}{2}\Big(|e_1\otimes f_1\rangle\langle e_1\otimes f_1| + |e_1\otimes f_1\rangle\langle e_2\otimes f_2| + |e_2\otimes f_2\rangle\langle e_1\otimes f_1| + |e_2\otimes f_2\rangle\langle e_2\otimes f_2|\Big),$$

whereas

$$|W_{22}\rangle\langle W_{22}| = |e_1\otimes f_1\rangle\langle e_1\otimes f_1|.$$

Therefore

$$C = \left(\frac{1}{2}\lambda_1\mu_1 + \lambda_2\mu_2\right)|e_1\otimes f_1\rangle\langle e_1\otimes f_1| + \frac{1}{2}\lambda_1\mu_1|e_1\otimes f_1\rangle\langle e_2\otimes f_2|$$
$$+\frac{1}{2}\lambda_1\mu_1|e_2\otimes f_2\rangle\langle e_1\otimes f_1| + \frac{1}{2}\lambda_1\mu_1|e_2\otimes f_2\rangle\langle e_2\otimes f_2|.$$

Hence the Schmidt–coherence matrix is

$$\Gamma = \begin{pmatrix} \frac{1}{2}\lambda_1\mu_1 + \lambda_2\mu_2 & \frac{1}{2}\lambda_1\mu_1 \\ \frac{1}{2}\lambda_1\mu_1 & \frac{1}{2}\lambda_1\mu_1 \end{pmatrix}.$$

In particular,

$$\Gamma_{12} = \frac{1}{2}\lambda_1\mu_1.$$

Therefore, by Proposition 2, the normalized DCM state is entangled whenever

$$\lambda_1\mu_1 > 0.$$

Thus any nonzero Bell component already forces entanglement, even in the presence of an additional product contribution. This example may be interpreted as a Bell state perturbed by a separable Gaussian contribution.

These examples illustrate that Proposition 2 yields a broad class of entangled Gaussian DCM states that are qualitatively different from the rank-one realization theorem. In the realization theorem, all nonzero tensors $W_{k\ell}$ are proportional to a single vector and the resulting state is pure. In contrast, the present examples involve several linearly independent tensors sharing a common Schmidt basis, so that the DCM state is mixed. Entanglement is then encoded in the off-diagonal entries of the Schmidt–coherence matrix Γ.

Example 3 (Temporal-sector realization of a Bell-type mixed state)**.** *We return to the Bell-type mixed family considered above, namely*

$$W_{11} = e_1\otimes f_1 + e_2\otimes f_2, \qquad W_{22} = e_1\otimes f_1 - e_2\otimes f_2,$$

and $W_{k\ell} = 0$ for all other pairs (k,ℓ), and consider examples of the corresponding temporal sector realizations.

Let the temporal Hilbert space admit an orthogonal decomposition

$$L^2(\Delta) = F_1 \oplus F_2 \oplus F_\perp,$$

where F_1 and F_2 are closed mutually orthogonal subspaces. Choose functions

$$\chi_1^A, \chi_1^B \in F_1, \qquad \chi_2^A, \chi_2^B \in F_2,$$

and define the sector overlap constants

$$\eta_r = \frac{1}{|\Delta|}\int_\Delta \chi_r^A(t)\overline{\chi_r^B(t)}\,dt, \qquad r = 1,2.$$

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Assume for simplicity that

$$\eta_1 = \eta_2 = 1.$$

For instance, this holds if one chooses normalized profiles with $\chi_r^A = \chi_r^B$, but this equality is not required in the general framework.

Define the temporal modes by

$$u_1(t) = \chi_1^A(t)e_1 + \chi_2^A(t)e_2, \qquad v_1(t) = \chi_1^B(t)f_1 + \chi_2^B(t)f_2,$$

$$u_2(t) = \chi_1^A(t)e_1 + \chi_2^A(t)e_2, \qquad v_2(t) = \chi_1^B(t)f_1 - \chi_2^B(t)f_2.$$

Equivalently, the nonzero amplitudes are

$$\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = 1,$$

$$\beta_{11} = 1, \qquad \beta_{12} = 1, \qquad \beta_{21} = 1, \qquad \beta_{22} = -1.$$

By Corollary 2,

$$W_{11} = \eta_1\alpha_{11}\overline{\beta_{11}}\, e_1 \otimes f_1 + \eta_2\alpha_{12}\overline{\beta_{12}}\, e_2 \otimes f_2 = e_1 \otimes f_1 + e_2 \otimes f_2,$$

and

$$W_{22} = \eta_1\alpha_{21}\overline{\beta_{21}}\, e_1 \otimes f_1 + \eta_2\alpha_{22}\overline{\beta_{22}}\, e_2 \otimes f_2 = e_1 \otimes f_1 - e_2 \otimes f_2.$$

Thus the Bell-type mixed state is realized by using the same two temporal sectors for the two Schmidt directions and flipping the sign of the second sector in the second B-mode.

Example 4 (Temporal-sector realization of a Bell-plus-product family). *We now realize the family*

$$W_{11} = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2), \qquad W_{22} = e_1 \otimes f_1,$$

and $W_{k\ell} = 0$ for all other pairs (k, ℓ).

Let

$$L^2(\Delta) = F_1 \oplus F_2 \oplus F_\perp,$$

with $F_1 \perp F_2$, and choose functions

$$\chi_1^A, \chi_1^B \in F_1, \qquad \chi_2^A, \chi_2^B \in F_2.$$

Define

$$\eta_r = \frac{1}{|\Delta|}\int_\Delta \chi_r^A(t)\overline{\chi_r^B(t)}\, dt, \qquad r = 1, 2,$$

and assume again for simplicity that

$$\eta_1 = \eta_2 = 1.$$

Define the temporal modes by

$$u_1(t) = \frac{1}{\sqrt{2}}\chi_1^A(t)e_1 + \frac{1}{\sqrt{2}}\chi_2^A(t)e_2, \qquad v_1(t) = \chi_1^B(t)f_1 + \chi_2^B(t)f_2,$$

$$u_2(t) = \chi_1^A(t)e_1, \qquad v_2(t) = \chi_1^B(t)f_1.$$

Equivalently, the nonzero amplitudes are

$$\alpha_{11} = \frac{1}{\sqrt{2}}, \qquad \alpha_{12} = \frac{1}{\sqrt{2}}, \qquad \beta_{11} = 1, \qquad \beta_{12} = 1,$$

$$\alpha_{21} = 1, \qquad \alpha_{22} = 0, \qquad \beta_{21} = 1, \qquad \beta_{22} = 0.$$

Again by Corollary 2,

$$W_{11} = \eta_1 \alpha_{11} \overline{\beta_{11}}\, e_1 \otimes f_1 + \eta_2 \alpha_{12} \overline{\beta_{12}}\, e_2 \otimes f_2 = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2),$$

while

$$W_{22} = \eta_1 \alpha_{21} \overline{\beta_{21}}\, e_1 \otimes f_1 + \eta_2 \alpha_{22} \overline{\beta_{22}}\, e_2 \otimes f_2 = e_1 \otimes f_1.$$

Hence the Bell-plus-product family is obtained by suppressing the second temporal sector in the second pair of modes.

These examples illustrate concretely the mechanism behind Proposition 3, Corollary 2, and Proposition 2. The orthogonal temporal sectors F_r determine the Schmidt directions $e_r \otimes f_r$, while the amplitudes $\alpha_{kr}, \beta_{\ell r}$, together with the sector overlap constants η_r, determine the coefficients

$$a_r^{(k\ell)} = \eta_r\, \alpha_{kr} \overline{\beta_{\ell r}}.$$

Consequently,

$$\Gamma_{rs} = \eta_r \overline{\eta_s} \sum_{k,\ell} \lambda_k \mu_\ell\, \alpha_{kr} \overline{\alpha_{ks}}\, \overline{\beta_{\ell r}} \beta_{\ell s}.$$

Thus the temporal-sector decomposition provides a concrete constructive route from Gaussian processes to a broad family of mixed entangled DCM states, well beyond the rank-one pure-state realization theorem.

For simplicity, we realize these examples within the special framework of Corollary, where all mode components in a given sector are proportional to fixed profiles $\chi_r^A(t)$ and $\chi_r^B(t)$. This is not essential for the existence of the examples, but it yields a particularly transparent construction. Below we present schematically the general framework.

Different temporal profiles in the same sector.

The paired-profile assumption of Corollary 2 is only a convenient sufficient condition for obtaining explicit coefficient formulas. It is not necessary for the realization of entangled families $\{W_{k\ell}\}$.

To illustrate this, suppose that two temporal sectors $F_1, F_2 \subset L^2(\Delta)$ are fixed. Choose functions

$$\phi_{11}, \phi_{21} \in F_1, \qquad \psi_{11}, \psi_{21} \in F_1,$$

and

$$\phi_{12}, \phi_{22} \in F_2, \qquad \psi_{12}, \psi_{22} \in F_2.$$

Define the temporal modes by

$$u_1(t) = \phi_{11}(t)e_1 + \phi_{12}(t)e_2, \qquad u_2(t) = \phi_{21}(t)e_1 + \phi_{22}(t)e_2,$$

and

$$v_1(t) = \psi_{11}(t)f_1 + \psi_{12}(t)f_2, \qquad v_2(t) = \psi_{21}(t)f_1 + \psi_{22}(t)f_2.$$

Then Proposition 3 yields

$$W_{k\ell} = a_1^{(k\ell)}\, e_1 \otimes f_1 + a_2^{(k\ell)}\, e_2 \otimes f_2,$$

where the coefficients are given by the temporal overlaps

$$a_1^{(11)} = \frac{1}{|\Delta|} \int_\Delta \phi_{11}(t) \overline{\psi_{11}(t)}\, dt, \qquad a_2^{(11)} = \frac{1}{|\Delta|} \int_\Delta \phi_{12}(t) \overline{\psi_{12}(t)}\, dt,$$

 doi:10.20944/preprints202607.1148.v1

$$a_1^{(22)} = \frac{1}{|\Delta|} \int_\Delta \phi_{21}(t)\overline{\psi_{21}(t)}\, dt, \qquad a_2^{(22)} = \frac{1}{|\Delta|} \int_\Delta \phi_{22}(t)\overline{\psi_{22}(t)}\, dt.$$

Hence, in order to realize the Bell-type family

$$W_{11} = e_1 \otimes f_1 + e_2 \otimes f_2, \qquad W_{22} = e_1 \otimes f_1 - e_2 \otimes f_2,$$

it is enough to choose these four overlaps so that

$$a_1^{(11)} = 1, \qquad a_2^{(11)} = 1, \qquad a_1^{(22)} = 1, \qquad a_2^{(22)} = -1.$$

There is no requirement that the temporal profiles coincide across the different modes. In particular, one does not need

$$\phi_{11} = \phi_{21}, \qquad \phi_{12} = \phi_{22},$$

nor

$$\psi_{11} = \psi_{21}, \qquad \psi_{12} = \psi_{22}.$$

The price of allowing such freedom is that one loses the simple factorization from Corollary 2. In the paired-profile setting, the coefficients factor as

$$a_r^{(k\ell)} = \eta_r\, \alpha_{kr}\overline{\beta_{\ell r}},$$

so that each sector contributes a rank-one coefficient array in the indices (k, ℓ). In the present more general setting, each coefficient is instead an independent temporal overlap,

$$a_r^{(k\ell)} = \frac{1}{|\Delta|} \int_\Delta \phi_{kr}(t)\overline{\psi_{\ell r}(t)}\, dt = \frac{1}{|\Delta|} \langle \psi_{\ell r}, \phi_{kr} \rangle_{L^2(\Delta)}.$$

Thus the paired-profile ansatz provides a particularly transparent rank-one factorization in each sector, whereas the general temporal-sector framework allows a much larger class of coefficient arrays.

This is a good place to emphasize once again that, in DCM, large classes of classical states (stochastic processes) may give rise to the same quantum state.

11. Conclusions and Outlook

The Double Covariance Model (DCM) was introduced as a classical probabilistic framework in which quantum states of composite systems are generated from classical stochastic processes by means of a *double covariance* construction, that is, as covariances of random temporal covariances. In its full generality, this construction is genuinely fourth order: the DCM state is determined by the covariance of the random tensor

$$C_\Delta(\omega) = \frac{1}{|\Delta|} \int_\Delta X_t(\omega) \otimes Y_t(\omega)\, dt,$$

or, equivalently, by the covariance of the corresponding random operator

$$\widehat{C}_\Delta(\omega) = \frac{1}{|\Delta|} \int_\Delta |X_t(\omega)\rangle\langle Y_t(\omega)|\, dt.$$

Thus the DCM naturally lives in a fourth-order statistical framework.

The main result of the present paper is that, for jointly Gaussian processes, this fourth-order structure admits a complete reduction to second-order covariance data. By the Isserlis–Wick theorem, every fourth-order moment entering the DCM construction is uniquely expressed in terms of pair covariances. Consequently, the double-covariance operator, and therefore the associated quantum density operator, is completely determined by the second-order covariance kernel of the joint Gaussian process. In this sense, the Gaussian reduction transforms the DCM from a fourth-order statistical model into a second-order one without changing the resulting quantum state.

 doi:10.20944/preprints202607.1148.v1

This reduction is not merely a technical simplification. It leads to a concrete and flexible Gaussian realization theory for quantum states generated by the DCM. In particular, by working with the Karhunen–Loève expansions of the subquantum Gaussian processes, one obtains an explicit description of the time-averaged tensor amplitudes

$$W_{k\ell} = \frac{1}{|\Delta|}\int_{\Delta} u_k(t) \otimes v_\ell(t)\, dt,$$

where u_k and v_ℓ are the Karhunen–Loève modes of the two processes. These tensors form the basic deterministic objects of the Gaussian DCM construction. In the independent Gaussian case, the DCM covariance operator takes the form

$$\widehat{C} = \sum_{k,\ell} \lambda_k \mu_\ell \, |W_{k\ell}\rangle\langle W_{k\ell}|,$$

so that the resulting quantum state is completely determined by the geometry of the tensor family $\{W_{k\ell}\}$.

This point of view reveals a feature of the DCM that is both mathematically striking and conceptually counterintuitive from the perspective of standard quantum-information intuition: quantum entangled states can be generated by *independent* jointly Gaussian processes. Thus, within the DCM, the origin of entanglement need not lie in ordinary second-order statistical dependence between the subquantum processes X and Y. In the Gaussian framework, the decisive ingredient is instead the temporal organization of the Karhunen–Loève modes and, more specifically, the structure of the integrated tensors $W_{k\ell}$.

For pure-state realizations, this temporal organization is naturally encoded by the decomposition of the temporal Hilbert space into mutually orthogonal sectors. The temporal-sector construction developed in this paper shows that the Schmidt structure of the target quantum state may be realized through an appropriate allocation of the Gaussian modes to orthogonal temporal sectors. In this way, the temporal-sector decomposition emerges as a basic constructive mechanism for the generation of entanglement in the Gaussian DCM.

The same Gaussian framework also gives a new interpretation of concurrence. For the pure-state realizations considered in this paper, the Schmidt coefficients are directly related to the expected distribution of the subquantum signal energy among the temporal sectors. Hence concurrence can be expressed in terms of the relative sector energies. We stress that this "energy" is not the physical energy observable of the quantum system itself. Rather, it is the energy of the classical random signals at the subquantum time scale, that is, an internal quantity of the DCM representation. Within this representation, entanglement becomes linked to a specific pattern of energy redistribution among orthogonal temporal sectors.

Beyond the rank-one realization theorem for pure states, the Gaussian reduction also suggests a broader general theory of Gaussian DCM states. In the independent Gaussian case, the entire DCM state is encoded by the tensor family $\{W_{k\ell}\}$, and this shifts the emphasis from covariance kernels themselves to the geometry of time-averaged tensor modes. This perspective led us to the study of families $\{W_{k\ell}\}$ admitting a common Schmidt decomposition and to the corresponding Schmidt–coherence matrix. In that framework, the temporal-sector decomposition provides a natural sufficient condition for the existence of a common Schmidt decomposition and hence a constructive route to broad classes of mixed entangled Gaussian DCM states.

At the same time, the present results should be interpreted with appropriate caution. Neither the DCM itself nor its Gaussian reduction can at present be regarded as a complete classical probabilistic foundation of quantum mechanics. What has been established so far is more modest, though still substantial. On the one hand, the DCM provides a mathematically explicit classical mechanism for the generation of quantum entangled states, including states produced by independent Gaussian processes. On the other hand, earlier work has shown that the same framework can be extended to

 doi:10.20944/preprints202607.1148.v1

quantum Markovian dynamics. These results capture important structural aspects of quantum theory, but they are far from constituting a full reconstruction of quantum mechanics.

Nevertheless, the DCM and its Gaussian reduction open a promising direction for research on the interplay between classical and quantum probability. Their main novelty lies in the use of a genuinely double-covariance construction based on two distinct time scales and on the interplay between temporal and statistical covariances. From this perspective, quantum states are not represented by ordinary second-order classical statistics, but by statistical covariances of random temporal covariances. The Gaussian reduction shows that this fourth-order framework can nevertheless be controlled by second-order covariance data, while still retaining a rich entanglement structure. We expect that further development of this viewpoint may lead to a broader geometric and probabilistic theory of Gaussian DCM states and, more generally, to new insights into the classical probabilistic structures underlying quantum phenomena.

Acknowledgments: This study was stimulated by a remark made by Lev Murokh during the discussion following my talk at the conference QIP26 (Växjö, Sweden, June 2026). He appreciated the novelty of the DCM as a fourth-order statistical framework, but pointed out that in the Gaussian case it should admit a reduction to second-order statistics. I am grateful to him for this insightful observation, which motivated me to write the present paper. I would also like to thank the other participants of QIP26 for valuable discussions and helpful comments.

Appendix: Isserlis–Wick Reduction for Circular Gaussian Processes

On the Historical and Structural Equivalence of Isserlis's and Wick's Theorems.

In the physics literature, the algebraic reduction of higher-order statistical moments to combinatoric sums of products of second-order covariances is almost universally referred to as *Wick's theorem*. In its original 1950 formulation [41], Wick's theorem is strictly an operator-theoretic result designed for quantum field theory, dictating how products of non-commuting creation and annihilation operators on a Fock space decompose into sums of pairwise contractions.

However, later mathematical physicists recognized a profound structural parallel: the combinatorics governing quantum vacuum expectation values of field operators are formally identical to the combinatorics of calculating higher-order moments of classical Gaussian random variables (see, e.g., [20,38]). Because physicists utilize quantum field theory daily, the name *Wick's theorem* evolved into a ubiquitous blanket term within the physics community for any algebraic procedure that partitions a fourth-order (or higher) product into pairwise contractions—regardless of whether the underlying system is classical or quantum. Consequently, even in purely classical domains such as statistical mechanics or stochastic signal processing, physicists heavily favor the invocation of "Wick's theorem."

Despite this widespread convention, DCM is established within a strictly classical probabilistic framework governed by Kolmogorov probability spaces and commuting, complex-valued stochastic processes. To maintain mathematical precision and emphasize the purely classical foundations of our model, we invoke the proper historical designation from multivariate statistics: *Isserlis's theorem* (originally proved by Leon Isserlis in 1918 [19,33]).

Crucially, when Isserlis's theorem is restricted to the *centered circular complex* Gaussian processes used in this work, phase-invariance symmetry forces all pure non-conjugated and pure conjugated pseudocovariances ($\mathbb{E}[X_t X_s] = 0$) to vanish identically. This classical probabilistic condition isolates a set of surviving cross-contractions that is combinatorially identical to the behavior of field operators in quantum mechanics, where non-matching operator pairings vanish ($\langle 0|aa|0\rangle = 0$). Utilizing Isserlis's theorem under circularity thus preserves the purely classical foundation of our model while cleanly recovering the algebraic reduction rules required to generate entangled quantum states, showing that this specific contraction structure is an intrinsic feature of circular Gaussianity rather than an exclusively quantum phenomenon.

Theorem 3. *Let X_t and Y_t be centered circular Gaussian Hilbert-space-valued stochastic processes. Then the fourth-order tensor kernel $\hat{K}(t,s) = \mathbb{E}[|X_t \otimes Y_t\rangle\langle X_s \otimes Y_s|]$ admits a complete Wick reduction to the second-order covariance kernels given by:*

$$\hat{K}(t,s) = \hat{R}_{XX}(t,s) \otimes \hat{R}_{YY}(t,s) + \hat{E}(t,s), \tag{10}$$

where $\hat{E}(t,s)$ is the exchange operator uniquely defined by its matrix elements:

$$\langle u \otimes v, \hat{E}(t,s)(u' \otimes v')\rangle = \langle u, \hat{R}_{XY}(t,s)v'\rangle\langle v, \hat{R}_{YX}(t,s)u'\rangle. \tag{11}$$

Proof. Let $u, u' \in H_A$ and $v, v' \in H_B$ be arbitrary vectors. The matrix elements of the fourth-order operator $\hat{K}(t,s)$ are given by the statistical expectation:

$$\langle u \otimes v, \hat{K}(t,s)(u' \otimes v')\rangle = \mathbb{E}[\langle u, X_t\rangle\langle v, Y_t\rangle\overline{\langle u', X_s\rangle\langle v', Y_s\rangle}]. \tag{12}$$

We map this expression onto the classical Isserlis–Wick theorem for complex random variables by setting:

$$\begin{aligned} A &= \langle u, X_t\rangle, & B &= \langle v, Y_t\rangle, \\ \bar{C} &= \overline{\langle u', X_s\rangle}, & \bar{D} &= \overline{\langle v', Y_s\rangle}. \end{aligned}$$

The full combinatorial pairing expansion under the Wick theorem yields exactly three distinct structural combinations:

$$\mathbb{E}[AB\bar{C}\bar{D}] = \mathbb{E}[A\bar{C}]\mathbb{E}[B\bar{D}] + \mathbb{E}[A\bar{D}]\mathbb{E}[B\bar{C}] + \mathbb{E}[AB]\mathbb{E}[\bar{C}\bar{D}]. \tag{13}$$

We analyze each pairing separately using the definition of the second-order covariance kernels:

1. **Pure Subsystem Pairings:** The first term pairs the non-conjugated component of each subsystem with its own macroscopic conjugated counterpart:

$$\mathbb{E}[A\bar{C}] = \mathbb{E}[\langle u, X_t\rangle\overline{\langle u', X_s\rangle}] = \langle u, \hat{R}_{XX}(t,s)u'\rangle, \tag{14}$$

$$\mathbb{E}[B\bar{D}] = \mathbb{E}[\langle v, Y_t\rangle\overline{\langle v', Y_s\rangle}] = \langle v, \hat{R}_{YY}(t,s)v'\rangle. \tag{15}$$

Taking their product directly reconstructs the standard tensor product operator representation:

$$\mathbb{E}[A\bar{C}]\mathbb{E}[B\bar{D}] = \langle u \otimes v, \big(\hat{R}_{XX}(t,s) \otimes \hat{R}_{YY}(t,s)\big)(u' \otimes v')\rangle. \tag{16}$$

2. **Cross-Subsystem Pairings:** The second term pairs the temporal state of system A with system B, describing the spatial correlation structure:

$$\mathbb{E}[A\bar{D}] = \mathbb{E}[\langle u, X_t\rangle\overline{\langle v', Y_s\rangle}] = \langle u, \hat{R}_{XY}(t,s)v'\rangle, \tag{17}$$

$$\mathbb{E}[B\bar{C}] = \mathbb{E}[\langle v, Y_t\rangle\overline{\langle u', X_s\rangle}] = \langle v, \hat{R}_{YX}(t,s)u'\rangle. \tag{18}$$

By utilizing the definition of the cross-system exchange operator $\hat{E}(t,s)$, this structural block evaluates to:

$$\mathbb{E}[A\bar{D}]\mathbb{E}[B\bar{C}] = \langle u \otimes v, \hat{E}(t,s)(u' \otimes v')\rangle. \tag{19}$$

3. **Pseudocovariance Pairings:** The third possible pairing collects terms of matching conjugation types:

$$\mathbb{E}[AB] = \mathbb{E}[\langle u, X_t\rangle\langle v, Y_t\rangle], \quad \text{and} \quad \mathbb{E}[\bar{C}\bar{D}] = \mathbb{E}[\overline{\langle u', X_s\rangle}\,\overline{\langle v', Y_s\rangle}].$$

doi:10.20944/preprints202607.1148.v1

Because the underlying processes are explicitly defined as circular Gaussian under Definition 1, all pure non-conjugated and pure conjugated pairings collapse identically to zero due to phase-invariance symmetry:

$$\mathbb{E}[AB] = 0 \quad \text{and} \quad \mathbb{E}[\bar{C}\bar{D}] = 0.$$

Consequently, this entire combinatorial channel drops out of the expectation profile:

$$\mathbb{E}[AB]\mathbb{E}[\bar{C}\bar{D}] = 0 \cdot 0 = 0. \tag{20}$$

Substituting Equations (16), (19), and (20) back into the Wick structural expansion (13) simplifies the matrix element expression to:

$$\langle u \otimes v, \hat{K}(t,s)(u' \otimes v') \rangle = \langle u \otimes v, \big(\hat{R}_{XX}(t,s) \otimes \hat{R}_{YY}(t,s) + \hat{E}(t,s)\big)(u' \otimes v') \rangle.$$

Since this identity holds tightly for all test vectors on the arbitrary dense domains of $H_A \otimes H_B$, it implies the operator identity $\hat{K}(t,s) = \hat{R}_{XX}(t,s) \otimes \hat{R}_{YY}(t,s) + \hat{E}(t,s)$. $\square$

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