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Deferred f-Statistical Convergence of Generalized Difference Sequences of Order α

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Abstract: Studies on difference sequences was introduced in the 1980s and then many mathematicians studied on these kind of sequences and obtained some generalized difference sequence spaces. In this paper, using the generalized difference operator, we introduce the concept of deferred $f-$statistical convergence of generalized difference sequences of order α and give some inclusion relations between deferred $f-$statistical convergence of generalized difference sequences and deferred $f-$statistical convergence of generalized difference sequences of order α. Our results are more general than the corresponding results in the existing literature

Keywords: difference sequence; deferred statistical convergence; statistical convergence of order α

MSC: 39A70; 47B39; 40A05

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1. Introduction, Definitions and Preliminaries

The concept of statistical convergence was introduced by Steinhaus [37] and Fast [19], then reintroduced independently by Schoenberg [36], and the notion was associated with summability theory by Bhardwaj et al. ([10],[21]), Braha et al. [11], Çolak [13], Connor [14], Et et al. ([17],[18]), Fridy [20], Işık et al. ([3],[22,23],[24]), Küçükaslan and Yılmaztürk [27], León-Saavedra et al. ([28],[29],[32]), Salat [34], Temizsu et al. ([38],[39]) and many others.

The natural density of subsets of $\mathbb{N}$ plays a critical role in the definition of statistical convergence. For a subset A of natural numbers if the following limit exists

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:k\in A\}|$$

then this unique limit is called *the density of* A and mostly abbreviated by $\delta(A)$, where $|\{k\leq n:k\in A\}|$ is the number of members of A not exceeding n.

A sequence $x=(x_k)$ statistically converges to L provided that

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:|x_k-L|\geq\varepsilon\}|=0$$

for each $\varepsilon>0$. It is written by $S-\lim x_k=L$. If $L=0$ then x is a statistically null sequence.

The study of difference sequences reveals patterns inherent in natural growth processes. By understanding the convergence models applied to these sequences, we can make predictions and identify anomalies. In essence, summability methods, when applied to difference sequence spaces, offer a powerful tool for obtaining highly useful insights. Difference sequence spaces, a recent development in Summability Theory, were first introduced by Kızmaz in the 1980s and have since been extensively studied by mathematicians. The difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ were introduced by Kızmaz [26], as the domain of the forward difference matrix Δ^F, transforming a sequence $x=(x_k)$ to the difference sequence $\Delta^F x = (x_k - x_{k+1})$, in the classical spaces ℓ_∞, c and c_0 of bounded, convergent and null sequences, respectively. Quite recently, the difference space bv_p was introduced as the domain of the backward difference matrix Δ^B, transforming a sequence $x = (x_k)$ to the difference sequence $\Delta^B x = (x_k - x_{k-1})$, in the space ℓ_p of absolutely p-summable sequences for $1 < p < \infty$ by Altay and Başar [5], and for $1 \le p < \infty$ by Başar and Altay [8]. For more information on ℓ_p-type spaces see [6],[7] and [43]. The reader can refer to the monographs [9] and [31] for the background on the normed and paranormed sequence spaces, and summability theory and related topics. The idea of difference sequences was generalized by Et and Çolak [15] as follows:

Given a sequence space X and a number $m \in \mathbb{N}$, the space $\Delta^m(X)$ is defined as

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so $\Delta^m x_k = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$\begin{aligned} x_k &= \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \\ y_{1-n} &= y_{2-n} = \cdots = y_0 = 0 \end{aligned} \tag{1}$$

for sufficiently large k, for instance $k > 2m$. Recently, a large amount of work has been carried out by several mathematicians regarding various generalizations of difference sequence spaces. For a detailed account of difference sequence spaces one may refer to ([4],[12],[16],[25],[35],[40],[41],[42]).

Deferred Cesàro mean of real valued sequences $x = (x_k)$ is defined by Agnew [1]. Taking into account Agnew's approach, Küçükaslan and Yılmaztürk [27] introduced the concept of deferred statistical convergence as follows:

A real valued sequence $x = (x_k)$ is called *deferred statistically convergent* to a number L provided for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{p_n < k \le q_n : |x_k - L| \ge \varepsilon\}|}{q_n - p_n} = 0$$

where $p = (p_n)$ and $q = (q_n)$ are sequences of non-negative integers satisfying the conditions

$$\lim_{n\to\infty} q_n = \infty \text{ and } p_n < q_n \text{ for all } n \in \mathbb{N}. \tag{2}$$

This is a mathematical concept that offers a more nuanced and flexible approach to studying the convergence of sequences and series. Unlike traditional methods, which analyze the entire sequence or series at once, deferred convergence allows us to focus on parts of the sequence. By examining specific parts, we can identify finer convergence patterns that might be hidden when looking at the entire sequence. Throughout the paper, we preassume that the sequences (p_n) and (q_n) satisfy (2) and additionally $\lim\limits_{n\to\infty}(q_n - p_n) = \infty$. We denote the set of all such (p, q) pairs by Ω. Some restrictions on (p, q) will be imposed if needed.

Modulus functions, introduced by Nakano [33], serve to bridge the gap between ordinary and statistical convergence. A modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \le f(x) + f(y)$ for all $x, y \ge 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

Hence f must be continuous everywhere on $[0, \infty)$. A modulus may be unbounded or bounded. For example, $f(x) = x^t$ $(0 < t \le 1)$ is unbounded, but $f(x) = \frac{x}{x+1}$ is bounded.

2. Δ_f^m-Deferred Statistically Convergence of Order α

Let f be an unbounded modulus, $(p,q)\in\Omega$, $\alpha\in(0,1]$, A be a subset of $\mathbb{N}$ and $A_{p,q}(n)$ denote the set $\{k : p_n<k\le q_n, k\in A\}$. The $(D^{f,\alpha})-$density of A is defined by

$$\delta_{p,q}^{f,\alpha}(A)=\lim_{n\to\infty}\frac{1}{[f(q_n-p_n)]^{\alpha}}f(|A_{p,q}(n)|)$$

provided the limit exists.

Remark 2.1
i) If $\delta_{p,q}^{f,\alpha}(A)=0$ then A is said to be a $(D^{f,\alpha})-$null set.

ii) If $x=(x_k)$ is a sequence such that x_k holds property $P(k)$ for all k except a $(D^{f,\alpha})-$null set, then we say that x_k holds $P(k)$ for "almost all k according $D^{f,\alpha}$" and we denote this by "*a.a.k* $(D^{f,\alpha})$ ".

The proof of each of the following results is straightforward, so we choose to state these results *without* proof.

Proposition 2.1 Let f be an unbounded modulus, $(p,q)\in\Omega$ and $0<\alpha\le\beta\le1$. Then $\delta_{p,q}^{f,\beta}(A)\le\delta_{p,q}^{f,\alpha}(A)$ for any $A\subset\mathbb{N}$.

Proposition 2.2 $A\subset B$ implies $\delta_{p,q}^{f,\alpha}(A)\le\delta_{p,q}^{f,\alpha}(B)$ for any unbounded modulus f, $(p,q)\in\Omega$ and $0<\alpha\le1$.

Proposition 2.3 $\delta_{p,q}^{f,\alpha}(A)=\delta_{p,q}^{f,\alpha}(B)=0$ implies $\delta_{p,q}^{f,\alpha}(A\cap B)=\delta_{p,q}^{f,\alpha}(A\cup B)=0$.

Definition 2.1 Let f be an unbounded modulus, $(p,q)\in\Omega$ be given and $\alpha\in(0,1]$. A sequence $x=(x_k)$ is said to be Δ_f^m-deferred statistically convergent of order α to L if there is a real number L such that for each $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{[f(q_n-p_n)]^{\alpha}}f(|\{p_n<k\le q_n : |\Delta^m x_k-L|\ge\varepsilon\}|)=0.$$

In this case we write $S_{p,q}^{\alpha}(\Delta_f^m)-\lim x_k=L$. The set of all Δ_f^m-deferred statistically convergent sequences of order α is denoted by $S_{p,q}^{\alpha}(\Delta_f^m)$. If $q_n=n$, $p_n=0$ for all $n\in\mathbb{N}$ and $\alpha=1$, then $S_{p,q}^{\alpha}(\Delta_f^m)=S(\Delta_f^m)$ and $q_n=n$, $p_n=0$ for all $n\in\mathbb{N}$, then $S_{p,q}^{\alpha}(\Delta_f^m)=S^{\alpha}(\Delta_f^m)$. If $f(x)=x$, we have $S_{p,q}^{\alpha}(\Delta_f^m)=S_{p,q}^{\alpha}(\Delta^m)$. In case of $m=0$ we have $S_{p,q}^{\alpha}(\Delta_f^m)=S_{p,q}^{f,\alpha}$.

Δ_f^m-deferred statistical convergence of order α is not well defined for $\alpha>1$. The following example confirms this.

Example 2.1 Let f be an unbounded modulus, $(p,q)\in\Omega$, $\alpha>1$ and a sequence $x=(x_j)$ be defined by

$$x_j=\begin{cases}0 & 1\le j\le 3\\ x_{j-1}+\frac{j-2}{2} & j=2n, n\ge 2\\ x_{j-1}+\frac{j-3}{2} & j=2n+1, n\ge 2\end{cases}\quad n=1,2,3,\ldots.$$

Taking $m=2$ we get

$$\Delta^2 x_j=\begin{cases}1, & j=2n\\ 0, & j\ne 2n\end{cases}\quad n=1,2,3,\ldots.$$

Then, for each $\varepsilon>0$, we have

$$\frac{f(|\{p_n<k\le q_n : |\Delta^m x_k-1|\ge\varepsilon\}|)}{[f(q_n-p_n)]^{\alpha}}\le\frac{f(q_n-p_n)}{[f(q_n-p_n)]^{\alpha}}$$

and

$$\frac{f(|\{p_n < k \le q_n : |\Delta^m x_k - 0| \ge \varepsilon\}|)}{[f(q_n - p_n)]^\alpha} \le \frac{f(q_n - p_n)}{[f(q_n - p_n)]^\alpha}$$

which means $S^\alpha_{p,q}(\Delta^2_f) - \lim x_j = 0$ and $S^\alpha_{p,q}(\Delta^2_f) - \lim x_j = 1$ for $\alpha > 1$.

We continue our work by giving some results without proof.

Theorem 2.1 Let f be an unbounded modulus, $(p,q) \in \Omega$, $\alpha \in (0,1]$ and $x = (x_k)$, $y = (y_k)$ be sequences of real numbers, then the following is true.

(i) If $S^\alpha_{p,q}(\Delta^m_f) - \lim x_k = L$ and $c \in \mathbb{R}$, then $S^\alpha_{p,q}(\Delta^m_f) - \lim cx_k = cL$.

(ii) If $S^\alpha_{p,q}(\Delta^m_f) - \lim x_k = L_1$ and $S^\alpha_{p,q}(\Delta^m_f) - \lim y_k = L_2$, then $S^\alpha_{p,q}(\Delta^m_f) - \lim(x_k + y_k) = L_1 + L_2$.

Theorem 2.2 Let f be an unbounded modulus, $(p,q) \in \Omega$, $\alpha \in (0,1]$. Then the inclusion $S^\alpha_{p,q}(\Delta^m_f) \subset S^\alpha_{p,q}(\Delta^{m+1}_f)$ strictly holds for $m \in \mathbb{N}$.

Corollary 2.1 Let f be an unbounded modulus, $(p,q) \in \Omega$, $\alpha \in (0,1]$. For all $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$ the inclusion $S^\alpha_{p,q}(\Delta^{m_1}_f) \subset S^\alpha_{p,q}(\Delta^{m_2}_f)$ is strict.

Theorem 2.3 Let f be an unbounded modulus, $(p,q) \in \Omega$ and $\alpha, \beta \in (0,1]$ with $\alpha < \beta$.
Then the inclusion $S^\alpha_{p,q}(\Delta^m_f) \subseteq S^\beta_{p,q}(\Delta^m_f)$ is strict.

Proof. The inclusion part of the proof is straightforward. To show the strictness of the inclusion, let us consider the sequence $y = (y_j)$ by

$$y_j = \begin{cases} 1 & j = i^2 \\ 0 & j \ne i^2 \end{cases}$$

such that $\Delta^m x_j = y_j$ for some $x = (x_j)$ by (1). Employing the modulus $f(x) = x^t$ $(0 < t \le 1)$, $p_n = n^2$, $q_n = 4n^2$ we observe that for each $n \in \mathbb{N}$ and $\varepsilon > 0$

$$\left|\left\{n^2 < j \le 4n^2 : |\Delta^m x_j| \ge \varepsilon\right\}\right| \le n$$

and so

$$\frac{f(|n^2 < j \le 4n^2 : |\Delta^m x_j| \ge \varepsilon|)}{[f(4n^2 - n^2)]^\beta} \le \frac{n^t}{3^{\beta t} n^{2\beta t}}.$$

Then taking limit as $n \to \infty$ we have $S^\beta_{p,q}(\Delta^m_f) - \lim x_j = 0$ where $\frac{1}{2} < \beta \le 1$. On the other hand, picking $\varepsilon = \frac{1}{3}$ and observing Fatih

$$\left|\left\{n^2 < j \le 4n^2 : |\Delta^m x_j| \ge \frac{1}{3}\right\}\right| = n$$

for each $n \in \mathbb{N}$, we have the following equality

$$\frac{f\left(\left|n^2 < j \le 4n^2 : |\Delta^m x_j| \ge \frac{1}{3}\right|\right)}{[f(4n^2 - n^2)]^\alpha} = \frac{n^t}{3^{\alpha t} n^{2\alpha t}}$$

which yields that $S^\alpha_{p,q}(\Delta^m_f) - \lim x_j \ne 0$ where $0 < \alpha \le \frac{1}{2}$.

Theorem 2.4 Let f be unbounded modulus, $(p,q) \in \Omega$ and $\alpha \in (0,1]$. Then every Δ^m-convergent sequence is Δ^m_f-deferred statistically convergent of order α, but the converse does not need to hold.

Proof. The inclusion follows from the fact that the set $\{k \in \mathbb{N} : |\Delta^m x_k - L| \ge \varepsilon\}$ is finite for each $\varepsilon > 0$ assuming $\lim \Delta^m x_k = L$. To show the converse does not hold for some particular cases, let us choose $p_n = n$ and $q_n = 2^n$, $f(x) = x^t$ $(0 < t \le 1)$ and a sequence $x = (x_k)$ such that

$$\Delta^m x_k = \begin{cases} n, & k = 2^n \\ \frac{1}{5}, & \text{else} \end{cases} \quad n = 1,2,3,\ldots \text{ by (1).}$$

It is obvious that

$$\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right| \leq n$$

for each $n \in \mathbb{N}$ and $\varepsilon > 0$. Therefore we have

$$\frac{f(\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right|)}{[f(2^n - n)]^\alpha} \leq \frac{f(n)}{[f(2^n - n)]^\alpha} = \frac{n^t}{(2^n - n)^{t\alpha}}$$

which results in $S^\alpha_{p,q}(\Delta^m_f) - \lim x_k = \frac{1}{5}$ for $\alpha \in \left(\frac{1}{2}, 1\right]$. However, it is clear that x is not Δ^m-convergent.

Theorem 2.5 Let f be an unbounded modulus, $(p,q) \in \Omega$ and $\alpha \in (0,1]$. Then every Δ^m_f-deferred statistically convergent sequence of order α is Δ^m-deferred statistically convergent, not conversely.

Proof. Let $x = (x_k)$ be Δ^m_f-deferred statistically convergent to L of order α. That is, for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{f(|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^\alpha} = 0.$$

Then for each $r \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ so that $n \geq n_0$ implies

$$f(|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|) \leq \frac{1}{r}[f(q_n - p_n)]^\alpha \leq \frac{1}{r} f(q_n - p_n)$$

Moreover, due to subadditiveness of f, we get

$$\frac{1}{r} f(q_n - p_n) = \frac{1}{r} f\left(r \frac{q_n - p_n}{r}\right) \leq f\left(\frac{q_n - p_n}{r}\right).$$

This follows that

$$\frac{|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|}{q_n - p_n} \leq \frac{1}{r}$$

since f is increasing. Thus x is Δ^m-deferred statistically convergent.

The sequence used in Theorem 2.4 can be reissued to see the converse of this result need not hold. Aforementioned sequence $x = (x_k)$ is Δ^m-deferred statistically convergent to $\frac{1}{5}$ where $p_n = n$ and $q_n = 2^n$. However, we observe the inequality

$$\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right| \geq n - \left[\sqrt{n}\right] - 1 \text{ for each } n \in \mathbb{N},$$

where $[.]$ denotes the integral part of the enclosed number. Considering the modulus $g(x) = \ln(x+1)$ and $\alpha \in (0,1]$ we have

$$\begin{aligned} \frac{g(\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right|)}{[g(2^n - n)]^\alpha} &\geq \frac{g(n - [\sqrt{n}] - 1)}{[g(2^n - n)]^\alpha} = \frac{\ln(n - [\sqrt{n}])}{[\ln(2^n - n + 1)]^\alpha} \\ &\geq \frac{\ln(n - [\sqrt{n}])}{[\ln(2^n + 1)]^\alpha} \geq \frac{\ln(n - [\sqrt{n}])}{\ln(2^n + 1)} \\ &> \frac{\ln(n - [\sqrt{n}])}{\ln(n^3 + 1)} = b_n. \end{aligned}$$

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This implies

$$\lim_{n\to\infty} \frac{g\left(\left|\left\{n < k \leq 2^n : \left|\Delta^m x_k - \frac{1}{5}\right| \geq \varepsilon\right\}\right|\right)}{[g(2^n - n)]^\alpha} \neq 0$$

since $\lim\limits_{n\to\infty} b_{n^2} = \lim\limits_{n\to\infty} \frac{\ln(n^2-n)}{\ln(n^6+1)} = \frac{1}{3}$. Thus $S^\alpha_{p,q}(\Delta^m_g) - \lim x_k \neq \frac{1}{5}$.

Theorem 2.6 Let f be an unbounded modulus, $(p,q) \in \Omega$ be given and α be a fixed real number such that $\alpha \in (0,1]$. If the sequence $\left\{\frac{[f(q_n)]^\alpha}{[f(q_n - p_n)]^\alpha}\right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m_f-statistically convergent sequence of order α is Δ^m_f-deferred statistically convergent of order α.

Proof. If (x_k) is a Δ^m_f-statistically convergent sequence of order α, there exists $L > 0$ such that

$$\lim_{n\to\infty} \frac{f(|\{k \leq n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(n)]^\alpha} = 0$$

Then, due to $\lim_{n\to\infty} q_n = \infty$ the sequence

$$\left\{\frac{f(|\{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n)]^\alpha}\right\}_{n\in\mathbb{N}}$$

is a null sequence. Furthermore the inclusion $\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\} \subseteq \{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}$ implies

$$\begin{aligned}\frac{f(|\{p_n < k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^\alpha} &\leq \frac{[f(q_n)]^\alpha}{[f(q_n - p_n)]^\alpha}\frac{f(|\{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n)]^\alpha} \\ &\leq M\frac{f(|\{k \leq q_n : |\Delta^m x - L| \geq \varepsilon\}|)}{[f(q_n)]^\alpha}\end{aligned}$$

for some $M > 0$. Taking limit as $n \to \infty$ yields that x is Δ^m_f-deferred statistically convergent to L of order α.

From Theorem 2.6 we get the following results.

Corollary 2.2 Let f be an unbounded modulus, $(p,q) \in \Omega$ be given and α be a fixed real number such that $\alpha \in (0,1]$. If $q_n < n$ for all $n \in \mathbb{N}$ and the sequence $\left\{\frac{[f(n)]^\alpha}{[f(q_n - p_n)]^\alpha}\right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m_f-statistically convergent sequence of order α is Δ^m_f-deferred statistically convergent of order α.

Corollary 2.3 Let f be an unbounded modulus, $(p,q) \in \Omega$ be given and α be a fixed real number such that $\alpha \in (0,1]$. If $\lim\limits_{n} \frac{[f(q_n - p_n)]^\alpha}{[f(n)]^\alpha} = a > 0$ $(a \in \mathbb{R})$ and $q_n < n$, then every Δ^m_f-statistically convergent sequence of order α is Δ^m_f-deferred statistically convergent of order α.

Corollary 2.4

i) Let f be an unbounded modulus, $(p,q) \in \Omega$. If the sequence $\left\{\frac{[f(q_n)]}{[f(q_n - p_n)]}\right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m_f-statistically convergent sequence is Δ^m_f-deferred statistically convergent.

ii) Let $(p,q) \in \Omega$ be given and α be a fixed real number such that $\alpha \in (0,1]$. If the sequence $\left\{\frac{(q_n)^\alpha}{(q_n - p_n)^\alpha}\right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m-statistically convergent sequence of order α is Δ^m-deferred statistically convergent of order α.

iii) Let $(p,q) \in \Omega$ be given. If the sequence $\left\{\frac{q_n}{q_n - p_n}\right\}_{n\in\mathbb{N}}$ is bounded, then every Δ^m-statistically convergent sequence is Δ^m-deferred statistically convergent.

In the following theorem, by changing the conditions on the sequences (p_n) and (q_n) we give the same relations as in Corollary 2.4 (ii).

Theorem 2.7 Let $m \in \mathbb{N}, (p,q) \in \Omega$ and α be a fixed real number such that $0 < \alpha \leq 1$, and $\liminf_n \dfrac{q_n}{p_n} > 1$. Then every Δ^m-statistically convergent sequence of order α is Δ^m-deferred statistically convergent of order α.

Proof. Let $\liminf_n \dfrac{q_n}{p_n} > 1$, then we can find a number $s > 0$ such that $\dfrac{q_n}{p_n} > 1 + s$ for sufficiently large n, which implies that

$$\frac{q_n - p_n}{q_n} \geq \frac{s}{1+s} \Longrightarrow \left(\frac{q_n - p_n}{q_n}\right)^\alpha \geq \left(\frac{s}{1+s}\right)^\alpha \Longrightarrow \frac{1}{{q_n}^\alpha} \geq \frac{s^\alpha}{(1+s)^\alpha}\frac{1}{(q_n-p_n)^\alpha}.$$

Since

$$\begin{aligned} \frac{|\{1 < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|}{{q_n}^\alpha} &\geq \frac{|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|}{q_n^\alpha} \\ &\geq \frac{s^\alpha}{(1+s)^\alpha}\frac{|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|}{(q_n-p_n)^\alpha} \end{aligned}$$

we have $x = (x_k)$ is deferred Δ^m-statistically convergent of order α

In the sequel results $S^\beta_{p,q}(\Delta^m_f)$ and $S^\alpha_{r,s}(\Delta^m_f)$ will be compared under the following conditions for $(p,q), (r,s) \in \Omega$ and

$$p_n < r_n < s_n < q_n \text{ for all } n \in \mathbb{N}. \tag{3}$$

Theorem 2.8 Let $m \in \mathbb{N}$, (p,q), $(r,s) \in \Omega$ and α, β be two fixed real numbers such that $0 < \alpha \leq \beta \leq 1$,

(i) If

$$\lim_{n\to\infty} \frac{[f(s_n - r_n)]^\alpha}{[f(q_n - p_n)]^\beta} = s > 0 \tag{4}$$

then $S^\beta_{p,q}(\Delta^m_f) \subset S^\alpha_{r,s}(\Delta^m_f)$,

(ii) If

$$\lim_{n\to\infty} \frac{q_n - p_n}{[f(s_n - r_n)]^\beta} = \frac{1}{f(1)} \tag{5}$$

then $S^\alpha_{r,s}(\Delta^m_f) \subseteq S^\beta_{p,q}(\Delta^m_f)$.

Proof. $i)$ Let $x = (x_k) \in S^\beta_{p,q}(\Delta^m_f)$. Since (3) is provided, for a given $\varepsilon > 0$ we have

$$\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\} \supseteq \{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}$$

and also we have the following inequality:

$$\begin{aligned} \frac{f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^\beta} &\geq \frac{f(|\{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^\beta} \\ &= \frac{[f(s_n - r_n)]^\alpha}{[f(q_n - p_n)]^\beta}\frac{f(|\{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(s_n - r_n)]^\alpha}. \end{aligned}$$

So we have $x \in S^\alpha_{r,s}(\Delta^m_f)$ provided (4) holds.

(ii) Let (5) be satisfied and $x \in S^{\alpha}_{r,s}(\Delta^m_f)$. Then for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^{\beta}} \\
&= \frac{f(|\{p_n < k \leq r_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^{\beta}} \\
&+ \frac{f(|\{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^{\beta}} + \frac{f(|\{s_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(q_n - p_n)]^{\beta}} \\
&\leq \frac{(q_n - p_n) - (s_n - r_n)}{[f(s_n - r_n)]^{\beta}} + \frac{f(|\{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(s_n - r_n)]^{\alpha}} \\
&\leq \frac{(q_n - p_n) - (s_n - r_n)^{\beta}}{[f(s_n - r_n)]^{\beta}} + \frac{f(|\{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(s_n - r_n)]^{\alpha}} \\
&\leq \frac{q_n - p_n - \frac{[f(s_n - r_n)]^{\beta}}{f(1)}}{[f(s_n - r_n)]^{\beta}} + \frac{f(|\{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(s_n - r_n)]^{\alpha}} \\
&= \left(\frac{q_n - p_n}{[f(s_n - r_n)]^{\beta}} - \frac{1}{f(1)} \right) + \frac{f(|\{r_n < k \leq s_n : |\Delta^m x_k - L| \geq \varepsilon\}|)}{[f(s_n - r_n)]^{\alpha}}
\end{aligned}$$

Therefore, $x \in S^{\beta}_{p,q}(\Delta^m_f)$.

From Theorem 2.8 we get the following results.

Corollary 2.5 $i)$ Let $m \in \mathbb{N}, (p,q), (r,s) \in \Omega$ and $0 < \alpha \leq 1$. If

$$\lim_{n\to\infty} \left[\frac{f(s_n - r_n)}{f(q_n - p_n)} \right]^{\alpha} = s > 0$$

then $S^{\alpha}_{p,q}(\Delta^m_f) \subset S^{\alpha}_{r,s}(\Delta^m_f)$.

$ii)$ Let $m \in \mathbb{N}, (p,q), (r,s) \in \Omega$ and $0 < \alpha \leq 1$. If

$$\lim_{n\to\infty} \frac{[f(s_n - r_n)]^{\alpha}}{f(q_n - p_n)} = s > 0$$

then $S_{p,q}(\Delta^m_f) \subset S^{\alpha}_{r,s}(\Delta^m_f)$.

$iii)$ Let $m \in \mathbb{N}, (p,q), (r,s) \in \Omega$. If

$$\lim_{i\to\infty} \frac{f(s_n - r_n)}{f(q_n - p_n)} = s > 0$$

then $S_{p,q}(\Delta^m_f) \subset S_{r,s}(\Delta^m_f)$.

Corollary 2.6 Let $m \in \mathbb{N}, (p,q), (r,s) \in \Omega$, If

$$\lim_{n\to\infty} \frac{f(q_n - p_n)}{f(s_n - r_n)} = 1$$

then $S_{r,s}(\Delta^m_f) \subseteq S_{p,q}(\Delta^m_f)$.

Proof. Omitted.

3. Strong Δ^m_f-Deferred Cesàro Summability of Order α

Now we introduce strong Δ^m_f-deferred Cesàro summability of order α and give some relations between strong Δ^m_f-deferred Cesàro summability of order α and strong Δ^m_f-deferred Cesàro summability of order β, where α and β are fixed real numbers such that $\beta \geq \alpha > 0$.

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Definition 3.1 Let f be a modulus and α be a positive real number. We define

$$\begin{aligned}
w_{p,q}^{\alpha,0}\left(\Delta_f^m\right) &= \left\{x \in w : \lim_{n\to\infty} \frac{1}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) = 0\right\},\\
w_{p,q}^{\alpha}\left(\Delta_f^m\right) &= \left\{x \in w : \lim_{n\to\infty} \frac{1}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) = 0 \text{ for some number } L\right\},\\
w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right) &= \left\{x \in w : \sup_n \frac{1}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) < \infty\right\}.
\end{aligned}$$

If $x \in w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ we shall say the sequence $x = (x_k)$ is strongly Δ_f^m-deferred Cesàro summable of order α to L (or strongly $w_{p,q}^{\alpha}\left(\Delta_f^m\right)-$Cesàro summable to L).

Some spaces are obtained by specializing f, α and pair of (p,q).

i) In the case $f(x) = x$, we write $w_{p,q}^{\alpha,0}(\Delta^m)$, $w_{p,q}^{\alpha}(\Delta^m)$ and $w_{p,q}^{\alpha,\infty}(\Delta^m)$ instead of $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, respectively,

ii) In the case $\alpha = 1$, we write $w_{p,q}^{0}\left(\Delta_f^m\right)$, $w_{p,q}\left(\Delta_f^m\right)$ and $w_{p,q}^{\infty}\left(\Delta_f^m\right)$ instead of $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, respectively,

iii) In the special cases $f(x) = x$ and $\alpha = 1$, we write $w_{p,q}^{0}(\Delta^m)$, $w_{p,q}(\Delta^m)$ and $w_{p,q}^{\infty}(\Delta^m)$ instead of $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, respectively,

iv) If $q_n = n$ and $p_n = 0$ (for all $n \in \mathbb{N}$) then we write we write $w^{\alpha,0}\left(\Delta_f^m\right)$, $w^{\alpha}\left(\Delta_f^m\right)$ and $w^{\alpha,\infty}\left(\Delta_f^m\right)$ instead of $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$, respectively.

Theorem 3.1 (i) For any modulus f and positive α, $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right) \subset w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$,

(ii) For any modulus f and $\alpha \geq 1$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right) \subset w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$.

Proof. (ii) Let $x \in w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $\alpha \geq 1$. Since f is subadditive and increasing we have

$$\begin{aligned}
\frac{1}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) &\leq \frac{1}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) + \frac{f(|L|)}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} 1\\
&= \frac{1}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) + \frac{f(|L|)(q_n-p_n)}{(q_n-p_n)^\alpha}\\
&= \frac{1}{(q_n-p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) + \frac{f(|L|)}{(q_n-p_n)^{\alpha-1}}
\end{aligned}$$

and since $\alpha \geq 1$ we have $x \in w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$.

Theorem 3.2 For any modulus f and $\alpha \geq 1$, we have

i) $w_{p,q}^{\alpha,0}(\Delta^m) \subset w_{p,q}^{\alpha,0}\left(\Delta_f^m\right)$,

ii) $w_{p,q}^{\alpha}(\Delta^m) \subset w_{p,q}^{\alpha}\left(\Delta_f^m\right)$,

iii) $w_{p,q}^{\alpha,\infty}(\Delta^m) \subset w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$.

Proof. We consider only the last inclusion, the others can be proved in the same way. Let $x \in w_{p,q}^{\alpha,\infty}(\Delta^m)$, then there exists a number $M > 0$ such that

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$$\frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} |\Delta^m x_k| < M, \text{ for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 < t \leq \delta$. We can write

$$\frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) = \frac{1}{(q_n - p_n)^\alpha} \sum_{\substack{p_n+1 \\ |\Delta^m x_k| \leq \delta}}^{q_n} f(|\Delta^m x_k|) + \frac{1}{(q_n - p_n)^\alpha} \sum_{\substack{p_n+1 \\ |\Delta^m x_k| > \delta}}^{q_n} f(|\Delta^m x_k|).$$

For $|\Delta^m x_k| \leq \delta$ we have

$$\frac{1}{(q_n - p_n)^\alpha} \sum_{\substack{p_n+1 \\ |\Delta^m x_k| \leq \delta}}^{q_n} f(|\Delta^m x_k|) \leq \frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} \varepsilon = \frac{\varepsilon}{(q_n - p_n)^{\alpha-1}}. \tag{6}$$

For $|\Delta^m x_k| > \delta$ we first use the inequality $|\Delta^m x_k| < \frac{|\Delta^m x_k|}{\delta} < 1 + \left[\frac{|\Delta^m x_k|}{\delta}\right]$ where $[.]$ denotes the integral part of the enclosed number and then by definition of modulus function, we can write

$$f(|\Delta^m x_k|) \leq \left(1 + \left[\frac{|\Delta^m x_k|}{\delta}\right]\right) f(1) \leq 2f(1) \frac{|\Delta^m x_k|}{\delta}$$

and so

$$\frac{1}{(q_n - p_n)^\alpha} \sum_{\substack{p_n+1 \\ |\Delta^m x_k| > \delta}}^{q_n} f(|\Delta^m x_k|) \leq 2f(1)\delta^{-1} \frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} |\Delta^m x_k| \tag{7}$$

From (6) and (7), we have

$$\frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|) \leq \frac{\varepsilon}{(q_n - p_n)^{\alpha-1}} + 2f(1)\delta^{-1} \frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} |\Delta^m x_k|.$$

Since $\alpha \geq 1$ and $x \in w_{p,q}^{\alpha,\infty}(\Delta^m)$, we have $x \in w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$ and the proof is complete.

We pause to recall that Maddox [30] proved for any modulus f, $\lim_{t\to\infty} \frac{f(t)}{t}$ exists and equals $\eta = \inf\{f(t)/t; t > 0\}$ such that $0 \leq \eta \leq f(1)$. In the next theorem, we show that the reciprocals of the inclusions in Theorem 3.2 also hold under a restriction on the modulus f.

Theorem 3.3 Let f be a modulus and α be a positive real number. If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right) \subset w_{p,q}^{\alpha,0}(\Delta^m)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right) \subset w_{p,q}^{\alpha}(\Delta^m)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right) \subset w_{p,q}^{\alpha,\infty}(\Delta^m)$.

Proof. Suppose $x \in w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right)$ and $\lim_{t\to\infty} \frac{f(t)}{t} = \eta = \inf\{f(t)/t; t > 0\} > 0$. Then we have $f(t) \geq \eta t$ which yields $t \leq \eta^{-1} f(t)$ for all $t \geq 0$. This gives rise to inequality

$$\frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} |\Delta^m x_k| \leq \eta^{-1} \frac{1}{(q_n - p_n)^\alpha} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k|).$$

Thus $x \in w_{p,q}^{\alpha,\infty}(\Delta^m)$. The proofs of the other inclusions are analogous, so we omit them.

Theorem 3.2 and Theorem 3.3 yield the next result.

Theorem 3.4 Let f be any modulus such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\alpha \geq 1$. Then we have $w_{p,q}^{\alpha,0}\left(\Delta_f^m\right) = w_{p,q}^{\alpha,0}(\Delta^m)$, $w_{p,q}^{\alpha}\left(\Delta_f^m\right) = w_{p,q}^{\alpha}(\Delta^m)$ and $w_{p,q}^{\alpha,\infty}\left(\Delta_f^m\right) = w_{p,q}^{\alpha,\infty}(\Delta^m)$.

In the next result, we compare sequence spaces $w_{p,q}^{\alpha}\left(\Delta_f^m\right)$ and $w_{p,q}^{\beta}\left(\Delta_f^m\right)$ without any restriction on modulus f and $(p,q) \in \Omega$.

Theorem 3.5 Let f be a modulus, $(p,q) \in \Omega$ and $\beta \geq \alpha > 0$. Then $w^{\alpha}_{p,q}\left(\Delta^m_f\right) \subset w^{\beta}_{p,q}\left(\Delta^m_f\right)$ and the inclusion may be particularly strict for certain specific choices of α and β.

Proof. The inclusion part of the proof is straightforward. To show that the inclusion may be strict, let f be a modulus, $q_n = 2n^2$ and $p_n = n^2$ (for all $n \in \mathbb{N}$) and consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1-n, & (n-1)^2+1 \leq k \leq n^2 \\ 0, & k = 1 \end{cases} \quad n = 2,3,\ldots$$

Observe that Δx_k equals 1 when k is a square and 0 when k is a non-square. Therefore, using the fact that $f(0) = 0$, for every $n \in \mathbb{N}$, we have

$$\frac{1}{(2n^2-n^2)^{\beta}} \sum_{p_n+1}^{q_n} f(|\Delta x_k - 0|) \leq \frac{n f(1)}{n^{2\beta}} \to 0, \text{ as } n \to \infty$$

so $x \in w^{\beta}_{p,q}\left(\Delta_f\right)$ for $\beta > \frac{1}{2}$. On the other hand,

$$\frac{1}{(2n^2-n^2)^{\alpha}} \sum_{p_n+1}^{q_n} f(|\Delta x_k - 0|) \geq \frac{\sqrt{n}-1}{n^{2\alpha}} f(1) \to \infty, \text{ as } n \to \infty$$

which implies that $x \notin w^{\alpha}_{p,q}\left(\Delta_f\right)$ for $0 < \alpha < \frac{1}{4}$.

Finally, we give a fairly general relation between strong Δ^m_f-deferred Cesàro summability of order α and Δ^m_f-deferred statistical convergence of order α.

Theorem 3.6 Let $0 < \alpha \leq \beta \leq 1$, f be an unbounded modulus such that there exists a positive constant c such that $f(xy) \geq c f(x) f(y)$ for all $x \geq 0, y \geq 0$ and $\lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Δ^m_f-deferred Cesàro summable of order α to L then it is Δ^m_f-deferred statistically convergent of order β to L.

Proof. Let $x = (x_k) \in w^{\alpha}_{p,q}\left(\Delta^m_f\right)$ and $\varepsilon > 0$, using the definition of modulus function, we have

$$\begin{aligned} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) &\geq f\left(\sum_{p_n+1}^{q_n} |\Delta^m x_k - L|\right) \\ &\geq f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|\varepsilon) \\ &\geq c f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|) f(\varepsilon) \end{aligned}$$

and since $\alpha \leq \beta$,

$$\begin{aligned} \frac{1}{(q_n-p_n)^{\alpha}} \sum_{p_n+1}^{q_n} f(|\Delta^m x_k - L|) &\geq \frac{c f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|) f(\varepsilon)}{(q_n-p_n)^{\alpha}} \\ &\geq \frac{c f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|) f(\varepsilon)}{(q_n-p_n)^{\beta}} \\ &= \frac{c f(|\{p_n < k \leq q_n : |\Delta^m x_k - L| \geq \varepsilon\}|) f(\varepsilon) [f(q_n-p_n)]^{\beta}}{[f(q_n-p_n)]^{\beta} (q_n-p_n)^{\beta}}. \end{aligned}$$

This completes the proof.

The following results are derivable from Theorem 3.6.

Corollary 3.1 Let $0 < \alpha \leq 1$, f be an unbounded modulus such that there exists a positive constant c such that $f(xy) \geq c f(x) f(y)$ for all $x \geq 0, y \geq 0$ and $\lim\limits_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly

Δ_f^m-deferred Cesàro summable of order α to L then it is Δ_f^m-deferred statistically convergent of order α to L.

Corollary 3.2 Let f be an unbounded modulus such that there exists a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0, y \geq 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Δ_f^m-deferred Cesàro summable to L then it is Δ_f^m-deferred statistically convergent to L.

By combining Theorem 3.3 of this article and Theorem 2.10 of Temizsu et al. [38] for the cases $\alpha = \beta$ and $r = 1$, we immediately obtain the next theorem.

Theorem 3.7 Let f be a modulus function such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\alpha \in (0,1]$. If a sequence is strongly Δ_f^m-deferred Cesàro summable of order α to L, then it is Δ^m-deferred statistically convergent of order α to L.

Specializing f and α in Theorem 3.7, we derive the following results.

Corollary 3.3 Let f be a modulus function such that $\lim_{t\to\infty} \frac{f(t)}{t} > 0$. If a sequence is strongly Δ_f^m-deferred Cesàro summable to L, then it is Δ^m-deferred statistically convergent to L.

Corollary 3.4 If a sequence is strongly Δ^m-deferred Cesàro summable to L, then it is Δ^m-deferred statistically convergent to L.

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