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GENERALIZATIONS OF THE BELL NUMBERS AND POLYNOMIALS AND THEIR PROPERTIES

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Abstract. In the paper, the authors present unified generalizations for the Bell numbers and polynomials, establish explicit formulas and inversion formulas for these generalizations in terms of the Stirling numbers of the first and second kinds with the help of the Faà di Bruno formula, properties of the Bell polynomials of the second kind, and the inversion theorem connected with the Stirling numbers of the first and second kinds, construct determinantal and product inequalities for these generalizations with aid of properties of the completely monotonic functions, and derive the logarithmic convexity for the sequence of these generalizations.

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1. Motivations

In combinatorial mathematics, the Bell numbers B_n for $n \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, count the number of ways a set with n elements can be partitioned into disjoint and nonempty subsets. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, a Scottish-born mathematician and science fiction writer who lived in the United States for most of his life and wrote about B_n in the 1930s. The Bell numbers B_n for $n \ge 0$ can be generated by

$$e^{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = 1 + t + t^2 + \frac{5}{6}t^3 + \frac{5}{8}t^4 + \frac{13}{30}t^5 + \frac{203}{720}t^6 + \frac{877}{5040}t^7 + \cdots \quad (1.1)$$

and the first eight Bell numbers B_n for $0 \le n \le 7$ are

$$\begin{aligned} &B_0 = 1, && B_1 = 1, && B_2 = 2, && B_3 = 5, \\ &B_4 = 15, && B_5 = 52, && B_6 = 203, && B_7 = 877. \end{aligned}$$

For detailed information on the Bell numbers B_n, please refer to [2, 4, 5, 6, 16, 17, 21] and plenty of references therein.

As well-known generalizations of the Bell numbers B_n for $n \ge 0$, the Bell polynomials $B_n(x)$ for $n \ge 0$ can be generated by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = 1 + xt + \frac{1}{2}x(x+1)t^2 + \frac{1}{6}x\bigl(x^2+3x+1\bigr)t^3 + \frac{1}{24}x\bigl(x^3+6x^2+7x+1\bigr)t^4 + \frac{1}{120}x\bigl(x^4+10x^3+25x^2+15x+1\bigr)t^5 + \cdots$$

and the first seven Bell polynomials $B_n(x)$ for $0 \le n \le 6$ are

$$1, \quad x, \quad x(x+1), \quad x\bigl(x^2+3x+1\bigr), \quad x\bigl(x^3+6x^2+7x+1\bigr),$$
$$x\bigl(x^4+10x^3+25x^2+15x+1\bigr), \quad x\bigl(x^5+15x^4+65x^3+90x^2+31x+1\bigr).$$

In the paper [19] it was pointed out that there have been studies in [9, 10, 11] on interesting applications of the Bell polynomials $B_n(x)$ in soliton theory, including links with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions. Therefore, applications of the Bell polynomials $B_n(x)$ to integrable nonlinear equations are greatly expected and any amendment on multilinear forms of soliton equations, even on exact solutions, would be beneficial to interested audiences in the research community. For more information about the Bell polynomials $B_n(x)$, please refer to [7, 19, 20] and closely related references therein.

For $n \ge 2$, let

$$\exp_n(t) = \overbrace{\exp(\exp(\exp\cdots\exp(\exp}^{n}(t)))) = \sum_{k=0}^{\infty} B_n(k)\frac{t^k}{k!}.$$

In [1], the quantities

$$b_n(k) = \frac{B_n(k)}{\exp_n(0)}, \quad k \ge 0$$

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were called the Bell numbers of order $n \ge 2$ and were deeply studied. In [1, Section 3] and [3, 8], the Bell numbers $b_n(k)$ of order n were applied to white noise distribution theory.

The Bell numbers $b_2(n)$ of order 2 for $n \ge 0$ are just the Bell numbers B_n generated in (1.1). By virtue of the software Mathematica, we can obtain

$$\exp_3(t) = e^e + e^{1+e}t + \frac{1}{2}e^{1+e}(2+e)t^2 + \frac{1}{6}e^{1+e}\bigl(5+6e+e^2\bigr)t^3 + \cdots$$

which implies that the first four Bell numbers $b_3(k)$ of order 3 for $0 \le k \le 3$ are

$$1, \quad e, \quad e(2+e), \quad e\bigl(5+6e+e^2\bigr).$$

Since not all these numbers are integers, we can say that the Bell numbers $b_n(k)$ of order n are not good generalizations of the Bell numbers $B_k = b_2(k)$ for $k \ge 0$.

In this paper, we will present better and unified generalizations $B_{m,n}(\boldsymbol{x}_m)$ for the Bell numbers B_n and the Bell polynomials $B_n(x)$, establish explicit formulas and inversion formulas for these generalizations $B_{m,n}(\boldsymbol{x}_m)$ in terms of the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$ with the help of the Faà di Bruno formula, properties of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, and the inversion theorem connected with the Stirling numbers $s(n,k)$ and $S(n,k)$, construct determinantal and product inequalities for these generalizations $B_{m,n}(\boldsymbol{x}_m)$ with aid of properties of the completely monotonic functions, and derive the logarithmic convexity for the sequence of these generalizations $B_{m,n}(\boldsymbol{x}_m)$.

2. Generalizations of the Bell numbers and polynomials

For $m \in \mathbb{N}$, let

$$\begin{aligned} f(t;x_1,x_2,\dots,x_{m-1},x_m) &= e^{x_1\left(e^{x_2\left(e^{\cdot^{\cdot^{\cdot^{x_{m-1}\left(e^{x_m(e^t-1)}-1\right)}}}}-1\right)}-1\right)} \\ &= \exp(x_1[\exp(x_2[\exp(\cdots x_{m-1}[\exp(x_m[\exp(t)-1])-1])-1])-1]) \\ &= \sum_{n=0}^{\infty} B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)\frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

In other words, the generating function $f(t;x_1,x_2,\dots,x_{m-1},x_m)$ of the quantities $B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)$ for $n \ge 0$ is a self-composite function of the function $g(t) = e^t - 1$ in the manner

$$\begin{aligned} f(t;x_1,x_2,\dots,x_{m-1},x_m) &= \exp(x_1 g(x_2(\cdots x_{m-1}g(x_m g(t))))) \\ &= g(x_1 g(x_2(\cdots x_{m-1}g(x_m g(t))))) + 1. \end{aligned} \tag{2.2}$$

It is easy to see that $B_{1,n}(x) = B_n(x)$ for $n \ge 0$,

$$f(0;x_1,x_2,\dots,x_{m-1},x_m) = 1, \quad m \in \mathbb{N}, \tag{2.3}$$

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and, by virtue of the software MATHEMATICA,

$$\begin{gathered}
B_{2,0}(x,y)=1, \quad B_{2,1}(x,y)=xy, \quad B_{2,2}(x,y)=xy(1+y+xy), \\
B_{2,3}(x,y)=xy\bigl(1+3y+y^2+3xy+3xy^2+x^2y^2\bigr), \quad B_{3,0}(x,y,z)=1, \\
B_{3,1}(x,y,z)=xyz, \quad B_{3,2}(x,y,z)=xyz(1+z+yz+xyz).
\end{gathered} \tag{2.4}$$

The equality (2.3) means that $B_{m,0}(x_1,x_2,\dots,x_{m-1},x_m)=1$ for all $m\in\mathbb{N}$.

For conveniently referring to the quantities $B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)$ for $n\ge0$, we need a name for them. We would like to recommend a name for them: the Bell polynomials of m variables $x_1,x_2,\dots,x_{m-1},x_m$. For the sake of simplicity and uniqueness, in what follows, we call them the Bell–Qi polynomials.

Let $\boldsymbol{x}_m=(x_1,x_2,\dots,x_m)$. Occasionally for shortness in symbols, we use the notation $B_{m,n}(\boldsymbol{x}_m)$ for $B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)$.

When $x_1=x_2=\dots=x_m=1$, we call $B_{m,n}(\overbrace{1,1,\dots,1}^{m})$ for $n\ge0$ the Bell–Qi numbers or the Bell numbers of m multiples and denote them by $B_{m,n}$.

It is not difficult to see that we can take t and x_k for $1\le k\le m$ in $B_{m,n}(\boldsymbol{x}_m)$ on the complex plane $\mathbb{C}$.

3. Explicit formulas for the Bell–Qi numbers and polynomials

Roughly judging from concrete expressions in (2.4), the Bell–Qi polynomial $B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)$ for $m\in\mathbb{N}$ and $n\ge0$ should be a polynomial of m variables $x_1,x_2,\dots,x_{m-1},x_m$ with degree $m\times n$ and positive integer coefficients. This guess will be verified by the following theorem.

Theorem 3.1. *For $m\in\mathbb{N}$ and $n\ge0$, the Bell–Qi polynomials $B_{m,n}(\boldsymbol{x}_m)$ can be computed explicitly by*

$$B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)=\prod_{q=1}^{m}\sum_{\ell_q=0}^{\ell_{q-1}}\prod_{q=1}^{m}S(\ell_{q-1},\ell_q)\prod_{q=1}^{m}x_q^{\ell_{m-q+1}}, \tag{3.1}$$

where $\ell_0=n$ and $S(n,k)$ for $n\ge k\ge0$, which can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},$$

represent the Stirling numbers of the second kind. Consequently, the Bell–Qi numbers $B_{m,n}$ can be computed explicitly by

$$B_{m,n}=\prod_{q=1}^{m}\sum_{\ell_q=0}^{\ell_{q-1}}\prod_{q=1}^{m}S(\ell_{q-1},\ell_q). \tag{3.2}$$

Proof. In combinatorial analysis, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{\ell_1,\dots,\ell_n\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n\ge k\ge0$, see [4, p. 134, Theorem A], and satisfy identities

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{3.3}$$

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and

$$\mathrm{B}_{n,k}(1,1,\dots,1)=S(n,k), \tag{3.4}$$

see [4, p. 135], where a and b are any complex numbers. The Faà di Bruno formula for computing higher order derivatives of composite functions can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(x)=\sum_{k=0}^{n}f^{(k)}(h(x))\mathrm{B}_{n,k}\bigl(h'(x),h''(x),\dots,h^{(n-k+1)}(x)\bigr), \tag{3.5}$$

see [4, p. 139, Theorem C]. Therefore, making use of (3.5), (3.3), and (3.4) in sequence and considering the composite relation (2.2) inductively yield

$$\frac{\partial^n f(t;x_1,\dots,x_m)}{\partial^n t}=\sum_{\ell_1=0}^{n}\frac{\partial^{\ell_1}f(u_1;x_1,\dots,x_{m-1})}{\partial^{\ell_1}u_1}\mathrm{B}_{n,\ell_1}\bigl(x_me^t,\dots,x_me^t\bigr)$$

$$=\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\frac{\partial^{\ell_2}f(u_2;x_1,\dots,x_{m-2})}{\partial^{\ell_2}u_2}\mathrm{B}_{\ell_1,\ell_2}\bigl(x_{m-1}e^{u_1},\dots,x_{m-1}e^{u_1}\bigr)x_m^{\ell_1}e^{\ell_1t}S(n,\ell_1)$$

$$=\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\frac{\partial^{\ell_2}f(u_2;x_1,\dots,x_{m-2})}{\partial^{\ell_2}u_2}x_{m-1}^{\ell_2}e^{\ell_2u_1}S(\ell_1,\ell_2)x_m^{\ell_1}e^{\ell_1t}S(n,\ell_1)$$

$$\cdots\cdots\cdots\cdots$$

$$=\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_{m-2}=0}^{\ell_{m-3}}\sum_{\ell_{m-1}=0}^{\ell_{m-2}}\frac{\partial^{\ell_{m-1}}f(u_{m-1};x_1)}{\partial^{\ell_{m-1}}u_{m-1}}\mathrm{B}_{\ell_{m-2},\ell_{m-1}}\bigl(x_2e^{u_{m-2}},\dots,x_2e^{u_{m-2}}\bigr)$$
$$\times x_3^{\ell_{m-2}}e^{\ell_{m-2}u_{m-3}}S(\ell_{m-3},\ell_{m-2})\cdots x_{m-1}^{\ell_2}e^{\ell_2u_1}S(\ell_1,\ell_2)x_m^{\ell_1}e^{\ell_1t}S(n,\ell_1)$$

$$=\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_{m-2}=0}^{\ell_{m-3}}\sum_{\ell_{m-1}=0}^{\ell_{m-2}}\sum_{\ell_m=0}^{\ell_{m-1}}\frac{\mathrm{d}^{\ell_m}e^{u_m}}{\mathrm{d}^{\ell_m}u_m}\mathrm{B}_{\ell_{m-1},\ell_m}\bigl(x_1e^{u_{m-1}},\dots,x_1e^{u_{m-1}}\bigr)$$
$$\times x_2^{\ell_{m-1}}e^{\ell_{m-1}u_{m-2}}S(\ell_{m-2},\ell_{m-1})x_3^{\ell_{m-2}}e^{\ell_{m-2}u_{m-3}}S(\ell_{m-3},\ell_{m-2})$$
$$\cdots x_{m-1}^{\ell_2}e^{\ell_2u_1}S(\ell_1,\ell_2)x_m^{\ell_1}e^{\ell_1t}S(n,\ell_1)$$

$$=\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_{m-2}=0}^{\ell_{m-3}}\sum_{\ell_{m-1}=0}^{\ell_{m-2}}\sum_{\ell_m=0}^{\ell_{m-1}}e^{u_m}x_1^{\ell_m}e^{\ell_mu_{m-1}}S(\ell_{m-1},\ell_m)$$
$$\times x_2^{\ell_{m-1}}e^{\ell_{m-1}u_{m-2}}S(\ell_{m-2},\ell_{m-1})x_3^{\ell_{m-2}}e^{\ell_{m-2}u_{m-3}}S(\ell_{m-3},\ell_{m-2})$$
$$\cdots x_{m-1}^{\ell_2}e^{\ell_2u_1}S(\ell_1,\ell_2)x_m^{\ell_1}e^{\ell_1t}S(n,\ell_1)$$

$$=\prod_{q=1}^{m}\sum_{\ell_q=0}^{\ell_{q-1}}e^{u_m}\exp\left(\sum_{q=1}^{m}\ell_qu_{q-1}\right)\prod_{q=1}^{m}S(\ell_{q-1},\ell_q)\prod_{q=1}^{m}x_q^{\ell_{m-q+1}},$$

where $u_0=u_0(t)=t$ and

$$u_q=u_q(u_{q-1})=x_{m-q+1}(e^{u_{q-1}}-1),\quad 1\le q\le m.$$

When $t\to0$, it follows that $u_q\to0$ for all $0\le q\le m$. As a result, by the definition in (2.1), we have

$$B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)=\lim_{t\to0}\frac{\partial^n f(t;x_1,\dots,x_m)}{\partial^n t}$$

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$$= \prod_{q=1}^{m}\sum_{\ell_q=0}^{\ell_{q-1}}\prod_{q=1}^{m}S(\ell_{q-1},\ell_q)\prod_{q=1}^{m}x_q^{\ell_{m-q+1}}.$$

The formula (3.1) is thus proved.

The formula (3.2) follows from taking $x_1 = x_2 = \cdots = x_m = 1$ in (3.1). The proof of Theorem 3.1 is complete. □

Remark 3.1. When letting $m = 1, 2, 3$ in (3.1), we can recover and find explicit formulas

$$B_n(x) = B_{1,n}(x) = \sum_{k=0}^{n} S(n,k)x^k, \tag{3.6}$$

$$B_{2,n}(x,y) = \sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1} S(n,\ell_1)S(\ell_1,\ell_2)x^{\ell_2}y^{\ell_1}, \tag{3.7}$$

and

$$B_{3,n}(x,y,z) = \sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{\ell_1}\sum_{\ell_3=0}^{\ell_2} S(n,\ell_1)S(\ell_1,\ell_2)S(\ell_2,\ell_3)x^{\ell_3}y^{\ell_2}z^{\ell_1} \tag{3.8}$$

for $n \ge 0$. The formula (3.6) was also recovered in [18, Theorem 3.1]. The formulas (3.7) and (3.8) coincide with those special values in (2.4). This convinces us that Theorem 3.1 and its proof in this paper are correct.

Theorem 3.2. *For $m \in \mathbb{N}$ and $n \ge 0$, the Bell–Qi polynomials $B_{m,n}(\boldsymbol{x}_m)$ satisfy*

$$\prod_{q=1}^{m}\left[\frac{1}{x_q^{\ell_{q-1}}}\sum_{\ell_q=0}^{\ell_{q-1}} s(\ell_{q-1},\ell_q)\right]B_{m,\ell_m}(\boldsymbol{x}_m) = 1, \tag{3.9}$$

where $\ell_0 = n$ and $s(n,k)$ for $n \ge k \ge 0$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x| < 1,$$

stand for the Stirling numbers of the first kind. Consequently, the Bell–Qi numbers $B_{m,n}$ satisfy the identity

$$\prod_{q=1}^{m}\left[\sum_{\ell_q=0}^{\ell_{q-1}} s(\ell_{q-1},\ell_q)\right]B_{m,\ell_m} = 1. \tag{3.10}$$

Proof. The formula (3.1) can be rearranged as

$$B_{m,n}(\boldsymbol{x}_m) = \sum_{\ell_1=0}^{n} S(n,\ell_1)x_m^{\ell_1}\sum_{\ell_2=0}^{\ell_1} S(\ell_1,\ell_2)x_{m-1}^{\ell_2} \cdots \sum_{\ell_{m-1}=0}^{\ell_{m-2}} S(\ell_{m-2},\ell_{m-1})x_2^{\ell_{m-1}}\sum_{\ell_m=0}^{\ell_{m-1}} S(\ell_{m-1},\ell_m)x_1^{\ell_m}. \tag{3.11}$$

In [22, p. 171, Theorem 12.1], it is stated that, if b_α and a_k are a collection of constants independent of n, then

$$a_n = \sum_{\alpha=0}^{n} S(n,\alpha)b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^{n} s(n,k)a_k.$$

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Applying this inversion theorem to (3.11) consecutively leads to

$$\sum_{\ell_2=0}^{n} S(n,\ell_2)x_{m-1}^{\ell_2}\cdots\sum_{\ell_{m-1}=0}^{\ell_{m-2}} S(\ell_{m-2},\ell_{m-1})x_2^{\ell_{m-1}}\sum_{\ell_m=0}^{\ell_{m-1}} S(\ell_{m-1},\ell_m)x_1^{\ell_m} = \frac{1}{x_m^n}\sum_{\ell_1=0}^{n} s(n,\ell_1)B_{m,\ell_1}(\boldsymbol{x}_m),$$

$$\frac{1}{x_{m-1}^n}\sum_{\ell_2=0}^{n} s(n,\ell_2)\frac{1}{x_m^{\ell_2}}\sum_{\ell_1=0}^{\ell_2} s(\ell_2,\ell_1)B_{m,\ell_1}(\boldsymbol{x}_m) = \sum_{\ell_3=0}^{n} S(n,\ell_3)\cdots\sum_{\ell_{m-1}=0}^{\ell_{m-2}} S(\ell_{m-2},\ell_{m-1})x_2^{\ell_{m-1}}\sum_{\ell_m=0}^{\ell_{m-1}} S(\ell_{m-1},\ell_m)x_1^{\ell_m},$$

and, inductively,

$$\sum_{\ell_{m-1}}^{n} s(n,\ell_{m-1})\frac{1}{x_3^{\ell_{m-1}}}\sum_{\ell_{m-2}}^{\ell_{m-1}}\cdots\frac{1}{x_{m-1}^{\ell_3}}\sum_{\ell_2=0}^{\ell_3} s(\ell_3,\ell_2)\frac{1}{x_m^{\ell_2}}\sum_{\ell_1=0}^{\ell_2} s(\ell_2,\ell_1)B_{m,\ell_1}(\boldsymbol{x}_m) = x_2^n\sum_{\ell_m=0}^{n} S(n,\ell_m)x_1^{\ell_m}$$

which can be further rewritten as

$$x_1^n = \sum_{\ell_1=0}^{n}\frac{s(n,\ell_1)}{x_2^{\ell_1}}\sum_{\ell_2=0}^{\ell_1}\frac{s(\ell_1,\ell_2)}{x_3^{\ell_2}}\sum_{\ell_3=0}^{\ell_2}\frac{s(\ell_2,\ell_3)}{x_4^{\ell_3}} \cdots\frac{s(\ell_{m-3},\ell_{m-2})}{x_{m-1}^{\ell_{m-2}}}\sum_{\ell_{m-1}=0}^{\ell_{m-2}}\frac{s(\ell_{m-2},\ell_{m-1})}{x_m^{\ell_{m-1}}}\sum_{\ell_m=0}^{\ell_{m-1}} s(\ell_{m-1},\ell_m)B_{m,\ell_m}(\boldsymbol{x}_m)$$

and the identity (3.9).

The identity (3.10) follows from taking $x_1 = x_2 = \cdots = x_m = 1$ in (3.9). The proof of Theorem 3.2 is complete. □

Remark 3.2. When letting $m = 1, 2, 3$ in (3.9), we can recover and derive

$$\sum_{k=0}^{n} s(n,k)B_k(x) = x^n, \tag{3.12}$$

$$\sum_{\ell_1=0}^{n}\frac{s(n,\ell_1)}{y^{\ell_1}}\sum_{\ell_2=0}^{\ell_1} s(\ell_1,\ell_2)B_{2,\ell_2}(x,y) = x^n, \tag{3.13}$$

and

$$\sum_{\ell_1=0}^{n}\frac{s(n,\ell_1)}{y^{\ell_1}}\sum_{\ell_2=0}^{\ell_1}\frac{s(\ell_1,\ell_2)}{z^{\ell_2}}\sum_{\ell_3=0}^{\ell_2} s(\ell_2,\ell_3)B_{3,\ell_3}(x,y,z) = x^n \tag{3.14}$$

for $n \ge 0$. The identity (3.12) was also obtained in [18, Theorem 3.1]. The identites (3.13) for $n = 0, 1, 2, 3$ and (3.14) for $n = 0, 1, 2$ can be easily verified by special values in (2.4).

4. Inequalities for the Bell–Qi polynomials

It is seemingly true that there have been more identities than inequalities in combinatorial mathematics. Now we start out to construct some determinantal and product inequalities for the Bell–Qi numbers $B_{m,n}$ and the Bell–Qi polynomials $B_{m,n}(\boldsymbol{x}_m)$. Consequently, we can derive that the sequences of the Bell–Qi numbers $B_{m,n}$ and the Bell–Qi polynomials $B_{m,n}(\boldsymbol{x}_m)$ are logarithmically convex.

Theorem 4.1. *Let $q\ge1$ be a positive integer, let $|e_{ij}|_q$ denote a determinant of order q with elements e_{ij}, and let $x_k>0$ for $1\le k\le m$.*

(1) *If a_i for $1\le i\le q$ are non-negative integers, then*
$$\left|(-1)^{a_i+a_j}B_{m,a_i+a_j}(\boldsymbol{x}_m)\right|_q\ge0 \tag{4.1}$$
and
$$\left|B_{m,a_i+a_j}(\boldsymbol{x}_m)\right|_q\ge0. \tag{4.2}$$

(2) *If $a=(a_1,a_2,\dots,a_q)$ and $b=(b_1,b_2,\dots,b_q)$ are non-increasing q-tuples of non-negative integers such that $\sum_{i=1}^k a_i\ge\sum_{i=1}^k b_i$ for $1\le k\le q-1$ and $\sum_{i=1}^q a_i=\sum_{i=1}^q b_i$, then*
$$\prod_{i=1}^q B_{m,a_i}(\boldsymbol{x}_m)\ge\prod_{i=1}^q B_{m,b_i}(\boldsymbol{x}_m). \tag{4.3}$$

Proof. Recall from [14, Chapter XIII] and [26, Chapter IV] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t)\ge0$ for $t\in I$ and $k\in\mathbb{N}$. Recall from [14, Chapter XIII], [23, Chapter 1], and [26, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $(-1)^kf^{(k)}(x)\ge0$ on I for all $k\ge0$. Theorem 2b in [26, p. 145] reads that, if $f_1(x)$ is absolutely monotonic and $f_2(x)$ is completely monotonic on their defined intervals, then their composite function $f_1(f_2(x))$ is completely monotonic on its defined interval. Therefore, since e^t and e^{-t} are respectively absolutely and completely monotonic on $[0,\infty)$, by induction, it is not difficult to reveal that, when $x_1,x_2,\dots,x_m>0$, the generating function $f(-t;x_1,\dots,x_m)$ is completely monotonic with respect to $t\in[0,\infty)$. Moreover, by (2.1), it is obvious that
$$B_{m,n}(x_1,x_2,\dots,x_{m-1},x_m)=(-1)^n\lim_{t\to0}\frac{\partial^nf(-t;x_1,\dots,x_m)}{\partial^nt}.$$
For simplicity, in what follows, we use $f(\pm t,\boldsymbol{x}_m)$ for $f(\pm t;x_1,\dots,x_m)$.

In [13] and [14, p. 367], it was proved that if $f(t)$ is completely monotonic on $[0,\infty)$, then
$$\left|f^{(a_i+a_j)}(t)\right|_q\ge0 \tag{4.4}$$
and
$$\left|(-1)^{a_i+a_j}f^{(a_i+a_j)}(t)\right|_q\ge0. \tag{4.5}$$
Applying $f(t)$ to the generating function $f(-t,\boldsymbol{x}_m)$ in (4.4) and (4.5) and taking the limit $t\to0^+$ give
$$\lim_{t\to0^+}\left|\left[f(-t,\boldsymbol{x}_m)\right]_t^{(a_i+a_j)}\right|_q=\left|(-1)^{a_i+a_j}B_{m,a_i+a_j}(\boldsymbol{x}_m)\right|_q\ge0$$
and
$$\lim_{t\to0^+}\left|(-1)^{a_i+a_j}\left[f(-t,\boldsymbol{x}_m)\right]_t^{(a_i+a_j)}\right|_q=\left|B_{m,a_i+a_j}(\boldsymbol{x}_m)\right|_q\ge0$$

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The determinantal inequalities (4.1) and (4.2) follow.

In [14, p. 367, Theorem 2], it was stated that if $f(t)$ is a completely monotonic function on $[0,\infty)$, then

$$\prod_{i=1}^{q}\bigl[(-1)^{a_i}f^{(a_i)}(t)\bigr]\ge\prod_{i=1}^{q}\bigl[(-1)^{b_i}f^{(b_i)}(t)\bigr]. \tag{4.6}$$

Applying $f(t)$ to the generating function $f(-t,\boldsymbol{x}_m)$ in (4.6) and taking the limit $t\to0^+$ result in

$$\lim_{t\to0^+}\prod_{i=1}^{q}\Bigl[(-1)^{a_i}(f(-t,\boldsymbol{x}_m))_t^{(a_i)}\Bigr]=\prod_{i=1}^{q}B_{m,a_i}(\boldsymbol{x}_m)$$

$$\ge\lim_{t\to0^+}\prod_{i=1}^{q}\Bigl[(-1)^{b_i}(f(-t,\boldsymbol{x}_m))_t^{(b_i)}\Bigr]=\prod_{i=1}^{q}B_{m,b_i}(\boldsymbol{x}_m).$$

The product inequality (4.3) follows. The proof of Theorem 4.1 is complete. □

Corollary 4.1. *Let $x_k>0$ for $1\le k\le m$. If $\ell\ge0$ and $q\ge k\ge0$, then*

$$[B_{m,q+\ell}(\boldsymbol{x}_m)]^k[B_{m,\ell}(\boldsymbol{x}_m)]^{q-k}\ge[B_{m,k+\ell}(\boldsymbol{x}_m)]^q.$$

Proof. This follows from taking

$$a=(\overbrace{q+\ell,\dots,q+\ell}^{k},\overbrace{\ell,\dots,\ell}^{q-k})\quad\text{and}\quad b=(k+\ell,k+\ell,\dots,k+\ell)$$

in the inequality (4.3). The proof of Corollary 4.1 is complete. □

Corollary 4.2. *Let $x_k>0$ for $1\le k\le m$. Then the sequence $\{B_{m,n}(\boldsymbol{x}_m)\}_{n\ge0}$ is logarithmically convex.*

Proof. In [14, p. 369] and [15, p. 429, Remark], it was obtained that if $f(t)$ is a completely monotonic function such that $f^{(k)}(t)\ne0$ for $k\ge0$, then the sequence

$$\ln\bigl[(-1)^{k-1}f^{(k-1)}(t)\bigr],\quad k\ge1 \tag{4.7}$$

is convex. Applying this conclusion to the generating function $f(-t,\boldsymbol{x}_m)$ figures out that the sequence

$$\ln\bigl[(-1)^{k-1}(f(-t,\boldsymbol{x}_m))_t^{(k-1)}\bigr]\to\ln B_{m,k-1}(\boldsymbol{x}_m),\quad t\to0^+$$

for $k\ge1$ is convex. Equivalently, the sequence $\{B_{m,n}(\boldsymbol{x}_m)\}_{n\ge0}$ is logarithmically convex.

Alternatively, letting

$$\ell\ge1,\quad n=2,\quad a_1=\ell+2,\quad a_2=\ell,\quad\text{and}\quad b_1=b_2=\ell+1$$

in the inequality (4.3) leads to

$$B_{m,\ell}(\boldsymbol{x}_m)B_{m,\ell+2}(\boldsymbol{x}_m)\ge B_{m,\ell+1}^2(\boldsymbol{x}_m)$$

which means that the sequence $\{B_{m,n}(\boldsymbol{x}_m)\}_{n\ge1}$ is logarithmically convex. The proof of Corollary 4.2 is complete. □

Theorem 4.2. *Let $x_k>0$ for $1\le k\le m$. For $q\ge0$ and $n\in\mathbb{N}$, we have*

$$\left[\prod_{\ell=0}^{n}B_{m,q+2\ell}(\boldsymbol{x}_m)\right]^{1/(n+1)}\ge\left[\prod_{\ell=0}^{n-1}B_{m,q+2\ell+1}(\boldsymbol{x}_m)\right]^{1/n}. \tag{4.8}$$

Proof. If $f(t)$ is a completely monotonic function on $(0,\infty)$, then, by the convexity of the sequence (4.7) and Nanson's inequality listed in [12, p. 205, 3.2.27],

$$\left[\prod_{\ell=0}^{n}(-1)^{q+2\ell+1}f^{(q+2\ell+1)}(t)\right]^{1/(n+1)} \ge \left[\prod_{\ell=1}^{n}(-1)^{q+2\ell}f^{(q+2\ell)}(t)\right]^{1/n}$$

for $q\ge0$. Replacing $f(t)$ by $f(-t,\boldsymbol{x}_m)$ in the above inequality results in

$$\left[\prod_{\ell=0}^{n}(-1)^{q+2\ell+1}(f(-t,\boldsymbol{x}_m))_t^{(q+2\ell+1)}\right]^{1/(n+1)} \ge \left[\prod_{\ell=1}^{n}(-1)^{q+2\ell}(f(-t,\boldsymbol{x}_m))_t^{(q+2\ell)}\right]^{1/n}$$

for $q\ge0$. Letting $t\to0^+$ in the above inequality leads to (4.8). The proof of Theorem 4.2 is complete. □

Theorem 4.3. *Let $x_k>0$ for $1\le k\le m$. If $\ell\ge0$, $n\ge k\ge q$, $2k\ge n$, and $2q\ge n$, then*

$$B_{m,k+\ell}(\boldsymbol{x}_m)B_{m,n-k+\ell}(\boldsymbol{x}_m)\ge B_{m,q+\ell}(\boldsymbol{x}_m)B_{m,n-q+\ell}(\boldsymbol{x}_m). \tag{4.9}$$

Proof. In [24, p. 397, Theorem D], it was recovered that if $f(t)$ is a completely monotonic function on $(0,\infty)$ and if $n\ge k\ge q$, $k\ge n-k$, and $q\ge n-q$, then

$$(-1)^nf^{(k)}(t)f^{(n-k)}(t)\ge(-1)^nf^{(q)}(t)f^{(n-q)}(t).$$

Replacing $f(t)$ by the function $(-1)^\ell[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}$ in the above inequality leads to

$$\begin{aligned}(-1)^n[f(-t,\boldsymbol{x}_m)]_t^{(k+\ell)}&[f(-t,\boldsymbol{x}_m)]_t^{(n-k+\ell)}\\ &\ge(-1)^n[f(-t,\boldsymbol{x}_m)]_t^{(q+\ell)}[f(-t,\boldsymbol{x}_m)]_t^{(n-q+\ell)}.\end{aligned}$$

Further taking $t\to0^+$ finds the inequality (4.9). The proof of Theorem 4.3 is complete. □

Theorem 4.4. *Let $x_k>0$ for $1\le k\le m$. For $\ell\ge0$ and $q,n\in\mathbb{N}$, let*

$$\begin{aligned}\mathcal{G}_{m,\ell,q,n}&=B_{m,\ell+2q+n}(\boldsymbol{x}_m)[B_{m,\ell}(\boldsymbol{x}_m)]^2-B_{m,\ell+q+n}(\boldsymbol{x}_m)B_{m,\ell+q}(\boldsymbol{x}_m)B_{m,\ell}(\boldsymbol{x}_m)\\&\quad-B_{m,\ell+n}(\boldsymbol{x}_m)B_{m,\ell+2q}(\boldsymbol{x}_m)B_{m,\ell}(\boldsymbol{x}_m)+B_{m,\ell+n}(\boldsymbol{x}_m)[B_{m,\ell+q}(\boldsymbol{x}_m)]^2,\\ \mathcal{H}_{m,\ell,q,n}&=B_{m,\ell+2q+n}(\boldsymbol{x}_m)[B_{m,\ell}(\boldsymbol{x}_m)]^2-2B_{m,\ell+q+n}(\boldsymbol{x}_m)B_{m,\ell+q}(\boldsymbol{x}_m)B_{m,\ell}(\boldsymbol{x}_m)\\&\quad+B_{m,\ell+n}(\boldsymbol{x}_m)[B_{m,\ell+q}(\boldsymbol{x}_m)]^2,\\ \mathcal{I}_{m,\ell,q,n}&=B_{m,\ell+2q+n}(\boldsymbol{x}_m)[B_{m,\ell}(\boldsymbol{x}_m)]^2-2B_{m,\ell+n}(\boldsymbol{x}_m)B_{m,\ell+2q}(\boldsymbol{x}_m)B_{m,\ell}(\boldsymbol{x}_m)\\&\quad+B_{m,\ell+n}(\boldsymbol{x}_m)[B_{m,\ell+q}(\boldsymbol{x}_m)]^2.\end{aligned}$$

Then

$$\begin{gathered}\mathcal{G}_{m,\ell,q,n}\ge0,\quad \mathcal{H}_{m,\ell,q,n}\ge0,\\ \mathcal{H}_{m,\ell,q,n}\lesseqgtr\mathcal{G}_{m,\ell,q,n}\quad\text{when } q\lesseqgtr n,\\ \mathcal{I}_{m,\ell,q,n}\ge\mathcal{G}_{m,\ell,q,n}\ge0\quad\text{when } n\ge q.\end{gathered} \tag{4.10}$$

Proof. In [25, Theorem 1 and Remark 2], it was obtained that if $f(t)$ is completely monotonic on $(0,\infty)$ and

$$\begin{aligned}G_{q,n}(t)&=(-1)^n\bigl\{f^{(n+2q)}(t)f^2(t)-f^{(n+q)}(t)f^{(q)}(t)f(t)\\&\quad-f^{(n)}(t)f^{(2q)}(t)f(t)+f^{(n)}(t)\bigl[f^{(q)}(t)\bigr]^2\bigr\},\\ H_{q,n}(t)&=(-1)^n\bigl\{f^{(n+2q)}(t)f^2(t)-2f^{(n+q)}(t)f^{(q)}(t)f(t)+f^{(n)}(t)\bigl[f^{(q)}(t)\bigr]^2\bigr\},\end{aligned}$$

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$$I_{q,n}(t)=(-1)^n\bigl\{f^{(n+2q)}(t)f^2(t)-2f^{(n)}(t)f^{(2q)}(t)f(t)+f^{(n)}(t)\bigl[f^{(q)}(t)\bigr]^2\bigr\}$$

for $n,q\in\mathbb{N}$, then

$$\begin{gathered}G_{q,n}(t)\ge0,\quad H_{q,n}(t)\ge0,\\ H_{q,n}(t)\lesseqqgtr G_{q,n}(t)\quad\text{when } q\lesseqgtr n,\\ I_{q,n}(t)\ge G_{q,n}(t)\ge0\quad\text{when } n\ge q.\end{gathered}\tag{4.11}$$

Replacing $f(t)$ by $(-1)^\ell[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}$ in $G_{q,n}(t)$, $H_{q,n}(t)$, and $I_{q,n}(t)$ and simplifying produce

$$\begin{aligned}G_{q,n}(t)&=(-1)^{\ell+n}\bigl\{[f(-t,\boldsymbol{x}_m)]_t^{(\ell+2q+n)}\bigl[[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}\bigr]^2\\&\quad-[f(-t,\boldsymbol{x}_m)]_t^{(\ell+q+n)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell+q)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}\\&\quad-[f(-t,\boldsymbol{x}_m)]_t^{(\ell+n)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell+2q)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}\\&\quad+[f(-t,\boldsymbol{x}_m)]_t^{(\ell+n)}\bigl[[f(-t,\boldsymbol{x}_m)]_t^{(\ell+q)}\bigr]^2\bigr\},\\H_{q,n}(t)&=(-1)^{\ell+n}\bigl\{[f(-t,\boldsymbol{x}_m)]_t^{(\ell+2q+n)}\bigl[[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}\bigr]^2\\&\quad-2[f(-t,\boldsymbol{x}_m)]_t^{(\ell+q+n)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell+q)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}\\&\quad+[f(-t,\boldsymbol{x}_m)]_t^{(\ell+n)}\bigl[[f(-t,\boldsymbol{x}_m)]_t^{(\ell+q)}\bigr]^2\bigr\},\\I_{q,n}(t)&=(-1)^{\ell+n}\bigl\{[f(-t,\boldsymbol{x}_m)]_t^{(\ell+2q+n)}\bigl[[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}\bigr]^2\\&\quad-2[f(-t,\boldsymbol{x}_m)]_t^{(\ell+n)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell+2q)}[f(-t,\boldsymbol{x}_m)]_t^{(\ell)}\\&\quad+[f(-t,\boldsymbol{x}_m)]_t^{(\ell+n)}\bigl[[f(-t,\boldsymbol{x}_m)]_t^{(\ell+q)}\bigr]^2\bigr\}.\end{aligned}$$

Further taking $t\to0^+$ reveals

$$\lim_{t\to0^+}G_{q,n}(t)=\mathcal{G}_{m,\ell,q,n},\quad \lim_{t\to0^+}H_{q,n}(t)=\mathcal{H}_{m,\ell,q,n},\quad \lim_{t\to0^+}I_{q,n}(t)=\mathcal{I}_{m,\ell,q,n}.$$

Substituting these quantities into (4.11) and simplifying bring about inequalities in (4.10). The proof of Theorem 4.4 is complete. □

Remark 4.1. When taking $x_1=x_2=\dots=x_m=1$, all results in this section become those for the Bell–Qi numbers $B_{m,n}$ for $n\ge0$.

Remark 4.2. When taking $m=1$ and $x_1=1,x$ respectively, all results in this section become those corresponding ones in the papers [17, 18].

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