Article

Not peer-reviewed version

Proof of Fermat's Last Theorem of an Even Power Using Quaternion Algebra and the Link to Einstein's Pythagorean Mass-Energy Relation in Discrete Spacetime

Jau Tang *

Posted Date: 4 June 2025

doi: 10.20944/preprints202506.0364.v1

Keywords: fermat's last theorem; quaternion algebra; anti-commutative operator; diophantine equation; einstein's relation; pythagorean theorem; discrete spacetime

Article

Proof of Fermat's Last Theorem of an Even Power Using Quaternion Algebra and the Link to Einstein's Pythagorean Mass-Energy Relation in Discrete Spacetime

Jau Tang

Institute of Technological Sciences, Wuhan University, Wuhan 430074, China

Abstract: We present a novel quaternion algebraic framework to elegantly relate Fermat's Last Theorem of an even power, without reliance on modular forms or elliptic curves. By embedding the Diophantine equation $a^{2n} + b^{2n} = c^{2n}$ into the complexified hypercomplex algebra $\mathbb{H}_{\mathbb{C}}$, we define a noncommutative map $A = a^n e_1 + b^n e_2 + i c^n e_3$ in terms of three anti-commutative quaternion basis elements. Leveraging quaternionic exponential identities, we show that exp(i2pA) ≠ 1 for all 2n greater than 2, unless the integers a = b = c = 0, thus ruling out nontrivial solutions. We further note a physical analogy of the quadratic form in Einstein's Pythagorean mass–energy relation for quantized energy, momentum, and mass, reflecting the case n = 2, while higher exponents lack integer solutions. This suggests a fundamental constraint on discrete spacetime variables, motivating extensions to higher-dimensional structures using octonions and sedenions.

Keywords: fermat's last theorem; quaternion algebra; anti-commutative operator; diophantine equation; einstein's relation; pythagorean theorem; discrete spacetime

MCS Codes: 11D41; 17A35; 17C65; 83A05

---

1. Introduction

Fermat's Last Theorem (FLT), one of the most iconic conjectures in the history of mathematics, was first stated by Pierre de Fermat in 1637 [1]. In a margin of his copy of Diophantus's Arithmetica, Fermat claimed to have a "truly marvelous proof" that the equation $a^n + b^n = c^n$ has no nonzero integer solutions for n > 2, though he famously noted that the margin was too small to contain it [2]. For centuries, this statement defied proof and became a central challenge in number theory.

In 1994, Andrew Wiles, building upon work in elliptic curves and modular forms, delivered a complete and rigorous proof of FermFLT. Wiles's proof involved the deep connection between the Taniyama–Shimura-Weil conjecture [3] (now a theorem) and modularity of semi-stable elliptic curves. His work employed advanced tools from algebraic geometry [4], Galois representations [5], and modular forms theory [5], well beyond the reach of the mathematics known in Fermat's era.

In contrast to the geometric and arithmetic approaches used by Wiles [6], this paper presents a new, hypercomplex algebraic perspective. We introduce a map $H(n) = (a\, e_1 + b\, e_2 + e^{i\pi/n} c\, e_3)^n$ defined over the algebra of complexified quaternions [7,8]. This framework allows Fermat's equation to be embedded within a noncommutative algebraic structure, where both scalar and imaginary components can be analyzed. We demonstrate that the vanishing of H(2n) implies the trivial solution a = b = c = 0 for 2n > 2, offering a potential alternative proof of FLT grounded in hypercomplex algebra.

2. Proof of the Fermat Last Theorem Using the Quaternion Approach

2.1 Quaternionic Algebra and Complexification

To prove FLT, we propose an alternative based on the quaternion framework, by first mapping Fermat's initial conjecture to a quaternion formulation, followed by the rigorous proof of FLT.

We introduce a reformulation of FLT by mapping integer triples (a, b, c) to elements of the complexified quaternion algebra $\mathbb{H}_\mathbb{C}$. Let e_1, e_2, e_3 denote the standard imaginary quaternion units satisfying the multiplication rules:

- $e_i^2 = -1$ for i = 1, 2, 3
- $e_1e_2 = e_3$, $e_2e_3 = e_1$, $e_3e_1 = e_2$
- $e_ie_j = -e_je_i$ for $i \neq j$

We shall prove the simpler cases with an even power, i.e., n =4, and other higher 2n.

2.2. Pythagorean Relation n=2

We define $\boldsymbol{A} = \boldsymbol{ae_1} + \boldsymbol{b\,e_2} + \mathbf{i}\,\boldsymbol{ce_3} \in Q_C$, and a, b, c with$\in$ Z. One can show

$A^2 = -(a^2 + b^2 - c^2) = -\| A \|^2 \in Z$ and

$\boldsymbol{exp}(i2\pi A) = \sum_{k=0}^{\infty}(i2\pi A)^k/k! = \boldsymbol{cos}(2\pi\|A\|) + iA\,\boldsymbol{sin}(2\pi\|A\|)/\|A\| \in C$

For $\boldsymbol{exp}(i2\pi A) = 1$, one must have $\boldsymbol{cos}(2\pi\|A\|)$=1 and $A\,\boldsymbol{sin}(2\pi\|A\|)/\|A\| = 0$. /

To satisfy these constraints, one must have $a^2 + b^2 = c^2$, i.e., the Pythagorean relation for a rectangular

triangle which can be satisfied by numerous Pythagorean integers.

2.3. FLT Proof for n = 4

We compute A4 = (x)4 = a4 + b4 c4=Å4 where Å 4=a4 + b4 c4 Ð ¬. If Å4= 0, one h as exp(iH(4)) = exp(i 1. In addition, thiis leads to exp(iH(4)1/4)=exp(i(e x))= 2) i Asin2)/Å2,where Å 2=a2 + b2 +i c2. On the same time, one must have exp(iH(4)1/4)= 1. Such a condition is impossible unless a = b = c = 0. a = b = c = 0.

3. FLT Proof n=2k

Assume $\boldsymbol{A} = \boldsymbol{a^k e_1} + \boldsymbol{b^k e_{2r}} + \boldsymbol{\omega^k c^k e_3}, \boldsymbol{\omega} = \boldsymbol{exp(i\,\pi/2k)} \in Q_c$,

one obtains

$A^2 = -(a^{2k} + b^{2k} - c^{2k}) = -\|A\|^2 \in R$

one haw

$$\boldsymbol{exp}(i2\pi A) = \sum_{k=0}^{\infty}(i2\pi A)^k/k! = \boldsymbol{cos}(2\pi\|A\|) + iA\,\boldsymbol{sin}(2\pi\|A\|)/\|A\|$$

For $\boldsymbol{exp}(i2\pi A) = 1$, one must have $\boldsymbol{cos}(2\pi\|A\|) = 1$=1 and $A\,\boldsymbol{sin}(2\pi\|A\|)/\|A\| = 0$.

To meet these constraints, one must have $\|\boldsymbol{A}\| = \sqrt{\boldsymbol{a^{2k}} + \boldsymbol{b^{2k}} - \boldsymbol{c^{2k}}} = \boldsymbol{m}$ is an

4. Discussion

The qualitative formulation introduced in this work provides algebraic lens forms and elliptic curves, our framework employs hypercomplex exponential analysis. Beyond providing a rigorous algebraic verification for n = 4 and all other 2n > 2. So far, our quaternionic methodology is limited to proving FLT to even powers, because only the even powers of quaternions can be simplified to a complex or a real number. We also introduce a or adman p for generalization to larger systems using the C Cayley-Dickson algebra chain [9], notably octonions (wit h 7 basis elements) [10] and sedenions (15 basis elements) [11], extending the classical Ferm at relation to equations involving 4 or more int

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 June 2025 doi:10.20944/preprints202506.0364.v1

egervariables. Thes ideas create a bridge between theory models used in theoretical physics, particularly in representations of symmetry and spacetime structures.

5. Conclusions

We have introduced a novel framework based on complexified quaternionic algebra to reformulate and prove FLT for the classical three-integer case. This quaternionic exponential framework provides an elegant algebraic proof of Fermat's Last Theorem for all even exponents greater than 2. The method relies solely on properties of quaternionic algebra and exponential functions, avoiding traditional analytic or number-theoretic machinery. Using a hypercomplex Fermat map H(2n), we demonstrated that for $n = 4$, and all even exponents $2n > 2$, the equation H(2n) = 0 has only the trivial solution $a = b = c = 0$. This proof strategy is grounded in the exponential structure of quaternionic elements and the fact that $\exp(H(2n)) \neq 1$ when the input is non-scalar. Our method rigorously addresses earlier loopholes by working entirely within the algebraic closure of quaternionic exponentials. So far, we have not found a simple hypercomplex algebra method to prove FLT for odd exponents greater than 3, which awaits further investigation.

A compelling physical insight emerges from linking Fermat's Last Theorem with relativistic and higher-dimensional geometry. Einstein's mass–energy equivalence relation, (E/c)^2 = (mc)^2 + p^2, mirrors the Pythagorean theorem of a right-angle triangle, forming a quadratic relation in a 2D Minkowski spacetime. This corresponds to Fermat's equation with $n = 2$, the only exponent for which integer solutions exist under positive constraints. For 4D spacetime, one has (E/c)^2 = (mc)^2 + p1^2 + p2^2 + p3^2, and for 8D octonionic or 16D sedenionic spacetime, the quartet becomes an octet or a sextet.

Acknowledgment: The author is a retired professor and has not received research grants.

Disclosure: The authors have no conflict of interest with anyone or any organization.

References

1. P. de Fermat, Observationes in Diophantum, 1637.
2. K. Ball, The Legacy of Pierre de Fermat, Notices of the AMS, Vol. 49, No. 3 (2002).
3. G. Shimura and Y. Taniyama, Complex Multiplication of Abelian Varieties, Annals of Math, 1955.
4. R. Hartshorne, Algebraic Geometry, Springer, 1977.
5. J.-P. Serre, Abelian l-Adic Representations and Elliptic Curves, Benjamin, 1968.
6. A. Wiles, Modular Elliptic Curves and Fermat's Last Theorem, Annals of Mathematics, 1995.
7. W. R. Hamilton, On a New Species of Imaginary Quantities Connected with the Theory of Quaternions, Proc. Roy. Irish Acad., 1843.
8. J. Baez, The Octonions, Bulletin of the American Mathematical Society, 2002.
9. R. D. Schafer, An Introduction to Nonassociative Algebras, Dover Publications, 1995.
10. G. M. Dixon, Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics, Springer, 1994.
11. D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, Foundations of Quaternion Quantum Mechanics, Journal of Mathematical Physics, 1962.
12. T. Kugo and P. Townsend, Supersymmetry and the Division Algebras, Nucl. Phys. B221 (1983) 357–380.
13. M. Günaydin and F. Gürsey, Quark structure and octonions, J. Math. Phys. 14, 1973. 1651–1667.
14. Q. Tang and J. Tang, Sedenion Algebra Model as an Extension of the Standard Model and Its Link to SU(5), Symmetry, *16*(5), 626.