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On the Received View vs. the Alternative View Controversy about Quantum (Non)individuality

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Abstract: Some philosophers of physics have addressed criticisms of the so-called Received View (RV) of non-individual quantum objects, also called the *orthodox view*. Dennis Dieks made a very good resume of these criticisms in [1] and Tomasz Bigaj in [2] has a more detailed account. In considering (mainly) these works and with some additional mentions, we hope to dissipate some misunderstandings about the RV and clarify what is happening with such a view. According to Dieks, the RV doesn't fit the *practice of physics* since in some situations the physicist assumes that quantum objects can be treated individually, imitating standard objects (individuals) of classical physics. Dieks also proposes an Alternative View (AV), generally called the *heterodox view*, which would give a view of the fundamental ontology of quantum physics and which would be more by the way physicists usually proceed. In my view, the AV cannot be viewed as *the* fundamental ontological theory despite being suitable for practical purposes. Furthermore, we think that it does not conflate the RV, but is complementary to it, substituting it when quantum objects are sufficiently apart and can be treated in conformity with the classical way. From our point of view, in the practice of physics, we can adopt AV. Still, the RV is more adequate when we are looking for logical and foundational analyses, at least when the supposed metaphysics comprises non-individuals.

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1. Introduction

Philosophers of physics such as T. Bigaj [2,3], F. Berto [4], M. Dorato and M. Morganti [5], A. Caulton [6] and others have offered a series of criticisms to the so-called Received View (RV) of quantum entities as devoid of identity conditions, *identity* being understood as that notion prescribed by classical 'great' logic.[1] One of the most recent works in this respect is Dennis Dieks' [1]. In considering his paper and by adding some further references, we think it is possible to dissipate what I think are misunderstandings about the RV and in doing so I think that their criticisms can be put between parentheses since they arise (in my opinion) due to the lack of noticing the *real* finality of the RV. In doing so, I hope to contribute to the discussion which surely increase the logical and metaphysical analyses of quantum theories.

The main justification for supposing that the *standard notion of identity* (henceforth, just 'standard identity') should be avoided regarding quantum entities is the simple fact that, according to this notion, in being more than one, things are necessarily *distinct* (not identical) and this implies the existence of some difference among them either a 'physical' or a 'logical' one, as we shall see below. But, as is part of the 'quantum folklore', called the *orthodox view* by these authors, quantum systems can be *completely indiscernible* and not just similar to each other; for instance, bosons can share all their quantum properties and, when forming (say) a bosonic condensate, *cannot be discerned in any way*. In

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[1] 'Great logic', or *Logica Magna*, in the words of E. W. Beth, is "a logical system which enables us to deduce the totality of pure mathematics without an appeal to any specific axioms" [7, p.230]. Of course, we can add a standard set theory such as the ZFC system (here considered) under such an epithet.

such a situation, we have a clear violation of standard logic. 'Classical objects' are *individuals* (see below); as N. da Costa has stressed, "[classical] logic arises from a static doctrine of the real" [8, p.77].

In this paper, we revise some of Dieks' main points, which go in the same direction as those of the philosophers mentioned above, and try to make clear what is going on with the RV, arguing that his view (termed the 'Alternative View', AV) is not *against* the RV but can be considered as *complementary* to it, being applied in certain situations more in accordance, as he says, with the practice of physics while, in contrast, the RV would hold in general from the logical and foundational point of view.

Concerning the expression 'the practice of physics', it was used by some of these philosophers to emphasize that *in some situations* the physicist can discern between electrons located in different positions (the case of bosons is still left as inconclusive) so that they can be described using certain symmetric combinations of projectors so that *all happens as if* they have distinct identities. In our opinion, this can be done FAPP ('for all practical purposes'), but deserves considerations we make in the next sections. For bosons, it is still agreed that their complete indiscernibility is a quantum fact [3].

One of the main criticisms of the RV is what Adan Caulton [6], termed *factorism*. In short, the RV is charged with taking particle labels, something that would distinguish them. Let us see one of the references to this notion. As French and Bigaj say,

> " When we write down a state of a system of two same-type particles in its symmetric or antisymmetric form, this state is supposed to occupy a tensor product of two identical Hilbert spaces: $\mathcal{H}_1 \otimes \mathcal{H}_2$. Factorism claims that these Hilbert spaces represent states and properties of the particles constituting the system. Consequently, labels 1 and 2 we use to distinguish the two components in the factorisation of the total Hilbert space can be also used to refer to individual particles. Indeed this is what is typically done when interpreting symmetric/antisymmetric states of two particles: the labels used in the description of such states, and concerning which the requirement of symmetry/antisymmetry is imposed, are assumed to refer directly to the particles occupying these states." [3, §5]

I don't agree with this interpretation. The labels do not refer to the particles as they were individuals, so they cannot be regarded as indicating that the labels refer *ostensively* to the particles (I suppose that by this they mean that the labels act as proper names). The standard anti-symmetric state (to cope with the case of fermions)

$$|\psi\rangle_{LR} = \frac{1}{\sqrt{2}}\left(|\psi\rangle_L \otimes |\varphi\rangle_R - |\varphi\rangle_L \otimes |\psi\rangle_R\right), \tag{1}$$

telling that there is an electron located at L (left) and another at R (right) but it *is not* identifying *which* electron is at each position, say by saying that electron Peter is on the L and electron Paul is at the R as if they were endowed with identities, that is, as if 'Peter' and 'Paul' could name electrons as rigid designators. Of course, I think that no physicist or philosopher would accept that since we know that any permutation of the electrons conduces to exactly the same expectation values, that is, it is indifferent to which electron is Peter and which one is Paul.

The labelling of the particles is a necessity of the language, for we have no other way to express that we are dealing with two entities. But the important thing concerning quantum physics is that these labels do not represent proper names to them, and this is the reason we use symmetric states. As Dalla Chiara and Toraldo di Francia sat, "quantum physics is the land of anonymity" [9]. But, let me insist, that anonymity can be achieved in a standard framework (say in a standard set theory) only if we turn a blind eye to the identity of the particles, something that arises from logic. Let us see this point in more detail.

2. Standard Identity

The controversy concerning the notion of identity is as old as philosophy. Surely we will not enter the historical details here; there are quite good texts in the literature which discuss the notion both

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in general and in formal terms; for instance, [10, Chap. 3], [11]. Here we keep with STI, the Standard Theory of Identity of classical logic (and of the most logical systems).

The need to go to logic is justified since the informal dictum that identity is *something* an object has which distinguishes it from something else is vague and redundant to be used in logical analyses. Such an informal characterisation that identity would be *that thing* an individual has that makes it what it is in distinction to any *other* thing, or that it shares just with itself, is also quite vague and apparently begs the question, since needs to refer to 'other', which seems to presuppose an a priori distinction. Furthermore, we repute attempts such as Geach's *relative identity* [12] and Quine's 'definition' (borrowed from Hilbert and Bernays [**?** , §5]) as nothing more than indiscernibility relative to the predicates of the language [14, Chap. 6]. Identity requires *all* predicates and relations, haecceities, quids (if they are supposed to exist), etc. Identity is *sameness*, *the same, just one*. This applies to the standard 'definition' of elementary particles in quantum mechanics, put by J. M. Jauch this way: "Two elementary particles are identical if they agree in all their intrinsic properties." [15, p.275] These are those properties that do not depend on the state of the system. It is quite obvious that this definition is not of the *logical* identity of quantum entities, but just indiscernibility restricted to intrinsic properties.

STI is formalised in first-order languages with a primitive binary relation '=' by two postulates, namely, reflexivity ($\forall x(x = x)$) and substitutivity, that is, for *any* formula α, $\forall x \forall y(x = y \to (\alpha(x) \leftrightarrow \alpha(y)))$, with the usual restrictions (see [16, p.95]). If our theory is some 'standard' set theory such as the ZFC system, we add an axiom of extensionality $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$, and a further axiom $\forall x \forall y(\forall z(x \in z \leftrightarrow y \in z) \to x = y)$ if there are atoms involved (the ZFA set theory — see [17]).

In higher-order languages, identity can be defined in the Whitehead-Russell's style, here done for second-order languages. This is Leibniz Law: $x = y := \forall F\,(F(x) \leftrightarrow F(y))$, where x and y are individual variables and F is a variable for properties of individuals. Notice that none form of *haecceity* is assumed, that is, all we have are the individual's properties and relations.[2]

This has consequences, as suggested above. Every entity described by such a mathematical framework encompassing STI is an individual in the above sense and can be discerned from anything else by a monadic property, that is, *absolutely*. In other words, every element of any model of the theory is *definable* in the sense of being the unique object satisfying some property, such as I_a above. Thus, if we wish to admit the existence of completely indiscernible things, that is, things that cannot be discerned in any way, we cannot use a mathematical framework encompassing STI (but see the subsection 8.2 for alternatives). Of course, one should justify why we should consider such indiscernible things and quantum mechanics comes in our help. Of course, there are interpretations, or versions, in which the basic entities are considered as having identity, but in our opinion, these are alternatives that can give suitable physical results but do not cover indistinguishability in the strong sense, which we regard as one of the most important notions of quantum physics. As the Nobel Prize winner Frank Wilczek says, "in the microworld, we need uniformity in the strong kind: complete indistinguishability." [19, p.135] It would be interesting to recall that some recent experiments such as the Hong-Ou-Mandel Effect require the complete indiscernibility of photons or atoms (in [**?**] this is discussed with an emphasis on the importance of considering indistinguishability seriously). This means that we need to look at the indistinguishability of the involved entities, which cannot be supposed to be even minimally discernible. The problem is to find a mathematical way to treat such situations *without* entering in mathematical artifices such as *ad hoc* postulates of symmetry or the confinement to deformable structures.[3]

To start, let us say what we understand by identity, a concept shared by Dieks himself as we can see when he presents the AV saying that "according to [the AV], quantum particles possess an identity grounded in physical properties, like classical particles, so that they can be handled by standard set

[2] Concerning the role (or the absence of any) of haecceities in quantum mechanics, see [18].
[3] See below section 8.2.

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theory." (op.cit., p.13) In the sequence, he questions the assumption of the RV that quantum particles don't obey the standard theory of identity of a standard set theory. In his words,

> "One might even wonder whether the notion of a particle without identity is not a contradiction in terms: the very concept of a particle, and more generally an object, in daily life and in the practice of physics is bound up with identifiability." (id.ibid.)

We answer that such a question can make sense only if the mathematics is classic, or some other framework sharing the same theory of identity. Then of course something that does not have identity entails a contradiction, since *every entity* in such theories do have identities. It is precisely to fulfil this gap that quasi-set theory was proposed. Summing up, we say that an object *has identity*, or that it is an *individual* if it obeys STI (see below and our informal definition of an individual in the next section). This implies at least the following things that are important for the discussion: (i) any two or more objects obeying STI are *different* and this entails that (ii) being different, the objects can be distinguished by some *monadic* property.

Notice that the property (or properties) which provide the differences among the entities having identity do not necessarily need to be *physical* but can be *logical*. For instance, the identity of the object a from a collection of finite objects (which seems to be the only case considered by physics) is given by the predicate $I_a(x) := x \in \{a\}$.[4] When objects can be distinguished by a monadic property, we call them *absolutely discernible* according to the usual literature; when they cannot be discerned in any way, we say that they are *completely indiscernible*.

For certain discussions, such as those considered here, it is important to make clear which is the metamathematical framework; since nothing is usually said in this respect, we assume a standard set theory such as the ZFC system, which encompasses STI, so *any* object can be discerned absolutely by its identity. It seems that this is a reasonable hypothesis. Soon we shall discuss why some philosophers don't accept the predicate I_a as providing the identity of a; (iii) fundamentally, if in a context we exchange an object by another, even quite 'similar' to it (in some sense of this word), the context changes. We can call this third criterion 'extensionality'.

Of course, when speaking of the identity (or of the lack of) of some objects, someone would explain what she understands by such a concept. Let us fix some terminology and suppositions. Dieks agrees that the RV is grounded on a non-classical mathematics termed *quasi-set theory*, '$\mathfrak{Q}$' for short. Using this theory, we can express in a more precise way what the RV claims. This theory encompasses two kinds of ur-elements, the M-atoms, which behave as the ur-elements of ZFA (the Zermelo-Fraenkel set theory with Atoms [17]), and the m-atoms, to which STI does not apply. This means that if either x or y are m-atoms, then the expressions of the form $x = y$ or $x \neq y$ have no meaning; in particular, the theory does not provide any meaning for $x = x$ if x stands for a m-atom.[5] There may exist m-atoms of 'different kinds', which may be *distinguishable* among them by characteristics provided by the physical theory (electrons from protons and neutrons, etc.), $x \not\equiv y$ in symbols. The basic primitive notions are membership ('$\in$') and indistinguishability ('$\equiv$') and this one has the properties of an equivalence relation, but it is not a congruence, since $x \in y$ and $x \equiv z$ does not entail that $z \in y$ (for details, see [**? ?**]). The universe is populated by such atoms and the quasi-sets ('qsets' for short) and the postulates extend those of the theory ZFA. A concept of identity, termed 'extensional identity', '$=_E$', is defined for both M-atoms, when they belong to the same qsets and qsets when the elements of one of them are also elements of the other. Some qsets may have a cardinal, termed the *quasi-cardinal* of the qset, in a way that the existence of a quasi-cardinal does not entail that the elements of the qset are discernible (this will be considered later — see section 8.4). The Axiom of Weak Extensionality (WEA) says that if

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4 In the finite case, every object can be named, say a, so the formula $\alpha(x) := x \in \{y\}$, once proven that there exists just one thing obeying α, *defines* a (which can be the name of the defined entity) [20].

5 This does not entail that we cannot define an identity for the m-atoms, as we show in subsection 8.3, but $\mathfrak{Q}$ does not assume that.

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qsets A and B have 'the same quantity' (expressed using the quasi-cardinals) of 'elements of the same sort' (that is, indiscernible among them), then they are indiscernible ($A \equiv B$). Extensionally identical things (or just 'identical things') are indiscernible, but not the other way around. From now on we shall use these notions. For details about this theory, see [**? ?**].

3. The Received View

The RV is not occupied with 'the practice of physics' in the same sense that the definitions of limit and continuity in Calculus were not occupied with the practice of the engineer or the applied mathematician when informally using the notion of infinitesimal (say when she speaks of an 'infinitesimal element of volume' for instance). Below we revise this analogy.

The RV treats some entities as *individuals*, and due to the importance of this notion for our discussion, let us summarise it here again; an individual is informally characterised as being something that (i) is a unity of a kind, say a table, a person, a pen; (ii) it has *genidentity*, that is, it can be re-identified *as such* individual in different contexts; we can say that it has *material genidentity* in the sense of Reichenbach [21, §26]. An alternative way is to say that individuals have *diachronic identities*, that is, we can say that individual 1 at time t_1 *is the same* (diachronically identical) as individual 2 at time t_2.[6]

We could say that an individual carries a passport. This second requisite is fundamental and usually forgotten by philosophers (see for instance [6]). Julius Caesar was a man and *the same man* when in Rome and when in Egypt, at least we usually accept that.[7] If something obeys STI, it is an individual in this sense and can be said *to have an identity*; the natural number two (once defined for instance in the sense of von Neumann) is the same natural number two when we list it as an element of the set of the even natural numbers and when we refer to the set-theoretical successor of one, that is, the set $1 \cup \{1\}$, being $1 = \{\emptyset, \{\emptyset\}\}$.[8]

Quantum objects can be *isolated* for instance in quantum traps, but this does not make them individuals in the above characterisation. As David Hume has reminded us, "One single object conveys the idea of unity, not that of identity." [**?**] (p.200). Quantum objects lack (ii); once one has left the trap, it will never more be identified as *that* entity of before. Usually, the notions of *identity*, *individuality* and *individuation* (or 'isolation') are conflated and taken as being synonymous while they should be discerned on each other (see 'Isolation' below).[9] For more details about these notions and their distinctions, see [**?**].

The RV has been accused of entering *factorism*, a notion to be discussed below. Anticipating something in this respect, we observe that, roughly speaking, factorism is characterised by assuming that the labels we use in the quantum formalism name particles and not only the Hilbert spaces which comprise their state vectors. This is a wrong reading of the RV. As we can see at [**?**](p.153), the RV assumes that in an entangled state such as the anti-symmetric state for two fermions, such as the 'Bell state' for the two electrons of a Helium atom, namely, $|\psi_1 2\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_2\rangle - |\psi_2\rangle||\psi_1\rangle)$, each electron partake both states at once, which entails that the particles in the allowed state have also all the state-dependent properties in common, so PII is violated; see also [25]. Thus the labels '1' and '2' name the states or, equivalently, the Hilbert spaces. Anyhow, we assume that there is something behind the state; as Sunny Auyang has claimed, "physical theories are about things" [26, p.], obviously, *physical things*, so, independently whether the initial entities are fields or strings, the target entities,

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6 Since any object 'changes' with time, to keep the Principle of the Identity of Indiscernibles holding, sometimes philosophers accept that diachronic identity exists when the objects are numerically distinct in different times but still connected by such an identity. See [22].

7 David Hume, for instance, claims that we *attribute* identity to an object when we observe that there is a continual succession of perceptions in our mind [**?** , p.65], [24, §2.7]. His Principle of Individuation "is nothing but the *invariableness* and *uninterruptedness* of any object"' — Hume *idem*, p.201.

8 This remark is posed because, as is known, there are different and non-equivalent definitions of natural numbers.

9 Hans Reichenbach says that they don't have 'material genidentity', but only 'functional genidentity' (op.cit.).

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those that enter in certain states and are accelerated in the particle accelerators, are treated as *particles* (in this broad sense), although this notion (that of a 'particle') has suffered *metamorphoses* from classical physics to the Standard Model of particle physics, as reported by B. Falkenburg [27, Chap.6]. Summing up, the labels 1 and 2 used in the above vector do not name particular particles but just refer to the fact that there are two of them without considering their distinction.

Dieks also assumes that in his view, the talk of particles is not appropriate to describe the quantum world. According to him, particles 'emerge' in special situations and in these special situations they possess individual identities in the same way as classical particles (STI) [28]. The AV discerns the particles not by their labels, but by employing suitable *projectors*, that is, projection operators over certain Hilbert spaces. In my view, this is of course a licit move and it is interesting because it maintains standard logic at least for fermions; the case of bosons is still open. But let me insist that in denying that before the 'emergence' of the particles the formalism is empty looks quite strange to me; I still think that, as S. Auyang recalls, we are speaking of something.

For the sake of clarity and to define our concepts, some general remarks would be recalled:

1. **The objectives of the RV —** The RV is occupied mainly with the logical and metaphysical foundations of quantum theories. The main book where such a view is outlined has a subtitle 'A historical, philosophical, and formal analysis' [**?**]. This should be enough to show its main finalities: a formal (logical) analysis of the view that quantum objects can be viewed as *non-individuals*. Notice in addition that the RV does not preach this vision radically, accepting that there are alternatives in considering quantum objects. In the same mentioned book, it is argued that quantum objects can also be viewed as *individuals*, provided that only some states are available. This is more or less in the direction Dieks suggests the physicist goes in her usual 'practice'. But, as we shall enlighten below, the RV aims also to be careful with the underlying logic, yet such details can be dispensed with in the first moment during applications.

Notwithstanding, the RV has applications. Beyond the possibility of expressing formally a metaphysics of non-individuals, it can sustain prolongments in several directions; let us mention some of them: (i) in [**?**], it was proposed that, since the Ontological Structural Realism (OSR) [29] lacks an adequate definition of *structure*, essential for the main idea that all there is are structures, and to them, a structure should be formed by relations but without mentioning the relata. In the mentioned paper, it was proposed that the relata are not 'fixed' in the relations, but can be substituted by indiscernible ones so that the philosophy will not depend on *particular* relata, but of, let us say, kinds of them. For instance, if we are dealing with electrons, what is important is that they are electrons and not particular electrons. This idea was embraced by S. French in his [30]. This is, of course, a program to be followed. Another possible application concerns mereology, quantum mereology. Standard 'extensional' mereologies [31] assume that a whole is composed of parts and two things are identical when they have *the same* parts. But what happens if these 'parts' are things like atoms or quantum particles? In standard mereologies, if a part of some whole is exchanged by another part whatever, the whole changes, and becomes something *different*, but not in quantum physics; the wholes become *indistinguishable*, which is not the same thing. Furthermore, in quantum physics, an entangled state cannot be 'factorised' in states for each component of the whole; so, how can we speak of the parts in isolation? In [**? ?**] these and other questions about the possibility of a 'quantum mereology' are discussed.

2. **Why to question STI —** The criticism of the STI assumed by the RV has a reason. According to this theory, whenever we have more than one object, we need to acknowledge that they are *different* ('$a \neq b$') and this means that a and b can be discerned by some monadic property and not only by some relation. That is if $a \neq b$, then there exists P such that $P(a)$ but $\neg P(b)$ or the other way around. There is no escape to this conclusion, which is imposed by logic, once one assumes a logic encompassing STI. But we think that today no one disputes the importance of the notion of completely indiscernible particles in quantum physics and its applications. Several examples have shown the necessity of assuming that in certain situations there cannot be even in principle a distinction among the considered quantum

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entities, as mentioned above (the Hong-On-Mandel effect [?], among others). For a specific case, take bosons in a bosonic condensate, a BEC (Bose-Einstein Condensate). They may be milliards, all in the same quantum state. It is assumed by the quantum theory that *there are no differences among them*, and as far as quantum mechanics works, no differences among them can be found. The obedience to quantum statistics (in this case, Bose-Einstein statistics) provides also an argument favouring the RV: without assuming substratum or something else beyond the properties, how can something obey such statistics under the validity of STI? In our opinion, this makes no sense: if it is assumed that bosons (or other quantum entities) cannot be discerned *in any way* (not even in principle), how can STI hold to them? Notice that we agree with the claim that the physicist can, *momentaneously*, treat them as individuals endowed with identity, but such an identity has no sense in the wide aspect and should be understood as just a *mock identity*, as advanced (and acknowledged by Dieks) by Toraldo di Francia [32,33] (see below).

3. **Weak discernibility and discerning properties —** As said before, some philosophers have proposed that in certain situations quantum entities (both bosons and fermions) cannot be discerned 'absolutely', that is, by a monadic predicate, but just by an irreflexive and symmetric relation. They call it *weak discernibility* (see [34] for references and discussion). We have discussed elsewhere such a proposal and will not revise it here, but just make some general remarks; for details, see [?]. The fact is that if the underlying logic encompasses STI, as classical logic these philosophers are assuming does, there is no escape to the conclusion that for any object y (say in the universe of sets and atoms) we can define the property 'identity of y' by positing (now without any individual constant) something like $I_y(x) := x \in \{y\}$. Since the unitary set $\{y\}$ can be assumed to exist for every y, then y is the only object having such a property. There is no surprise concerning this, for it is exactly what STI says. However, the mentioned philosophers are not satisfied with such a logical imposition. They refuse that a 'logical property' can individuate a thing, saying that the discerning property must be 'physical', something 'measurable'. This supposition introduces a lot of other difficult questions, such as (a) what is a 'physical property'? (b) what is a 'measurable property'? (c) why such a bias to avoid logical implications in a theory? — see below, section 4. There are no clear answers to these questions. We think that by 'measurable property' we can understand whatever property to which we can ascribe the epithet 'true' or 'false' when compared with some value or set of values (for instance, whether a certain observable has a value in some specific Borelian set). For instance, 'the volume of that portion of water is half a litre' can be true or false depending on the portion of water. But also 'the spin of the electron in the z direction is UP' and 'x belongs to the unitary set of y' are 'measurable' in this sense. So, we don't see any reason to restrict the properties (hence the formulas) involved in the axioms of STI. As Shoenfield insists, "the symbol '$\forall$' [used in the definition of identity in STI] means *for all*" [35, p.13], and we could add 'and not *for some*'. Thus, when we say (even if in the metalanguage) that in being identical x and y satisfy all the same formulas, we are not restricting the phrase to 'some formulas' (or predicates).

On the contrary, the theory of quasi-sets may be the right place to formalise the weak discernibility claim. We can suppose a qset with two indiscernible m-atoms and with quasi-cardinal two so that there is an irreflexive but symmetric relation holding between them. No monadic property is supposed to exist. In the classical example of the two electrons of a neutral Helium atom, the property 'to have spin opposite to' exemplifies the case. The idea that electrons are so discerned but are not discerned by a monadic property can be formalised in $\mathfrak{Q}$, but not in ZFC.

4. **Isolation —** We have remarked already that with regard to quantum objects being distinct in certain situations conflates the notions of identity and *isolation*, or *individuation*. Identity is a logical notion, given by some 'theory of identity' such as STI; individuation is an epistemological notion of taking something in a situation that can give us *the impression* that we are facing an isolated individual with an identity. For instance, a case also explored by Dieks speaks of isolation and not of identity. Quantum objects located in traps in distant places (say the South Pole and the North Pole) are *isolated* or *individuated*, being able to be momentaneously treated as individuals, but they are not individuals

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since they do not satisfy re-identification, something required by something 'having identity'. Later we shall further analyse this example.

5. **Factorism —** Dieks criticizes the RV also because this view presupposes that quantum objects are the basic tenet of the theory and labels such as '#1', '#2', etc are given to these entities, causing what he calls the *factorizing* account (a term borrowed from A. Caulton [6]). According to this view, labels such as '#1' and '#2' refer to particles and this would cause them to be able to be *absolutely discerned* by a monadic property. As Dieks and Lubberdink say, "the factor space labels [in the N-fold tensor product of Hilbert spaces] should *not* be thought of as referring to single particles" and the tensor product, in their interpretation, does not have the form of a concatenation of one-particle states [34]. They attribute such a 'complication' to the symmetrization postulates:

> "As the symmetrization postulates apply universally and globally to all particles of a given kind (...), 'factorists' must hold that each single electron [say] is equally present at all positions in the universe."

Not at all! As anticipated already above, 'identical' particles share the Bell state (their equation (1))

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|L_1|R\rangle_2 \pm |R\rangle_1|L\rangle_2) \quad \text{(Bell)}$$

are not both at L (left) together with all other particles of the universe (of the same kind). They share that state *while a measurement is not made*, but as a vector of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, this does not imply that we are identifying *that* particle (say, 'Peter') as the particle whose states are represented by vectors at $\mathcal{H}_1$ and which is at L. No, the tensor product just says that we have *one* particle in each position (or around it) without saying which one is it. The identification makes no sense, as is well known. They add that "[i]t is therefore impossible to individuate the 'factorist particles' via different physical characteristics." The defenders of the AV claim that, when dealing with (say) two quantum systems of the same kind, let $\mathcal{H}$ be the Hilbert space for the states of one of the systems. For the joint system, the states are represented in the tensor product $\mathcal{H} \otimes \mathcal{H}$ and them "the first and the second factors in the product correspond to individual particles," ([2, p.117]) suggesting that they are being individualized. But the RV does not say that; the fact that the states are in the tensor product does not compel us to discern the particles, as it would be obvious.

While sharing the Bell state, we know that there is no way to individuate them; as explained one of the main references of the heterodox view, a paper by Ghirardi and colleagues [36], "when one is dealing with assemblies of identical quantum systems it is simply meaningless to try to 'name' them in a way or another," and also that (when in an entangled state), "one cannot pretend that a particular one of them has properties." Only the whole system has properties, as the quantum theory says. In the above case, the identification, say that one of the particles is at L, comes only with the measurement. Maybe this emphasises the role of the underlying logic (or mathematics); the two particles are *two* and even if there are no *physical* characteristics that discern them, there are the *logical* ones once we remain with STI. But if we wish, as it seems clear, to maintain that *before measurement* the particles are 'really' indiscernible, then STI must be placed aside.

In general, the authors forget that, even implicitly, they assume something like ZFC since they seem to reason with the classical logic canons. Thus, in a system of N quantum systems, the state space is the 'factored' tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \mathcal{H}_N$ of the Hilbert spaces of each system. The indices $1, \ldots, N$ are labelling both the H-spaces and the particles. But this is not so in the $\mathfrak{Q}$-spaces [**? ?**] built in $\mathfrak{Q}$. There, there are no particle labels that significantly, tag the particles. Maybe this would satisfy Dieks et al.

The RV intends to consider entities of the kind just referred to; it assumes that, whatever their origin, they are better characterised as non-individuals, even if resulting from quantum fields or strings. We reinforce that the RV does not refuse the alternatives that the literature presents us, Dieks' inclusive, but

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proposes a different approach. If Dieks argues that the RV is *wrong*, then there is no other alternative than to say that it is so wrong as the approach he proposes.
6. **Isolation does not entail identity —** In the RV, 'identical particles' (in the physicist's jargon), that is, indistinguishable particles of the same species, can have different properties, say when localised in different traps, or when having different directions of spin. As seen before, Dieks suggests that the RV entails that all such particles have exactly the same properties. We never found where French and Krause (some of the main proponents of the RV) are saying that or making such a hypothesis. The two electrons in a neutral Helium atom are fermions and have opposite values of spin in a given direction, so cannot have all the same properties and values. Their indistinguishability results because they share the same entangled state, a state that cannot be separated into two states, one in each factor of the tensor product, as the Bell state above. If there is some quantum system (a positron, say) in a quantum trap, this should be regarded as an *isolated* entity, not as an individual. The notion of 'mock' (or 'fake') identity is useful here. The non-individuality is linked more with the impossibility of re-identification, but *isolation* is possible, as discussed in the case of Hans Dehmelt's positron 'Priscilla' (see [1**?**]). Just to summarize, Priscilla was a positron trapped in Dehmelt's laboratory and supposedly, she had an identity because of that. But this is not so; recalling David Hume once more, a single object gives us the idea of unity, not of identity [**?** , p.200]. If Priscilla were an individual, she would carry her identity whenever she goes, but this is not what happens; as we know from quantum theory, quantum entities don't have *genidentity*, something acknowledged by Dieks himself. Individuals can have proper names which serve to re-identify them in other contexts; these proper names act as *rigid designators* in Kripke's sense, something not available in quantum physics [9]. 'Priscilla' was the name given to the trapped positron, but she has nothing special concerning any other positron beyond the fact that 'she' is in the laboratory; she is not like Cleopatra for sure, who had an identity.
6. **The bank account analogy —** Dieks goes further. He criticises the theory of quasi-sets by stating that to use such a theory to discuss the bank account analogy would be "an extravagant decision". Not at all! We think that quasi-sets can explain nicely what happens with such an analogy, something that a standard set theory cannot do. Let us explain, even if briefly. The bank account analogy was introduced by Schrödinger [37] and used for instance by Paul Teller [38] to exemplify quantum non-individuality. Suppose I have € 100 in my bank account. Is there a sense in asking for my particular euros, that is, to suppose that there are in the bank some particular euros that are exactly my hundred euros? Not at all, and here enters an important feature of the RV: what import are the *qualities* (of the things, that is, their *kinds*), say 'euros', and their *quantities*, say hundred. No particular euro exists as being mine. The same happens in chemistry and, in a more general situation, with quantum theory. In a water molecule H_2O, what import is that we have *two* Hydrogen atoms and *one* Oxygen atom distributed in a certain way, and not which particular atoms we have. *Kinds and quantities.* It is precisely this that the theory of quasi-sets enables us to consider. Symbolically, we could write in terms of quasi-sets the above molecule as $\langle H, O; 2, 1\rangle$, while my account could be associated with $\langle euros; 100\rangle$. If we have two accounts in two distinct banks, one in New York with \$ 200 and another in Paris with € 100, we can write $\langle dollars, euros; 200, 100\rangle$. No identification, no identity is attributed to particular dollars and euros.

Quasi-set theory enlarges standard set theory (say the ZFA system) by enabling us to consider collections (quasi-sets) with a cardinal, but with no associated ordinal, as explained already. It is a theory that goes in the direction of the problem posed by Yuri Manin when he asked for a 'more general' theory of collections to cope with indistinguishable quantum objects (see [39**?**] and Section 8.4 below).
8. **Permutations —** In a comment on quasi-sets, Dieks claims that "if labels cannot be defined, permutations as ordinarily defined makes no sense". This is correct; a quasi-set with N indistinguishable elements of kind k is *indistinguishable* (and not 'identical') to any quasi-set with N indistinguishable elements of the same kind k. They would be identical iff there were no *two* collections, but just one, according to the definition of extensional identity. The idea can be extended

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to the situation with more kinds of things so that we can say that two sulphur acid molecules are indistinguishable, $H_2SO_4 \equiv H_2SO_4$ in the quasi-sets notation. A permutation can modify things in this case; although Dieks did not comment on this point, we suppose it is relevant and deserves mention. In the acid molecule, of course, there is no sense in permuting Oxygen atoms among them or Hydrogen atoms among them. But we can modify the *format* of some molecules by re-arranging their components and getting different things. This is the case, for instance, when isomers are considered, that is, substances that have the same molecular formulas with the same number of atoms of each element, but the atoms are arranged differently in space. To consider the *form* of something constituted by different kinds of things and their respective quantities, our suggestion is to develop a *quantum mereology*, something not achieved yet (see [40**? ?**] and below). Thus, by specifying how a whole is formed by its parts, maybe we arrive at a way to approach the form of an entity.

4. The Role of the Underlying Logic

Any theory has an underlying logic, even if it is not made explicit. Usually, scientists assume that what is called *classical logic*, or at least a part of it is their basic logic, perhaps because the ways we reason are more conformed to such a logic (of course we should say the opposite: classical logic was created to cope with the standard ways we reason). By classical logic we understand the standard 'classical' first-order predicate logic with or without equality, some subsystem of this calculus such as the classical propositional logic and even those system of *Magna Logica*, encompassing higher-order 'classical' systems and set theories. Categorical logic ([41]) can also be included in this schema. What characterises a logical system as 'classical' is the obedience to some basic principles, such as the following ones: the excluded middle law, the law of non-contradiction, the explosion rule, the principle of identity, the double negation rule, Peirce's law, the 'classical' reductio at absurdum (in distinction to the intuitionistic reduction at absurdum, which by the way holds also in classical logic), some form of Leibniz's Principle of the Identity of Indiscernibles, an axiom of extensionality in set theory, compositionality (the truth value of a complex sentence is a function of the truth values of its component formulas), etc. In particular, classical logic (in whatever form) encompasses a *theory of identity* which we term the Standard Theory of Identity (STI) discussed earlier. Hence it is a theorem of any theory based on classical logic that if $a \neq b$, then there exists F (either a predicate or a set) such that $F(a)$ ($a \in F$ in the case of set theories) and $\neg F(b)$. Thus, a and b can always be discerned *absolutely*, if by this term we understand the existence of a *monadic* predicate obeying the indicated conditions. So, one should pay attention to this, to be emphasised below: even if two particles represented in a mathematical scheme comprising STI cannot be discerned by what philosophers wish to call 'physical properties', they are discerned (absolutely) by 'logical' properties. Since the theorems of the underlying logic are also theorems of the physical theory, we need to conclude that, with STI, even 'identical' particles are discerned absolutely.

The role of the theory's underlying logic is to give an account of the theory's acceptable inferences (deductions in the case of deductive logics), and all theorems of the underlying logic are theorems of the theory itself. So, if the theory's underlying logic is classical logic, for any formulas A and B, the formula $A \to (B \to A)$ is a theorem, but it is not if the logic is some suitable quantum logic [42].[10] So, we may say that the theorems of a theory T can be divided up into two classes: (i) the *logical* theorems, which do not make use of any specific notion of the theory, and (ii) the *specific* theorems, which are typical of the considered theory. So, take T as classical particle mechanics as formulated by McKinsey, Sugar and Suppes in 1953 (see [43]). Kepler's laws are theorems of T and belong to the second group of theorems, but the above $A \to (B \to A)$ is also a theorem of such mechanics since it is supposed

[10] The reason is easy to explain. Suppose that A means 'the spin of the particle was measured in the z-direction and found UP', while B means 'the spin of the particle was measured in the x-direction and found DOWN'. Now, measure the spin in the z-direction again. Will we find A again? Quantum physics says that there are 50% of chances only.

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to be settled on classical set theory (hence encompassing the classical propositional calculus). What respect to quantum mechanics, let us assume a formulation such as that presented in some standard book like [44] but let us put it in a 'logical mode'.

In this sense, a theory T can be viewed as an ordered pair $T = \langle F, M \rangle$ [45, p.60], where F is a 'formalism', which means the mathematical counterpart of T, and M is a class of mathematical structures, the *models* of T. We are not considering here 'models' other than set-theoretical structures, such as toy models or mockups. Furthermore, we assume that at least in principle every scientific theory can be axiomatised by a set-theoretical predicate [43] so explicitly defining its models.

Thus we may assume that F is formulated in the language of a set theory such as the ZFC system (or ZFA), which is enough for most of the physical theories that are important here, enlarged by specific (proper of the theory) concepts, such as 'electron', 'wave-function', and so on. Being mathematical structures, the models of T must be built *in some place*. That is, we need a meta-theory to give rise to models; after all, $T = \langle F, M \rangle$ is a *set* in some set theory. If we are modelling also ZFC (ZFA), what would happen if we assume that the theory's logic is precisely ZFC (ZFA), then the models need to be constructed in a strong theory, such as the KM (Kelley-Morse) system [46,47]. This is due to Gödel's second incompleteness theorem; if consistent, since it is an axiomatic theory,[11] it is subjected to the fact that it cannot prove its own consistency, something achieved by showing a model; see [48].

The choice of the theory's underlying logic, as the choice of the theory's primitive notions and axioms, is a task of the scientist. But once chosen, the axioms, both of the logic and of the theory, become *normative*: they determine what can be accepted as legitimate at least concerning deductions. If something goes wrong, the scientist needs to revise her assumptions. But before this step, the scientist is free to choose the logic and the postulates she wants, and despite there being no specific rules in this direction, in general, there is an underlying metaphysics being assumed, even if unconsciously. For instance, in classical physics one makes assumptions, such as (i) determinism, (ii) individuals, (iii) impenetrability, (iv) the existence of trajectories, (v) locality, (vi) separability. all of this in a certain way subsumed in a 'classical' setting. As N. da Costa said, "logic [and we could extend this to theories in general] is grounded on a metaphysical view of the world." [8, p.77]

5. The Infinitesimals Analogy

It is well documented that Newton's fluxions and fluents and Leibniz's infinitesimals led to contradictions [49,50]. Let us summarise a little using updated language. The idea is to calculate what later was called the derivative of a function $y = f(x)$ (a fluent). In Leibniz's terms, this would require calculating the quotient of the increments given to the independent variable x and that one got by the dependent variable, that is, $\frac{y}{x}$. Take $y = x^2$ as a paradigmatic case. Given an infinitesimal increment dx to x, we get $y + dy = (x + dx)^2$, that is, $y + dy = x^2 + 2xdx + (dx)^2$, so $dy = 2xdx + (dx)^2$. Dividing both terms by dx (which is supposed to be not null since it is an increment), we get $\frac{y}{x} = 2x + dx$. And here is the tricking step: since dx is *arbitrarily small*, we can dispense it and arrive at the derivative, $\frac{y}{x} = 2x$. Of course, there is a contradiction since dx was taken both as not zero and as zero. The right way to provide such a calculation came only later with Cauchy and Weierstrass with the notion of limit.

The practical results of the Calculus were in general correct despite this fact. Still today an engineer may use infinitesimal elements of areas or volumes in her reasoning in the old style, sometimes even by ignoring the right definitions in terms of limits or the existence of Non-Standard Analysis, introduced in 1960 by A. Robinson, who gave a re-birth to the notion of infinitesimals without the old problems (see the above references). Berkeley, who made a serious criticism of the use of infinitesimals, acknowledged that the results of the Calculus are right and did not criticise them, *but its logical deficiencies*. This

[11] A theory is axiomatic if we can effectively know — by a computer program, say — whether a given formula is an axiom; see [16, p.34].

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is important to our case, as we shall see now. Anyhow, it is clear that to neglect the infinitesimals constitutes a logical mistake.

Dieks argues that sometimes physicists consider two quantum entities as described by independent wave functions ψ_1 and ψ_2 when they are sufficiently apart. In this case, as he recalls, *all happens* as if they were two distinct and isolated entities behaving as classical physics says. Let us summarise the argument given in [9] with a simplified example. We shall not use the bra-ket notation here for simplicity. Suppose we have two elementary particles of the same kind located at different points A and B, say the North Pole and the South Pole of Earth. Being x_1 and x_2 their coordinates,[12] let $\psi_A(x_1)$ and $\psi_B(x_2)$ the wave-functions of the particles. Then the joint probability amplitude for finding the first particle at x_1 and the second at x_2 might be thought as being done by the tensorial product $\psi_A(x_1)\psi_B(x_2)$, but it is not! Since the particles are indiscernible, nothing different would get if they are exchanged, that is that the joint probability amplitude would be $\psi_A(x_2)\psi_B(x_1)$, which is *different* from the first product once the tensor product is not commutative. As Dalla Chiara and Toraldo di Francia said, "this would go against the indistinguishability principle".[13]

The acknowledged right vector for describing the joint probability amplitude is

$$\psi_{12} = \frac{1}{\sqrt{2}}(\psi_A(x_1)\psi_B(x_2) \pm \psi_A(x_2)\psi_B(x_1)),$$

where the plus sign holds for bosons and the minus sign holds for fermions. The joint probability density is then given by

$$||\psi_{12}||^2 = \frac{1}{2}\left(||\psi_A(x_1)||^2||\psi_B(x_2)||^2 + ||\psi_A(x_2)||^2||\psi_B(x_1)||^2 \pm 2\mathrm{Re}\left(\langle\psi_A(x_1)\psi_B(x_2)|\psi_A(x_2)\psi_B(x_1)\rangle\right)\right).$$

where the last term $2Re(\ldots)$ is the *interference term*. This term, for the 'practice of physics', can be eliminated, since the overlap of the two wave functions becomes appreciable only when the distance between the particles is not much larger than the de Broglie's wavelength. As Dalla Chiara and Toraldo di Francia emphasize,

> "This is the reason why an engineer, when discussing a drawing, can *temporarily* make an exception to the anonymity principle[14] and say for instance: 'Electron a issued from point S will hit the screen at P while electron b issued from T hits it at Q.' But this mock individuality of the particles has very brief duration. When the electron hits the screen (...) it meets with other electrons with substantial overlapping, and the individuality is lost. In fact the de Broglie wavelength of an electron inside an atom is on the same order of magnitude as the atomic diameter."

This shows that the supposition that the interference term can be neglected is similar to the supposition that infinitesimals can be dispensed with. The results in quantum physics, so as those in the Calculus, are right (as far as we know), but from the logical foundational point of view the logical mistake is evident.

We emphasize here, again, that the Received View is not occupied with the practice of physics, but with its logical foundations. This is why we think that the interference term cannot be dispensed with in such an analysis.

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[12] Notice that the coordinates do not provide identity to the particles, but just say that one is in the North Pole while the other is in the South Pole; the identity of the particles don't matter, mainly if they are of the same kind.

[13] This principle states that for all vectors (states) $|\psi\rangle$, all operators $\hat{A}$, and all particle label permutation operators P, we have $|\hat{A}|\psi = |\hat{A}|P\psi$, that is, the expectation values are the same before and after a permutation.

[14] [According to them, quantum physics is the land of anonymity, where proper names make no sense since they to not play the role of rigid designators, as it would be if the involved entities were individuals.]

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The Alternative View finds another way to express the state vector of a joint system of particles of the same kind using certain symmetric projectors (see [2]), inspired in [36]. We accept this alternative but see it as it is: an alternative, *another way* of writing the relevant mathematical facts. But, in general, there is nothing against those who continue to proceed as in some standard textbook on quantum mechanics.

6. Dieks' Proposal: the Alternative View

Dieks et al. propose an Alternative View (AV) to substitute the RV which, they say, is more in conformity with the practice of physics. In this section, we revise his main claims that conduce to such a view and add some 'comments'.

Dieks accuses the RV of using labels to name the particles (in a join system with N of them). The labels $1, 2, \ldots, N$ serve to name the corresponding Hilbert spaces in the tensor product but also name the particles. He considers anti-symmetrized states (we use Dieks' numeration for the equation below):

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|\psi\rangle_1|\phi\rangle_2 - |\phi\rangle_1|\psi\rangle_2\right) \tag{2.1}$$

He says that such equations (this one and similar) are so that "the symmetry of [the equations] implies that each of those particles is in exact the same state, a mixed state that can be obtained by the procedure of 'partial tracing'.[15] This sameness of physical states entails that switching the labels has no significance: all statistical predictions of quantum mechanics are invariant under permutations of particle labels." [1] (p.15).

This is the Indistinguishability Postulate, a core notion in quantum theory. French and Krause conclude that the labels are 'otiose', and Dieks explores that, by insisting that "all quantum particles of the same sort are in exactly the same physical state and possess exactly the same physical properties and then cannot be distinguished and individuated by any physical process."

Comment —

Fermions cannot share the same quantum states; this is Pauli's Exclusion Principle. For instance, the two electrons of a neutral He atom share the same entangled state, but given a certain direction, one of them has spin UP and the other has spin DOWN. They do not have 'exactly' the same properties; the case is that when they share an entangled state such as (2.1), we cannot specify which is which, that is, to know *which* electron has spin UP, something that will be revealed only after a measurement. The 'which is which' of this phrase means that, as said before, the component systems do not have particular properties while in the entangled state, but only the entire system has properties. So, there is no sense in asking for a particular component. In this sense, we cannot speak of the electrons as isolated individuals; it seems that this is a quantum fact due to the non-factorised assumption of the entangled state. So, neither quantum mechanics nor the RV assumes that *all* the particles (of the universe) of the same kind are in the same quantum state and have the same properties.

Dieks takes for granted that when we follow the path of a particle in a bubble chamber, or when a single particle is trapped in a potential well or even when just one electron is fired and later detected on a screen, we are assigning them identities. We have discussed the case of trapped particles elsewhere (see [**?**]).

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[15] The partial tracing is an operation used when the join system is described by a density operator and enables us to consider the trace (which gives us the expectation values) of some component of the total system even if the whole state is entangled. But it should be remarked that if the systems are of the same kind, it is not relevant which is the particular system we are taking into apart.

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Comment —

Again, the claim depends on what one understands by identity. What does Dieks mean by that? Of course, we suppose anyone will agree, that the particles do not obey STI for in this case they would carry 'permanent identities', which in physical terms might be thought of as *genidentity*, or diachronic identity. The above assumption can be made *provisorily* by the physicists, but one should agree that it is an extrapolation of what quantum mechanics says. The trapped positron, as said before, was *isolated* but the isolation does not confer it an identity due to the lack of re-identification.

In proposing the Alternative View, Dieks says that the particles are not defined by reference to their labels but by observable physical properties which give them their states. He recalls Schrödinger in that it is the state that confers a particle its *momentaneous* (our add) individuality. So, according to him, states such as (2.1) represent two individual particles possessing well-defined individual properties, one characterised by the state $|\phi\rangle$ and the other by the orthogonal state $|\psi\rangle$. He reports to Ghirardi et al. [36] in which they have analysed such a situation where the particles behave *in many respects* like product states representing individual particles (our emphasis). This is the core of the AV: to be closer to the practice of physics; really, Ghirardi et al. say explicitly that they are considering what happens "in certain situations." A nice example helps in clarifying the issue. Dieks supposes two particles, one at the left L and another at the right R of a certain apparatus whose details do not import here. The entangled state is similar to (2.1) with $|L\rangle$ for $|\psi\rangle$ and $|R\rangle$ for $|\phi\rangle$. Then he states that "the states $|L\rangle$ and $|R\rangle$ do the job of identifying the particles in this alternative approach. (...) The particles are clearly distinguishable."

Comment —

Being in $|L\rangle$ or being in $|R\rangle$ do not provide *identity* to the particles. If we close our eyes for a moment and an evil genius appears and says that he *probably* has permuted the particles, how could we know whether he is telling the truth? Any measurement will do the same result independently of *which* particle is in the left (right), something that is typical from quantum physics. This is not compatible with the identity ascribed by STI; if someone says that in a football team he has exchanged Lionel Messi by Cristiano Ronaldo, we all will notice the difference. But, in the case of quantum entities, we will never be able to know if the genius is telling the truth.

Dieks suggests that the attribution of identities to the particles (without specifying the identity he is considering) "is in accordance with how states of this kind *are interpreted in the physical practice*." (our emphasis). "By the contrary, he continues, according to the RV the state (2.1) represents two particles with exactly the same location, 'smeared out' overly over L and R."

Comment —

Not at all! The RV never said or assumed that! As said already, the RV assumes that there is *one* particle at L and *one* at R, but quantum physics cannot give them identities in the sense of STI. In our opinion, the practice of physics requires precisely this: one at left and one at right, but their identities do not matter (being of the same kind): *kinds and quantities*, not individualities. This is well exemplified by Dieks himself, but we also refer to the above mentioning of Dalla Chiara and Toraldo di Francia.

7. Conclusions

The RV and the AV are not conflicting interpretations. Instead, we regard them as complementary. Once one assumes that quantum entities can be viewed as non-individuals, failing to obey STI, then an underlying mathematics such as that provided by quasi-set theory is in order. But to account with the practice of physics, *one can reason as if* the particles can be localised and have (momentary) identities so that *all happens* as if they were individuals. But we insist: on neglecting the interference term, as mentioned above, we are committing the same error as in the early calculus when they have neglected the infinitesimal increments.

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Dieks et al. analyses are of course good, and their points are clever. But we insist that the RV is not proposed to cope with the practice of physics but with its logical foundations. If the AV is also proposed as a view on the fundamental ontology of quantum theory, then we need to agree that it postulates that quantum entities are individuals endowed with 'classical' identity, hence re-identifiable in different contexts. Can we accept that if not for practical purposes only?

Thus, as the quantum logicians have agreed, if one wishes to preserve the non-distributivity law needs to go outset a Boolean structure, we argue that if someone wishes to consider 'legitimate' (and not 'fake') indistinguishable things she needs to go out of a mathematical framework that encompasses STI. Quasi-set theory is of course an alternative.

8. Appendix

In this Appendix, we recall some other questions posed by philosophers of physics who seem to be against the RV, some of them endorsed by Dieks himself. The comments are just to enlighten the main points, which are developed in other works mentioned below.

8.1. The Relevance of the Notion of Identity

We have seen that contrary to what Otávio Bueno, Francisco Berto and others say, something endorsed by Dieks et al., the notion of identity is not 'essential' for the meaning of the concept of an entity.[16] By an entity, we understand everything that can be referred to by a suitable language either by a proper name or by some description. An electron is an entity, and so are all quantum 'particles'.

We really can suppose the existence of *entitites* to which (at least) the standard notion of identity (given by STI) does not hold. There is no logical contradiction in supposing that, except if the theory's logic says the opposite. However, since in the general discussions in the philosophy of physics, the logic being used is rarely made explicit, we are free to suppose that there are also non-individuals in our metaphysical pantheon.

Quantum entities seem to provide a paradigmatic example of non-individuals (entities that fail to obey STI). From a logical point of view, this was also assumed by F. P. Ramsey when he criticised Whitehead and Russell's definition of identity (Leibniz Law) saying that it is logically possible the existence of more than one thing sharing all their properties [51] (p.31). There is no logical contradiction here once we leave STI.

Perhaps what is involved in an atavistic criterion, is the belief that *we need* identity to construct our theories, so to say that identity can be dispensed with would be a fallacy. Here we should distinguish between identity in the object language, that is, comprising the entities the theory is compromised with, and identity in the meta-level, in the meta-language we use to elaborate our theory. In this second level, we can accept that we share an intuitive idea of identity, except if also the meta-theory is formalised, which apparently would require an intuitive identity in the meta-meta level. But, at the level of the language of the theory, identity can be dispensed with. To understand this point, an analogy seems to be interesting. Think of the construction of paraconsistent or intuitionistic systems, which reject respectively the full validity of the principles of Non-Contradiction and the Excluded Middle. But, in the metalanguage, we assume the validity of these principles; surely no one would say that a sequence of primitive symbols is a formula and that it is not a formula in the same context, nor would negate that some sequence is a formula or that it is not. The same happens with identity; we usually *assume* an intuitive identity to start with for distinguishing symbols and other basic things in writing, but later we can dispense the very logical notion of identity for the objects the theory is dealing with in favour of a theory encompassing non-individuals. So, the sense according to which identity would be indispensable should not cover the object theory.

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16 A response to Bueno is presented in [?]. Again recurring to the example of a BEC, would the *entities* that form a BEC (atoms, say) not be 'entities' of some kind? If you accept this claim, how can you say that they do have identities?

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Summing up, we regard identity as a useful concept (perhaps even a necessary one) to conceptualise *individuals*, but of course not for *any* kind of entity.

8.2. Emulating non-individuals

STI is incompatible with (completely) indistinguishable but not identical things. This does not entail that we cannot *mimic* them within a theory encompassing STI, as the ZFC system. This is the case of taking deformable or non-rigid structures. A set-theoretical structure $\mathfrak{A}$ is *rigid* if its only automorphism is the identity function, the trivial automorphism, otherwise, it is deformable. For instance, the additive group of the integers, $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$ is deformable, since the application $h(x) = -x$ is an automorphism, as it is easy to prove. So, *within* $\mathcal{Z}$, the integers 2 and -2 are indistinguishable. Thus, in a deformable structure, we can make things happen *as if* some individuals were indistinguishable by the canons of the structure, as it is supposed to happen when we opt by eliminating surplus structures in favour of just symmetric and anti-symmetric ones (see [**?** , Chap.4]). With this move, we can work inside a 'standard' logical framework as usually done; by the way, all standard books of quantum physics are developed with standard mathematics by 'mimicking' indistinguishability by the use of symmetry postulates.[17]

But every structure in ZFC (and in similar theories) can be extended to a rigid structure. So, even if two elements are indiscernible inside a structure, in the extended one they can be realised to be individuals. Furthermore, the whole universe of sets is rigid [53, p.66]. This entails that once we 'leave out' the considered structure, we can realise that the considered things do have an identity. That is, *everything* represented in a theory like ZFC is an individual and we do not need any argument other than logical ones to state that.

8.3. Identity for m-atoms

Can we define identity for m-atoms in the theory $\mathfrak{Q}$? Of course, we can. So, why not do that? The answer is similar to that one a paraconsistent logician might give when asked why she does not accept the universal validity of the Principle of Non-Contradiction or then the answer given by an intuitionistic logician when asked why she does not accept the universal validity of the Principle of the Excluded Middle: the reason is that we don't wish to do it! We are presenting a logical system to support a metaphysical view comprising non-individuals, understood as entities that fail to obey STI. So, no relation that turns out to be equivalent to the identity of STI is to be allowed, as those presented below (without the proofs), and the reasons were put before.

Let us suggest some ways to define an identity among m-atoms so that the theory $\mathfrak{Q}$ would turn equivalent to ZFA. Let x and y be m-atoms. Then we pose

Definition 1. $x =_a y := \forall z(x \in z \leftrightarrow y \in z)$

Definition 2. $x =_b y := x \in [[y]] \wedge y \in [[x]]$,

where $[[w]]$ is a strong singleton of w, let us recall, a qset whose quasi-cardinal is one and whose only element is indistinguishable from w.[18] This definition can be put another way as follows

Definition 3. $x =_c y := [[x]] =_E [[y]]$,

[17] We remember that Yuri Manin says that quantum mechanics (and quantum physics in general) has no 'proper' language, making use of a fragment of the standard functional calculus [52, p.84], of course with the introduction of criteria that enable us to mimic indiscernible things.

[18] But recall once more that we cannot assert that such an element *is* w for to say that we need identity: the element of the qset is just 'identical' to w.

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being $=_E$ the extensional identity introduced earlier. All these alternatives conduce to the identity of STI as it seems to be immediate but, as said before, we do not intend to introduce either of these (or other) identities for m-atoms which turn out to be 'classical identity' (given by STI).

8.4. Quantity by Not Ordering

Francisco Berto for instance [4], claims something accepted by most of the philosophers we are considering, namely, that we cannot have a collection of entities with a definite cardinality if these entities do not possess self-identity that makes them different from each other; Dieks and other philosophers agree [54,55]. We remark that the reasoning of these people applies to 'classical' things, embedded in a standard set theory encompassing STI.[19] Here we sketch a way to do it with a qset of N indiscernible things devoid of self-identity (N being a *quasi-cardinal* ('q-cardinal' for short); the infinite case will be not touched here).

What we shall do is enlarge the theory $\mathfrak{Q}$ with additional axioms that give us the q-cardinals. The step theory of q-cardinals mimics Peano's Arithmetics in its language and axioms, but q-cardinals would not be confounded with natural numbers. Of course, we have a copy of the standard model of Peano Arithmetics (PA) in $\mathfrak{Q}$, but the q-cardinals we shall consider came not from the PA in $\mathfrak{Q}$, or from any model built in $\mathfrak{Q}$; they come from the axioms given below we *add* to $\mathfrak{Q}$ as a step-theory, in the same sense that, say, the theory of complex numbers are added to the formalism of quantum mechanics or thensor calculus is added to Einstein's general relativity. If we take q-cardinals from any model of PA in $\mathfrak{Q}$, they would be ordinals, something we wish to avoid. Let us call $\mathfrak{Q}'$ this new theory, whose postulates are those of $\mathfrak{Q}$ plus those below, say with a signature $\langle \overline{0}, S, \oplus, \otimes, = \rangle$. Hence q-cardinals are not considered as ordinals such as $\emptyset$, $\{\emptyset\}$ $\{\emptyset, \{\emptyset\}\}$ etc. (in von Neumann's sense), but just $\overline{0}$, $S\overline{0}$, $SS\overline{0}$ and so on, where S stands for the successor function and $\overline{0}$ is the q-cardinal 'zero', '$\oplus$' is the addition and '$\otimes$' the multiplication (given recursively). Notice that identity holds for q-cardinals using the primitive relation '='. Let us define the q-cardinals $\overline{1}, \overline{2}, \ldots$ in an obvious way: $\overline{1} := S\overline{0}$, $\overline{2} := SS\overline{0}$; we use $m, n, p, q, \ldots$ as variables ranging over the q-cardinals $\overline{0}$, $\overline{1}$, $\overline{2}$, etc. and denote the successor of the q-cardinal n by $n \oplus \overline{1}$. We recall once more that despite the above axioms being exactly those of Peano's Arithmetics, the q-cardinals are not qsets, in particular, they are not ordinals.

The postulates are the following (see also [?]):

(1) $\forall m(m = m)$
(2) $\forall m \forall n(m = n \rightarrow (\alpha(m) \rightarrow \alpha(n))$, where the usual restrictions are assumed.
(3) $\forall m \forall n \forall p(m = n \rightarrow (m = p \rightarrow n = p))$
(4) $\forall m \forall n(m = n \rightarrow Sm = Sm)$
(5) $\forall m(\overline{0} \neq Sm)$
(6) $\forall m \forall n(Sm = Sn \rightarrow m = n)$
(7) $\forall m(\overline{0} \oplus m = m)$
(8) $\forall m \forall n(m \oplus Sn = Sm \oplus n))$
(9) $\forall m(\overline{0} \otimes m = \overline{0})$
(10) $\forall m \forall n(m \otimes Sn = (m \otimes n) \oplus m)$
(11) If P is a property for q-cardinals, then

$$P(\overline{0}) \wedge \forall m(P(m) \rightarrow P(Sm)) \rightarrow \forall m P(m).$$

(12) $\forall m(m^{\overline{0}} = \overline{1})$[20]
(13) $\forall m \forall n(m^{Sn} = m^n \otimes m)$

19 In [56], Bueno suggests giving up any set theory to treat non-individuals; the idea looks interesting, but he speaks in the general the idea still deserves consideration, for instance, which would be the mathematical framework to do that.

20 Notice that this axiom is postulating that $\overline{0}^{\overline{0}} = \overline{1}$.

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Now we can assume the existence of a binary predicare symbol K and write $K(x, n)$ to mean that the qset x has q-cardinal $\overline{n}$.[21] The postulates of this new notion are the following, where the variables $x, y, \ldots$ range over qsets:

1. $\forall x(K(x, \overline{0}) \leftrightarrow \neg\exists y(y \in x)))$
2. $\forall x \forall y \forall n(K(x, n) \wedge K(y, \overline{1}) \wedge \neg\exists z(z \in x \cap y) \rightarrow K(x \cup y, n \oplus \overline{1}))$
3. $\forall x \forall n(K(x, n) \rightarrow K(\mathcal{P}(x), \overline{2}^n))$
4. $\forall x(K(x, n) \rightarrow \forall m(m < n \rightarrow \exists y(y \in \mathcal{P}(x) \wedge K(y, m)))$, where $m < n := \exists p(m \oplus p = n)$.

The way we attribute a q-cardinal n to a qset x is not a logical problem, being left to the physical theory. For instance, chemistry has a way of attributing q-cardinals to the atom's energy levels (seen as qsets) [58, Chap.10]; physicists have ways of at least estimating the approximate number of atoms in a BEC, and so on.[22] The important thing here is that we can assume that a certain qset has a finite number of elements.

So, $\mathfrak{Q}'$ shows that we really can consider collections of completely indiscernible things with a q-cardinal. For the sake of making things clear, consider the third axiom and let $K(x, \overline{4})$. Then $K(\mathcal{P}(x), \overline{16})$. So, *we can reason as if* there is one subqset with no element, four subqsets with one element each, six subqsets with two elements, also four with three elements and one with the four elements, although we cannot discern among the subqsets with the same q-cardinality (by WEA).

Notice that in $\mathfrak{Q}'$, so as in $\mathfrak{Q}$, the fact that we cannot discern either the elements of some qsets or the qsets themselves, does not make them identical as STI would require.

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9. On 'Physical' Properties

According to Muller and Saunders [?], the properties and relations that serve to discern among quantum entities should be *physical*, and one should not appeal to the theory's mathematical formalism for that. In order to discuss this point, let us distinguish between the *logical part* of a theory T, let us call it 'L', and its *physical part*, let us call it 'P'. If we add to this schema the possible *interpretations*, coped in the theory's *models*, encapsulated in a class M of structures [43], we can state that T is a triple

$$T = \langle L, P, M \rangle. \tag{2}$$

This is what Caulton calls 'the folklore'. Usually, physical theories are axiomatized (at least ideally) within a set theory, so it would be normal to assume that we have a membership predicate '$\in$' as available. This entails that L must be what is called *Magna Logica*, encompassing set theory. We assume that our theory T can be axiomatised by a set-theoretical predicate in the sense of Suppes [43]. Thus, the vocabulary of the language of T can be divided up into its *logical part* and its *specific part*. The first involves the logical connectives, quantifiers, auxiliary symbols, equality and membership (we will avoid admitting atoms or ur-elements by now). The specific vocabulary has all the symbols to cope with the 'physical' notions. Notice that this is not so simple. Usually, the additional symbols of a theory are defined by *nomimal definitions* in the metalanguage [?], that is, a new concept is defined as

[21] See also the alternative approach proposed by E. Wajch in [57].
[22] For instance, the online journal Review of Scientific Instruments 77, 023106 (2006); doi: 10.1063/1.2163977 reported the creation of a BEC with circa 20×10^6 atoms; https://shorturl.at/kPW35

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an abbreviation of some (object-) language's expression, as in the case of the concept of subset, which reads $A \subseteq B := \forall x(x \in A \to x \in B)$, being ':=' the metalinguistic symbol for 'equal by definition'. But the mentioned authors apparently wish to introduce the new 'physical' concepts by symbols *in the language*, so enlarging the language of set theory with new symbols which are not in the metalanguage, but in the language itself. In this case, it is known that the definitions of the new symbols must obey Leśniewski's conditions of eliminability and non-creativity [?] [Chap.8], something never made clear by the mentioned authors. By the way, they also do not explain what they mean by a 'physical property' except by suggesting the quite vague notion that they are *something that can be measured* (see below).

So, what are 'physical properties'? The apparent only answer is that they are linked among the *specific* predicates of the language, that is, among its *non-logical symbols*. Really, here is Adam Caulton about 'mathematical versus physical discernment':

> Of course, it is crucial that the properties and relations used to discern the particles be physical; we cannot appeal to elements of the theory's mathematical formalism that have no representational function. Thus, for example, we cannot discern two particles in an assembly merely by appealing to the fact that the Hilbert space for that assembly is a tensor product of two copies of a factor Hilbert space. For all we know, this representative structure may be redundant; there may in fact only be one particle. So we must instead appeal to quantities in the formalism which genuinely represent physical quantities. Like Muller, Saunders, and Seevinck [a third author that has contributed to the issue], I call this sort of legitimate discernment 'physical discernment'. I call instances of spurious discernment 'mathematical discernment' – Muller and Saunders instead use the phrase 'lexicon discernment', but it is important to distinguish between mathematical objects (like Hilbert spaces) and mathematical language. Thus, I restrict (HB) above [this is the Hilbert-Bernays' definition of identity] to contain only physical predicates; mathematical predicate (such as set membership '∈') are not to be included." [?]

The 'HB' definition states that $a = b$ iff they share all the predicates of the language; if we have one unary predicate P and one binary predicate Q, then

$$a = b := P(a) \leftrightarrow P(b)) \wedge \forall x((Q(x,a) \leftrightarrow Q(x,b)) \wedge (Q(a,x) \leftrightarrow Q(b,x))). \tag{3}$$

We emphasise from the outset that this is not a definition of the identity of a and b in the sense that they are *the same object*, but it is just *indiscernibility* relative to the language's predicates. Quine himself, who used this definition [14] [p.] acknowledges that (see for instance [?] [p.55], when he says that a and b could be discerned by a predicate not belonging to the primitive language but by a 'background theory').[23]

Let us consider this point in a purely logical way. We have a language (our object language) which contains a stock of primitive predicate symbols. According to the above ideas, these predicates would be discerned between 'physical' and 'not physical'. How to do such a distinction? There are some alternatives, let us see one: we can use a two-sorted language and denote by $P_1, P_2, \ldots$ the physical predicates and by $Q_1, Q_2, \ldots$ the others. To state the distinction, we need to present postulates for characterizing what would be a physical predicate. None of the mentioned authors has presented them. So, we can go to semantic considerations, by assuming that the physical predicates (let us take just monadic ones) are interpreted distinctly from the remaining ones. But what would be such a semantics? According to Scott and Suppes [?], a physical property is a property that can be measurable, but here enters another difficult concept: measurement. Suppes and collaborators have developed

[23] This is similar to go to an extended structure in the sense to be explained below in the Conclusions.

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a huge 'theory of measurement', but for sure it is not what the philosophers we are referring to are considering. I suppose that they take the notion from an informal point of view, more or less in a semantic account. So, we need to *interpret* our above language. Let us try.

If the language is of first order (something difficult to consider when physical theories are being considered — see []),[24] we can proceed as follows, by standardly assuming an interpretation of the language. We chose some group of relations (for unary predicates, just subsets of the domain) over the domain to interpret each group of predicates, but the choice is arbitrary, not fixed by logic. Notwithstanding, it may be done by 'physical' means, yet a lot of considerations should be made here. For instance, suppose that two objects share all their physical properties. Hence they must belong to the intersection of all subsets of the domain which stand for the physical properties they share. Are they indiscernible? Not at all. As two elements of a set (the domain), they are distinct, hence distinguished by what Quine called a *background theory*; in our case, just two disjoined open sets centred in the points (usually, the topology is Hausdorff).

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[24] We shall not enter in the discussion whether set theory, formulated in a first-order language, remains within the scope of first-order theories since its capacity of expression is greater than that.

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