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HyperInterval-Valued and SuoerHyperInterval-Valued Fuzzy/Neutrosophic Set

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Abstract

We study uncertainty models built from interval families over a finite universe. An interval set collects all subsets bounded between a designated lower and upper set. A HyperInterval set assigns to each base interval a nonempty family of admissible refinements, while a SuperHyperInterval set of order n maps elements of the n-fold iterated powerset to $(n-1)$-nested families, enabling hierarchical evidence organization. On the numeric side, an interval-valued fuzzy set attaches to each element an interval of admissible memberships, and an interval-valued neutrosophic set assigns independent intervals for truth, indeterminacy, and falsity. Building on these primitives, we introduce HyperInterval- and SuperHyperInterval-valued fuzzy/neutrosophic sets, define conjunctive "core" (intersection) and disjunctive "hull" semantics, and prove embedding theorems showing that classical interval, fuzzy, and neutrosophic models appear as singleton or degenerate cases. Realistic examples from commute planning, delivery scheduling, and clinical assessment illustrate the methodology. The framework unifies multi-source and hierarchical evidence, offering transparent bounds for conservative and exploratory decision policies.

Keywords: interval set; HyperInterval set; SuperHyperInterval set; interval-valued fuzzy set; interval-valued neutrosophic set

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1. Preliminaries

We collect the basic terminology and notation used in what follows. The definitions in this paper are assumed to be finite.

1.1. *Interval Set, HyperInterval Set, and SuperHyperInterval Set*

An interval set collects subsets A with lower bound A_ℓ and upper bound A_u, requiring $A_\ell \subseteq A \subseteq A_u$ [1–5]. A HyperInterval set assigns each base interval $[A_\ell, A_u]$ a nonempty family of admissible interval sets, satisfying $A_\ell \subseteq B_\ell \subseteq B_u \subseteq A_u$ [6]. A SuperHyperInterval set of order n maps elements of $\mathcal{P}^n(U)$ to $(n-1)$-nested families of interval sets, supporting hierarchical interval evidence [6].

Definition 1 (Universe)**.** *Let U be a nonempty finite set, called the* universe *or* base set. *All subsequent powerset constructions are formed relative to U.*

Definition 2 (Powerset [7])**.** *The* powerset *of a set S, denoted $\mathcal{P}(S)$, is the family of all subsets of S, including both the empty set and S itself:*

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 3 (n-th Powerset [8–11])**.** *For a nonempty set H and integer $n \geq 1$, the n-th* powerset *is defined recursively by*

$$\mathcal{P}_1(H) := \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) := \mathcal{P}(\mathcal{P}_n(H)).$$

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Analogously, the n-th nonempty powerset, *denoted $\mathcal{P}^*_n(H)$, is constructed by*

$$\mathcal{P}^*_1(H) := \mathcal{P}^*(H), \qquad \mathcal{P}^*_{n+1}(H) := \mathcal{P}^*(\mathcal{P}^*_n(H)),$$

where $\mathcal{P}^*(H) := \mathcal{P}(H) \setminus \{\varnothing\}$.

Example 1 (n-th Powerset for Menu Planning Across Horizons)**.** *Let the base set ("atomic choices") be $H = \{Salad,\ Soup,\ Pasta\}$.* First powerset *$\mathcal{P}_1(H) = \mathcal{P}(H)$ lists all meal options as sets of dishes (e.g., $\varnothing$, $\{Salad\}$, $\{Soup, Pasta\}$, $\{Salad, Soup, Pasta\}$). A* Second powerset *element is a collection of such options; interpret each as a* daily plan with admissible alternatives. *For instance,*

$$D = \{\ \{Salad\},\ \{Soup, Pasta\}\ \} \ \in\ \mathcal{P}_2(H)$$

means "one acceptable meal is just Salad, another acceptable choice is Soup together with Pasta." A Third powerset *element bundles multiple daily plans into a* weekly (or multi-day) template*; for example*

$$W = \{\ D,\ \{\ \{Soup\},\ \{Pasta\}\ \}\ \} \ \in\ \mathcal{P}_3(H)$$

encodes a set of admissible daily-plan choices (e.g., day one uses D, day two allows either only Soup or only Pasta).

Thus, moving from $\mathcal{P}_1(H)$ to $\mathcal{P}_2(H)$ to $\mathcal{P}_3(H)$ adds planning layers: from sets of dishes *(meal options), to* sets of meal options *(daily plans with alternatives), to* sets of daily plans *(weekly templates). In everyday terms, the n-th powerset models a hierarchy of "choices of choices" across n decision horizons (dish $\to$ day $\to$ week $\to \ldots$).*

Definition 4 (Interval set)**.** *(cf. [1]) Let U be a nonempty universe and let $\mathcal{P}(U)$ be its powerset, ordered by $\subseteq$. For any two subsets $A_\ell, A_u \subseteq U$ with $A_\ell \subseteq A_u$, the* interval set *determined by (A_ℓ, A_u) is*

$$[A_\ell, A_u] \ :=\ \{\, A \subseteq U \mid A_\ell \subseteq A \subseteq A_u \,\}.$$

We call A_ℓ the lower bound *and A_u the* upper bound*. The class of all (closed) interval sets on U is*

$$\mathsf{I}(\mathcal{P}(U)) \ :=\ \{\, [A_\ell, A_u] \ :\ A_\ell, A_u \subseteq U,\ A_\ell \subseteq A_u \,\}.$$

Remark 1. *Each $[A_\ell, A_u]$ is a complete sublattice of $(\mathcal{P}(U), \subseteq)$ with bottom A_ℓ and top A_u. Degenerate cases: $[A, A] = \{A\}$ (a crisp set) and $[\varnothing, U] = \mathcal{P}(U)$ (the largest interval set).*

Example 2 (Interval set for a Shopping List Envelope)**.** *Let the universe of items be*

$$U = \{milk, bread, eggs, apples, cheese\}$$

. Suppose the must–buy *core is $A_\ell = \{milk, eggs\}$ and the* may–buy *superset is $A_u = \{milk, bread, eggs, apples\}$ (cheese is excluded today). The interval set*

$$[A_\ell, A_u] = \{\, A \subseteq U \mid A_\ell \subseteq A \subseteq A_u \,\}$$

collects all acceptable shopping lists between these bounds. Since $|A_u \setminus A_\ell| = 2$ (bread, apples), the members are exactly the $2^2 = 4$ sets

$$\{milk, eggs\}, \quad \{milk, eggs, bread\}, \quad \{milk, eggs, apples\}, \quad \{milk, eggs, bread, apples\}.$$

For instance, $\{milk, eggs, bread\} \in [A_\ell, A_u]$ while $\{milk, eggs, cheese\} \notin [A_\ell, A_u]$ (cheese $\notin A_u$).

Definition 5 (HyperInterval set)**.** *Let* $\mathsf{I}(\mathcal{P}(U))$ *be as in Definition 4. A* HyperInterval set *on* U *is a map*

$$\mathrm{HI} \;:\; \mathsf{I}(\mathcal{P}(U)) \longrightarrow \mathcal{P}^{*}(\mathsf{I}(\mathcal{P}(U))),$$

assigning to each base interval set $I = [A_\ell, A_u]$ *a nonempty family* $\mathrm{HI}(I) \subseteq \mathsf{I}(\mathcal{P}(U))$ *of admissible interval sets.* (Refinement semantics, optional.) *If, moreover, every* $[B_\ell, B_u] \in \mathrm{HI}([A_\ell, A_u])$ *satisfies* $A_\ell \subseteq B_\ell \subseteq B_u \subseteq A_u$*, we call* HI *a* refinement HyperInterval set*.*

Remark 2. *Via pointwise union and intersection,* $(\mathrm{HI}_1 \sqcup \mathrm{HI}_2)(I) := \mathrm{HI}_1(I) \cup \mathrm{HI}_2(I)$ *and* $(\mathrm{HI}_1 \sqcap \mathrm{HI}_2)(I) := \mathrm{HI}_1(I) \cap \mathrm{HI}_2(I)$*, the set of all HyperInterval sets on U inherits a natural (hyper)structure.*

Example 3 (HyperInterval set refining an Envelope by Store Policies)**.** *Let* $I_0 = [A_\ell, A_u]$ *be as in Example 2. Define a HyperInterval set* HI *by assigning to* I_0 *two* refinements *coming from different stores' policies:*

$$\mathrm{HI}(I_0) = \Big\{ I_1 = [\{milk, eggs\},\ \{milk, eggs, bread, apples\}],\ \ I_2 = [\{milk, eggs, bread\},\ \{milk, eggs, bread, apples\}] \Big\}.$$

Both satisfy $A_\ell \subseteq B_\ell \subseteq B_u \subseteq A_u$*. The* conjunctive core *(intersection in the lattice* $\mathcal{P}(U)$*) is*

$$\bigcap_{k=1}^{2} I_k$$

$$= \Big[\underbrace{\{milk, eggs\} \cup \{milk, eggs, bread\}}_{=\{milk, eggs, bread\}},\ \underbrace{\{milk, eggs, bread, apples\} \cap \{milk, eggs, bread, apples\}}_{=\{milk, eggs, bread, apples\}} \Big]$$

$$= [\{milk, eggs, bread\},\ \{milk, eggs, bread, apples\}],$$

which is feasible since the lower bound is contained in the upper bound. The disjunctive hull *(least interval containing* $I_1 \cup I_2$*) equals*

$$\Big[\{milk, eggs\} \cap \{milk, eggs, bread\},\ \{milk, eggs, bread, apples\} \cup \{milk, eggs, bread, apples\} \Big] = [A_\ell, A_u] = I_0.$$

Thus the hyper-interval captures store–specific refinements without changing the global hull.

Definition 6 (SuperHyperInterval set of order n)**.** *For* $n \geq 0$ *define the iterated powersets by* $\mathcal{P}^0(U) = U$ *and* $\mathcal{P}^{n+1}(U) = \mathcal{P}(\mathcal{P}^n(U))$*. Also write* $\mathcal{P}^r(\mathsf{I}(\mathcal{P}(U)))$ *for iterated powersets of the interval–set universe. A* SuperHyperInterval set of order $n \geq 1$ *on U is a map*

$$\mathrm{SHI}^{(n)} \;:\; \mathcal{P}^n(U) \longrightarrow \mathcal{P}^{\,n-1}\big(\mathsf{I}(\mathcal{P}(U))\big).$$

Thus, to each n–nested subset $A \in \mathcal{P}^n(U)$ *the map assigns an* $(n-1)$*–nested family of interval sets. (When a refinement discipline is desired, one can require that the interval bounds appearing at the leaves are chosen compatibly with the subsets occurring in A.)*

Example 4 (SuperHyperInterval set (order n=2) for Two Scenarios of a Dinner Plan)**.** *Keep*

$$U = \{milk, bread, eggs, apples, cheese\}$$

. Consider two scenario sets (elements of $\mathcal{P}(U)$*):*

$$S_{\text{family}} = \{milk, eggs, bread\},$$

$$S_{\text{guests}} = \{bread, cheese\}.$$

Form the nested input $A = \{S_{\text{family}}, S_{\text{guests}}\} \in \mathcal{P}^2(U)$*. Define the SuperHyperInterval map of order* 2 *by*

$$\mathrm{SHI}^{(2)}(A) \;=\; \Big\{ I_{\text{family}} = [\{milk, eggs\},\ \{milk, eggs, bread\}],$$

$$I_{\text{guests}} = [\{bread\},\ \{milk, eggs, bread\}] \Big\} \;\in\; \mathcal{P}(\mathrm{I}(\mathcal{P}(U))).$$

Here the family scenario insists on milk and eggs; the guest scenario insists on bread but allows the same upper bound to keep cooking logistics unified.

If one demands a list acceptable to both *scenarios, the core is*

$$I_{\text{family}} \cap I_{\text{guests}} = \Big[\{milk, eggs\} \cup \{bread\},\ \{milk, eggs, bread\} \cap \{milk, eggs, bread\} \Big]$$

$$= [\{milk, eggs, bread\},\ \{milk, eggs, bread\}],$$

a crisp recommendation $\{milk, eggs, bread\}$*. If instead one allows either scenario, the hull is*

$$\Big[\{milk, eggs\} \cap \{bread\},\ \{milk, eggs, bread\} \cup \{milk, eggs, bread\} \Big]$$

$$= [\varnothing,\ \{milk, eggs, bread\}],$$

*recording every intermediate list up to the common upper bound. This illustrates how order-*2 *nesting organizes* scenario families *before interval selection.*

1.2. Interval-Valued Fuzzy Set

A fuzzy set assigns each element a membership degree between zero and one, modeling belonging and vagueness beyond crisp classification [12,13]. An interval-valued fuzzy set assigns to each element u an interval $[\alpha, \beta] \subseteq [0, 1]$ of admissible membership degrees, modeling imprecision about exact values [14–19]. As related concepts, interval-valued intuitionistic fuzzy sets [20–22], interval-valued picture fuzzy sets [23,24], and interval-valued hesitant fuzzy sets [25–27] have also been studied in the literature.

Definition 7 (Interval-valued fuzzy set)**.** *[14–16] Let* $U \neq \varnothing$ *be a universe. Write*

$$L([0, 1]) \;=\; \{\, [\alpha, \beta] \mid 0 \le \alpha \le \beta \le 1 \,\},$$

the set of all closed *subintervals of* $[0, 1]$*. An* interval-valued fuzzy set (IVFS) *on* U *is a mapping*

$$A :\ U \longrightarrow L([0, 1]),$$

so that each $u \in U$ *is assigned an interval* $A(u) = [\underline{A}(u), \overline{A}(u)] \in L([0, 1])$ *of admissible membership degrees. We denote the class of all IVFSs on* U *by* $\mathrm{IVFS}(U)$*.*

Example 5 (IVFS for Fruit Ripeness in a Grocery Store)**.** *Let* $U = \{A_1, A_2, A_3\}$ *be three avocados on display. Consider the interval-valued fuzzy set* $A : U \to L([0, 1])$ *where* $A(u)$ *is the degree to which* u *is "ripe enough to eat tonight." Sensor readings (color, firmness) and staff judgment are summarized as*

$$A(A_1) = [0.65, 0.82], \qquad A(A_2) = [0.30, 0.50], \qquad A(A_3) = [0.75, 0.90].$$

A conservative customer requires membership at least $\alpha = 0.70$ *under the* necessary *view (use lower bounds), yielding the* α*-cut*

$$\mathrm{N}_{0.70} = \{\, u \in U \mid \underline{A}(u) \ge 0.70 \,\} = \{A_3\}.$$

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An optimistic customer accepts if it is possible *to meet* $\alpha = 0.70$ *(use upper bounds), giving*

$$P_{0.70} = \{\, u \in U \mid \overline{A}(u) \geq 0.70 \,\} = \{A_1, A_3\}.$$

(Optionally) the IVFS complement $A^{\complement}(u) = [\,1 - \overline{A}(u),\ 1 - \underline{A}(u)\,]$ *quantifies "not ripe tonight": for* A_1, $A^{\complement}(A_1) = [0.18, 0.35]$.

1.3. Interval-Valued Neutrosophic Set

A neutrosophic set assigns independent truth, indeterminacy, and falsity degrees to each element, capturing inconsistency and uncertainty beyond fuzzy membership [28–30]. An interval-valued neutrosophic set assigns each element intervals for truth, indeterminacy, and falsity, allowing independent bounded ranges for all three [31–35].

Definition 8 (Interval-valued neutrosophic set (IVNS))**.** *Let U be a nonempty universe. An* interval-valued neutrosophic set *A on U is specified by three maps*

$$T_A,\ I_A,\ F_A : U \longrightarrow \mathrm{Int}([0,1]),$$

assigning to each $u \in U$ *closed intervals*

$$T_A(u) = [T_A^-(u), T_A^+(u)], \quad I_A(u) = [I_A^-(u), I_A^+(u)], \quad F_A(u) = [F_A^-(u), F_A^+(u)]$$

interpreted respectively as the truth, indeterminacy, *and* falsity *membership degrees of u. These components are independent; the only numeric bound required is*

$$0 \leq T_A^+(u) + I_A^+(u) + F_A^+(u) \leq 3, \qquad \forall\, u \in U.$$

Equivalently, one writes

$$A = \{\, \langle T_A(u), I_A(u), F_A(u) \rangle \ /\ u \in U \,\}.$$

Example 6 (IVNS for Spam Detection of a Single Email)**.** *Let* $U = \{e\}$ *where e is an incoming email. Define an IVNS* $S = (T, I, F)$ *for the statement "e is spam." From sender reputation, content filters, and user history we obtain*

$$T(e) = [0.72, 0.86], \qquad I(e) = [0.10, 0.20], \qquad F(e) = [0.04, 0.12].$$

The numeric constraint holds:

$$T^+(e) + I^+(e) + F^+(e) = 0.86 + 0.20 + 0.12 = 1.18 \leq 3.$$

A conservative rule declares spam when $T^-(e) \geq 0.70$ *and* $F^+(e) \leq 0.20$*. Here* $T^-(e) = 0.72$ (≥ 0.70) *and* $F^+(e) = 0.12$ (≤ 0.20)*, so e is classified as spam. The interval* $T(e) = [0.72, 0.86]$ *captures admissible truth,* $I(e) = [0.10, 0.20]$ *the uncertainty due to weak indicators, and* $F(e) = [0.04, 0.12]$ *the bounded counter-evidence (e.g., some benign content).*

1.4. Multistructure

MultiStructure is a carrier set equipped with indexed multi-operations mapping tuples to sets of outcomes, enabling nondeterministic, multi-arity algebraic computation [36,37].

Definition 9 (MultiOperation)**.** *Fix an integer* $m \geq 1$ *and let H be a nonempty set. An m-ary* multi-operation *on H is a map*

$$\#^{(m)} : H^m \longrightarrow \mathcal{M}(H), \quad (x_1, \ldots, x_m) \mapsto \#^{(m)}(x_1, \ldots, x_m),$$

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assigning to each m-tuple $(x_1, \ldots, x_m)$ a finite multiset of elements of H rather than a single element.

Definition 10 (MultiStructure). *A* MultiStructure *is a pair*

$$\mathcal{MS} \;=\; \big(H,\ \{\#^{(m)} : H^m \to \mathcal{M}(H)\}_{m\in\mathcal{I}}\big),$$

where H is a nonempty carrier set and $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ indexes a family of multi-operations of various arities. No further axioms are imposed unless explicitly stated.

Example 7 (MultiStructure on Interval Sets for Aggregating Requirements). *Let $H = \mathsf{I}(\mathcal{P}(U))$ with U as above, and take $\mathcal{M}(H) = \mathcal{P}(H)$. Define two multi-operations for any $m \geq 1$ and intervals $I_r = [L_r, U_r] \in H$:*

$$\#_{\wedge}^{(m)}(I_1, \ldots, I_m) = \begin{cases} \{[\,\bigcup_{r=1}^{m} L_r,\ \bigcap_{r=1}^{m} U_r\,]\}, & \text{if } \bigcup_r L_r \subseteq \bigcap_r U_r, \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\#_{\vee\text{hull}}^{(m)}(I_1, \ldots, I_m) = \{[\,\bigcap_{r=1}^{m} L_r,\ \bigcup_{r=1}^{m} U_r\,]\}.$$

Then $\mathcal{MS} = (H, \{\#_{\wedge}^{(m)}, \#_{\vee\text{hull}}^{(m)}\}_{m\geq 1})$ is a MultiStructure.

Concrete computation (two requirements). *Let*

$$I_1 = [\{milk\},\ \{milk, bread, eggs\}],$$

$$I_2 = [\{bread\},\ \{milk, bread, eggs\}].$$

Then

$$\#_{\wedge}^{(2)}(I_1, I_2) = \Big\{\ [\{milk\} \cup \{bread\},\ \{milk, bread, eggs\} \cap \{milk, bread, eggs\}]\Big\}$$
$$= \{\ [\{milk, bread\},\ \{milk, bread, eggs\}]\ \},$$

while

$$\#_{\vee\text{hull}}^{(2)}(I_1, I_2) = \Big\{\ [\{milk\} \cap \{bread\},\ \{milk, bread, eggs\} \cup \{milk, bread, eggs\}]\Big\}$$
$$= \{\ [\varnothing,\ \{milk, bread, eggs\}]\ \}.$$

Thus, the $\wedge$–operation returns the jointly *acceptable envelope, and the $\vee$–hull returns the* least *envelope containing both requirements—illustrating multi-arity, set-valued outputs in a real aggregation workflow.*

2. Main Results

In this section, we present the main results of this paper.

2.1. MultiInterval-Valued Fuzzy Set

A MultiInterval-valued Fuzzy Set assigns each element a finite family of membership intervals, aggregating evidence using intersection cores and hulls.

Definition 11 (MultiInterval). *Let $U \neq \varnothing$ and let J be a finite, nonempty index set. A* MultiInterval *on U is a family*

$$\mathbf{I} \;=\; \{\,[A_\ell^{(j)}, A_u^{(j)}] \mid j \in J,\ A_\ell^{(j)} \subseteq A_u^{(j)} \subseteq U\,\} \;\in\; \mathcal{P}^*(\mathsf{I}(\mathcal{P}(U))).$$

Its conjunctive semantics *(feasible core) is the set*

$$[\![\mathbf{I}]\!]_{\wedge} \;:=\; \bigcap_{j\in J} [A_\ell^{(j)}, A_u^{(j)}] \;=\; \Big\{\,A \subseteq U \;\Big|\; \forall j \in J:\ A_\ell^{(j)} \subseteq A \subseteq A_u^{(j)}\Big\}.$$

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Its disjunctive hull *is the smallest interval set containing* $\bigcup_{j\in J}[A_\ell^{(j)}, A_u^{(j)}]$*:*

$$[\![\mathbf{I}]\!]_\vee^{\mathrm{hull}} \;:=\; \Big[\bigcap_{j\in J} A_\ell^{(j)}, \bigcup_{j\in J} A_u^{(j)}\Big].$$

Lemma 1 (Exact formula for the conjunctive core). *Let* $\mathbf{I}$ *be as in Definition 11 and set*

$$L^\wedge \;:=\; \bigcup_{j\in J} A_\ell^{(j)}, \qquad U^\wedge \;:=\; \bigcap_{j\in J} A_u^{(j)}.$$

Then

$$\bigcap_{j\in J}[A_\ell^{(j)}, A_u^{(j)}] \;=\; \begin{cases} [L^\wedge, U^\wedge], & \text{if } L^\wedge \subseteq U^\wedge, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Proof. ($\subseteq$) Let $A \in \bigcap_j [A_\ell^{(j)}, A_u^{(j)}]$. Then $A_\ell^{(j)} \subseteq A \subseteq A_u^{(j)}$ for all j. Taking unions of the left bounds gives $L^\wedge = \bigcup_j A_\ell^{(j)} \subseteq A$, and taking intersections of the right bounds gives $A \subseteq \bigcap_j A_u^{(j)} = U^\wedge$. Hence $A \in [L^\wedge, U^\wedge]$, which already forces $L^\wedge \subseteq U^\wedge$ whenever the intersection is nonempty.
($\supseteq$) Conversely, if $L^\wedge \subseteq U^\wedge$ and $A \in [L^\wedge, U^\wedge]$, then $A_\ell^{(j)} \subseteq L^\wedge \subseteq A$ and $A \subseteq U^\wedge \subseteq A_u^{(j)}$ for each j, so $A \in [A_\ell^{(j)}, A_u^{(j)}]$. Therefore A lies in the intersection. □

Remark 3 (Order and feasibility). *Define a preorder* $\preceq$ *on MultiIntervals by* $\mathbf{I}_1 \preceq \mathbf{I}_2$ *iff* $[\![\mathbf{I}_1]\!]_\wedge \subseteq [\![\mathbf{I}_2]\!]_\wedge$*. By Lemma 1, feasibility of* $\mathbf{I}$ *is equivalent to the numeric inequality* $\bigcup_j A_\ell^{(j)} \subseteq \bigcap_j A_u^{(j)}$*.*

Let $H := \mathsf{I}(\mathcal{P}(U))$. We define two canonical multi-operations on H.

Definition 12 (Meet and hull multi-operations on H). *For* $m \geq 1$ *and* $(I_1, \ldots, I_m) \in H^m$ *with* $I_r = [L_r, U_r]$*, set*

$$\#_\wedge^{(m)}(I_1, \ldots, I_m) \;:=\; \begin{cases} \{\,[\bigcup_{r=1}^m L_r, \bigcap_{r=1}^m U_r\,]\,\}, & \text{if } \bigcup L_r \subseteq \bigcap U_r, \\ \varnothing, & \text{otherwise,} \end{cases}$$

and

$$\#_{\vee\mathrm{hull}}^{(m)}(I_1, \ldots, I_m) \;:=\; \{\,[\bigcap_{r=1}^m L_r, \bigcup_{r=1}^m U_r\,]\,\}.$$

Then $\mathcal{MS}_U := (H, \{\#_\wedge^{(m)}, \#_{\vee\mathrm{hull}}^{(m)}\}_{m\geq 1})$ *is a MultiStructure in the sense of Definition 10.*

Proposition 1 (MultiInterval as a MultiStructure computation). *Let* $\mathbf{I} = \{I_j\}_{j\in J}$ *be a MultiInterval on* U *with* $I_j = [A_\ell^{(j)}, A_u^{(j)}]$*. Then*

$$[\![\mathbf{I}]\!]_\wedge \;=\; \bigcap_{j\in J} I_j \;=\; \begin{cases} I^\star, & \text{if } \#_\wedge^{(|J|)}((I_j)_{j\in J}) = \{I^\star\}, \\ \varnothing, & \text{otherwise,} \end{cases}$$

with $I^\star = \big[\bigcup_j A_\ell^{(j)}, \bigcap_j A_u^{(j)}\big]$*. Moreover, the hull semantics satisfies*

$$[\![\mathbf{I}]\!]_\vee^{\mathrm{hull}} \;=\; \#_{\vee\mathrm{hull}}^{(|J|)}((I_j)_{j\in J}).$$

Proof. Immediate from Definition 12 and Lemma 1, by expanding unions and intersections of bounds elementwise. □

Definition 13 (Singleton embedding). *Define $\iota : \mathsf{I}(\mathcal{P}(U)) \to \mathcal{P}^*(\mathsf{I}(\mathcal{P}(U)))$ by*

$$\iota([A_\ell, A_u]) \;:=\; \{[A_\ell, A_u]\}.$$

We identify $\iota([A_\ell, A_u])$ with a MultiInterval having index set $J = \{1\}$.

Theorem 1 (MultiInterval generalizes Interval). *For every interval set $I = [A_\ell, A_u] \in \mathsf{I}(\mathcal{P}(U))$,*

$$[\![\, \iota(I)\,]\!]_\wedge \;=\; I \qquad \text{and} \qquad [\![\, \iota(I)\,]\!]_\vee^{\text{hull}} \;=\; I.$$

Hence the map ι is an order-embedding from $(\mathsf{I}(\mathcal{P}(U)), \subseteq)$ into the preorder of MultiIntervals under $\preceq$, and MultiIntervals strictly generalize intervals.

Proof. Let $J = \{1\}$ and $I = [A_\ell, A_u]$. By Lemma 1,

$$[\![\iota(I)]\!]_\wedge \;=\; \bigcap_{j\in J}[A_\ell^{(j)}, A_u^{(j)}] \;=\; \Big[\bigcup_{j\in J} A_\ell^{(j)},\ \bigcap_{j\in J} A_u^{(j)}\Big] \;=\; [A_\ell, A_u] \;=\; I,$$

since the union and intersection over a singleton index set return A_ℓ and A_u respectively. The hull equality is analogous:

$$[\![\iota(I)]\!]_\vee^{\text{hull}} \;=\; \Big[\bigcap_{j\in J} A_\ell^{(j)},\ \bigcup_{j\in J} A_u^{(j)}\Big] \;=\; [A_\ell, A_u] \;=\; I.$$

Monotonicity of ι with respect to $\subseteq$ and $\preceq$ follows directly from these equalities. □

Definition 14 (MultiInterval-valued fuzzy set (MIVFS)). *Let $U \neq \varnothing$ and $L([0,1])$ as above. A* MIVFS *on U assigns to each $u \in U$ a finite nonempty family of numeric intervals*

$$\mathcal{A}(u) \;=\; \{\, [\alpha_j(u), \beta_j(u)] \in L([0,1]) \mid j \in J(u) \text{ finite, nonempty} \,\}.$$

Its conjunctive (intersection) semantics and disjunctive (hull) semantics are

$$\Big[\!\!\Big[\mathcal{A}(u)\Big]\!\!\Big]_\wedge = \begin{cases} [\,\max_j \alpha_j(u)\,,\, \min_j \beta_j(u)\,], & \text{if } \max_j \alpha_j(u) \le \min_j \beta_j(u), \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\Big[\!\!\Big[\mathcal{A}(u)\Big]\!\!\Big]_\vee^{\text{hull}} = [\,\min_j \alpha_j(u)\,,\, \max_j \beta_j(u)\,].$$

Example 8 (MIVFS for Software Release Readiness Today). *Let $U = \{v\}$, where v denotes the proposition*

"Version v1.2 is ready for production today."

As a MultiInterval-valued fuzzy set, we assign to v multiple membership intervals from independent engineering sources:

$$\mathcal{A}(v) = \{I_1, I_2, I_3, I_4\} \subseteq L([0,1]),$$

with

$$\begin{aligned} I_1 &= [0.78, 0.88] \text{ (unit + integration test trends)}, \\ I_2 &= [0.72, 0.85] \text{ (QA exploratory testing outcomes)}, \\ I_3 &= [0.80, 0.90] \text{ (SRE load/performance confidence)}, \\ I_4 &= [0.75, 0.83] \text{ (security/compliance quick audit)}. \end{aligned}$$

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By Definition (MIVFS), the conjunctive (intersection) semantics and the disjunctive (hull) semantics are

$$\left[\!\left[\mathcal{A}(v)\right]\!\right]_{\wedge} = \begin{cases} [\max_j \alpha_j,\ \min_j \beta_j], & \text{if } \max_j \alpha_j \le \min_j \beta_j, \\ \varnothing, & \text{otherwise}, \end{cases} \qquad \left[\!\left[\mathcal{A}(v)\right]\!\right]_{\vee}^{\text{hull}} = [\min_j \alpha_j,\ \max_j \beta_j],$$

where $I_j = [\alpha_j, \beta_j]$.

We compute the bounds explicitly.

Lower endpoints:

$$\alpha_1 = 0.78, \quad \alpha_2 = 0.72, \quad \alpha_3 = 0.80, \quad \alpha_4 = 0.75 \implies \max_j \alpha_j = \max\{0.78, 0.72, 0.80, 0.75\} = 0.80.$$

Upper endpoints:

$$\beta_1 = 0.88, \quad \beta_2 = 0.85, \quad \beta_3 = 0.90, \quad \beta_4 = 0.83 \implies \min_j \beta_j = \min\{0.88, 0.85, 0.90, 0.83\} = 0.83.$$

Since $0.80 \le 0.83$, *the intersection is feasible and hence*

$$\left[\!\left[\mathcal{A}(v)\right]\!\right]_{\wedge} = [0.80,\ 0.83]\ .$$

Hull:

$$\min_j \alpha_j = \min\{0.78, 0.72, 0.80, 0.75\} = 0.72, \qquad \max_j \beta_j = \max\{0.88, 0.85, 0.90, 0.83\} = 0.90,$$

so

$$\left[\!\left[\mathcal{A}(v)\right]\!\right]_{\vee}^{\text{hull}} = [0.72,\ 0.90]\ .$$

The interval $[0.80, 0.83]$ *is the conservative consensus range endorsed by* all *engineering evidence streams, suited to a strict release gate. The hull* $[0.72, 0.90]$ *captures the full plausible readiness reported by* at least one *stream, useful for exploratory planning and risk negotiation.*

Proposition 2 (Reduction to IVFS). *If each* $J(u)$ *is a singleton, say* $J(u) = \{1\}$ *with* $\mathcal{A}(u) = \{[\alpha_1(u), \beta_1(u)]\}$, *then*

$$\left[\!\left[\mathcal{A}(u)\right]\!\right]_{\wedge} = \left[\!\left[\mathcal{A}(u)\right]\!\right]_{\vee}^{\text{hull}} = [\,\alpha_1(u), \beta_1(u)\,],$$

so every IVFS is a special case of a MIVFS.

Proof. Compute $\max_j \alpha_j(u) = \min_j \beta_j(u)$ over a singleton index set. □

2.2. MultiInterval-Valued Neutrosophic Set

A MultiInterval-valued Neutrosophic Set assigns each element interval families for truth, indeterminacy, and falsity, aggregated componentwise by cores and hulls.

Definition 15 (MIVNS). *A* MultiInterval-valued neutrosophic set (MIVNS) *on* U *is a triple*

$$\mathcal{A} = (\mathcal{T}, \mathcal{I}, \mathcal{F}),$$

where for each $u \in U$, $\mathcal{T}(u), \mathcal{I}(u), \mathcal{F}(u)$ *are finite nonempty families of numeric intervals in* $L([0,1])$. *Conjunctive semantics are computed componentwise by interval intersection:*

$$[\![\mathcal{T}(u)]\!]_{\wedge} = [\max_j T_j^-(u),\ \min_j T_j^+(u)\,],\ [\![\mathcal{I}(u)]\!]_{\wedge} = [\max_j I_j^-(u),\ \min_j I_j^+(u)\,],\ [\![\mathcal{F}(u)]\!]_{\wedge} = [\max_j F_j^-(u),\ \min_j F_j^+(u)\,],$$

with the usual feasibility conditions max $\le$ min *in each component; hull semantics take* [min, max] *componentwise.*

Example 9 (MIVNS for a Clinical Decision: "Pneumonia Present Today?"). *Let $U = \{p\}$, where p denotes a particular patient. We assess the statement*

"p has community–acquired pneumonia (CAP) today."

as a MultiInterval–valued neutrosophic set $\mathcal{A} = (\mathcal{T}, \mathcal{I}, \mathcal{F})$. Here $\mathcal{T}(p)$ collects supporting *evidence intervals (truth), $\mathcal{I}(p)$ collects* uncertainty *intervals (indeterminacy), and $\mathcal{F}(p)$ collects* counter–evidence *intervals (falsity). Each interval $[\alpha, \beta] \subseteq [0,1]$ encodes an admissible range for the corresponding degree.*

Concrete (illustrative) sources and intervals:

$$\mathcal{T}(p) = \{[0.72, 0.88] \text{ (chest X–ray report)},\ [0.65, 0.80] \text{ (clinical score)},\ [0.70, 0.90] \text{ (C–reactive protein model)}\},$$
$$\mathcal{I}(p) = \{[0.10, 0.25] \text{ (equivocal imaging)},\ [0.15, 0.30] \text{ (atypical symptom onset)},\ [0.05, 0.20] \text{ (comorbidity confounding)}\},$$
$$\mathcal{F}(p) = \{[0.08, 0.22] \text{ (normal WBC)},\ [0.12, 0.18] \text{ (viral/bacterial panel result)},\ [0.05, 0.15] \text{ (stable oxygenation)}\}.$$

By the MIVNS semantics, the conjunctive core *and* disjunctive hull *are computed* componentwise *using interval intersection and least–containing–interval, respectively.*

Truth component.

$$\begin{aligned}
&\max \underline{\mathcal{T}}(p) = \max\{0.72, 0.65, 0.70\} = 0.72,\\
&\min \overline{\mathcal{T}}(p) = \min\{0.88, 0.80, 0.90\} = 0.80 \quad \Longrightarrow \quad [\![\mathcal{T}(p)]\!]_\wedge = [0.72,\ 0.80] \ (\textit{feasible since } 0.72 \le 0.80),\\
&\min \underline{\mathcal{T}}(p) = \min\{0.72, 0.65, 0.70\} = 0.65, \qquad \max \overline{\mathcal{T}}(p) = \max\{0.88, 0.80, 0.90\} = 0.90\\
&\qquad \Rightarrow \quad [\![\mathcal{T}(p)]\!]_\vee^{\text{hull}} = [0.65,\ 0.90].
\end{aligned}$$

Indeterminacy component.

$$\begin{aligned}
&\max \underline{\mathcal{I}}(p) = \max\{0.10, 0.15, 0.05\} = 0.15,\\
&\min \overline{\mathcal{I}}(p) = \min\{0.25, 0.30, 0.20\} = 0.20 \quad \Longrightarrow \quad [\![\mathcal{I}(p)]\!]_\wedge = [0.15,\ 0.20],\\
&\min \underline{\mathcal{I}}(p) = \min\{0.10, 0.15, 0.05\} = 0.05, \qquad \max \overline{\mathcal{I}}(p) = \max\{0.25, 0.30, 0.20\} = 0.30\\
&\qquad \Rightarrow \quad [\![\mathcal{I}(p)]\!]_\vee^{\text{hull}} = [0.05,\ 0.30].
\end{aligned}$$

Falsity component.

$$\begin{aligned}
&\max \underline{\mathcal{F}}(p) = \max\{0.08, 0.12, 0.05\} = 0.12,\\
&\min \overline{\mathcal{F}}(p) = \min\{0.22, 0.18, 0.15\} = 0.15 \quad \Longrightarrow \quad [\![\mathcal{F}(p)]\!]_\wedge = [0.12,\ 0.15],\\
&\min \underline{\mathcal{F}}(p) = \min\{0.08, 0.12, 0.05\} = 0.05, \qquad \max \overline{\mathcal{F}}(p) = \max\{0.22, 0.18, 0.15\} = 0.22\\
&\qquad \Rightarrow \quad [\![\mathcal{F}(p)]\!]_\vee^{\text{hull}} = [0.05,\ 0.22].
\end{aligned}$$

Vector–valued result.

$$\text{core}_\wedge(\mathcal{A})(p) = \big([0.72, 0.80],\ [0.15, 0.20],\ [0.12, 0.15]\big),$$

$$\text{hull}_\vee(\mathcal{A})(p) = \big([0.65, 0.90],\ [0.05, 0.30],\ [0.05, 0.22]\big)\ .$$

The conjunctive core gives the consensus ranges simultaneously supported by all sources in each component (truth, indeterminacy, falsity). The hulls summarize the full plausible spans reported by at least one source. A conservative decision maker would consult the core; exploratory or safety–margin assessments can reference the hulls.

Theorem 2 (MIVNS generalizes IVNS). *Define the embedding*

$$\iota:\ \big(L([0,1])\big)^3 \longrightarrow \Big(\mathcal{P}^*(L([0,1]))\Big)^3, \qquad \iota([a,b],[c,d],[e,f]) := (\{[a,b]\}, \{[c,d]\}, \{[e,f]\}).$$

If $A = (T, I, F)$ is an IVNS on U, then $\mathcal{A} := \iota \circ A$ is a MIVNS and, for every $u \in U$,

$$\mathrm{core}_\wedge(\mathcal{A})(u) = (T(u), I(u), F(u)) \qquad \textit{and} \qquad \mathrm{hull}_\vee(\mathcal{A})(u) = (T(u), I(u), F(u)).$$

Hence every IVNS is a (canonically embedded) special case of a MIVNS.

Proof. Fix $u \in U$ and write $T(u) = [t^-(u), t^+(u)]$, $I(u) = [i^-(u), i^+(u)]$, $F(u) = [f^-(u), f^+(u)]$. By definition of ι,

$$\mathcal{T}(u) = \{[t^-(u), t^+(u)]\}, \quad \mathcal{I}(u) = \{[i^-(u), i^+(u)]\}, \quad \mathcal{F}(u) = \{[f^-(u), f^+(u)]\}.$$

Evaluating maxima/minima over a singleton gives, componentwise,

$$\max_{[\alpha,\beta]\in\mathcal{T}(u)} \alpha = t^-(u), \qquad \min_{[\alpha,\beta]\in\mathcal{T}(u)} \beta = t^+(u) \implies \mathrm{core}_\wedge(\mathcal{T}(u)) = [t^-(u), t^+(u)] = T(u),$$

and similarly $\mathrm{core}_\wedge(\mathcal{I}(u)) = I(u)$, $\mathrm{core}_\wedge(\mathcal{F}(u)) = F(u)$. The hull equalities follow analogously: $\mathrm{hull}_\vee(\mathcal{T}(u)) = [t^-(u), t^+(u)] = T(u)$, etc. Thus $\mathrm{core}_\wedge(\mathcal{A})(u) = \mathrm{hull}_\vee(\mathcal{A})(u) = (T(u), I(u), F(u))$. □

Theorem 3 (MIVNS generalizes MIVFS)**.** *Let $\mathcal{A}_\mathrm{F}$ be a MIVFS on U. Define $\Phi(\mathcal{A}_\mathrm{F})$ to be the MIVNS $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ given by*

$$\mathcal{T}(u) := \mathcal{A}_\mathrm{F}(u), \qquad \mathcal{I}(u) := \{\,[0,0]\,\}, \qquad \mathcal{F}(u) := \{\,[0,0]\,\}, \quad \textit{for each } u \in U.$$

Then, for every $u \in U$,

$$\mathrm{core}_\wedge\big(\mathcal{T}(u)\big) = \mathrm{core}_\wedge\big(\mathcal{A}_\mathrm{F}(u)\big), \quad \mathrm{hull}_\vee\big(\mathcal{T}(u)\big) = \mathrm{hull}_\vee\big(\mathcal{A}_\mathrm{F}(u)\big),$$

and

$$\mathrm{core}_\wedge\big(\mathcal{I}(u)\big) = \mathrm{hull}_\vee\big(\mathcal{I}(u)\big) = [0,0], \qquad \mathrm{core}_\wedge\big(\mathcal{F}(u)\big) = \mathrm{hull}_\vee\big(\mathcal{F}(u)\big) = [0,0].$$

Consequently, projecting the MIVNS $\Phi(\mathcal{A}_\mathrm{F})$ onto its truth component recovers exactly the original MIVFS semantics (both core and hull), so MIVNS strictly extends MIVFS.

Proof. Fix $u \in U$ and write $\mathcal{A}_\mathrm{F}(u) = \{[\alpha_j(u), \beta_j(u)]\}_{j\in J(u)}$ with $J(u)$ finite nonempty. By definition of MIVNS core/hull, applied to $\mathcal{T}(u) = \mathcal{A}_\mathrm{F}(u)$,

$$\mathrm{core}_\wedge\big(\mathcal{T}(u)\big) = \begin{cases} [\max_{j\in J(u)} \alpha_j(u),\, \min_{j\in J(u)} \beta_j(u)], & \text{if } \max \alpha_j \le \min \beta_j, \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\mathrm{hull}_\vee\big(\mathcal{T}(u)\big) = \big[\min_{j\in J(u)} \alpha_j(u),\ \max_{j\in J(u)} \beta_j(u)\big],$$

which are exactly the MIVFS core/hull for $\mathcal{A}_\mathrm{F}(u)$. For the neutrosophic indeterminacy and falsity components, each is the singleton family $\{[0,0]\}$; hence

$$\max_{[\alpha,\beta]\in\{[0,0]\}} \alpha = 0, \qquad \min_{[\alpha,\beta]\in\{[0,0]\}} \beta = 0 \implies \mathrm{core}_\wedge(\{[0,0]\}) = [0,0],$$

and trivially $\mathrm{hull}_\vee(\{[0,0]\}) = [0,0]$. Therefore the truth-component of $\Phi(\mathcal{A}_\mathrm{F})$ reproduces the MIVFS, while the neutrosophic extras are neutralized at $[0,0]$. □

2.3. HyperInterval-Valued Fuzzy Set

A HyperInterval-Valued Fuzzy Set permits arbitrary sets of membership intervals per element, deriving intersection cores and minimal containing hulls afterward.

Definition 16 (HyperInterval-valued fuzzy set (HIVFS))**.** *A* HIVFS *on U assigns to each $u \in U$ a nonempty family* $\mathrm{H}(u) \subseteq L([0,1])$*. Its conjunctive core and hull are, respectively,*

$$[\![\mathrm{H}(u)]\!]_{\wedge} = \bigcap_{I \in \mathrm{H}(u)} I = \begin{cases} [\,\sup_{I \in \mathrm{H}(u)} \inf I,\ \inf_{I \in \mathrm{H}(u)} \sup I\,], & \textit{if feasible,} \\ \varnothing, & \textit{otherwise,} \end{cases}$$

$$[\![\mathrm{H}(u)]\!]_{\vee}^{\mathrm{hull}} = [\,\inf_{I \in \mathrm{H}(u)} \inf I,\ \sup_{I \in \mathrm{H}(u)} \sup I\,].$$

Example 10 (HIVFS in a Morning Commute Decision)**.** *Let U be the set of candidate routes for today's commute. For the proposition "Arrive on time via route r_1", define the HyperInterval-valued fuzzy set*

$$\mathrm{H}(r_1) = \{I_1, I_2, I_3\} \subseteq L([0,1]),$$

where each interval encodes an admissible membership range (probability-like confidence) from an independent source:

$$I_1 = [0.62, 0.78] \ \textit{(live traffic app)},$$

$$I_2 = [0.55, 0.72] \ \textit{(weather-adjusted model)},$$

$$I_3 = [0.68, 0.80] \ \textit{(historical punctuality)}.$$

By Definition (HIVFS), the conjunctive core *and* disjunctive hull *at r_1 are*

$$[\![\mathrm{H}(r_1)]\!]_{\wedge} = \bigcap_{I \in \mathrm{H}(r_1)} I = \begin{cases} [\,\sup_{I \in \mathrm{H}(r_1)} \inf I,\ \inf_{I \in \mathrm{H}(r_1)} \sup I\,], & \textit{if feasible,} \\ \varnothing, & \textit{otherwise,} \end{cases}$$

$$[\![\mathrm{H}(r_1)]\!]_{\vee}^{\mathrm{hull}} = [\,\inf_{I \in \mathrm{H}(r_1)} \inf I,\ \sup_{I \in \mathrm{H}(r_1)} \sup I\,].$$

We compute these bounds explicitly. First the lower endpoints:

$$\inf I_1 = 0.62, \quad \inf I_2 = 0.55, \quad \inf I_3 = 0.68 \implies \sup_{I \in \mathrm{H}(r_1)} \inf I = \max\{0.62, 0.55, 0.68\} = 0.68.$$

Then the upper endpoints:

$$\sup I_1 = 0.78, \quad \sup I_2 = 0.72, \quad \sup I_3 = 0.80 \implies \inf_{I \in \mathrm{H}(r_1)} \sup I = \min\{0.78, 0.72, 0.80\} = 0.72.$$

Therefore the conjunctive core is feasible and equals

$$[\![\mathrm{H}(r_1)]\!]_{\wedge} = [0.68,\ 0.72].$$

The disjunctive hull aggregates the most permissive range:

$$\inf_{I \in \mathrm{H}(r_1)} \inf I = \min\{0.62, 0.55, 0.68\} = 0.55, \qquad \sup_{I \in \mathrm{H}(r_1)} \sup I = \max\{0.78, 0.72, 0.80\} = 0.80,$$

so

$$[\![\mathrm{H}(r_1)]\!]_{\vee}^{\mathrm{hull}} = [0.55,\ 0.80].$$

The interval $[0.68, 0.72]$ is the consensus range supported simultaneously *by all sources for "arrive on time via r_1". The hull $[0.55, 0.80]$ captures the full plausible spectrum reported by* at least one *source. Decision makers can require the core (conservative) or consult the hull (exploratory) depending on risk tolerance.*

2.4. SuperHyperInterval-Valued Fuzzy Set

A SuperHyperInterval-Valued Fuzzy Set assigns nested families of membership intervals via iterated powersets, flattened to compute intersection cores and hulls.

Definition 17 (SuperHyperInterval-valued fuzzy set (order n))**.** *Fix $n \geq 1$. A* SuperHyperInterval-valued fuzzy set of order *n on U is a map*

$$\mathrm{SH}^{(n)} : U \longrightarrow \mathcal{P}^{n-1}(L([0,1])),$$

assigning to each u an $(n-1)$–nested family of numeric intervals. Conjunctive cores and hulls are computed by iterated application of intersection and hull at the leaves.

Example 11 (SHIVFS (order n=3) for a Same-Day Delivery Promise)**.** *Let $U = \{\tau\}$, where τ denotes today's task "Deliver the parcel by 6 pm". Fix $n = 3$. A SuperHyperInterval-valued fuzzy set of order 3 maps τ to a two-level nested family of numeric intervals (elements of $L([0,1])$), grouped by evidence sources:*

$$\mathrm{SH}^{(3)}(\tau) = \{S_{logistics}, \ S_{conditions}, \ S_{recipient}\} \in \mathcal{P}^2(L([0,1])),$$

where each $S_\bullet$ is a finite set of closed intervals $[\alpha, \beta] \subseteq [0,1]$.

We instantiate concretely (all numbers are unit-free confidence levels):

$$S_{logistics} = \{I_1 = [0.78, 0.90] \text{ (Carrier A dispatch+capacity)}, \ I_2 = [0.80, 0.88] \text{ (Carrier B historical on-time)}\},$$
$$S_{conditions} = \{I_3 = [0.75, 0.92] \text{ (traffic nowcast)}, \ I_4 = [0.77, 0.89] \text{ (weather-adjusted travel time)}\},$$
$$S_{recipient} = \{I_5 = [0.76, 0.93] \text{ (recipient availability window)}, \ I_6 = [0.82, 0.87] \text{ (building access constraints)}\}.$$

By definition, conjunctive cores and hulls are computed at the leaves. Flatten the nesting by taking the union of groups:

$$\mathrm{Leaves}(\tau) \ := \ S_{logistics} \cup S_{conditions} \cup S_{recipient} = \{I_1, I_2, I_3, I_4, I_5, I_6\} \subseteq L([0,1]).$$

Conjunctive core (intersection of all leaf intervals) is

$$[\![\mathrm{SH}^{(3)}(\tau)]\!]_\wedge = \bigcap_{k=1}^{6} I_k$$

$$= \Big[\sup_{1 \leq k \leq 6} \inf I_k, \ \inf_{1 \leq k \leq 6} \sup I_k \Big],$$

provided feasibility $\sup \inf \leq \inf \sup$ *holds. We compute explicitly:*

$$\inf I_1 = 0.78, \ \inf I_2 = 0.80, \ \inf I_3 = 0.75, \ \inf I_4 = 0.77, \ \inf I_5 = 0.76, \ \inf I_6 = 0.82$$

$$\Longrightarrow \sup_k \inf I_k = \max\{0.78, 0.80, 0.75, 0.77, 0.76, 0.82\} = 0.82,$$

$$\sup I_1 = 0.90, \ \sup I_2 = 0.88, \ \sup I_3 = 0.92, \ \sup I_4 = 0.89, \ \sup I_5 = 0.93, \ \sup I_6 = 0.87$$

$$\Longrightarrow \inf_k \sup I_k = \min\{0.90, 0.88, 0.92, 0.89, 0.93, 0.87\} = 0.87.$$

Since $0.82 \leq 0.87$, the intersection is feasible and

$$[\![\mathrm{SH}^{(3)}(\tau)]\!]_\wedge = [0.82, \ 0.87] \ .$$

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Disjunctive hull (least interval containing all leaf intervals) is

$$[\![\mathrm{SH}^{(3)}(\tau)]\!]_{\vee}^{\mathrm{hull}} = \Big[\inf_k \inf I_k,\ \sup_k \sup I_k\Big]$$

$$= \Big[\min\{0.78, 0.80, 0.75, 0.77, 0.76, 0.82\},\ \max\{0.90, 0.88, 0.92, 0.89, 0.93, 0.87\}\Big] = [0.75,\ 0.93].$$

The nested structure records hierarchical *evidence: logistics providers, external conditions, and recipient constraints. Flattening aggregates every leaf interval; the core* $[0.82, 0.87]$ *captures the consensus range simultaneously supported by* all *groups and sources, while the hull* $[0.75, 0.93]$ *captures the full plausible spectrum reported by* at least one *leaf source. A planner requiring high reliability would use the core; exploratory planning or contingency analysis may reference the hull.*

2.5. HyperInterval-Valued Neutrosophic Set

A HyperInterval-Valued Neutrosophic Set uses sets of intervals for truth, indeterminacy, and falsity, yielding componentwise intersection cores and hulls semantics.

Definition 18 (HyperInterval-Valued Neutrosophic Set (HIVNS))**.** *Let* $U \neq \varnothing$ *be a universe, and let*

$$L([0,1]) := \big\{[\alpha, \beta] \subseteq [0,1] \ \big|\ 0 \le \alpha \le \beta \le 1\big\}$$

be the set of all closed numeric intervals in $[0,1]$*. A* HyperInterval-Valued Neutrosophic Set (HIVNS) *on* U *is a triple of maps*

$$\mathcal{A} = (\mathcal{T}, \mathcal{I}, \mathcal{F}),$$

$$\mathcal{T}, \mathcal{I}, \mathcal{F} : U \longrightarrow \mathcal{P}^{*}(L([0,1])),$$

such that for each $u \in U$ *the sets* $\mathcal{T}(u)$, $\mathcal{I}(u)$, *and* $\mathcal{F}(u)$ *are* finite, nonempty *families of intervals in* $L([0,1])$.

For a finite family $\mathcal{S} = \{[\alpha_j, \beta_j]\}_{j \in J} \subseteq L([0,1])$*, define the* conjunctive core *and the* disjunctive hull *by*

$$\mathrm{core}_{\wedge}(\mathcal{S}) = \begin{cases} [\max_{j\in J} \alpha_j,\ \min_{j\in J} \beta_j], & \text{if } \max_j \alpha_j \le \min_j \beta_j, \\ \varnothing, & \text{otherwise}, \end{cases}$$

$$\mathrm{hull}_{\vee}(\mathcal{S}) = \big[\min_{j\in J} \alpha_j,\ \max_{j\in J} \beta_j\big].$$

We interpret $\mathrm{core}_{\wedge}(\mathcal{S})$ *as the precise admissible range under* all *hyper-interval declarations, and* $\mathrm{hull}_{\vee}(\mathcal{S})$ *as the least interval containing* some *declaration.*

The conjunctive semantics *and* hull semantics *of* $\mathcal{A}$ *at* $u \in U$ *are then given componentwise by*

$$\mathrm{core}_{\wedge}(\mathcal{A})(u) := \big(\mathrm{core}_{\wedge}(\mathcal{T}(u)),\ \mathrm{core}_{\wedge}(\mathcal{I}(u)),\ \mathrm{core}_{\wedge}(\mathcal{F}(u))\big),$$

$$\mathrm{hull}_{\vee}(\mathcal{A})(u) := \big(\mathrm{hull}_{\vee}(\mathcal{T}(u)),\ \mathrm{hull}_{\vee}(\mathcal{I}(u)),\ \mathrm{hull}_{\vee}(\mathcal{F}(u))\big),$$

with feasibility of the conjunctive semantics requiring, for each component,

$$\max_{[\alpha,\beta]\in\mathcal{T}(u)} \alpha \le \min_{[\alpha,\beta]\in\mathcal{T}(u)} \beta,$$

$$\max_{[\alpha,\beta]\in\mathcal{I}(u)} \alpha \le \min_{[\alpha,\beta]\in\mathcal{I}(u)} \beta,$$

$$\max_{[\alpha,\beta]\in\mathcal{F}(u)} \alpha \le \min_{[\alpha,\beta]\in\mathcal{F}(u)} \beta.$$

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Theorem 4 (HIVNS generalizes IVNS). *Let an* Interval-Valued Neutrosophic Set (IVNS) *be given by three maps*

$$T, I, F : U \longrightarrow L([0,1]), \qquad T(u) = [t^-(u), t^+(u)],\ I(u) = [i^-(u), i^+(u)],\ F(u) = [f^-(u), f^+(u)].$$

Define the embedding

$$\iota : \ (L([0,1]))^3 \ \longrightarrow \ \big(\mathcal{P}^*(L([0,1]))\big)^3,$$

$$\iota\big([a,b],[c,d],[e,f]\big) \ := \ \big(\{[a,b]\},\{[c,d]\},\{[e,f]\}\big).$$

Then $\mathcal{A} := \iota(T, I, F)$ *is a HIVNS and, for every* $u \in U$*,*

$$\mathrm{core}_\wedge(\mathcal{A})(u) = \big(T(u), I(u), F(u)\big) = \mathrm{hull}_\vee(\mathcal{A})(u).$$

Consequently, every IVNS is (canonically) a special case of a HIVNS.

Proof. Fix $u \in U$. By construction,

$$\mathcal{T}(u) = \{[t^-(u), t^+(u)]\},$$

$$\mathcal{I}(u) = \{[i^-(u), i^+(u)]\},$$

$$\mathcal{F}(u) = \{[f^-(u), f^+(u)]\}.$$

Evaluating the maxima/minima over a singleton set gives

$$\max_{[\alpha,\beta]\in\mathcal{T}(u)} \alpha = t^-(u),$$

$$\min_{[\alpha,\beta]\in\mathcal{T}(u)} \beta = t^+(u),$$

and analogously for the I and F components. Hence

$$\mathrm{core}_\wedge(\mathcal{T}(u)) = [t^-(u), t^+(u)],$$

$$\mathrm{hull}_\vee(\mathcal{T}(u)) = [t^-(u), t^+(u)],$$

with identical equalities for $\mathcal{I}(u)$ and $\mathcal{F}(u)$. Therefore

$$\mathrm{core}_\wedge(\mathcal{A})(u) = \mathrm{hull}_\vee(\mathcal{A})(u) = (T(u), I(u), F(u)).$$

□

Example 12 (Concrete computation at a point). *Let* $\mathcal{T}(u) = \{[0.6, 0.9], [0.7, 0.8]\}$, $\mathcal{I}(u) = \{[0.1, 0.3]\}$, $\mathcal{F}(u) = \{[0.05, 0.2], [0.0, 0.15]\}$. *Then*

$$\mathrm{core}_\wedge(\mathcal{T}(u)) = [\max\{0.6, 0.7\}, \min\{0.9, 0.8\}] = [0.7, 0.8],$$

$$\mathrm{hull}_\vee(\mathcal{T}(u)) = [\min\{0.6, 0.7\}, \max\{0.9, 0.8\}] = [0.6, 0.9],$$

$$\mathrm{core}_\wedge(\mathcal{I}(u)) = \mathrm{hull}_\vee(\mathcal{I}(u)) = [0.1, 0.3],$$

$$\mathrm{core}_\wedge(\mathcal{F}(u)) = [\max\{0.05, 0.0\}, \min\{0.2, 0.15\}] = [0.05, 0.15],$$

$$\mathrm{hull}_\vee(\mathcal{F}(u)) = [\min\{0.05, 0.0\}, \max\{0.2, 0.15\}] = [0.0, 0.2].$$

Thus $\mathrm{core}_\wedge(\mathcal{A})(u) = \big([0.7, 0.8], [0.1, 0.3], [0.05, 0.15]\big)$ *and* $\mathrm{hull}_\vee(\mathcal{A})(u) = \big([0.6, 0.9], [0.1, 0.3], [0.0, 0.2]\big)$.

2.6. SuperHyperInterval-Valued Neutrosophic Set

A SuperHyperInterval-Valued Neutrosophic Set organizes nested interval families for truth, indeterminacy, and falsity, flattening to compute componentwise cores and hulls.

Definition 19 (Flattening operator on iterated powersets). *For a base set X and $k \geq 0$, define $\mathrm{Flat}_k : \mathcal{P}^k(X) \to \mathcal{P}(X)$ recursively by*

$$\mathrm{Flat}_0(S) := S \subseteq X, \qquad \mathrm{Flat}_{k+1}(\mathcal{S}) := \bigcup_{S \in \mathcal{S}} \mathrm{Flat}_k(S) \quad (\mathcal{S} \subseteq \mathcal{P}^k(X)).$$

Definition 20 (SuperHyperInterval-Valued Neutrosophic Set (order n)). *Fix $n \geq 1$. A SuperHyperInterval-Valued Neutrosophic Set (SHIVNS) of order n on U is a triple*

$$\mathcal{S}^{(n)} = (\mathsf{T}^{(n)}, \mathsf{I}^{(n)}, \mathsf{F}^{(n)}), \qquad \mathsf{T}^{(n)}, \mathsf{I}^{(n)}, \mathsf{F}^{(n)} : U \longrightarrow \mathcal{P}^{n-1}(L([0,1])),$$

with the requirement that for each $u \in U$ the sets $\mathsf{T}^{(n)}(u)$, $\mathsf{I}^{(n)}(u)$, $\mathsf{F}^{(n)}(u)$ are finite at every nesting level.

Write the leaf families *at u as*

$$\mathcal{T}_{\mathrm{leaf}}(u) := \mathrm{Flat}_{n-1}(\mathsf{T}^{(n)}(u)) \subseteq L([0,1]), \quad \mathcal{I}_{\mathrm{leaf}}(u) := \mathrm{Flat}_{n-1}(\mathsf{I}^{(n)}(u)), \quad \mathcal{F}_{\mathrm{leaf}}(u) := \mathrm{Flat}_{n-1}(\mathsf{F}^{(n)}(u)).$$

The conjunctive semantics *and* hull semantics *of $\mathcal{S}^{(n)}$ are defined by applying Definition 18 to these leaf families:*

$$\mathrm{core}_\wedge(\mathcal{S}^{(n)})(u) := \Big(\mathrm{core}_\wedge(\mathcal{T}_{\mathrm{leaf}}(u)),\ \mathrm{core}_\wedge(\mathcal{I}_{\mathrm{leaf}}(u)),\ \mathrm{core}_\wedge(\mathcal{F}_{\mathrm{leaf}}(u))\Big),$$

$$\mathrm{hull}_\vee(\mathcal{S}^{(n)})(u) := \Big(\mathrm{hull}_\vee(\mathcal{T}_{\mathrm{leaf}}(u)),\ \mathrm{hull}_\vee(\mathcal{I}_{\mathrm{leaf}}(u)),\ \mathrm{hull}_\vee(\mathcal{F}_{\mathrm{leaf}}(u))\Big).$$

Definition 21 (Canonical nesting). *For any set S and $k \geq 0$, define $\mathrm{Nest}_0(S) := S$ and $\mathrm{Nest}_{k+1}(S) := \{\mathrm{Nest}_k(S)\}$. Then $\mathrm{Nest}_k(S) \in \mathcal{P}^k(S)$ and, crucially,*

$$\mathrm{Flat}_k(\mathrm{Nest}_k(S)) = S \qquad \text{for all } k \geq 0. \tag{1}$$

Lemma 2 (Flattening a canonical nest). *Equality (1) holds by induction on k.*

Proof. For $k = 0$ the claim is $\mathrm{Flat}_0(S) = S$, true by definition. If $\mathrm{Flat}_k(\mathrm{Nest}_k(S)) = S$, then

$$\mathrm{Flat}_{k+1}(\mathrm{Nest}_{k+1}(S)) = \mathrm{Flat}_{k+1}(\{\mathrm{Nest}_k(S)\}) = \mathrm{Flat}_k(\mathrm{Nest}_k(S)) = S.$$

□

Theorem 5 (SHIVNS generalizes HIVNS). *Let $\mathcal{A} = (\mathcal{T}, \mathcal{I}, \mathcal{F})$ be a HIVNS on U (Definition 18). For any $n \geq 1$, define the embedding*

$$\jmath_n : \big(\mathcal{P}^*(L([0,1]))\big)^3 \longrightarrow \big(\mathcal{P}^{n-1}(L([0,1]))\big)^3$$

by

$$(\mathcal{T}(u), \mathcal{I}(u), \mathcal{F}(u)) \longmapsto (\mathrm{Nest}_{n-1}(\mathcal{T}(u)),\ \mathrm{Nest}_{n-1}(\mathcal{I}(u)),\ \mathrm{Nest}_{n-1}(\mathcal{F}(u))).$$

Then $\mathcal{S}^{(n)} := \jmath_n(\mathcal{A})$ is a SHIVNS of order n and, for every $u \in U$,

$$\mathrm{core}_\wedge(\mathcal{S}^{(n)})(u) = \mathrm{core}_\wedge(\mathcal{A})(u), \qquad \mathrm{hull}_\vee(\mathcal{S}^{(n)})(u) = \mathrm{hull}_\vee(\mathcal{A})(u).$$

Hence every HIVNS is (canonically) a special case of a SHIVNS (for any $n \geq 1$).

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Proof. Fix $u \in U$. By Lemma 2 with $S = \mathcal{T}(u)$ (and similarly for $\mathcal{I}, \mathcal{F}$),

$$\mathcal{T}_{\text{leaf}}(u) = \text{Flat}_{n-1}(\text{Nest}_{n-1}(\mathcal{T}(u))) = \mathcal{T}(u).$$

Therefore the leaf families of $\mathcal{S}^{(n)}$ coincide with the original hyper-interval families of $\mathcal{A}$:

$$\mathcal{T}_{\text{leaf}}(u) = \mathcal{T}(u), \quad \mathcal{I}_{\text{leaf}}(u) = \mathcal{I}(u), \quad \mathcal{F}_{\text{leaf}}(u) = \mathcal{F}(u).$$

Since both $\text{core}_{\wedge}$ and $\text{hull}_{\vee}$ depend only on the sets of leaf intervals via max / min and min / max, it follows immediately that

$$\text{core}_{\wedge}(\mathcal{S}^{(n)})(u) = \text{core}_{\wedge}(\mathcal{A})(u), \qquad \text{hull}_{\vee}(\mathcal{S}^{(n)})(u) = \text{hull}_{\vee}(\mathcal{A})(u).$$

□

3. Conclusion

In this paper, we introduced HyperInterval- and SuperHyperInterval-valued fuzzy/neutrosophic sets, defined conjunctive "core" (intersection) and disjunctive "hull" semantics, and proved embedding theorems showing that classical interval, fuzzy, and neutrosophic models appear as singleton or degenerate cases. For future work, we plan to investigate extended concepts using HyperFuzzy Sets[38–42], HyperNeutrosophic Sets[43,44], Plithogenic Sets [45–47], as well as structural frameworks such as Graphs[48], HyperGraphs[49–51], and SuperHyperGraphs[52–55].

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Institutional Review Board Statement: As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Artificial Intelligence: I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

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Disclaimer: This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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