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Not peer-reviewed version

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Posted Date: 9 September 2025

doi: 10.20944/preprints202509.0663.v1

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Article

The $3\mathbb{D}$ Field of Complex Vectors

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Abstract

Based on the isomorphic algebraic structures of the $2\mathbb{D}$ Euclidean field of complex vectors $V_{\mathbb{C}}$ and the field of complex numbers \mathbb{C} , in terms of identical geometric products of the elements of both fields, this paper brings the algebraic structure of a $3\mathbb{D}$ field of complex vectors, as well as the corresponding fundamental integral identities in those vector fields.

Keywords: three dimensional complex vectors; scalar and vector fields

1. Introduction

A geometric (*Clifford*) algebra is an extension of elementary algebra to work with geometrical objects such as vectors. It is built out of two fundamental operations: addition and geometric product, [1]. The multiplication of vectors results in objects called multivectors. Compared with any other formalism for manipulating vectors, *Clifford* algebra alone supports dividing by vectors. The geometric product was first mentioned by *Grassmann*, who founded the so-called external algebra, [2]. After that, *Clifford* himself greatly expanded upon *Grassmann's* work, to form geometric algebra, named after him *Clifford* algebra [1], by unifying both *Grassmann's* algebra and *Hamilton's* quaternion algebra (\mathbb{H}). In the middle of the 20th century, *Hestenes* repopularized the term geometric algebra [4,5].

On the other hand, although rarely used explicitly, a geometric representation of complex numbers is implicitly based on its structure of the *Euclidean* 2-dimensional vector space. If the binary operation of the product $\bar{w}z$, of two complex numbers \bar{w} and z , is considered as the sum of the inner product $w \circ z = (\bar{w}z + w\bar{z})/2 = \mathbf{1}Re(\bar{w}z)$ and outer product $w \wedge z = (\bar{w}z - w\bar{z})/2 = \mathbf{i}Im(\bar{w}z)$, where $\mathbf{1} = (1, i0)$ and $\mathbf{i} = (0, i)$, and i is an imaginary unit, it can be said that $\bar{w}z$ is in the form of a geometric product of two *ivectors* (two complex numbers), as two geometric objects belonging to the *ivector* field \mathbb{C} (to the field of complex numbers \mathbb{C}). For any complex number z , its absolute value $|z|$ is its *Euclidean* norm denoted by ρ , and the argument $\arg z$ of z is the polar angle φ . Since ordered pairs represent both complex numbers and vectors, the binary operation of the product of two complex numbers (two ordered pairs), in the form of the geometric product $\bar{w}z$ ($\bar{w}z = w \circ z + w \wedge z$), will be the basis for modifying *Grassmann's* geometric product of vectors, which is defined as the sum of the inner (scalar) and outer (vector) products of two vectors. By this modification, the geometric product of two vectors becomes commutative, similar to the product of complex numbers themselves, which still supports vector division.

It is an old and interesting problem to obtain a natural extension of complex numbers, especially to three-dimensional complex numbers. *Hamilton* interpreted complex numbers, via couple numbers (ordered pairs of two real numbers), as points in the *Euclidean* plane [3], and he was looking for a way to do the same for points in three dimensional space. Points in space are triples of numbers. *Hamilton* had known how to add and subtract triples. However, for a long time, he had been stuck on the problem of multiplication and division. In fact, *Frobenius* proved, in 1877., that for a division algebra over the real numbers to be finite dimensional and associative, it cannot be three dimensional. There are only three such division algebras: \mathbb{R} , \mathbb{C} and \mathbb{H} , which have dimension: 1, 2 and 4, respectively. *Segre* [15] described bicomplex numbers, as points in a 4-dimensional space. Unlike *Hamilton's* quaternions, which are non commutative and form a division algebra, bicomplex numbers are commutative and do not form a division algebra.

All told, many famous mathematicians have studied how to define multicomplex numbers and the corresponding function theory. Following this, and on the basis of the redefined geometric product of two vectors in the *Euclidean* plane, this paper presents the algebraic structure of the $3\mathbb{D}$ field of complex vectors, as well as the corresponding integral identities.

The idea, on the basis of which the algebraic structure of the $3\mathbb{D}$ field of complex vectors is defined, is as follows: At the beginning, a $2\mathbb{D}$ field of complex vectors $\mathbf{V}_{\mathbb{C}}$ is introduced, which corresponds to the field of complex numbers \mathbb{C} , defined as a *Cartesian* product of one $1\mathbb{D}$ real and one $1\mathbb{D}$ i -real vector space with a commutative geometric product of elements. Following the same pattern, in the next section a $3\mathbb{D}$ field of complex vectors is introduced, which corresponds to the *Cartesian* product of one $3\mathbb{D}$ real and one $3\mathbb{D}$ i -real vector space with a commutative geometric product of elements, defined as the sum of the geometric products of the elements in three complex planes. Each of these complex planes is the *Cartesian* products of one $1\mathbb{D}$ real and one $1\mathbb{D}$ i -real vector space, which are vector subspaces of $3\mathbb{D}$ real and $3\mathbb{D}$ i -real vector space, respectively. Clearly, one can take the *Cartesian* product of one $2\mathbb{D}$ real and one $2\mathbb{D}$ i -real vector space in the same way. In that case, the geometric products of the elements in two complex planes are summed up, in order to obtain the commutative geometric product of the elements in this vector space.

2. The $2\mathbb{D}$ Field of Complex Vectors $\mathbf{V}_{\mathbb{C}}$

The ordered pairs $\mathbf{1} = (1, i0)$ and $\mathbf{i} = (0, i)$ form the basis of a 2-dimensional real/real vector ($2\mathbb{D}$ i -vector) space $V_{\mathbb{C}}$, [6,9]. It is obvious that $V_{\mathbb{C}}$ is the *Cartesian* product $V_{\mathbb{R}} \times iV_{\mathbb{R}}$ of a one-dimensional real vector space and a one-dimensional i -real vector space, and such as it is an additive *Abelian* (commutative) group of i -vectors $(x, iy) = \mathbf{1}x + \mathbf{i}y$. On the other hand, it is possible to complete the i -vector space $V_{\mathbb{C}}$ with a binary operation of the product of two i -vectors (a, ib) and (c, id) , which corresponds to the matrix product, in such a manner that

$$\begin{aligned} (a, ib)(c, id) &\Leftrightarrow \begin{bmatrix} a & ib \\ ib & a \end{bmatrix} \begin{bmatrix} c & id \\ id & c \end{bmatrix} = \\ &= \begin{bmatrix} ac - bd & i(ad + bc) \\ i(ad + bc) & ac - bd \end{bmatrix} \Leftrightarrow (ac - bd, i(ad + bc)). \end{aligned} \quad (1)$$

The inverse element $(x, iy)^{-1} = (x, -iy)/(x^2 + y^2)$ of the space $V_{\mathbb{C}}^* = V_{\mathbb{C}} \setminus \{(0, i0)\}$ corresponds to the inverse matrix $\begin{bmatrix} x & iy \\ iy & x \end{bmatrix}^{-1}$. Here, both the commutative and associative axioms of multiplication and the distributive axiom are satisfied. Thus, the i -vector space $V_{\mathbb{C}}$ becomes the field of complex numbers (i -vectors) \mathbb{C} , to which the field of complex vectors $\mathbf{V}_{\mathbb{C}}$ corresponds. The ordered pairs of both a real and an imaginary vector: $\mathbf{e} = (\mathbf{1}_x, i\mathbf{0}_y)$ and $\hat{\mathbf{e}} = (\mathbf{0}_x, i\mathbf{1}_y)$, such that the unit vectors $\mathbf{1}_x$ ($\mathbf{0}_x = \mathbf{0}\mathbf{1}_x$) and $\mathbf{1}_y$ ($\mathbf{0}_y = \mathbf{0}\mathbf{1}_y$) are orthonormal basis vectors of the real plane \mathbf{V}_2 , form an orthonormal basis vectors of the field $\mathbf{V}_{\mathbb{C}}$, so that $\mathbf{1} \Leftrightarrow \mathbf{e}$ and $\mathbf{i} \Leftrightarrow \hat{\mathbf{e}}$. The complex vectors $\% = x\mathbf{e} + y\hat{\mathbf{e}}$, as elements of $\mathbf{V}_{\mathbb{C}}$, correspond to the complex numbers $z \in \mathbb{C}$. If $(x, -iy) \Leftrightarrow x\mathbf{e} - y\hat{\mathbf{e}} = \bar{\%}$, then $q = \|\bar{\%}\| = \|(x, -iy)\| = \sqrt{x^2 + y^2}$ is the norm on the fields $\mathbf{V}_{\mathbb{C}}$ and \mathbb{C} . In addition, $\bar{\%}\% \Leftrightarrow (x, -iy)(x, iy) = q^2\mathbf{1} \Leftrightarrow q^2\mathbf{e}$ and $\% = q^2\%^{-1}$. It is quite clear that the inverse element $\%^{-1}$ allows division by a vector in $\mathbf{V}_{\mathbb{C}}$. On the other hand, on the basis of the above-mentioned geometric product $(a, -ib)(c, id) = (ac + bd, i(ad - bc))$, of two complex numbers (i -vectors) $(a, -ib)$ and (c, id) , the corresponding geometric product of two complex vectors $\bar{\%}_1 = a\mathbf{e} - b\hat{\mathbf{e}}$ and $\%_2 = ce + d\hat{\mathbf{e}}$ in $\mathbf{V}_{\mathbb{C}}$, can be defined as follows

$$\begin{aligned} \bar{\%}_1\%_2 &= \frac{1}{2}(\bar{\%}_1\%_2 + \%_1\bar{\%}_2) + \frac{1}{2}(\bar{\%}_1\%_2 - \%_1\bar{\%}_2) := \\ &:= (\bar{\%}_1 \cdot \%_2)\mathbf{e} + (\%_1 \times \%_2) \times \mathbf{e} = (\%_1 \cdot \bar{\%}_2)\mathbf{e} - (\bar{\%}_1 \times \%_2) \times \mathbf{e}, \end{aligned} \quad (2)$$

where $\bar{\%}_1 \cdot \%_2 = \|\%_1\| \|\%_2\| \cos \theta$ and $(\%_1 \times \%_2) \times \mathbf{e} = \|\%_1\| \|\%_2\| \sin \theta \hat{\mathbf{e}}$, which is obviously commutative. Here,

$$\frac{1}{2}(\bar{\%}_1 \%_2 + \%_1 \bar{\%}_2) := (\bar{\%}_1 \cdot \%_2) \mathbf{e} = \%_1 \circ \%_2 \Leftrightarrow (a, ib) \circ (c, id) \text{ and} \quad (3)$$

$$\frac{1}{2}(\bar{\%}_1 \%_2 - \%_1 \bar{\%}_2) := (\%_1 \times \%_2) \times \mathbf{e} = \%_1 \wedge \%_2 \Leftrightarrow (a, ib) \wedge (c, id). \quad (4)$$

So, the dot and cross products of two complex vectors $\%_1 = a\mathbf{e} + b\hat{\mathbf{e}}$ and $\%_2 = c\mathbf{e} + d\hat{\mathbf{e}}$ in $\mathbf{V}_{\mathbb{C}}$ are as follows

$$\bar{\%}_1 \cdot \%_2 = ac + bd \text{ and } \%_1 \times \%_2 = (ad - bc)\mathbf{n}, \quad (5)$$

where $\mathbf{n} = \mathbf{e} \times \hat{\mathbf{e}}$. Since $(a, -ib)(c, id) = (a, ib) \circ (c, id) + (a, ib) \wedge (c, id)$, it follows that

$$\begin{aligned} & [(a, ib) \circ (c, id)]^2 - [(a, ib) \wedge (c, id)]^2 = \\ & = (a, -ib)(c, id)(c, -id)(a, ib) = \|(a, ib)\|^2 \|(c, id)\|^2 \mathbf{1}. \end{aligned} \quad (6)$$

3. The $3\mathbb{D}$ Field of Complex Vectors

Let $\mathbf{1}_x, \mathbf{1}_y$ and $\mathbf{1}_z$ be orthonormal basis vectors of a $3\mathbb{D}$ space of real vectors \mathbf{V}_3 . Then, three pairs of ordered pairs: $\mathbf{e}_1 = (\mathbf{1}_x, i\mathbf{0}_y)$ and $\hat{\mathbf{e}}_1 = (\mathbf{0}_x, i\mathbf{1}_y)$, $\mathbf{e}_2 = (\mathbf{1}_y, i\mathbf{0}_z)$ and $\hat{\mathbf{e}}_2 = (\mathbf{0}_y, i\mathbf{1}_z)$, $\hat{\mathbf{e}}_3 = (\mathbf{1}_z, i\mathbf{0}_x)$ and $\hat{\mathbf{e}}_3 = (\mathbf{0}_z, i\mathbf{1}_x)$, form orthonormal bases of three $2\mathbb{D}$ fields of complex vectors $\%_{\alpha}$, which are component fields of a $3\mathbb{D}$ field of complex vectors with elements

$$\% = x_{\alpha} \mathbf{e}^{\alpha} + \hat{x}_{\alpha} \hat{\mathbf{e}}^{\alpha} = \%_{\alpha} \mathbf{e}^{\alpha}, \quad (7)$$

where $\%_{\alpha} = \langle x\mathbf{e} + \hat{x}\hat{\mathbf{e}} \rangle_{\alpha}$ are complex vectors in the component fields, and here, as in the following text, the index α ($\alpha = 1, 2, 3$), repeated as subscript and superscript in the products, represents the summation over the range of the index α , according to the *Einstein* summation convention.

The commutative geometric product of two $3\mathbb{D}$ complex vectors \mathbf{a} and \mathbf{b} is defined as the sum of the geometric products of the component vectors $\bar{\mathbf{a}}_{\alpha} = \langle a\mathbf{e} - \hat{a}\hat{\mathbf{e}} \rangle_{\alpha}$ and $\mathbf{b}_{\alpha} = \langle b\mathbf{e} + \hat{b}\hat{\mathbf{e}} \rangle_{\alpha}$, as follows

$$\bar{\mathbf{a}}\mathbf{b} = \mathbf{a} \circ \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \frac{\langle \bar{\mathbf{a}}\mathbf{b} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{\langle \mathbf{a} \circ \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} \text{ and} \quad (8)$$

$$\mathbf{a}\bar{\mathbf{b}} = \mathbf{a} \circ \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = \frac{\langle \mathbf{a}\bar{\mathbf{b}} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{\langle \bar{\mathbf{b}}\mathbf{a} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{\langle \mathbf{b} \circ \mathbf{a} + \mathbf{b} \wedge \mathbf{a} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}}. \quad (9)$$

Clearly, $\mathbf{a} \circ \mathbf{b} = (\bar{\mathbf{a}}\mathbf{b} + \mathbf{a}\bar{\mathbf{b}})/2$ and $\mathbf{a} \wedge \mathbf{b} = (\bar{\mathbf{a}}\mathbf{b} - \mathbf{a}\bar{\mathbf{b}})/2$. In addition, the vector

$$\mathbf{a}^{-1} = \frac{1}{\sqrt[2]{3}} \mathbf{a}^{-1} \mathbf{e}^{\alpha}, \quad (10)$$

where $\mathbf{a}^{-1} = \langle \bar{\mathbf{a}}/\|\mathbf{a}\|^2 \rangle_{\alpha}$ and $\|\mathbf{a}\|_{\alpha}^2 = (a^2 + \hat{a}^2)_{\alpha}$, such that

$$\mathbf{a}^{-1}\mathbf{a} = \frac{\langle \mathbf{a}^{-1}\mathbf{a} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{\mathbf{e}_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}}, \quad (11)$$

is the inverse vector of the $3\mathbb{D}$ complex vector \mathbf{a} , which allows division by the vector in the $3\mathbb{D}$ field of complex vectors. On the other hand, if $(\hat{a}\hat{b} - \hat{b}\hat{a})$ is denoted by the bracket $[a, \hat{b}]$, it follows that

$$\mathbf{a} \wedge \mathbf{b} = \frac{\langle \mathbf{a} \wedge \mathbf{b} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{\langle (\mathbf{a} \times \mathbf{b}) \times \mathbf{e} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{[a, \hat{b}]_{\alpha} \hat{\mathbf{e}}^{\alpha}}{\sqrt[2]{3}} \text{ and} \quad (12)$$

$$\mathbf{a} \circ \mathbf{b} = \frac{\langle \mathbf{a} \circ \mathbf{b} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{\langle (\bar{\mathbf{a}} \cdot \mathbf{b}) \mathbf{e} \rangle_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}} = \frac{(ab + \hat{a}\hat{b})_{\alpha} \mathbf{e}^{\alpha}}{\sqrt[2]{3}}. \quad (13)$$

3.1. Differential forms in $\mathbf{V}_{\mathbb{C}}$

To represent a complex vector $\% \in \mathbf{V}_{\mathbb{C}}$ in polar form, one introduces a vector analogue $\hat{\mathbf{c}}\mathbf{s}$ of the shorthand notation cis for the algebraic operator $\cos \cdot + i \sin \cdot$, as follows

$$\hat{\mathbf{c}}\mathbf{s} \cdot = \mathbf{e} \cos \cdot + \hat{\mathbf{e}} \sin \cdot. \quad (14)$$

Let $\%_0 = \hat{\mathbf{c}}\mathbf{s}\varphi = \exp(\hat{\mathbf{e}}\varphi)$, where $\exp(\hat{\mathbf{e}}\cdot)$ is the exponential form of the operator $\hat{\mathbf{c}}\mathbf{s} \cdot$, be a radial unit complex vector in $\mathbf{V}_{\mathbb{C}}$. Therefore, $\log \% = \ln \varrho\mathbf{e} + \varphi\hat{\mathbf{e}}$, where $\% = \varrho\%_0$ and $2\varphi\hat{\mathbf{e}} = 2 \log \%_0 = \log(\%_0/\bar{\%}_0)$, is a complex vector logarithmic function, and $\text{Log } \% = \ln \varrho\mathbf{e} + (\varphi \pm 2\pi n)\hat{\mathbf{e}}$, $n \in \mathbb{N}$. If $\%_{\perp} = \%\hat{\mathbf{e}}$ and $\hat{\%}_0 = -\%_{0\perp}^{-1} = \hat{\mathbf{s}}\mathbf{e}\mathbf{c}\varphi = \hat{\mathbf{e}}\bar{\%}_0$, then the ordered pair of unit vectors $(\bar{\%}_0, \hat{\%}_0)$ is the inverse orthonormal basis with respect to the orthonormal basis $(\%_0, \%_{0\perp})$ of the field of complex vectors $\mathbf{V}_{\mathbb{C}}$. For an arbitrary vector $\mathbf{a} \in \mathbf{V}_{\mathbb{C}}$, the vector $\mathbf{a}\%_0$ is the rotated vector \mathbf{a} , in the positive mathematical direction, by the angle φ , and the vector $\mathbf{a}\%_{0\perp}$ by the angle $\pi/2 + \varphi$. The geometric products of the vector \mathbf{a} with the inverse basis vectors $\bar{\%}_0$ and $\hat{\%}_0$ rotate \mathbf{a} by the angles $-\varphi$ and $\pi/2 - \varphi$, respectively, in the positive mathematical direction.

If $d = dr\partial_r + d\varphi\partial_{\varphi}$ is a differential operator, then $d\% = dr\%_0 + d\varphi\%_{\perp}$. Hence, $d\%_{\perp} = \hat{\mathbf{e}}d\% = dr\%_{0\perp} - d\varphi\%$ and $d\bar{\%} = \hat{\mathbf{e}}d\bar{\%} = \hat{\mathbf{e}}(dr\bar{\%}_0 - d\varphi\hat{\%}_0) = dr\hat{\%}_0 + d\varphi\bar{\%}_0$. Since $2\varrho \cos \varphi\mathbf{e} = \% + \bar{\%}$ and $2\varrho \sin \varphi\hat{\mathbf{e}} = \% - \bar{\%}$, the vector operators of partial derivatives are introduced, as a vector analogue of the *Virtinger* operators [16],

$$\bar{\partial}_{\%} = \partial_{\%}\varrho\partial_{\varrho} + \partial_{\%}\varphi\partial_{\varphi} = \frac{1}{2}(\bar{\%}_0\partial_{\varrho} - \frac{\hat{\%}_0}{\varrho}\partial_{\varphi}) \text{ and } \bar{\partial}_{\bar{\%}} = \bar{\partial}_{\bar{\%}} = \frac{1}{2}(\%_0\partial_{\varrho} + \frac{\%_{0\perp}}{\varrho}\partial_{\varphi}), \quad (15)$$

where $\partial_{\%}\varphi = \cos^2 \varphi \partial_{\%} \tan \varphi = [(\% + \bar{\%})/(2\varrho)]^2 [(-2\bar{\%})/(\% + \bar{\%})^2]\hat{\mathbf{e}} = -\hat{\%}_0/(2\varrho)$ and $2\partial_{\%}\varrho = \partial_{\%}\varrho^2/\varrho = \bar{\%}_0$. It is important to emphasize that when geometric products and geometric quotients are differentiated, the same rules apply as when ordinary products and quotients are differentiated, so that $d(\%/ \bar{\%}) = (\bar{\%}d\% - \%d\bar{\%})/\bar{\%}^2$. Let's prove this,

$$\begin{aligned} d\frac{\%}{\bar{\%}} &= (d\frac{1}{\varrho^2})\%^2 + \frac{1}{\varrho^2}d\%^2 = -2(\frac{\%^2}{\varrho^3}d\varrho - \frac{1}{\varrho^2}\%d\%) = \\ &= -2[\frac{\%^2}{2\varrho^4}(\bar{\%}d\% + \%d\bar{\%}) - \frac{1}{\varrho^2}\%d\%] = -2[\frac{1}{2\bar{\%}^2}(\bar{\%}d\% + \%d\bar{\%}) - \frac{1}{\bar{\%}^2}\%d\%] = \frac{\bar{\%}d\% - \%d\bar{\%}}{\bar{\%}^2}. \end{aligned} \quad (16)$$

Definition 1. The geometric product $d\%\bar{\partial}_{\%}$ is an operator of a $1\mathbb{D}$ differential form.

The symmetric and antisymmetric parts of the geometric product $d\%\bar{\partial}_{\%}$ are as follows

$$d\%\bar{\partial}_{\%} + d\bar{\%}\bar{\partial}_{\bar{\%}} = 2d\% \circ \bar{\partial}_{\bar{\%}} = \mathbf{e}(d\varrho\partial_{\varrho} + d\varphi\partial_{\varphi}) \text{ and} \quad (17)$$

$$d\%\bar{\partial}_{\%} - d\bar{\%}\bar{\partial}_{\bar{\%}} = -2d\% \wedge \bar{\partial}_{\bar{\%}} = \hat{\mathbf{e}}(\varrho d\varphi\partial_{\varrho} - \frac{1}{\varrho}d\varrho\partial_{\varphi}). \quad (18)$$

Therefore,

$$2\%_0d\%\bar{\partial}_{\%} = \%_0(d\varrho\partial_{\varrho} + d\varphi\partial_{\varphi}) + \%_{0\perp}(\varrho d\varphi\partial_{\varrho} - \frac{1}{\varrho}d\varrho\partial_{\varphi}) = \%_0d + \%_{0\perp}\hat{\mathbf{d}}, \quad (19)$$

where $\mathbf{d} = \%_0d$ and $\mathbf{d}_{\perp} = \%_{0\perp}\hat{\mathbf{d}}$ are the radial and transverse components of $d\%\bar{\partial}_{\%}$, respectively. Accordingly, since $2\varphi\hat{\mathbf{e}} = \log(\%/ \bar{\%})$, it follows that

$$\mathbf{d}\varphi = \%_0d\varphi = \%_0(\bar{\partial}_{\%}\varphi d\% + \bar{\partial}_{\bar{\%}}\varphi d\bar{\%}) = -\frac{\%_{0\perp}}{2}d \log\left(\frac{\%}{\bar{\%}}\right) \text{ and} \quad (20)$$

$$\mathbf{d}q = \%_0 dq = \%_0 (\partial_{\%_0} q d\% + \partial_{\bar{\%}_0} q d\bar{\%}) = \%_0 d(\% \bar{\%})^{\frac{1}{2}} = \frac{\%}{2} d \log(\% \bar{\%}), \quad (21)$$

which leads to the following vector differential identity

$$\begin{aligned} \%_{0\perp} d q d \varphi &= \frac{\%}{4} d \log(\% \bar{\%}) d \log(\frac{\%}{\bar{\%}}) = \frac{\%}{4 \varrho^4} (\bar{\%} d\% + \% d\bar{\%})(\bar{\%} d\% - \% d\bar{\%}) = \\ &= \frac{\%}{4 \varrho^2} (\bar{\mathbf{d}}\% + \mathbf{d}\bar{\%})(\bar{\mathbf{d}}\% - \mathbf{d}\bar{\%}) = \frac{\%}{4 \varrho^2} [(\bar{\mathbf{d}}\%)^2 - (\mathbf{d}\bar{\%})^2]. \end{aligned} \quad (22)$$

Consequently,

$$\frac{\%_0}{2} (\mathbf{d}\bar{\%} \wedge \bar{\mathbf{d}}\%) = \frac{\%}{4 \varrho} [(\bar{\mathbf{d}}\%)^2 - (\mathbf{d}\bar{\%})^2] = \%_{\perp} d q d \varphi, \quad (23)$$

since $\overline{\mathbf{d}\bar{\%}} = \bar{\%}_0 d\% = \bar{\mathbf{d}}\%$. The vector identity just derived can be obtained explicitly via the *Jacobian determinant* of the bijective mapping $\mathbf{V}_{\mathbb{C}} \rightarrow \mathbf{V}_{\mathbb{C}}$, defined by the system of vector equations $2 \ln q e = \log(\% \bar{\%})$ and $2 \varphi \hat{\mathbf{e}} = \log(\% / \bar{\%})$, as follows

$$\mathbf{J} = \begin{vmatrix} \partial_r \log(\% \bar{\%}) & \partial_\varphi \log(\% \bar{\%}) \\ \partial_r \log(\% / \bar{\%}) & \partial_\varphi \log(\% / \bar{\%}) \end{vmatrix} = \begin{vmatrix} 2 \varrho^{-1} \mathbf{e} & \mathbf{0} \\ \mathbf{0} & 2 \hat{\mathbf{e}} \end{vmatrix} = \frac{4}{\varrho} \hat{\mathbf{e}}. \quad (24)$$

In this case, $4 \%_{0\perp} d q d \varphi = \% \mathbf{J} d q d \varphi = \% d \log(\% \bar{\%}) d \log(\% / \bar{\%})$, which leads to (22). The complex vector $d\mathbf{S} = (\mathbf{d}\bar{\%} \wedge \bar{\mathbf{d}}\%) / 2 = \varrho d q d \varphi \%_{0\perp} \bar{\%}_0 = d\mathbf{S} \hat{\mathbf{e}}$, corresponding to the i -vector $\varrho d q d \varphi \hat{\mathbf{e}}$, is the *Lebesgue vector measure* of the infinitesimal surface in the field $\mathbf{V}_{\mathbb{C}}$.

Definition 2. Let $\partial_{\%_0}^2 \bar{\%}_0 = \partial_{\%_0} \partial_{\bar{\%}_0}$. Then, the geometric product $d\mathbf{S} \partial_{\%_0}^2 \bar{\%}_0$ is an operator of a $2\mathbb{D}$ differential form.

3.2. Fundamental Theorem of Integral Calculus in the Field $\mathbf{V}_{\mathbb{C}}$

Let γ be a closed smooth *Jordan curve*, bounding an arbitrary simply connected region G in $\mathbf{V}_{\mathbb{C}}$, and $\%_\gamma$ be a point on γ , surrounded by a circle $c(\%_\gamma, \varepsilon)$ with center at $\%_\gamma$ and arbitrarily small radius ε , which intersects the curve γ at the points $\%_{\gamma_1}$ and $\%_{\gamma_2}$. Similarly, an arbitrary point $\%_G$ inside G is surrounded by circle $c(\%_G, \varepsilon)$, which is connected to the circle $c(\%_\gamma, \varepsilon)$ by two parallel straight line segments $l_{\delta_\varepsilon}^1$ and $l_{\delta_\varepsilon}^2$, at a distance $\delta_\varepsilon \ll \varepsilon$ from each other. The region S , inside which are the points $\%_{\gamma_1}$ and $\%_{\gamma_2}$, and which is bounded by the segments $l_{\delta_\varepsilon}^1$ and $l_{\delta_\varepsilon}^2$ as well as the arc segments on the boundaries $\partial c(\%_\gamma, \varepsilon)$ and $\partial c(\%_G, \varepsilon)$ of the circles $c(\%_\gamma, \varepsilon)$ and $c(\%_G, \varepsilon)$, whose endpoints are the intersection points of the boundaries $\partial c(\%_\gamma, \varepsilon)$ and $\partial c(\%_G, \varepsilon)$ with the segments $l_{\delta_\varepsilon}^1$ and $l_{\delta_\varepsilon}^2$, is the so-called *residue region*. If $G_\varepsilon^+ = G \cup S$, the contour integration operators are as follows

$$\int_{\gamma_\varepsilon^+}^\circ d\% \partial_{\%_0} = \int_{\partial G_\varepsilon^+}^\circ d\% \partial_{\%_0} \text{ and } \int_{\gamma_\varepsilon^-}^\circ d\% \partial_{\%_0} = \int_{\partial G_\varepsilon^+}^\circ d\% \partial_{\%_0} - \int_{\partial c(\%_\gamma, \varepsilon)}^\circ d\% \partial_{\%_0}, \quad (25)$$

where $\partial G_\varepsilon^+ = \gamma_\varepsilon^+$ is the boundary of G_ε^+ . Based on the additive property of definite integrals, in the limit, as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned} vt \int_{\gamma^+}^\circ d\% \partial_{\%_0} &= \lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_\varepsilon^+}^\circ d\% \partial_{\%_0} = \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\partial G_{\delta_\varepsilon}}^\circ d\% \partial_{\%_0} + \int_{\partial S}^\circ d\% \partial_{\%_0} \right) = \\ &= vp \int_{\gamma}^\circ d\% \partial_{\%_0} + \lim_{\varepsilon \rightarrow 0^+} \int_{c(\%_\gamma, \varepsilon)}^{\%_{\gamma_2} \widehat{\%_{\gamma_1}}} d\% \partial_{\%_0} = vt \int_{\gamma^-}^\circ d\% \partial_{\%_0} + \lim_{\varepsilon \rightarrow 0^+} \int_{\partial c(\%_\gamma, \varepsilon)}^\circ d\% \partial_{\%_0}, \end{aligned} \quad (26)$$

where vp and vt denote the principal and total integral values [9]-[14], respectively, and $G_{\delta_\varepsilon} = G \setminus \text{int}.S$ ($\text{int}.S$ is the interior of S).

On the one hand, as a vector analogue of the so-called areolar derivative of complex analysis, defined by *Pompeiu* [8], the vector differential operator $\partial_{\%_0\%_1}^2$ is the gradient (the plane derivative) of $\partial_{\%_0}$

$$\lim_{G \rightarrow \%_G} \frac{1}{2\mathbf{S}_\gamma} \int_{\gamma}^{\circ} d\% \partial_{\%_0} = \partial_{\%_0\%_1}^2 |_{\%_G}, \quad (27)$$

where $2\mathbf{S}_\gamma = \int_{\gamma}^{\circ} \% \wedge d\%$, so that $\lim_{G \rightarrow \%_G} \int_{\gamma}^{\circ} d\% \partial_{\%_0} = 2d\mathbf{S} \partial_{\%_0\%_1}^2 |_{\%_G}$. On the other hand, on the basis of the result of the *Kelvin-Stokes* theorem (*Green's* theorem) [6],

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{\partial G_{\delta_\varepsilon}}^{\circ} d\% \partial_{\%_0} &= \lim_{\varepsilon \rightarrow 0^+} \int_{\partial G_{\delta_\varepsilon}}^{\circ} [(d\% \circ \text{grad}) - (d\% \wedge \text{grad})] = \\ &= \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \int_G \hat{\%} d\text{rd}\varphi (\%_0 \text{divgrad} + \%_0 \times \text{curlgrad}) = v p \int_G (\mathbf{d}\bar{\%} \wedge \bar{\mathbf{d}}\%) \partial_{\%_0\%_1}^2. \end{aligned} \quad (28)$$

Here,

$$\text{divgrad} = \frac{4}{\varrho^2} (\%_0 \partial_{\%_0} \cdot \bar{\%}_0 \partial_{\%_0}) = \frac{1}{\varrho^2} [\varrho \partial_\varrho (\varrho \partial_\varrho) + \partial_{\varphi^2}^2] \text{ and} \quad (29)$$

$$\text{curlgrad} = \frac{4}{\varrho^2} (\bar{\%}_0 \partial_{\%_0} \times \%_0 \partial_{\%_0}) = \frac{\%_0 \times \%_0 \perp}{\varrho^2} (\partial_{\varphi\varrho}^2 - \partial_{\varrho\varphi}^2) = \mathbf{0}, \quad (30)$$

so that

$$\begin{aligned} \partial_{\%_0\%_1}^2 &= \partial_{\%_0} \partial_{\%_0} = \frac{\bar{\%}_0 \%_0}{\varrho^2} (\%_0 \partial_{\%_0} \cdot \bar{\%}_0 \partial_{\%_0}) + \frac{\bar{\%}_0 \%_0}{\varrho^2} \times (\bar{\%}_0 \partial_{\%_0} \times \%_0 \partial_{\%_0}) = \\ &= \frac{\%_0}{4\varrho^2} [\%_0 [\varrho \partial_\varrho (\varrho \partial_\varrho) + \partial_{\varphi^2}^2] + \%_0 \perp (\partial_{\varphi\varrho}^2 - \partial_{\varrho\varphi}^2)] = \frac{\%_0}{4} (\%_0 \text{divgrad} + \%_0 \times \text{curlgrad}). \end{aligned} \quad (31)$$

Consequently,

$$v t \int_{\gamma^+}^{\circ} d\% \partial_{\%_0} = v t \int_{G^+} (\mathbf{d}\bar{\%} \wedge \bar{\mathbf{d}}\%) \partial_{\%_0\%_1}^2, \quad (32)$$

where $v t \int_{G^+} (\mathbf{d}\bar{\%} \wedge \bar{\mathbf{d}}\%) \partial_{\%_0\%_1}^2 = v p \int_G (\mathbf{d}\bar{\%} \wedge \bar{\mathbf{d}}\%) \partial_{\%_0\%_1}^2 + \lim_{\varepsilon \rightarrow 0^+} \int_{\partial S}^{\circ} d\% \partial_{\%_0}$. The vector integral operator

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\partial S}^{\circ} d\% \partial_{\%_0} = 2\pi \hat{\mathbf{e}} \text{Res} \partial_{\%_0} |_{\%_G} + \int_{I^+} d\% \partial_{\%_0} - \int_{I^-} d\% \partial_{\%_0} + 2\pi \hat{\mathbf{e}} \text{Res} \partial_{\%_0} |_{\%_\gamma}, \quad (33)$$

where

$$\int_{I^+} d\% \partial_{\%_0} = \lim_{\varepsilon \rightarrow 0^+} \int_{I_{\delta_\varepsilon}^+} d\% \partial_{\%_0}, \quad \int_{I^-} d\% \partial_{\%_0} = \lim_{\varepsilon \rightarrow 0^+} \int_{I_{\delta_\varepsilon}^-} d\% \partial_{\%_0}, \quad (34)$$

$$2\pi \hat{\mathbf{e}} \text{Res} \partial_{\%_0} |_{\%_\gamma} = \lim_{\varepsilon \rightarrow 0^+} \int_{\partial C(\%_\gamma, \varepsilon)}^{\circ} d\% \partial_{\%_0} = 2d\mathbf{S} \partial_{\%_0\%_1}^2 |_{\%_\gamma} \text{ and} \quad (35)$$

$$2\pi \hat{\mathbf{e}} \text{Res} \partial_{\%_0} |_{\%_G} = \lim_{\varepsilon \rightarrow 0^+} \int_{\partial C(\%_G, \varepsilon)}^{\circ} d\% \partial_{\%_0} = 2d\mathbf{S} \partial_{\%_0\%_1}^2 |_{\%_G}, \quad (36)$$

is the vector residue operator in G . Obviously, the operator $\text{Res} \partial_{\%_0}$ is a synonym for $d\mathbf{S} \partial_{\%_0\%_1}^2$.

By choosing two points, one on the contour γ and the other inside the region G , the generality of the previously obtained results is not lost, so that the fundamental theorem of integral calculus in $\mathbf{V}_\mathbb{C}$ can be formulated in what follows. Previously, let $\bullet_{\%_0}$ be a vector differential form $d\% \partial_{\%_0} \bullet$, obtained by applying $d\% \partial_{\%_0}$ to an arbitrary uniform scalar or vector field \bullet in $\mathbf{V}_\mathbb{C}$, [7].

Theorem 1. Let γ be a closed smooth Jordan curve, bounding an arbitrary simply connected region G in $\mathbf{V}_{\mathbb{C}}$. For an arbitrary uniform scalar or vector field \bullet in $\mathbf{V}_{\mathbb{C}}$, which is regular almost everywhere on G and whose vector differential form $!_{\%}$ is totally integrable on γ , there holds

$$vt \int_{\gamma^+}^{\circ} !_{\%} = 2\pi \hat{\mathbf{e}} \sum_{\%_i \in G} \text{Res} \partial_{\%} \bullet(\%_i) = vt \int_{G^+} !_{\mathbf{S}}, \quad (37)$$

where $!_{\mathbf{S}} = 2d\mathbf{S} \partial_{\%}^2 \bullet$.

Integral formula in the theorem formulated above is a slightly generalized vector analogue of the *Cauchy-Pompeiu* integral formula, [17]. The sum of $\text{Res} \partial_{\%} \bullet$, on the compact set of points $G \setminus E$, at which the field \bullet is regular ($\text{Res} \partial_{\%} \bullet$ is identically zero on the set $G \setminus E$), is $vp \int_G !_{\mathbf{S}}$. So, $vt \int_{G^+} !_{\mathbf{S}} = vp \int_G !_{\mathbf{S}} + 2\pi \hat{\mathbf{e}} \sum_{\%_i \in E} \text{Res} \partial_{\%} \bullet(\%_i)$, where $E \subset G$ is the singular set with *Lebesgue* measure zero of the field \bullet . The sum on the right-hand side of the last integral equality can be in the indeterminate form of the difference of two infinities, which is equal to the finite integral value on the left-hand side.

3.3. Integral Calculus Formulas in the 3D Field of Complex Vectors

Let G_{α} be simply connected regions bounded by closed smooth *Jordan* curves γ_i , such that they are projections of a smooth surface S onto the component fields of the 3D field of complex vectors. Then, the 3D differential forms are as follows

$$!_{\%} = \mathbf{F} d\bar{\%} = \frac{\langle \mathbf{F} d\bar{\%} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}}{\sqrt[2]{3}} \text{ and}$$

$$!_{\mathbf{S}} = -2 \partial_{\%} \mathbf{F} d\hat{\mathbf{S}} = -2 \frac{\langle \partial_{\%} \mathbf{F} d\hat{\mathbf{S}} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}}{\sqrt[2]{3}}, \quad (38)$$

where $\mathbf{F} = \mathbf{F}_{\alpha} (\bar{\%}_0 \%_0)^{\alpha} = \langle F\%_0 + F_{\perp} \%_{0\perp} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}$ is an arbitrary uniform vector field in the 3D field of complex vectors, $d\bar{\%} = d\bar{\%}_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}$ and $2d\hat{\mathbf{S}}_{\alpha} = \langle d\bar{\%} \wedge d\% \rangle_{\alpha}$. According to *Theorem 1.*, if the vector field \mathbf{F} is regular almost everywhere on S , and the vector differential forms $\bar{!}_{\%} = \bar{\mathbf{F}} d\bar{\%}$ and $!_{\bar{\%}}$ are totally integrable on ∂S , there holds

$$\langle vt \int_{\partial S^+}^{\circ} \mathbf{F} \circ d\bar{\%} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha} = -2 \langle vt \int_{G^+} (\partial_{\bar{\%}} \wedge \mathbf{F}) d\hat{\mathbf{S}} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha} \text{ and} \quad (39)$$

$$\langle vt \int_{\partial S^+}^{\circ} \mathbf{F} \wedge d\mathbf{r} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha} = 2 \langle vt \int_{G^+} (\partial_{\bar{\%}} \circ \mathbf{F}) d\hat{\mathbf{S}} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}, \quad (40)$$

so that

$$vt \int_{\partial S^+}^{\circ} \bar{!}_{\%} = vt \int_{\partial S^+}^{\circ} \bar{\mathbf{F}} d\bar{\%} = \frac{1}{\sqrt[2]{3}} [vt \int_{\gamma^+}^{\circ} \langle \bar{\mathbf{F}} d\bar{\%} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}] =$$

$$= \frac{2}{\sqrt[2]{3}} [vt \int_{G^+} \langle \partial_{\bar{\%}} \bar{\mathbf{F}} d\hat{\mathbf{S}} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}] = 2vt \int_{S^+} \partial_{\bar{\%}} \bar{\mathbf{F}} d\hat{\mathbf{S}} = vt \int_{S^+} \bar{!}_{\mathbf{S}} \text{ and} \quad (41)$$

$$vt \int_{\partial S^+}^{\circ} !_{\bar{\%}} = vt \int_{\partial S^+}^{\circ} \mathbf{F} d\% = \frac{1}{\sqrt[2]{3}} [vt \int_{\gamma^+}^{\circ} \langle \mathbf{F} d\% \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}] =$$

$$= -\frac{2}{\sqrt[2]{3}} [vt \int_{G^+} \langle \partial_{\%} \mathbf{F} d\hat{\mathbf{S}} \rangle_{\alpha} (\bar{\%}_0 \%_0)^{\alpha}] = -2vt \int_{S^+} \partial_{\%} \mathbf{F} d\hat{\mathbf{S}} = vt \int_{S^+} !_{\mathbf{S}}. \quad (42)$$

Furthermore, since

$$\langle (\bar{\partial}_{\%}\mathbf{F} - \partial_{\%}\mathbf{F})d\hat{\mathbf{S}} \rangle_{\alpha} \cdot (\bar{\%}\%_0)^{\alpha} = -\langle (\text{curl}\mathbf{F} \times \bar{\%}\%_0)d\hat{\mathbf{S}} \rangle_{\alpha} \cdot (\bar{\%}\%_0)^{\alpha} = \tag{43}$$

$$= \langle \bar{\%}\%_0 \times \text{curl}\mathbf{F} \rangle_{\alpha} \cdot d\hat{\mathbf{S}}_{\alpha} = \langle \text{curl}\mathbf{F} \rangle_{\alpha} \cdot \langle d\hat{\mathbf{S}} \times \bar{\%}\%_0 \rangle_{\alpha} = \text{curl}\mathbf{F} \cdot d\bar{\mathbf{S}} \text{ and}$$

$$(\bar{\%}\%_0)^{\alpha} \times \langle (\bar{\partial}_{\mathbf{r}}\bar{\mathbf{F}} + \partial_{\mathbf{r}}\mathbf{F})d\hat{\mathbf{S}}^{\perp} \rangle_{\alpha} = (\bar{\%}\%_0)^{\alpha} \times \langle (\text{div}\mathbf{F}\bar{\%}\%_0 \times d\hat{\mathbf{S}}) \times \bar{\%}\%_0 \rangle_{\alpha} = \tag{44}$$

$$= \langle \text{div}\mathbf{F}\bar{\%}\%_0 \rangle_{\alpha} \times d\hat{\mathbf{S}}_{\alpha},$$

where $\langle \text{div}\mathbf{F} \rangle_{\alpha} = \langle [\partial_{\varrho}(\varrho F) + \partial_{\varphi}F_{\perp}]/\varrho \rangle_{\alpha}$, $\langle \|\text{curl}\mathbf{F}\| \rangle_{\alpha} = \langle [\partial_{\varrho}(\varrho F_{\perp}) - \partial_{\varphi}F]/\varrho \rangle_{\alpha}$ and $2\langle \bar{\partial}_{\%}\mathbf{F} \rangle_{\alpha} = \langle \text{div}\mathbf{F}\bar{\%}\%_0 + \text{curl}\mathbf{F} \times \bar{\%}\%_0 \rangle_{\alpha}$, and in addition $d\bar{\mathbf{S}} = d\hat{\mathbf{S}}_{\alpha} \times (\bar{\%}\%_0)^{\alpha}$, the vector identities (41) and (42) leads to the complex generalized Stokes integral identity

$$vt \int_{\partial S^+}^{\circ} \bar{\mathbf{F}} \cdot d\% = 2vt \int_{S^+} (\bar{\partial}_{\%} \wedge \mathbf{F}) \cdot d\bar{\mathbf{S}}, \tag{45}$$

where $\bar{\mathbf{F}} \cdot d\% = \langle \mathbf{F} \circ d\mathbf{r} \rangle_{\alpha} \cdot (\bar{\%}\%_0)^{\alpha}$, as well as to the complex vector integral identity

$$vt \int_{\partial S^+}^{\circ} \mathbf{F} \times d\% = 2vt \int_{S^+} \langle \bar{\partial}_{\%} \circ \mathbf{F} \rangle_{\alpha} \times d\hat{\mathbf{S}}_{\alpha}, \tag{46}$$

where $\mathbf{F} \times d\% = (\bar{\%}\%_0)^{\alpha} \times \langle \mathbf{F} \wedge d\% \rangle_{\alpha}$. Thus, the complex vector integral identities (45) and (46) are explicit consequences of the fundamental theorem of integral calculus in the $3\mathbb{D}$ field of complex vectors, which follows.

Theorem 2. Let γ be a closed smooth spatial curve in the $3\mathbb{D}$ field of complex vectors, bounding an arbitrary simply connected spatial surface S . For an arbitrary uniform complex vector field \mathbf{F} , which is regular almost everywhere on S and whose $3\mathbb{D}$ differential form $\bar{\mathbf{F}}_{\%}$ is totally integrable on γ , there holds

$$vt \int_{\partial S^+}^{\circ} \bar{\mathbf{F}}_{\%} = 2\pi\hat{\mathbf{e}} \sum_{\%_i \in S} \text{Res}\bar{\mathbf{F}}(\%_i) = vt \int_{S^+} \bar{\mathbf{F}}_{\%}. \tag{47}$$

On the other hand, it is quite possible to formulate the theorem, as follows, using a procedure similar to that used to formulate Theorem 1., based on the result of Green's theorem, but now on the basis of Gauss-Ostrogradsky's theorem.

Theorem 3. Let S be a closed smooth surface in the $3\mathbb{D}$ field of complex vectors, bounding an arbitrary simply connected region V . For an arbitrary uniform complex vector field $\mathbf{F} = F\mathbf{e} + \hat{F}\hat{\mathbf{e}}$, which is regular almost everywhere on V and whose $3\mathbb{D}$ differential form $\bar{\mathbf{F}}_S = Fd\bar{\mathbf{S}}$ is totally integrable on S , there holds

$$vt \int_{\partial V}^{\circ} \bar{\mathbf{F}}_S = 2\pi\hat{\mathbf{e}} \sum_{\%_i \in V} \text{Res}\bar{\mathbf{F}}(\%_i) = vt \int_V \bar{\mathbf{F}}_S dV, \tag{48}$$

where $\bar{\mathbf{F}}_S = 2\bar{\partial}_{\%}F dV$.

Obviously, the integral formula (48) is an explicit consequence of the following integral identities

$$vt \int_{\partial V}^{\circ} \mathbf{F}_{\alpha}(\bar{\mathbf{n}}^{\alpha} \cdot d\mathbf{S}) = vt \int_{\partial V}^{\circ} \langle Fd\bar{\mathbf{S}} \rangle_{\alpha} \mathbf{e}^{\alpha} = \tag{49}$$

$$= vt \int_{\partial V}^{\circ} \langle \bar{\mathbf{F}} \circ d\mathbf{S} + \bar{\mathbf{F}} \wedge d\mathbf{S} \rangle_{\alpha} \mathbf{e}^{\alpha} = 2vt \int_V \langle \bar{\partial}_{\%} \circ F\mathbf{e} + \bar{\partial}_{\%} \wedge \hat{F}\hat{\mathbf{e}} \rangle_{\alpha} \mathbf{e}^{\alpha} dV \text{ and}$$

$$vt \int_{\partial V}^{\circ} \langle \bar{\mathbf{F}} \times \mathbf{n} \rangle_{\alpha}(\bar{\mathbf{n}}^{\alpha} \cdot d\mathbf{S}) = -vt \int_{\partial V}^{\circ} \langle Fd\hat{\mathbf{S}} \rangle_{\alpha} \mathbf{e}^{\alpha} = \tag{50}$$

$$= -vt \int_{\partial V}^{\circ} \langle \bar{\mathbf{F}} \circ d\hat{\mathbf{S}} + \bar{\mathbf{F}} \wedge d\hat{\mathbf{S}} \rangle_{\alpha} \mathbf{e}^{\alpha} = 2vt \int_V \langle \bar{\partial}_{\varphi_0} \circ \hat{\mathbf{F}} \hat{\mathbf{e}} + \bar{\partial}_{\varphi_0} \wedge \mathbf{F} \mathbf{e} \rangle_{\alpha} \mathbf{e}^{\alpha} dV,$$

where $\mathbf{n}_{\alpha} = \langle \mathbf{e} \times \hat{\mathbf{e}} \rangle_{\alpha}$, $d\mathbf{S}_{\alpha} = \langle (\mathbf{e} \times d\hat{\mathbf{S}}) \times \hat{\mathbf{e}} \rangle_{\alpha}$ and $d\bar{\mathbf{S}}_{\alpha} = \langle d\mathbf{S} - d\hat{\mathbf{S}} \rangle_{\alpha}$. Therefore, if $2\langle \bar{\partial}_{\varphi_0} \circ \mathbf{F} \rangle_{\alpha} \cdot \mathbf{e}^{\alpha} = \operatorname{div} \mathbf{F}$ and $\mathbf{e}^{\alpha} \times 2\langle \bar{\partial}_{\varphi_0} \wedge \mathbf{F} \rangle_{\alpha} = \operatorname{curl} \mathbf{F}$, then

$$vt \int_{\partial V}^{\circ} \mathbf{F} \cdot d\bar{\mathbf{S}} = vt \int_{V} \operatorname{div} \mathbf{F} dV \text{ and} \quad (51)$$

$$vt \int_{\partial V}^{\circ} d\mathbf{S} \times \mathbf{F} = vt \int_{V} \operatorname{curl} \mathbf{F} dV. \quad (52)$$

4. Conclusions

If one compares the integral identities in the fundamental theorems, formulated above, it can be concluded that they are basically the same integral identities, with the corresponding differential forms, so that they can be represented by one integral identity, as follows

$$vt \int_{\partial \Omega}^{\circ} ! = vt \int_{\Omega} d!, \quad (53)$$

where Ω is the corresponding compact set of points in a $N\mathbb{D}$ field of complex vectors ($N \in \{2, 3\}$) with boundary $\partial\Omega$ and

$$d! = \frac{1}{2\pi \hat{\mathbf{e}}} \lim_{\Omega \rightarrow \%} \int_{\partial \Omega}^{\circ} !. \quad (54)$$

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