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Investigating the Degree of Approximation of Fourier Series Through Product Means

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Abstract

In this paper, we have established a theorem concerning the degree of approximation of functions $f \in Lip\alpha$ by means of the product summability method $(E.1)(N.p.q)$ applied to the Fourier series associated with a function. The result offers new insights into the convergence behavior and approximation properties of such summation techniques within the Lipschitz class, highlighting the effectiveness of product summability in Fourier analysis.

Keywords: degree of approximation; lipschitz class functions; $(E, 1)$ mean; $(N.p.q)$ mean; $(E, 1).(N.p.q)$ product summability; fourier series; lebesgue integrable functions; big O; small O

1. Introduction

In mathematical analysis, the degree of approximation is crucial for understanding how infinite series behave. This concept goes beyond standard convergence theory, providing a more detailed understanding of series that might diverge or converge conditionally. Through this method, mathematicians can establish meaningful limits for such series, uncovering inherent patterns and order in what would otherwise seem complex or unsolvable expressions.

A vibrant area where these ideas find resonance is in the study of the Fourier series. These series provide elegant representations of periodic functions as infinite sums of *sines* and *cosines*, bridging the gap between pure mathematical theory and practical applications. From solving differential equations to signal processing and acoustics to quantum mechanics and electrical engineering, Fourier analysis plays a central role. However, the classical theory of the Fourier series is not without its limitations. Functions with discontinuities or irregular behavior often resist uniform convergence, presenting challenges that call for more nuanced analysis methods.

This is where the interplay between Banach summability and the Fourier series becomes both natural and fruitful. The framework of Banach summability offers powerful tools to investigate the convergence properties of Fourier series, especially in cases where classical convergence fails. By broadening the notion of limits, Banach summability methods allow for a deeper understanding of the behavior of series and open pathways to new applications in both theoretical and applied contexts.

The origins of summability theory can be traced back to the seminal work of Godfrey Harold Hardy [1], whose famous evergreen book "Divergent Series" (1970) laid the foundation for a rigorous treatment of divergent series. Hardy's work inspired future generations of mathematicians, including Stefan Banach [2], Salomon Bochner [3], Ram Chandran [4], and Shyam Lal Singh [5], among others. Of particular significance is the contribution of Stefan Banach [6], whose introduction of Banach limits and summability revolutionized the analysis of convergence and divergence in infinite series.

Building upon these foundations, researchers have developed more sophisticated summability methods tailored to specific mathematical contexts. For instance, S.K. Paikray et al. [6] introduced the notion of absolute indexed summability factors using quasi-monotone sequences. In contrast, R.K. Jati et al. [7] employed absolute indexed matrix summability to study infinite series in a more generalized

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setting. Further contributions by J.K. Mishra & M. Mishra [8], G.D. Dikshit [9], L. McFadden [10], and T. Pati [11] have advanced the field by extending various methods of absolute summability and exploring their applications to Fourier series. H.K. Nigam [12] found the way to the degree of approximation of product means. E.C. Titchmarch [13] contributed to the development of trigonometric theory, and A. Zygmund [14] revealed the development of trigonometric series.

This convergence of ideas from classical analysis to modern summability techniques underscores the evolving nature of mathematical inquiry. By integrating the tools of Banach summability with the rich structure of Fourier analysis, we not only deepen our theoretical understanding but also enhance our ability to address practical challenges in science and engineering.

2. Definitions

Definition 1. *Let (a_n) be a sequence of real or complex numbers. The Euler mean of order 1 is denoted $(E,1)$ and defined by*

$$(E,1) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}a_k \tag{1}$$

Further, if $\lim_{n\to\infty}(E,1) = s$, a finite number, then we say a_n is Euler summable of order 1 to s and written $\Sigma a_n = s(E,1)$.

Definition 2. *Let $\sum a_n$ be a sequence of real or complex numbers and s_n denotes its n^{th} partial sums. For two sequences $\{p_n\}$ and $\{q_n\}$, define $\{t_n\}$ by*

$$t_n = \frac{1}{r_n}\sum_{v=0}^{n} p_{n-v}q_v s_v \tag{2}$$

where $r_n = p_0q_n + p_1q_{n-1} + ... + p_nq_0$. If $\lim_{n\to\infty} t_n = S$, a finite number, then we say $\sum a_n$ is said to be (N,p,q) summable to s and written $\Sigma a_n = s(N,p,q)$.

Further, if the $(E,1)$ transform of the (N,p,q) transform $\{s_n\}$ is defined by $\tau_n = (E,1).(N,p.q)$, then

$$\tau_n = \left(\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}a_k\right)\left(\frac{1}{r_n}\sum_{v=0}^{n} p_{n-v}q_v S_v\right) \tag{3}$$

If $\tau_n \to s$ when $n\to\infty$, then we say $\sum a_n$ is said to be $(E,1).(N,p,q)$ summable to a finite number s.

Definition 3. *When we express $f(n) = O(g(n))$, it signifies that there exists a positive constant C and a threshold value n_0 such that for all n exceeding n_0 the relationship holds true:*

$$|f(n)| \le Cg(n), \tag{4}$$

where big 'O' notation stands for an upper limit on the growth of a function.

Example 1. *Let $f(n) = 5n^2 + 3n - 3$. We can say that $f(n) = O(n^2)$. For sufficiently large values of n, the expression $3n^2 + 2n + 1$ is bounded above by Cn^2 for some constant C. In the context of asymptotic notation, the small 'o' notation, denoted $f(n) = o(g(n))$, signifies that $f(n)$ grows strictly slower than $g(n)$ as $n\to\infty$. . Formally, this is defined by the limit:*

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0. \tag{5}$$

Example 2. *If $f(n) = n$ and $g(n) = n^2$, then $f(n) = o(g(n))$ because n becomes insignificant compared to n^2 as n increases.*

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Let $f(t)$ be a periodic function Lebesgue integrable in $(-\pi, \pi)$. Then the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x) \tag{6}$$

is referred to as the Fourier series of $f(t)$, where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \tag{7}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt \tag{8}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt \tag{9}$$

Let $\{s_n\}$ be the n^{th} partial sum of $\sum a_n$. Then L_∞ norm of a function $R \to R$ is defined by

$$||f||_\infty = \sup\{|f(x) : x \in R|\} \tag{10}$$

The L_v-norm is defined by

$$||f||_v = \left\{\int_0^{2\pi} |f(x)|^v dx\right\}^{\frac{1}{v}} \tag{11}$$

The degree of approximation $f : R \longrightarrow R$ defined by polynomial $p_n(x)$ of degree n under norm $||.||_\infty$ is defined by

$$||p_n - f||_\infty = \sup\{|p_n(x) - f(x)| : x \in R\} \tag{12}$$

Also, the degree of approximation of a function $f \in L_v$ is defined by

$$E_n(f) = \inf\{||p_n(x) - f||_v : x \in R\}. \tag{13}$$

A function f is said to be in the class $Lip\alpha$ if

$$|f(x+p) - f(x)| = O(|p|^\alpha) \text{ where } 0 < \alpha < 1. \tag{14}$$

We use the following notation throughout this paper:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x). \tag{15}$$

$$K_n(t) = \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{r_n}\sum_{v=0}^{k} p_{n-v}q_v \frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\} \tag{16}$$

3. Known Result

Theorem 1. *If f be a 2π periodic function of $Lip\alpha$, the degree of approximation by the product $(E.q)(N.P_n)$ summability means on its Fourier series (6) is given by $||\tau_n - f||_\infty = O\left\{\frac{1}{(n+1)^\alpha}\right\}$ where $0 < \alpha < 1$ and τ_n is defined by (3).*

Theorem 2. *If f is a periodic function of period 2π and class of $Lip\alpha$, then the degree of approximation by product Euler & Cesaro summability of its Fourier series (6) is given by $O\left\{\frac{1}{(n+1)^\alpha}\right\}$ where $0 < \alpha < 1$.*

4. Principal theorem

In this paper, we have proved the degree of approximation by product mean $(E,1)(N,p,q)$ of the Fourier series of a function of class $Lip\alpha$.

Theorem 3. *If f is a periodic function of period 2π of the class $Lip(\alpha, 1)$ then the degree of approximation by product $(E,1)(N,p,q)$ summability its Fourier series is given by $O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$ where $0 < \alpha < 1$.*

5. Required Lemmas

We require the following Lemma to prove the theorem.

Lemma 1. $|K_n(t)| = O\left(\frac{1}{t}\right)$ *for* $\frac{1}{n+1} \leq t \leq \pi$.

Proof. *Proof:* Since for $\frac{1}{n+1} \leq t \leq \pi$, $\sin(t/2) \geq \frac{t}{\pi} = \frac{t}{\pi}$ and $|\sin(nt)| \leq 1$.

$$\begin{aligned}
|K_n(t)| &= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{r_k} \sum_{v=0}^{k} p_{k-v} q_v \frac{\sin(v+1/2)t}{\sin(t/2)} \right\} \right| \\
&\ll \frac{1}{t.2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{r_k} \sum_{v=0}^{k} p_{k-v} q_v \right\} \right| \\
&= \frac{1}{t.2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \right| \quad \left(\text{assuming } \sum_{v=0}^{k} p_{k-v} q_v = r_k\right) \\
&= \frac{1}{t.2^{n+1}} .2^{n+1} \\
&= O\left(\frac{1}{t}\right)
\end{aligned}$$

□

Lemma 2. *For* $0 \leq t < \frac{1}{n+1}$, $K_n(t) = O(n)$.

Proof.

$$\begin{aligned}
K_n(t) &= \frac{1}{\pi.2^{n+1}} \left[\sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{r_k} \sum_{v=0}^{k} p_{k-v} q_v \frac{\sin(\nu+1/2)t}{\sin(t/2)} \right\} \right] \\
&\ll \frac{1}{\pi.2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{r_k} \sum_{v=0}^{k} p_{k-v} q_v (v+1/2) \frac{t}{t/2} \right\} \right| \quad (\text{since } \sin x \approx x \text{ for small } x) \\
&\ll \frac{1}{\pi.2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{r_k} \sum_{v=0}^{k} p_{k-v} q_v (2v+1) \right\} \right| \\
&\ll \frac{1}{\pi.2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} (2k+1) \right| \quad \left(\text{assuming } \sum_{v=0}^{k} p_{k-v} q_v (2v+1) \approx (2k+1) r_k\right) \\
&= \frac{1}{\pi.2^{n+1}} \left(2 \sum_{k=0}^{n} k \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k} \right) \\
&= \frac{1}{\pi.2^{n+1}} \left(2n2^{n-1} + 2^{n} \right) \\
&= \frac{1}{\pi.2^{n+1}} (n2^{n} + 2^{n}) \\
&= \frac{2^{n}(n+1)}{\pi.2^{n+1}} \\
&= O(n)
\end{aligned}$$

□

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6. Proof of Principal Theorem

From Riemann-Lebesgue theorem for the n^{th} partial sum of Fourier series of $f(x)$ and the following Titchmarch [13], we get

$$s_n(f,x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin(n+1/2)t}{\sin(t/2)}dt$$

Using (1), the (N,p,q) transform of $|t_n - f|$ is

$$|t_n - f| = t_n(f,x) - f(x) = \frac{1}{2\pi r_n}\int_0^{\pi} \phi(t)\sum_{v=0}^{n} p_{n-v}q_v\frac{\sin(v+1/2)t}{\sin(t/2)}dt$$

Denoting the product summability by $(E,1)(N,p,q)$, we have

$$\begin{aligned}
|\tau_n - f| &= \frac{1}{\pi.2^{n+1}}\left|\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{r_k}\sum_{\nu=0}^{k}p_{k-\nu}q_\nu\frac{\sin(\nu+1/2)t}{\sin(t/2)}\right\}dt\right| \\
&= \int_0^{\pi}\phi(t)K_n(t)dt \\
&= \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right)\phi(t)K_n(t)dt \\
&= I_1 + I_2 \quad \text{(say)}
\end{aligned}$$

Using Lemma 2 and the property of $Lip\alpha$ class, $\phi(t) = O(t^\alpha)$:

$$|I_1| = \frac{1}{\pi.2^{n+1}}\left|\int_0^{\frac{1}{n+1}}\phi(t)\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{r_k}\sum_{\nu=0}^{k}p_{k-\nu}q_\nu\frac{\sin(\nu+1/2)t}{\sin(t/2)}\right\}dt\right| \tag{17}$$

$$\leq O(n)\left|\int_0^{\frac{1}{n+1}}\phi(t)dt\right| \tag{18}$$

$$= O(n)\int_0^{\frac{1}{n+1}} t^\alpha dt \tag{19}$$

$$= O(n)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}} \tag{20}$$

$$= O(n)\frac{1}{(\alpha+1)(n+1)^{\alpha+1}} \tag{21}$$

$$= O\left(\frac{1}{(n+1)^\alpha}\right) \tag{22}$$

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Using Lemma 1 and the property of the $Lip\alpha$ class:

$$|I_2| = \left|\int_{\frac{1}{n+1}}^{\pi} \phi(t)K_n(t)dt\right| \tag{23}$$
$$\leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||K_n(t)|dt \tag{24}$$
$$\leq \int_{\frac{1}{n+1}}^{\pi} O(t^\alpha)O\left(\frac{1}{t}\right)dt \tag{25}$$
$$\leq \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1}dt \tag{26}$$
$$= \left[\frac{t^\alpha}{\alpha}\right]_{\frac{1}{n+1}}^{\pi} \tag{27}$$
$$= O(\pi^\alpha) - O\left(\frac{1}{(n+1)^\alpha}\right) \tag{28}$$
$$= O\left(\frac{1}{(n+1)^\alpha}\right) \tag{29}$$

Combining Equation (17) and Equation (23), we get the required result: $|\tau_n - f| = O\left(\frac{1}{(n+1)^\alpha}\right)$. Therefore, $||\tau_n - f||_\infty = O\left\{\frac{1}{(n+1)^\alpha}\right\}$.

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