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Article

Dynamics of a Class of Stochastic Parabolic Equations with Delay and Non-Autonomous Forcing

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Abstract

This paper is concerned with the dynamics of solutions of a class of stochastic parabolic equations with delay and deterministic non-autonomous forcing. Specifically, the existence of random attractors is proven, and the upper semicontinuity of random attractors is shown when the delay approaches zero. The key technique in the proof of the existence results of random attractors is to derive some uniform estimates of solutions, and the most important step in the proof of the upper semicontinuity of random attractors is to establish the convergence of solutions.

Keywords: random attractor; stochastic parabolic equation; delay; upper semicontinuity

1. Introduction

In this paper, we investigate the dynamics of solutions of stochastic delay parabolic equations with deterministic non-autonomous forcing defined on a bounded domain $\mathcal{O} \subset \mathbb{R}^n$ ($n \geq 2$):

$$\begin{aligned} \frac{du(t,x)}{dt} + \lambda u(t,x) - \operatorname{div}(\sigma(x)\nabla u(t,x)) = & F(x,u(t,x)) + f(x,u(t-\rho,x)) + g(t,x) \\ & + \sum_{j=1}^m h_j(x) \frac{dw_j(t)}{dt} \end{aligned} \tag{1}$$

for $t > \tau$ and $x \in \mathcal{O}$, with the boundary condition and initial condition

$$u(t,x) = 0, \text{ for } t > \tau, x \in \partial\mathcal{O}, \tag{2}$$

$$u_\tau(\xi,x) := u(\tau+\xi,x) = \phi(\xi,x), \text{ for } (\xi,x) \in [-\rho,0] \times \mathcal{O}, \tag{3}$$

where $\tau \in \mathbb{R}$, λ is a positive constant, ρ is the delay time of the system, σ is the diffusion coefficient, F is a superlinear source term, f is a nonlinearity capturing the time delay, g is a deterministic time-dependent forcing, $\{h_j\}_{j=1}^m$ are given functions defined on $\mathcal{O}$, and $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified later.

Stochastic differential equations of this type arise from physical evolution phenomena when time delay is taken into account. For the deterministic version of problem (1)-(3), the existence of attractors has been investigated in both autonomous [case](#) [17,18] and non-autonomous case [1]. To reveal the essential dynamics of random systems with wide fluctuations, the concept of random attractor was introduced in [7-10,12,14-16,20,22,26,28,30,32-36] as an extension of the theory of attractors for deterministic equations in [3,4,6,24]. The existence of random attractors has been investigated for stochastic PDEs without delay and time-dependent forcing in [11,13]. For equations with delays, the existence of attractors was also obtained in [19,29] in the autonomous stochastic case. The existence of random attractors was established in [21,27] for the non-autonomous systems without delay. However, there are very few works in the literature to address the problem whether random attractors exist for stochastic PDEs with delay and time-dependent forcing. What dynamics will random attractors

exhibit when the delay approaches zero? All that motivate us to investigate the dynamics of problem (1)-(3), explore the interaction of delay and time-dependent forcing under random perturbations.

More precisely, we will investigate the existence and upper semicontinuity of random attractors of problem (1)-(3). For this purpose, we first show that the systems generate continuous random dynamical systems. However, it is not known whether a stochastic parabolic equation with delay and time-dependent forcing generates a non-autonomous random dynamical system, which is a extremely difficult issue in the field of related research. To overcome the difficulty, we shall transform the stochastic equations into deterministic versions with random coefficients through Ornstein-Uhlenbeck transformation, exploit and develop the ideas from the theory of deterministic PDEs with polynomial nonlinearity and delay PDEs with Lipschitz nonlinearity. Then, we prove the existence and uniqueness of random attractors by establishing the asymptotic compactness and absorbing set of the equations. The critical technique is to derive uniform estimates of solutions with respect to delay and time-dependent forcing for the nonlinearity satisfying an arbitrary polynomial growth condition. Finally, we establish the upper semicontinuity of random attractors of problem (1)-(3) as the delay goes to zero. The most important step is to establish the convergence of solutions. It is worth mentioning that our main works improve the relevant results in [19] and our method is applicable for other non-autonomous stochastic delay PDEs.

The paper is organized as follows. In the next section, we establish the existence of a continuous non-autonomous random dynamical system. In Section 3, we conduct uniform estimates of solutions which are necessary for proving the pullback absorbing property and the pullback asymptotic compactness of the equation. Then we establish the existence of random attractors in Section 4. In Section 5, we obtain the upper semicontinuity of random attractors of problem (1)-(3) when the delay approaches zero. In the last section, we present the conclusion based on the results obtained.

2. Mathematical Preparation

In this section, our main goal is to establish the existence of a continuous non-autonomous random dynamical system generated by the stochastic delay parabolic equation (1).

We first recall some basic results about function spaces which will be used later. Let $\|\cdot\|_{L^2}$ and $(\cdot, \cdot)$ be the norm and inner product of $L^2(\mathcal{O})$, respectively. The norm of $L^p(\mathcal{O})$ is written as $\|\cdot\|_{L^p}$. Assume that $C([-\rho, 0], X)$ is the space of all continuous functions from $[-\rho, 0]$ to X with norm $\|\varphi\|_{C([-\rho, 0], X)} = \sup\{\|\varphi(\xi)\|_X : \xi \in [-\rho, 0]\}$ for $\varphi \in C([-\rho, 0], X)$. The natural energy space for problem (1)-(3) involves the space $\mathcal{D}_0^{1,2}(\mathcal{O}, \sigma)$ defined as the closure of $C_0^\infty(\mathcal{O})$ with respect to the norm

$$\|u\|_{\mathcal{D}_0^{1,2}} := \left(\int_{\mathcal{O}} \sigma(x) |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

The space $\mathcal{D}_0^{1,2}(\mathcal{O}, \sigma)$ is a Hilbert space with respect to the scalar product

$$(u, v)_\sigma := \int_{\mathcal{O}} \sigma(x) \nabla u \cdot \nabla v dx.$$

For convenience, we denote $\mathcal{C}_p = C([-\rho, 0], L^p(\mathcal{O}))$ and $\mathcal{C}_{\mathcal{D}} = C([-\rho, 0], \mathcal{D}_0^{1,2}(\mathcal{O}, \sigma))$ and their norms by $\|\cdot\|_{\mathcal{C}_p}$ and $\|\cdot\|_{\mathcal{C}_{\mathcal{D}}}$, respectively. The letter c stands for a general positive constant which may change its value from line to line or even in the same line. In what follows, to study problem (1)-(3) we introduce the following hypotheses:

- (A1) $\sigma : \mathcal{O} \rightarrow \mathbb{R}^+ \cup \{0\}$ is a non-negative measurable function such that $\sigma \in L_{loc}^1(\mathcal{O})$, and for some $\alpha \in (0, 2)$, $\liminf_{x \rightarrow z} |x - z|^{-\alpha} \sigma(x) > 0$ for every $z \in \overline{\mathcal{O}}$;
- (A2) $F : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which satisfies a dissipativeness and growth condition of polynomial type, i.e., there is a number $p \geq 2$ such that for all $x \in \mathcal{O}$ and $u \in \mathbb{R}$,

$$F(x, u)u \leq -\alpha_1 |u|^p + \beta_1(x), \quad (1)$$

$$|F(x, u)| \leq \alpha_2 |u|^{p-1} + \beta_2(x), \quad (2)$$

$$\frac{\partial}{\partial u} F(x, u) \leq \alpha_3, \quad (3)$$

$$\left| \frac{\partial}{\partial x} F(x, u) \right| \leq \beta_3(x), \quad (4)$$

where $\alpha_1, \alpha_2, \alpha_3$ are positive constants, β_1, β_2 and β_3 are nonnegative functions on $\mathcal{O}$ such that $\beta_1 \in L^{(2p-2)/p}(\mathcal{O})$, $\beta_2, \beta_3 \in L^2(\mathcal{O})$;

- (A3) $f : \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that for all $x \in \mathcal{O}$ and $u, v \in \mathbb{R}$,

$$|f(x, u) - f(x, v)| \leq \mathcal{C}_f |u - v|, \quad (5)$$

$$|f(x, u)|^2 \leq L_f^2 |u|^2 + |\eta(x)|^2, \quad (6)$$

where $\mathcal{C}_f, L_f$ are positive constants, $\eta \in L^2(\mathcal{O})$, and L_f satisfies $L_f < \frac{1}{4}\lambda$;

- (A4) $g \in L_{loc}^2(\mathbb{R}, L^2(\mathcal{O}))$ and satisfies the following conditions

$$\int_{-\infty}^{\tau} e^{ms} \|g(s, \cdot)\|_{L^2}^2 ds < \infty \quad \forall \tau \in \mathbb{R}, \quad (7)$$

$$\lim_{t \rightarrow -\infty} e^{ct} \int_{-\infty}^0 e^{ms} \|g(s+t, \cdot)\|_{L^2}^2 ds = 0 \quad \forall c > 0, \quad (8)$$

where m is a positive constant satisfying

$$m - \lambda + \frac{4L_f^2}{\lambda} e^{mp} < 0; \quad (9)$$

- (A5) The functions $h_j, j = 1, \dots, m$, belong to $L^{2p-2}(\mathcal{O}) \cap L^\infty(\mathcal{O}) \cap \text{Dom}(A) \cap D^p(A)$ for some $p \geq 2$, where $Au = -\text{div}(\sigma(x)\nabla u)$, $\text{Dom}(A) = \{u \in \mathcal{D}_0^{1,2}(\mathcal{O}, \sigma) : Au \in L^2(\mathcal{O})\}$, and $D^p(A) = \{u \in \mathcal{D}_0^{1,2}(\mathcal{O}, \sigma) : \int_{\mathcal{O}} |Au|^p dx < +\infty\}$.

Under condition (A5), the operator $A = -\text{div}(\sigma(x)\nabla)$ with the domain

$$\text{Dom}(A) = \{u \in \mathcal{D}_0^{1,2}(\mathcal{O}, \sigma) : Au \in L^2(\mathcal{O})\}$$

is positive and self-adjoint. The space $\text{Dom}(A)$ is a Hilbert space endowed with the usual graph scalar product. Therefore, there exists a complete orthonormal system of eigenvectors (e_j, λ_j) such that

$$(e_j, e_k) = \delta_{jk} \text{ and } -\text{div}(\sigma(x)\nabla e_j) = \lambda_j e_j, \quad j, k = 1, 2, 3, \dots,$$

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \lambda_j \rightarrow +\infty \text{ as } j \rightarrow +\infty.$$

Noting that

$$\lambda_1 = \inf \left\{ \frac{\|u\|_{\mathcal{D}_0^{1,2}}^2}{\|u\|_{L^2}^2} : u \in \mathcal{D}_0^{1,2}(\mathcal{O}, \sigma), u \neq 0 \right\},$$

we have

$$\|u\|_{\mathcal{D}_0^{1,2}}^2 \geq \lambda_1 \|u\|_{L^2}^2, \text{ for all } u \in \mathcal{D}_0^{1,2}(\mathcal{O}, \sigma).$$

In the sequel, we consider the canonical probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\},$$

and $\mathcal{F}$ is the Borel σ -algebra induced by the compact-open topology of Ω , while P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we will identify ω with

$$W(t) \equiv (w_1(t), w_2(t), \dots, w_m(t)) = \omega(t) \text{ for } t \in \mathbb{R}.$$

We define the time shift by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

Then, $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system.

We now associate a continuous random dynamical system with the stochastic parabolic equation over $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. To this end, we need to convert the stochastic equation containing white noise terms into a deterministic one with random coefficients.

Given $j = 1, \dots, m$, consider the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \lambda z_j dt = dw_j(t). \quad (10)$$

One may check that a solution to (10) is given by

$$z_j(t) = z_j(\theta_t \omega_j) \equiv -\lambda \int_{-\infty}^0 e^{\lambda s} (\theta_t \omega_j)(s) ds, \quad t \in \mathbb{R}.$$

In addition, the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t \omega_j)$ is P -a.e. continuous. Thus, it follows from Proposition 4.3.3 in [2] that there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^m \left(|z_j(\omega_j)|^2 + |z_j(\omega_j)|^p + |z_j(\omega_j)|^{2p-2} \right) \leq r(\omega), \quad (11)$$

where $r(\omega)$ satisfies, for P -a.e $\omega \in \Omega$,

$$r(\theta_t \omega) \leq e^{\frac{\lambda}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \quad (12)$$

Combining (11) and (12), it implies that for P -a.e $\omega \in \Omega$,

$$\sum_{j=1}^m \left(|z_j(\theta_t \omega_j)|^2 + |z_j(\theta_t \omega_j)|^p + |z_j(\theta_t \omega_j)|^{2p-2} \right) \leq e^{\frac{\lambda}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \quad (13)$$

Putting $z(\theta_t \omega) = \sum_{j=1}^m h_j z_j(\theta_t \omega_j)$, by (10) we have

$$dz + \lambda z dt = \sum_{j=1}^m h_j dw_j.$$

Since $h_j \in L^{2p-2}(\mathcal{O}) \cap \text{Dom}(A) \cap D^p(A)$, we have

$$p(\theta_t \omega) \leq ce^{\frac{\lambda}{2}|t|} r(\omega), \quad (14)$$

where $p(\omega) = \|z(\omega)\|_{\mathcal{D}_0^{1,2}}^2 + \|z(\omega)\|_{L^p}^p + \|z(\omega)\|_{L^{2p-2}}^{2p-2} + \|Az(\omega)\|_{L^2}^2 + \|Az(\omega)\|_{L^p}^p$.

The existence of a solution to stochastic delay parabolic equation (1) follows from [23]. To show that problem (1)-(3) generates a random dynamical system, we let $v(t) = u(t) - z(\theta_t \omega)$, where u is a solution of equation (1). Then v satisfies

$$\frac{dv}{dt} + Av + \lambda v = F(x, v + z(\theta_t \omega)) + f(x, v(t - \rho) + z(\theta_{t-\rho} \omega)) + g(t, x) - Az(\theta_t \omega), \quad (15)$$

with boundary condition

$$v(t, x) = 0, \text{ for } (t, x) \in (\tau, \infty) \times \partial\mathcal{O}, \quad (16)$$

and initial condition

$$v_\tau(\xi, x) = \phi(\xi, x) - z(\theta_{\tau+\xi}\omega) \triangleq \psi(\xi, x), \quad (\xi, x) \in [-\rho, 0] \times \mathcal{O}. \quad (17)$$

For the pathwise deterministic problem (15)-(17), by the Galerkin method as in [5] one can show that for all $\tau \in \mathbb{R}$, $\psi \in \mathcal{C}_2$ and P -a.e $\omega \in \Omega$, there is a unique solution $v(\cdot, \tau, \omega, \psi) \in C([\tau - \rho, \infty), L^2(\mathcal{O})) \cap L^2_{loc}(\tau, \infty; \mathcal{D}_0^{1,2}(\mathcal{O}, \sigma)) \cap L^p_{loc}(\tau, \infty; L^p(\mathcal{O}))$ if F and f satisfy assumptions (A2) and (A3). Moreover, this solution is continuous in $\psi \in \mathcal{C}_2$ and $v(\cdot, \tau, \cdot, \psi) : (\Omega, \mathcal{F}) \rightarrow (\mathcal{C}_2, \mathcal{B}(\mathcal{C}_2))$ is measurable. In terms of the solution v , we are able to introduce a continuous non-autonomous random dynamical system for the stochastic problem (1)-(3). Let $u(t, \tau, \omega, \phi) = v(t, \tau, \omega, \psi) + z(\theta_t\omega)$, then the process u is the solution of equation (1) with initial condition ϕ . We now define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{C}_2 \rightarrow \mathcal{C}_2$ by

$$\Phi(t, \tau, \omega, \phi)(\cdot) = u_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \phi) = v_{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \psi) + z(\theta_{t+\tau+\cdot}\omega), \quad (18)$$

where $u_{t+\tau}(\xi, \tau, \theta_{-\tau}\omega, \phi) = u(t + \tau + \xi, \tau, \theta_{-\tau}\omega, \phi)$ for $\xi \in [-\rho, 0]$. It is easy to show that Φ satisfies conditions (i)-(iv) of Definition 2.1 in [31], and hence it is a continuous non-autonomous random dynamical system on $\mathcal{C}_2$ associated with problem (1)-(3). Given a bounded nonempty subset D of $\mathcal{C}_2$, the Hausdorff semidistance between D and the origin in $\mathcal{C}_2$ is denoted by $\|D\|_{\mathcal{C}_2} = \sup_{\phi \in D} \|\phi\|_{\mathcal{C}_2}$. Let $\mathcal{D} = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of $\mathcal{C}_2$. Such a $\mathcal{D}$ is said to be tempered in $\mathcal{C}_2$ if for every $\gamma > 0$,

$$\lim_{t \rightarrow -\infty} e^{\gamma t} \|D(\tau + t, \theta_t\omega)\|_{\mathcal{C}_2} = 0.$$

Henceforth, we always assume that $\mathcal{D}$ is the collection of all families of tempered nonempty subsets of $\mathcal{C}_2$.

3. Uniform Estimates of Solutions

In this section, we conduct a variety of uniform estimates on the solutions of problem (1)-(3) by utilizing the converted random equation (15), to prove the existence of pullback absorbing set and the pullback asymptotic compactness of random dynamical system.

Lemma 1. Suppose that assumptions (A1)-(A5) hold. Let $\mathcal{D} = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then for every $\epsilon, \tau \in \mathbb{R}$ and P -a.e. $\omega \in \Omega$, there exists $\mathcal{T} = \mathcal{T}(\tau, \omega, D, \epsilon) > 0$ such that for all $t \geq \mathcal{T}$, the solution u of problem (1)-(3) satisfies

$$\begin{aligned} \|u_\epsilon(\cdot, \tau - t, \theta_{-\tau}\omega, \phi)\|_{\mathcal{C}_2}^2 &+ \int_{\tau-t}^{\epsilon} e^{m(s-\epsilon)} \|u(s, \tau - t, \theta_{-\tau}\omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds \\ &+ \int_{\tau-t}^{\epsilon} e^{m(s-\epsilon)} \|u(s, \tau - t, \theta_{-\tau}\omega, \phi)\|_{L^p}^p ds \\ &\leq R(\epsilon, \tau, \omega), \end{aligned} \quad (1)$$

where $\phi \in D(\tau - t, \theta_{-\tau}\omega)$, and $R(\epsilon, \tau, \omega)$ is determined by

$$R(\epsilon, \tau, \omega) = c \int_{-\infty}^{\epsilon-\tau} e^{m(s+\tau-\epsilon)} \|g(s+\tau)\|_{L^2}^2 ds + c(r(\omega) + 1), \quad (2)$$

with c being a positive deterministic constant independent of τ, ω and D .

Proof. Multiplying (15) by v and then integrating over $\mathcal{O}$, we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2}^2 + \|v(t)\|_{\mathcal{D}_0^{1,2}}^2 + \lambda \|v(t)\|_{L^2}^2 &= \int_{\mathcal{O}} F(x, u(t, x)) v dx + \int_{\mathcal{O}} f(x, u(t - \rho, x)) v dx \\ &+ \int_{\mathcal{O}} g(t, x) v dx - \int_{\mathcal{O}} Az(\theta_t \omega) v dx. \end{aligned} \quad (3)$$

We now estimate each term on the right-hand side of (3). For the first term, by (1), (2) and Young's inequality, we obtain

$$\begin{aligned} &\int_{\mathcal{O}} F(x, u(t, x)) v dx \\ &= \int_{\mathcal{O}} F(x, u(t, x)) u dx - \int_{\mathcal{O}} F(x, u(t, x)) z(\theta_t \omega) dx \\ &\leq -\alpha_1 \|u(t)\|_{L^p}^p + \|\beta_1\|_{L^1} + \alpha_2 \int_{\mathcal{O}} |u(t)|^{p-1} |z(\theta_t \omega)| dx + \int_{\mathcal{O}} \beta_2(x) |z(\theta_t \omega)| dx \\ &\leq -\frac{\alpha_1}{2} \|u(t)\|_{L^p}^p + \|\beta_1\|_{L^1} + \frac{p-1}{p} \|\beta_2\|_{L^{\frac{p}{p-1}}}^{\frac{p-1}{p}} + \left(\frac{1}{p} + \frac{2\alpha_2(p-1)}{\alpha_1 p^2} \right) \|z(\theta_t \omega)\|_{L^p}^p. \end{aligned} \quad (4)$$

By (6), we deduce that

$$\begin{aligned} &\int_{\mathcal{O}} f(x, u(t - \rho, x)) v dx \\ &\leq \frac{1}{\lambda} \int_{\mathcal{O}} \left(L_f^2 |u(t - \rho)|^2 + |\eta(x)|^2 \right) dx + \frac{\lambda}{4} \|v(t)\|_{L^2}^2 \\ &\leq \frac{2L_f^2}{\lambda} \|v(t - \rho)\|_{L^2}^2 + \frac{\lambda}{4} \|v(t)\|_{L^2}^2 + \frac{1}{\lambda} \|\eta(x)\|_{L^2}^2 + \frac{2L_f^2}{\lambda} \|z(\theta_{t-\rho} \omega)\|_{L^2}^2. \end{aligned} \quad (5)$$

The last two terms on the right-hand side of (3) are bounded by

$$\left| \int_{\mathcal{O}} g(t, x) v dx - \int_{\mathcal{O}} Az(\theta_t \omega) v dx \right| \leq \frac{\lambda}{4} \|v(t)\|_{L^2}^2 + \frac{2}{\lambda} \|g(t)\|_{L^2}^2 + \frac{2}{\lambda} \|Az(\theta_t \omega)\|_{L^2}^2. \quad (6)$$

Consequently, it follows from (14) and (4)-(6) that

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{L^2}^2 + \|v(t)\|_{\mathcal{D}_0^{1,2}}^2 + \alpha_1 \|u(t)\|_{L^p}^p &\leq -\lambda \|v(t)\|_{L^2}^2 + \frac{4L_f^2}{\lambda} \|v(t - \rho)\|_{L^2}^2 + \frac{4}{\lambda} \|g(t)\|_{L^2}^2 \\ &+ cp_1(\theta_t \omega) + c, \end{aligned} \quad (7)$$

where $p_1(\omega) = \|z(\omega)\|_{L^p}^p + \|z(\theta_{-\rho} \omega)\|_{L^2}^2 + \|Az(\omega)\|_{L^2}^2$. Then for m satisfying (9), we find from (7) that

$$\begin{aligned} &\frac{d}{dt} \left(e^{mt} \|v(t)\|_{L^2}^2 \right) + e^{mt} \|v(t)\|_{\mathcal{D}_0^{1,2}}^2 + \alpha_1 e^{mt} \|u(t)\|_{L^p}^p \\ &\leq e^{mt} \left(\frac{4L_f^2}{\lambda} \|v(t - \rho)\|_{L^2}^2 + \frac{4}{\lambda} \|g(t)\|_{L^2}^2 + cp_1(\theta_t \omega) + c \right) + e^{mt} (m - \lambda) \|v(t)\|_{L^2}^2. \end{aligned} \quad (8)$$

Considering time $\tau - t$ instead of τ , integrating (8) over $[\tau - t, \epsilon + \zeta]$ for any fixed $\zeta \in [-\rho, 0]$ with $\epsilon \geq \tau - t + \rho$, and then replacing ω by $\theta_{-\tau}\omega$ (i.e., v now denotes $v(\cdot, \tau - t, \theta_{-\tau}\omega, \psi)$), we obtain

$$\begin{aligned} & e^{m(\epsilon+\zeta)} \|v(\epsilon + \zeta, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 + \int_{\tau-t}^{\epsilon+\zeta} e^{ms} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{\mathcal{D}_0^{1,2}}^2 ds \\ & + \alpha_1 \int_{\tau-t}^{\epsilon+\zeta} e^{ms} \|u(s, \tau - t, \theta_{-\tau}\omega, \phi)\|_{L^p}^p ds \\ & \leq e^{m(\tau-t)} \|\psi(0)\|_{L^2}^2 + (m - \lambda) \int_{\tau-t}^{\epsilon+\zeta} e^{ms} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 ds \\ & + \frac{4}{\lambda} \int_{\tau-t}^{\epsilon+\zeta} e^{ms} \|g(s)\|_{L^2}^2 ds + c \int_{\tau-t}^{\epsilon+\zeta} e^{ms} (p_1(\theta_{s-\tau}\omega) + 1) ds \\ & + \frac{4L_f^2}{\lambda} \int_{\tau-t}^{\epsilon+\zeta} e^{ms} \|v(s - \rho, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 ds. \end{aligned} \quad (9)$$

For the last term on the right-hand side of (9), we have

$$\begin{aligned} & \int_{\tau-t}^{\epsilon+\zeta} e^{ms} \|v(s - \rho, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 ds \\ & = \int_{\tau-t-\rho}^{\epsilon+\zeta-\rho} e^{m(s+\rho)} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 ds \\ & \leq \frac{1}{m} e^{m(\tau-t+\rho)} \|\psi\|_{\mathcal{C}_2}^2 + \int_{\tau-t}^{\epsilon+\zeta} e^{m(s+\rho)} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 ds. \end{aligned} \quad (10)$$

It follows from (9) and (9)-(10) that

$$\begin{aligned} & \|v(\epsilon + \zeta, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 + \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon-\zeta)} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{\mathcal{D}_0^{1,2}}^2 ds \\ & + \alpha_1 \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon-\zeta)} \|u(s, \tau - t, \theta_{-\tau}\omega, \phi)\|_{L^p}^p ds \\ & \leq c e^{m(\tau-t-\epsilon+\rho-\zeta)} \|\psi\|_{\mathcal{C}_2}^2 + c \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon-\zeta)} \|g(s)\|_{L^2}^2 ds \\ & + c \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon-\zeta)} (p_1(\theta_{s-\tau}\omega) + 1) ds. \end{aligned} \quad (11)$$

Since $\zeta \in [-\rho, 0]$, we get from the above inequality that

$$\begin{aligned} & \|v(\epsilon + \zeta, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 + \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon)} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{\mathcal{D}_0^{1,2}}^2 ds \\ & + \alpha_1 \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon)} \|u(s, \tau - t, \theta_{-\tau}\omega, \phi)\|_{L^p}^p ds \\ & \leq c e^{2m\rho} e^{m(\tau-t-\epsilon)} \|\psi\|_{\mathcal{C}_2}^2 + c e^{m\rho} \int_{\tau-t}^{\epsilon} e^{m(s-\epsilon)} \|g(s)\|_{L^2}^2 ds \\ & + c e^{m\rho} \int_{\tau-t}^{\epsilon} e^{m(s-\epsilon)} (p_1(\theta_{s-\tau}\omega) + 1) ds. \end{aligned} \quad (12)$$

Note that $u(t, \tau, \theta_t \omega, \phi) = v(t, \tau, \theta_t \omega, \psi) + z(\theta_t \omega)$, where $\phi(\zeta, x) = \psi(\zeta, x) + z(\theta_{\tau+\zeta} \omega)$ for $\zeta \in [-\rho, 0]$. Therefore, by (12) we get that, for all $t \geq \tau$,

$$\begin{aligned} & \|u_\epsilon(\cdot, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{C}_2}^2 + \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon)} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds \\ & + \int_{\tau-t}^{\epsilon+\zeta} e^{m(s-\epsilon)} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{L^p}^p ds \\ & \leq c e^{m(\tau-t-\epsilon)} \left(\|\phi\|_{\mathcal{C}_2}^2 + \|z(\theta_{-\tau} \omega)\|_{\mathcal{C}_2}^2 \right) + c \int_{\tau-t}^{\epsilon} e^{m(s-\epsilon)} \|g(s)\|_{L^2}^2 ds \\ & + c \int_{\tau-t}^{\epsilon} e^{m(s-\epsilon)} p_2(\theta_{s-\tau} \omega) ds + c, \end{aligned} \quad (13)$$

where $p_2(\omega) = p_1(\omega) + \|z(\omega)\|_{\mathcal{D}_0^{1,2}}^2$. Since $\phi \in D(\tau - t, \theta_{-\tau} \omega)$ and $\|z(\omega)\|_{\mathcal{C}_2}^2$ is tempered, we deduce that

$$\limsup_{t \rightarrow \infty} e^{m(\tau-t-\epsilon)} \|\phi\|_{\mathcal{C}_2}^2 \leq \limsup_{t \rightarrow \infty} e^{m(\tau-t-\epsilon)} \|D(\tau - t, \theta_{-\tau} \omega)\|_{\mathcal{C}_2}^2 = 0,$$

and

$$\limsup_{t \rightarrow \infty} e^{m(\tau-t-\epsilon)} \|z(\theta_{-\tau} \omega)\|_{\mathcal{C}_2}^2 = 0.$$

On the other hand, it follows from (12) and (14) that

$$\int_{\tau-t}^{\epsilon} e^{m(s-\epsilon)} p_2(\theta_{s-\tau} \omega) ds \leq c \int_{\tau-t-\epsilon}^0 e^{ms - \frac{\lambda}{2}(s+\epsilon-\tau)} r(\omega) ds \leq cr(\omega).$$

Therefore, there exists $\mathcal{T} = \mathcal{T}(\tau, \omega, D, \epsilon) > 0$ such that (1) holds for all $t \geq \mathcal{T}$, which completes the proof of Lemma 1. $\square$

Lemma 2. Suppose that assumptions (A1)-(A5) hold. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then for every $\tau \in \mathbb{R}$ and P-a.e. $\omega \in \Omega$, there exists $\mathcal{T} = \mathcal{T}(\tau, \omega, D) \geq \rho + 1$ such that for all $t \geq \mathcal{T}$, the solution v of problem (15)-(17) with ω replaced by $\theta_{-\tau} \omega$ satisfies

$$\begin{aligned} & \|v_\tau(\cdot, \tau - t, \theta_{-\tau} \omega, \psi)\|_{\mathcal{C}_D}^2 + \int_{\tau-\rho}^{\tau} \|Av(s, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^2}^2 ds \\ & \leq c \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1), \end{aligned} \quad (14)$$

where c is a positive deterministic constant independent of τ, ω and D .

Proof. Taking the inner product of (15) with $Av = -\operatorname{div}(\sigma(x) \nabla v)$ in $L^2(\mathcal{O})$, we get that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{\mathcal{D}_0^{1,2}}^2 + \|Av\|_{L^2}^2 + \lambda \|v\|_{\mathcal{D}_0^{1,2}}^2 & = \int_{\mathcal{O}} F(x, u(t)) Av dx + \int_{\mathcal{O}} f(x, u(t - \rho)) Av dx \\ & + \int_{\mathcal{O}} g(t, x) Av dx - \int_{\mathcal{O}} Az(\theta_t \omega) Av dx. \end{aligned} \quad (15)$$

Now we estimate each term on the right-hand side of (15). For the first term, by (2)-(4) and Young's inequality, we have

$$\begin{aligned}
 & \int_{\mathcal{O}} F(x, u(t)) A v dx \\
 &= \int_{\mathcal{O}} F(x, u(t)) A u dx - \int_{\mathcal{O}} F(x, u(t)) A z(\theta_t \omega) dx \\
 &\leq \int_{\mathcal{O}} \frac{\partial F(x, u)}{\partial x} \sigma(x) \nabla u dx + \int_{\mathcal{O}} \frac{\partial F(x, u)}{\partial u} \sigma(x) |\nabla u|^2 dx + \left| \int_{\mathcal{O}} F(x, u(t)) A z(\theta_t \omega) dx \right| \\
 &\leq \int_{\mathcal{O}} \beta_3(x) |\sigma(x) \nabla u| dx + \alpha_3 \int_{\mathcal{O}} \sigma(x) |\nabla u|^2 dx + \alpha_2 \int_{\mathcal{O}} |u|^{p-1} |A z(\theta_t \omega)| dx \\
 &\quad + \int_{\mathcal{O}} |\beta_2(x) A z(\theta_t \omega)| dx \\
 &\leq \frac{1}{2} \int_{\mathcal{O}} (|\beta_3(x)|^2 \sigma + \sigma |\nabla u|^2) dx + \alpha_3 \|u\|_{\mathcal{D}_0^{1,2}}^2 + \frac{\alpha_2(p-1)}{p} \|u\|_{L^p}^p + \frac{\alpha_2}{p} \|A z(\theta_t \omega)\|_{L^p}^p \\
 &\quad + \frac{p-1}{p} \int_{\mathcal{O}} |\beta_2(x)|^{\frac{p}{p-1}} dx + \frac{1}{p} \int_{\mathcal{O}} |A z(\theta_t \omega)|^p dx \\
 &\leq c \left(\|u\|_{\mathcal{D}_0^{1,2}}^2 + \|u\|_{L^p}^p + \|A z(\theta_t \omega)\|_{L^p}^p + 1 \right).
 \end{aligned} \tag{16}$$

By (6) and Young's inequality, we obtain

$$\begin{aligned}
 \int_{\mathcal{O}} f(x, u(t-\rho)) A v dx &\leq \int_{\mathcal{O}} |f(x, u(t-\rho))|^2 dx + \frac{1}{4} \int_{\mathcal{O}} |A v|^2 dx \\
 &\leq L_f^2 \|u(t-\rho)\|_{L^2}^2 + \|\eta(x)\|_{L^2}^2 + \frac{1}{4} \|A v\|_{L^2}^2.
 \end{aligned} \tag{17}$$

In addition, we deduce that

$$\left| \int_{\mathcal{O}} g(t, x) A v dx - \int_{\mathcal{O}} A z(\theta_t \omega) A v dx \right| \leq 2 \|g(t)\|_{L^2}^2 + 2 \|A z(\theta_t \omega)\|_{L^2}^2 + \frac{1}{4} \|A v\|_{L^2}^2. \tag{18}$$

Consequently, it follows from (16)-(18) that

$$\frac{d}{dt} \|v\|_{\mathcal{D}_0^{1,2}}^2 + \|A v\|_{L^2}^2 \leq c \left(\|u\|_{\mathcal{D}_0^{1,2}}^2 + \|u\|_{L^p}^p + \|u(t-\rho)\|_{L^2}^2 + \|g(t)\|_{L^2}^2 + p_3(\theta_t \omega) + 1 \right), \tag{19}$$

where $p_3(\omega) = \|A z(\omega)\|_{L^p}^p + \|A z(\omega)\|_{L^2}^2$. Let $\tau \in \mathbb{R}$, $t \geq \rho + 1$, $\omega \in \Omega$, $\epsilon \in (\tau + \zeta - 1, \tau + \zeta)$, and $\zeta \in [-\rho, 0]$. Integrating (19) over $(\epsilon, \tau + \zeta)$, we have

$$\begin{aligned}
 & \|v(\tau + \zeta, \tau - t, \omega, \psi)\|_{\mathcal{D}_0^{1,2}}^2 \\
 &\leq \|v(\epsilon, \tau - t, \omega, \psi)\|_{\mathcal{D}_0^{1,2}}^2 + c \int_{\tau-\rho-1}^{\tau} \|u(s - \rho, \tau - t, \omega, \phi)\|_{L^2}^2 ds \\
 &\quad + c \int_{\tau-\rho-1}^{\tau} \|u(s, \tau - t, \omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds + c \int_{\tau-\rho-1}^{\tau} \|u(s, \tau - t, \omega, \phi)\|_{L^p}^p ds \\
 &\quad + c \int_{\tau-\rho-1}^{\tau} \|g(s)\|_{L^2}^2 ds + c \int_{\tau-\rho-1}^{\tau} p_3(\theta_s \omega) ds + c.
 \end{aligned} \tag{20}$$

Integrating the above inequality with respect to ϵ over $(\tau + \zeta - 1, \tau + \zeta)$ and noticing $u(t, \tau - t, \omega, \phi) = v(t, \tau - t, \omega, \psi) + z(\theta_t \omega)$, we obtain

$$\begin{aligned} & \|v(\tau + \zeta, \tau - t, \theta_{-\tau} \omega, \psi)\|_{\mathcal{D}_0^{1,2}}^2 \\ & \leq c \int_{\tau-\rho-1}^{\tau} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds + c \int_{\tau-\rho-1}^{\tau} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{L^p}^p ds \\ & \quad + c \int_{\tau-\rho-1}^{\tau} \|u(s - \rho, \tau - t, \theta_{-\tau} \omega, \phi)\|_{L^2}^2 ds + c \int_{\tau-\rho-1}^{\tau} \|g(s)\|_{L^2}^2 ds \\ & \quad + c \int_{\tau-\rho-1}^{\tau} p_4(\theta_{s-\tau} \omega) ds + c, \end{aligned} \quad (21)$$

where $p_4(\omega) = p_3(\omega) + \|z(\omega)\|_{\mathcal{D}_0^{1,2}}^2$. By (1), there exists $\mathcal{T} = \mathcal{T}(\tau, \omega, D) \geq \rho + 1$ such that for all $t \geq \mathcal{T}$,

$$\begin{aligned} & e^{-m(\rho+1)} \left(\int_{\tau-\rho-1}^{\tau} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds + \int_{\tau-\rho-1}^{\tau} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{L^p}^p ds \right) \\ & \leq \int_{\tau-\rho-1}^{\tau} e^{m(s-\tau)} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds + \int_{\tau-\rho-1}^{\tau} e^{m(s-\tau)} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{L^p}^p ds \\ & \leq \int_{\tau-t}^{\tau} e^{m(s-\tau)} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds + \int_{\tau-t}^{\tau} e^{m(s-\tau)} \|u(s, \tau - t, \theta_{-\tau} \omega, \phi)\|_{L^p}^p ds \\ & \leq \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1). \end{aligned} \quad (22)$$

Similarly, by (1) we get for $t \geq \mathcal{T}$,

$$\begin{aligned} & \int_{\tau-\rho-1}^{\tau} \|u(s - \rho, \tau - t, \theta_{-\tau} \omega, \phi)\|_{L^2}^2 ds \\ & \leq \int_{\tau-\rho-1}^{\tau} \|u_s(\cdot, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{C}_2}^2 ds \\ & \leq (\rho + 1) \sup_{\tau-\rho-1 \leq s \leq \tau} \|u_s(\cdot, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{C}_2}^2 \\ & \leq c(\rho + 1) e^{m(\rho+1)} \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(\rho + 1) e^{m(\rho+1)} (r(\omega) + 1). \end{aligned} \quad (23)$$

Note that

$$\int_{\tau-\rho-1}^{\tau} \|g(s)\|_{L^2}^2 ds \leq e^{m(\rho+1)} \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds \quad (24)$$

and

$$\int_{\tau-\rho-1}^{\tau} p_4(\theta_{s-\tau} \omega) ds = \int_{-\rho-1}^0 p_4(\theta_s \omega) ds \leq c \int_{-\rho-1}^0 e^{-\frac{\lambda}{2}s} r(\omega) ds \leq cr(\omega), \quad (25)$$

which together with (21)-(23) imply that for all $\zeta \in [-\rho, 0]$ and $t \geq \mathcal{T}$,

$$\|v_{\tau}(\cdot, \tau - t, \theta_{-\tau} \omega, \psi)\|_{\mathcal{C}_{\mathcal{D}}}^2 \leq c \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1). \quad (26)$$

Finally, integrating (19) over $(\tau - \rho, \tau)$ and utilizing the same arguments as above, we obtain for all $t \geq \mathcal{T}$,

$$\begin{aligned}
 & \int_{\tau-\rho}^{\tau} \|Av(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^2}^2 ds \\
 & \leq \|v(\tau - \rho, \tau - t, \theta_{-\tau}\omega, \psi)\|_{\mathcal{D}_0^{1,2}}^2 + c \int_{\tau-\rho}^{\tau} \|u(s, \tau - t, \theta_{-\tau}\omega, \phi)\|_{\mathcal{D}_0^{1,2}}^2 ds \\
 & \quad + c \int_{\tau-\rho}^{\tau} \|u(s, \tau - t, \theta_{-\tau}\omega, \phi)\|_{L^p}^p ds + c \int_{\tau-\rho}^{\tau} \|u(s - \rho, \tau - t, \theta_{-\tau}\omega, \phi)\|_{L^2}^2 ds \\
 & \quad + c \int_{\tau-\rho}^{\tau} \|g(s)\|_{L^2}^2 ds + c \int_{\tau-\rho}^{\tau} p_3(\theta_{s-\tau}\omega) ds + c \\
 & \leq c \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1).
 \end{aligned} \tag{27}$$

This completes the proof of Lemma 2. $\square$

Lemma 3. Suppose that assumptions (A1)-(A5) hold. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then for every $\tau \in \mathbb{R}$ and P-a.e. $\omega \in \Omega$, there exists $\mathcal{T} = \mathcal{T}(\tau, \omega, D) \geq \rho + 1$ such that for all $t \geq \mathcal{T}$, the solution v of problem (15)-(17) with ω replaced by $\theta_{-\tau}\omega$ satisfies for all $\zeta \in [-\rho, 0]$,

$$\begin{aligned}
 & \|v(\tau + \zeta, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^p}^p + \int_{\tau-\rho}^{\tau} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^{2p-2}}^{2p-2} ds \\
 & \leq c \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1),
 \end{aligned} \tag{28}$$

where c is a positive deterministic constant independent of τ , ω and D .

Proof. Taking the inner product of (15) with $|v|^{p-2}v$ in $L^2(\mathcal{O})$, we obtain

$$\begin{aligned}
 & \frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} |v|^p dx + (p-1) \int_{\mathcal{O}} \sigma(x) |\nabla v|^2 |v|^{p-2} dx + \lambda \int_{\mathcal{O}} |v|^p dx \\
 & = \int_{\mathcal{O}} F(x, u(t)) |v|^{p-2} v dx + \int_{\mathcal{O}} f(x, u(t-\rho)) |v|^{p-2} v dx + \int_{\mathcal{O}} g(t, x) |v|^{p-2} v dx \\
 & \quad - \int_{\mathcal{O}} Az(\theta_t \omega) |v|^{p-2} v dx.
 \end{aligned} \tag{29}$$

It follows from (A2) and Young's inequality that

$$\begin{aligned}
 & \int_{\mathcal{O}} F(x, u(t)) |v|^{p-2} v dx \\
 & = \int_{\mathcal{O}} F(x, u(t)) |v|^{p-2} u dx - \int_{\mathcal{O}} F(x, u(t)) |v|^{p-2} z(\theta_t \omega) dx \\
 & \leq \int_{\mathcal{O}} [-\alpha_1 |u|^p + \beta_1(x)] |v|^{p-2} dx + \int_{\mathcal{O}} [\alpha_2 |u|^{p-1} + \beta_2(x)] |z(\theta_t \omega)| |v|^{p-2} dx \\
 & \leq \int_{\mathcal{O}} \left[-\frac{\alpha_1}{2} |u|^p + \beta_1(x) + \frac{p-1}{p} \beta_2^{\frac{p}{p-1}}(x) + \left(\frac{1}{p} + \frac{2\alpha_2(p-1)}{\alpha_1 p^2} \right) |z(\theta_t \omega)|^p \right] |v|^{p-2} dx \\
 & \leq \int_{\mathcal{O}} \left[-\frac{\alpha_1}{2^p} |v|^p + \beta_1(x) + \frac{p-1}{p} \beta_2^{\frac{p}{p-1}}(x) + \left(\frac{1}{p} + \frac{2\alpha_2(p-1)}{\alpha_1 p^2} \right) |z(\theta_t \omega)|^p \right] |v|^{p-2} dx \\
 & \leq -\frac{2\kappa}{p} \|v(t)\|_{L^{2p-2}}^{2p-2} + c \|z(\theta_t \omega)\|_{L^{2p-2}}^{2p-2} + c,
 \end{aligned} \tag{30}$$

where $\kappa = \frac{\alpha_1 p}{2^{p+2}}$. By (6), we obtain

$$\begin{aligned}
 & \int_{\mathcal{O}} f(x, u(t-\rho)) |v|^{p-2} v dx \leq \frac{\kappa}{2p} \int_{\mathcal{O}} |v|^{2p-2} dx + \frac{p}{2\kappa} \int_{\mathcal{O}} |f(x, u(t-\rho))|^2 dx \\
 & \leq \frac{\kappa}{2p} \|v(t)\|_{L^{2p-2}}^{2p-2} + c \|u(t-\rho)\|_{L^2}^2 + c.
 \end{aligned} \tag{31}$$

Utilizing Young's inequality, the last two terms on the right-hand side of (29) are bounded by

$$\begin{aligned} & \left| \int_{\mathcal{O}} g(t, x) |v|^{p-2} v dx - \int_{\mathcal{O}} Az(\theta_t \omega) |v|^{p-2} v dx \right| \\ & \leq \frac{\kappa}{2p} \|v(t)\|_{L^{2p-2}}^{2p-2} + c \|g(t)\|_{L^2}^2 + c \|Az(\theta_t \omega)\|_{L^2}^2 + c. \end{aligned} \quad (32)$$

Combining (29) and (30)-(32), we deduce that

$$\begin{aligned} \frac{d}{dt} \|v(t)\|_{L^p}^p + \kappa \|v(t)\|_{L^{2p-2}}^{2p-2} & \leq c \|u(t-\rho)\|_{L^2}^2 + c \|g(t)\|_{L^2}^2 + c \|z(\theta_t \omega)\|_{L^{2p-2}}^{2p-2} \\ & + c \|Az(\theta_t \omega)\|_{L^2}^2 + c. \end{aligned} \quad (33)$$

Let $\tau \in \mathbb{R}$, $t \geq \rho + 1$, $\omega \in \Omega$, and $\epsilon \in (\tau + \zeta - 1, \tau + \zeta)$ for $\zeta \in [-\rho, 0]$. Integrating (33) over $(\epsilon, \tau + \zeta)$, we have

$$\begin{aligned} \|v(\tau + \zeta, \tau - t, \omega, \psi)\|_{L^p}^p & \leq \|v(\epsilon, \tau - t, \omega, \psi)\|_{L^p}^p + c \int_{\tau-\rho-1}^{\tau} \|u(s-\rho, \tau - t, \omega, \phi)\|_{L^2}^2 ds \\ & + c \int_{\tau-\rho-1}^{\tau} \|g(s)\|_{L^2}^2 ds + c \int_{\tau-\rho-1}^{\tau} p_5(\theta_s \omega) ds + c, \end{aligned} \quad (34)$$

where $p_5(\omega) = \|z(\omega)\|_{L^{2p-2}}^{2p-2} + \|Az(\omega)\|_{L^2}^2$. Now integrating the above inequality with respect to ϵ over $(\tau + \zeta - 1, \tau + \zeta)$, we obtain

$$\begin{aligned} & \|v(\tau + \zeta, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p}^p \\ & \leq \int_{\tau-\rho-1}^{\tau} \|v(s, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p}^p ds + c \int_{\tau-\rho-1}^{\tau} \|u_s(\cdot, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{C}_2}^2 ds \\ & + c \int_{\tau-\rho-1}^{\tau} \|g(s)\|_{L^2}^2 ds + c \int_{\tau-\rho-1}^{\tau} p_5(\theta_{s-\tau} \omega) ds + c. \end{aligned} \quad (35)$$

Noting that $\|v(s, \tau - t, \omega, \psi)\|_{L^p}^p \leq 2^{p-1} \left(\|u(s, \tau - t, \omega, \psi)\|_{L^p}^p + \|z(\theta_s \omega)\|_{L^p}^p \right)$, we get

$$\begin{aligned} & \|v(\tau + \zeta, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p}^p \\ & \leq c \int_{\tau-\rho-1}^{\tau} \|u(s, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p}^p ds + c \int_{\tau-\rho-1}^{\tau} \|u_s(\cdot, \tau - t, \theta_{-\tau} \omega, \phi)\|_{\mathcal{C}_2}^2 ds \\ & + c \int_{\tau-\rho-1}^{\tau} \|g(s)\|_{L^2}^2 ds + c \int_{\tau-\rho-1}^{\tau} p_6(\theta_{s-\tau} \omega) ds + c, \end{aligned} \quad (36)$$

where $p_6(\omega) = p_5(\omega) + \|z(\omega)\|_{L^p}^p$ satisfies

$$\int_{\tau-\rho-1}^{\tau} p_6(\theta_{s-\tau} \omega) ds = \int_{-\rho-1}^0 p_6(\theta_s \omega) ds \leq c \int_{-\rho-1}^0 e^{-\frac{\lambda}{2}s} r(\omega) ds \leq cr(\omega). \quad (37)$$

Combining (22)-(24) and (37), we find that there exists $\mathcal{T} = \mathcal{T}(\tau, \omega, D) \geq \rho + 1$ such that for all $t \geq \mathcal{T}$,

$$\|v(\tau + \zeta, \tau - t, \theta_{-\tau} \omega, \psi)\|_{L^p}^p \leq c \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1). \quad (38)$$

Finally, integrating (33) over $(\tau - \rho, \tau)$ and utilizing the same arguments as above, we get for all $t \geq \mathcal{T}$,

$$\begin{aligned} & \kappa \int_{\tau-\rho}^{\tau} \|v(s, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^{2p-2}}^{2p-2} ds \\ & \leq \|v(\tau - \rho, \tau - t, \theta_{-\tau}\omega, \psi)\|_{L^p}^p + c \int_{\tau-\rho}^{\tau} \|u(s - \rho, \tau - t, \theta_{-\tau}\omega, \phi)\|_{L^2}^2 ds \\ & \quad + c \int_{\tau-\rho}^{\tau} \|g(s)\|_{L^2}^2 ds + c \int_{\tau-\rho}^{\tau} p_5(\theta_{s-\tau}\omega) ds + c \\ & \leq c \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1), \end{aligned} \quad (39)$$

which along with (38) yields (28). This completes the proof. $\square$

4. Existence of Random Attractors

In this section, we establish the existence of tempered pullback attractors for the random dynamical system Φ associated with the stochastic parabolic equation (1). For this purpose, we first show that Φ has a closed random absorbing set in $\mathcal{D}$.

Lemma 4. *Suppose that assumptions (A1)-(A5) hold. Then the non-autonomous random dynamical system Φ has a closed measurable $\mathcal{D}$ -pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ as defined by*

$$K(\tau, \omega) = \left\{ \varphi \in \mathcal{C}_2 : \|\varphi\|_{\mathcal{C}_2}^2 \leq R(\tau, \tau, \omega) \right\}, \quad (1)$$

where $R(\tau, \tau, \omega)$ is given by (2).

Proof. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$, it follows from Lemma 1 that for all $t \geq \mathcal{T}$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega). \quad (2)$$

This shows that $\{K(\tau, \omega)\}_{\omega \in \Omega}$ is a random absorbing set for Φ in $\mathcal{D}$. In what follows, we prove that K is tempered. By (2), we have

$$R(\tau, \tau, \omega) = c \int_{-\infty}^0 e^{ms} \|g(s + \tau)\|_{L^2}^2 ds + c(r(\omega) + 1). \quad (3)$$

Then for an arbitrary positive number γ we obtain

$$\begin{aligned} e^{\gamma t} \|K(\tau + t, \theta_t \omega)\|_{\mathcal{C}_2}^2 & \leq e^{\gamma t} R(\tau + t, \tau + t, \theta_t \omega) \\ & = c e^{\gamma t} \int_{-\infty}^0 e^{ms} \|g(s + \tau + t)\|_{L^2}^2 ds + c e^{\gamma t} (r(\omega) + 1). \end{aligned} \quad (4)$$

Let $t_1 = t + \tau$, then it follows from (8) that

$$\begin{aligned} \lim_{t \rightarrow -\infty} e^{\gamma t} \int_{-\infty}^0 e^{ms} \|g(s + \tau + t)\|_{L^2}^2 ds & = e^{-\gamma \tau} \lim_{t_1 \rightarrow -\infty} e^{\gamma t_1} \int_{-\infty}^0 e^{ms} \|g(s + t_1)\|_{L^2}^2 ds \\ & = 0. \end{aligned} \quad (5)$$

Meanwhile, the second term on the right-hand side of (4) converges to zero as $t \rightarrow -\infty$, which along with (4) and (5) shows that

$$\lim_{t \rightarrow -\infty} e^{\gamma t} \|K(\tau + t, \theta_t \omega)\|_{\mathcal{C}_2}^2 = 0.$$

Therefore, K is tempered and belongs to $\mathcal{D}$. Notice that for each $\tau \in \mathbb{R}$, $R(\tau, \tau, \cdot) : \Omega \rightarrow \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable. This implies that K is a closed measurable $\mathcal{D}$ -pullback absorbing set for Φ . $\square$

Next, we discuss the $\mathcal{D}$ -pullback asymptotic compactness of Φ , which will be proved by utilizing the Arzela-Ascoli theorem in [25].

Lemma 5. Suppose that assumptions (A1)-(A5) hold. Then the non-autonomous random dynamical system Φ of problem (1)-(3) is $\mathcal{D}$ -pullback asymptotically compact in $\mathcal{C}_2$.

Proof. It suffices to show that for every $\tau \in \mathbb{R}$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and P -a.e. $\omega \in \Omega$, if $t_n \rightarrow \infty$ and $\phi_n \in D(\tau - t_n, \theta_{-t_n}\omega)$, then the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, \phi_n)$ has a convergent subsequence in $\mathcal{C}_2$. To this end, we will utilize the Ascoli-Arzela theorem to prove that $u_\tau(\cdot, \tau - t_n, \theta_{-t_n}\omega, \phi_n)$ is precompact in $\mathcal{C}_2$.

First, we prove that $u_\tau(\cdot, \tau - t_n, \theta_{-t_n}\omega, \phi_n)$ is uniformly equicontinuous in $\mathcal{C}_2$. In fact, it follows from (15) and assumptions (A1)-(A5) that

$$\begin{aligned} & \int_{\tau-\rho}^{\tau} \left\| \frac{d}{ds} v(s, \tau - t_n, \theta_{-t_n}\omega, \psi_n) \right\|_{L^2}^2 ds \\ & \leq c \int_{\tau-\rho}^{\tau} \|Av(s, \tau - t_n, \theta_{-t_n}\omega, \psi_n)\|_{L^2}^2 ds + c \int_{\tau-\rho}^{\tau} \|v_s(\cdot, \tau - t_n, \theta_{-t_n}\omega, \psi_n)\|_{\mathcal{C}^2}^2 ds \\ & \quad + c \int_{\tau-\rho}^{\tau} \|u(s, \tau - t_n, \theta_{-t_n}\omega, \psi_n)\|_{L^{2p-2}}^{2p-2} ds + c \int_{\tau-\rho}^{\tau} \|u_s(\cdot, \tau - t_n, \theta_{-t_n}\omega, \psi_n)\|_{\mathcal{C}^2}^2 ds \\ & \quad + c \int_{-\rho}^0 p(\theta_{s-\tau}\omega) ds + c. \end{aligned}$$

In view of (12), and Lemmas 1-3, we can find large enough positive integer N such that for all $n \geq N$,

$$\int_{\tau-\rho}^{\tau} \left\| \frac{d}{ds} v(s, \tau - t_n, \theta_{-t_n}\omega, \psi_n) \right\|_{L^2}^2 ds \leq c(r(\omega) + 1).$$

Therefore, for every $n \geq N$ and $s_1, s_2 \in [-\rho, 0]$,

$$\begin{aligned} & \|v_\tau(s_2, \tau - t_n, \theta_{-t_n}\omega, \psi_n) - v_\tau(s_1, \tau - t_n, \theta_{-t_n}\omega, \psi_n)\|_{L^2} \\ & \leq |s_2 - s_1|^{\frac{1}{2}} \left(\int_{\tau+s_1}^{\tau+s_2} \left\| \frac{d}{ds} v(s, \tau - t_n, \theta_{-t_n}\omega, \psi_n) \right\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\ & \leq |s_2 - s_1|^{\frac{1}{2}} \left(\int_{\tau-\rho}^{\tau} \left\| \frac{d}{ds} v(s, \tau - t_n, \theta_{-t_n}\omega, \psi_n) \right\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\ & \leq c(r(\omega) + 1) |s_2 - s_1|^{\frac{1}{2}}. \end{aligned} \tag{6}$$

By (18) and (6) we obtain, for all $n \geq N$ and $s_1, s_2 \in [-\rho, 0]$,

$$\begin{aligned} & \|u_\tau(s_2, \tau - t_n, \theta_{-t_n}\omega, \phi_n) - u_\tau(s_1, \tau - t_n, \theta_{-t_n}\omega, \phi_n)\|_{L^2} \\ & \leq c |s_2 - s_1|^{\frac{1}{2}} + c \|z(\theta_{\tau+s_2}\omega) - z(\theta_{\tau+s_1}\omega)\|_{L^2}, \end{aligned}$$

which along with the continuity of ω implies the desired equicontinuity.

On the other hand, for each fixed $s \in [-\rho, 0]$, we can get from Lemma 2 that $u(\tau + s, \tau - t_n, \theta_{-t_n}\omega, \phi_n)$ is bounded in $\mathcal{D}_0^{1,2}(\mathcal{O}, \sigma)$. Then, it follows from the compactness of embedding $\mathcal{D}_0^{1,2}(\mathcal{O}, \sigma) \hookrightarrow L^2(\mathcal{O})$ that $u(\tau + s, \tau - t_n, \theta_{-t_n}\omega, \phi_n)$ is precompact in $\mathcal{C}_2$. Therefore, we deduce that $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, \phi_n)$ is asymptotically compact in $\mathcal{C}_2$. $\square$

We are now in a position to present the existence of $\mathcal{D}$ -pullback attractors for Φ in $\mathcal{C}_2$.

Theorem 1. Suppose that assumptions (A1)-(A5) hold. Then the non-autonomous random dynamical system Φ associated with problem (1)-(3) has a unique $\mathcal{D}$ -pullback attractor in $\mathcal{C}_2$. If, in addition, there exists $T > 0$ such that g is T -periodic in its first argument, then the attractor is also T -periodic.

Proof. Notice that the random dynamical system Φ has a closed random absorbing set $\{K(\tau, \omega)\}_{\omega \in \Omega}$ in $\mathcal{D}$ by Lemma 4, and is asymptotically compact in $\mathcal{C}_2$ by Lemma 5. Hence the existence of a unique $\mathcal{D}$ -pullback attractor for Φ follows from Proposition 2.1 in [31] immediately. If g is T -periodic in its first argument, then the random dynamical system Φ and the absorbing set K are also T -periodic, which implies the T -periodicity of the attractor. $\square$

5. Upper Semicontinuity of Random Attractors

In the section, we will consider the upper semicontinuity of random attractors of problem (1)-(3) as the delay approaches zero. Hereafter, we assume $\rho \in (0, 1]$ and denote the solution and non-autonomous random dynamical system of problem (1)-(3) as u^ρ and Φ_ρ , respectively. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let

$$B_\rho(\tau, \omega) = \{\varphi \in \mathcal{C}_2 : \|\varphi\|_{\mathcal{C}_2}^2 \leq R(\tau, \tau, \omega)\}, \quad (1)$$

where $R(\tau, \tau, \omega)$ is the number given by (2). It follows from (1) and Lemma 1 that for every $\rho \in (0, 1]$, $B_\rho = \{B_\rho(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a $\mathcal{D}$ -pullback absorbing set of Φ_ρ in $\mathcal{C}_2$. Let $\mathcal{A}_\rho$ be the attractor of Φ_ρ in $\mathcal{C}_2$. Then we deduce that for all $\tau \in \mathbb{R}$ and P -a.e. $\omega \in \Omega$

$$\mathcal{A}_\rho(\tau, \omega) \subseteq B_\rho(\tau, \omega). \quad (2)$$

Let $\rho = 0$, from (1)-(3) we have

$$\begin{aligned} \frac{du}{dt} + \lambda u(t, x) - \operatorname{div}(\sigma(x) \nabla u(t, x)) = & F(x, u(t, x)) + f(x, u(t, x)) + g(t, x) \\ & + \sum_{j=1}^m h_j(x) \frac{dw_j(t)}{dt} \end{aligned} \quad (3)$$

for $x \in \mathcal{O}$ and $t > \tau$, with boundary condition

$$u(t, x) = 0, \quad t > \tau, x \in \partial\mathcal{O}, \quad (4)$$

and initial condition

$$u(\tau, x) = \phi(x), \quad x \in \mathcal{O}. \quad (5)$$

Let Φ_0 be the non-autonomous random dynamical system generated by problem (3)-(5) in $L^2(\mathcal{O})$. Suppose that $\mathcal{D}_0$ is the collection of all tempered families of nonempty subsets of $L^2(\mathcal{O})$, i.e., for any $c > 0$,

$$\mathcal{D}_0 = \left\{ \{D(\tau, \omega) \subseteq L^2(\mathcal{O}) : \tau \in \mathbb{R}, \omega \in \Omega\} : \lim_{t \rightarrow -\infty} e^{ct} \|D(\tau + t, \theta_t \omega)\|_{L^2} = 0 \right\}.$$

We notice that all estimates of solutions in section 3 are valid for $\rho = 0$. Therefore, Φ_0 has a unique $\mathcal{D}_0$ -pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $L^2(\mathcal{O})$ and an absorbing set $K_0 = \{K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ given by

$$K_0(\tau, \omega) = \{\varphi \in L^2(\mathcal{O}) : \|\varphi\|_{L^2}^2 \leq R(\tau, \tau, \omega)\}. \quad (6)$$

From (1) and (6), it is trivial to deduce that for all $\tau \in \mathbb{R}$ and P -a.e. $\omega \in \Omega$,

$$\limsup_{\rho \rightarrow 0} \|B_\rho(\tau, \omega)\|_{\mathcal{C}_2} = \|K_0(\tau, \omega)\|_{L^2}. \quad (7)$$

We now prove the convergence of solutions of (1)-(3) as $\rho \rightarrow 0$. For this purpose, we introduce the following assumption:

- (A6) There exist $\alpha_4 > 0$ and $\beta_4(x) \in L^{p^*}(\mathcal{O})$ such that for all $x \in \mathcal{O}$ and $u \in \mathbb{R}$,

$$\left| \frac{\partial}{\partial u} F(x, u) \right| \leq \alpha_4 |u|^{p-2} + \beta_4(x), \quad (8)$$

where $p^* = \infty$ for $p = 2$, and $p^* = \frac{p}{p-2}$ for $p > 2$.

Lemma 6. Suppose that assumptions (A1)-(A6) hold, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Let u^ρ and u be the solutions of (1)-(3) and (3)-(5) with initial data ϕ^ρ and ϕ , respectively. Then for every $\eta \in (0, 1]$, there exists $\varrho \in (0, 1]$ depending on τ, ω, T , and η such that for all $\rho \leq \varrho$ and $t \in [\tau, \tau + T]$,

$$\sup_{-\rho \leq \zeta \leq 0} \|u^\rho(t + \zeta) - u(t)\|_{L^2}^2 \leq c\eta \left(\|\phi^\rho\|_{L^2}^2 + \|\phi\|_{L^2}^2 + r(\omega) + 1 \right) + c \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2,$$

where c is a positive constant independent of ρ and η .

Proof. Let $v(t, \tau, \omega, \psi) = u(t, \tau, \omega, \phi) - z(\theta_t \omega)$ and $v^\rho(t, \tau, \omega, \psi^\rho) = u^\rho(t, \tau, \omega, \phi^\rho) - z(\theta_t \omega)$, where $\phi(x) = \psi(x) + z(\theta_\tau \omega)$ and $\phi^\rho(\zeta) = \psi^\rho(\zeta) + z(\theta_{\tau+\zeta} \omega)$ for all $\zeta \in [-\rho, 0]$. Fix $\zeta \in [-\rho, 0]$, and let $\tilde{v}(t) = v^\rho(t + \zeta) - v(t)$. Then, for $t > \tau - \zeta$ and $\zeta \in [-\rho, 0]$, $\tilde{v}(t)$ satisfies

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} + A\tilde{v} + \lambda \tilde{v} &= F(x, u^\rho(t + \zeta)) - F(x, u(t)) + f(x, u^\rho(t + \zeta - \rho)) - f(x, u(t)) \\ &\quad + g(t + \zeta, x) - g(t, x) - Az(\theta_{t+\zeta} \omega) + Az(\theta_t \omega). \end{aligned} \quad (9)$$

Multiplying (9) by $\tilde{v}$ and then integrating over $\mathcal{O}$, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\tilde{v}(t)\|_{L^2}^2 + \|\tilde{v}(t)\|_{\mathcal{D}_0^{1,2}}^2 + \lambda \|\tilde{v}(t)\|_{L^2}^2 \\ &= \int_{\mathcal{O}} (F(x, u^\rho(t + \zeta)) - F(x, u(t))) \tilde{v} dx + \int_{\mathcal{O}} (f(x, u^\rho(t + \zeta - \rho)) - f(x, u(t))) \tilde{v} dx \\ &\quad + \int_{\mathcal{O}} (g(t + \zeta) - g(t)) \tilde{v} dx - \int_{\mathcal{O}} (Az(\theta_{t+\zeta} \omega) - Az(\theta_t \omega)) \tilde{v} dx. \end{aligned} \quad (10)$$

For the first term on the right-hand side of (10), by (3) and (8) we obtain

$$\begin{aligned} &\int_{\mathcal{O}} (F(x, u^\rho(t + \zeta)) - F(x, u(t))) \tilde{v} dx \\ &= \int_{\mathcal{O}} \frac{\partial F}{\partial u}(x, u) (u^\rho(t + \zeta) - u(t)) \tilde{v} dx \\ &= \int_{\mathcal{O}} \frac{\partial F}{\partial u}(x, u) \tilde{v}^2 dx + \int_{\mathcal{O}} \frac{\partial F}{\partial u}(x, u) (z(\theta_{t+\zeta} \omega) - z(\theta_t \omega)) \tilde{v} dx \\ &\leq \int_{\mathcal{O}} (\alpha_4 (|u^\rho(t + \zeta)| + |u(t)|)^{p-2} + \beta_4(x)) (|z(\theta_{t+\zeta} \omega)| + |z(\theta_t \omega)|) |\tilde{v}| dx + \alpha_3 \|\tilde{v}(t)\|_{L^2}^2 \\ &\leq c \left(\|u^\rho\|_{L^p}^p + \|u\|_{L^p}^p + \|\tilde{v}\|_{L^p}^p + \|z(\theta_{t+\zeta} \omega)\|_{L^p}^p + \|z(\theta_t \omega)\|_{L^p}^p + \|\beta_4\|_{L^{p^*}}^{p^*} \right) + \alpha_3 \|\tilde{v}\|_{L^2}^2. \end{aligned} \quad (11)$$

By (5), we have

$$\begin{aligned} &\int_{\mathcal{O}} (f(x, u^\rho(t + \zeta - \rho)) - f(x, u(t))) \tilde{v} dx \leq C_f \int_{\mathcal{O}} |u^\rho(t + \zeta - \rho) - u(t)| |\tilde{v}(t)| dx \\ &\leq c \|\tilde{v}(t)\|_{L^2}^2 + c \|u^\rho(t + \zeta - \rho) - u(t)\|_{L^2}^2. \end{aligned} \quad (12)$$

Note that

$$\int_{\mathcal{O}} (g(t + \zeta) - g(t)) \tilde{v} dx \leq \frac{1}{2} \|\tilde{v}(t)\|_{L^2}^2 + \frac{1}{2} \|g(t + \zeta) - g(t)\|_{L^2}^2, \quad (13)$$

and

$$\int_{\mathcal{O}} (Az(\theta_{t+\zeta} \omega) - Az(\theta_t \omega)) \tilde{v} dx \leq \frac{1}{2} \|\tilde{v}(t)\|_{L^2}^2 + \|Az(\theta_{t+\zeta} \omega)\|_{L^2}^2 + \|Az(\theta_t \omega)\|_{L^2}^2. \quad (14)$$

It follows from (11)-(14) that for $t > \tau - \zeta$ and $\zeta \in [-\rho, 0]$,

$$\begin{aligned} \frac{d}{dt} \|\tilde{v}(t)\|_{L^2}^2 &\leq c \left(\|u^\rho(t + \zeta)\|_{L^p}^p + \|u(t)\|_{L^p}^p + 1 \right) + c \|u^\rho(t + \zeta - \rho) - u(t)\|_{L^2}^2 \\ &\quad + c \|\tilde{v}(t)\|_{L^2}^2 + \|g(t + \zeta) - g(t)\|_{L^2}^2 + c p_7(\theta_t \omega), \end{aligned} \quad (15)$$

where $p_7(\omega) = \|z(\theta_\zeta \omega)\|_{L^p}^p + \|z(\omega)\|_{L^p}^p + \|Az(\theta_\zeta \omega)\|_{L^2}^2 + \|Az(\omega)\|_{L^2}^2$. Let $t \in [\tau, \tau + T]$ and $t \geq \tau - \zeta$. Integrating the above inequality over $(\tau - \zeta, t)$, we obtain

$$\begin{aligned} \|\tilde{v}(t)\|_{L^2}^2 &\leq \|\tilde{v}(\tau - \zeta)\|_{L^2}^2 + c \int_{\tau - \zeta}^t \|\tilde{v}(s)\|_{L^2}^2 ds + c \int_{\tau - \zeta}^t \left(\|u^\rho(s + \zeta)\|_{L^p}^p + \|u(s)\|_{L^p}^p + 1 \right) ds \\ &\quad + c \int_{\tau - \zeta}^t \|u^\rho(s + \zeta - \rho) - u(s)\|_{L^2}^2 ds + \int_{\tau - \zeta}^t \|g(s + \zeta) - g(s)\|_{L^2}^2 ds \\ &\quad + c \int_{\tau - \zeta}^t p_7(\theta_s \omega) ds. \end{aligned} \quad (16)$$

For the fourth term on the right-hand side of (16), we have

$$\begin{aligned} \int_{\tau - \zeta}^t \|u^\rho(s + \zeta - \rho) - u(s)\|_{L^2}^2 ds &= \int_{\tau - \zeta}^{\tau + \rho - \zeta} \|u^\rho(s + \zeta - \rho) - u(s)\|_{L^2}^2 ds \\ &\quad + \int_{\tau + \rho - \zeta}^t \|u^\rho(s + \zeta - \rho) - u(s)\|_{L^2}^2 ds. \end{aligned}$$

Meanwhile, we observe that

$$\begin{aligned} &\int_{\tau - \zeta}^{\tau + \rho - \zeta} \|u^\rho(s + \zeta - \rho) - u(s)\|_{L^2}^2 ds \\ &\leq 2 \int_{\tau - \zeta}^{\tau + \rho - \zeta} \|u^\rho(s + \zeta - \rho) - \phi\|_{L^2}^2 ds + 2 \int_{\tau - \zeta}^{\tau + \rho - \zeta} \|u(s) - \phi\|_{L^2}^2 ds \\ &\leq 2 \int_{\tau - \rho}^{\tau} \|u^\rho(s) - \phi\|_{L^2}^2 ds + 2 \int_{\tau - \zeta}^{\tau + \rho - \zeta} \|u(s) - \phi\|_{L^2}^2 ds, \end{aligned}$$

and

$$\begin{aligned} &\int_{\tau + \rho - \zeta}^t \|u^\rho(s + \zeta - \rho) - u(s)\|_{L^2}^2 ds \\ &= \int_{\tau - \zeta}^{t - \rho} \|u^\rho(s + \zeta) - u(s + \rho)\|_{L^2}^2 ds \\ &\leq \int_{\tau - \zeta}^t \|u^\rho(s + \zeta) - u(s + \rho)\|_{L^2}^2 ds \\ &\leq 2 \int_{\tau - \zeta}^t \|u^\rho(s + \zeta) - u(s)\|_{L^2}^2 ds + 2 \int_{\tau - \zeta}^t \|u(s + \rho) - u(s)\|_{L^2}^2 ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_{\tau - \zeta}^t \|u^\rho(s + \zeta - \rho) - u(s)\|_{L^2}^2 ds \\ &\leq 2\rho \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2 + 2 \int_{\tau - \zeta}^{\tau + \rho - \zeta} \|u(s) - \phi\|_{L^2}^2 ds + 4 \int_{\tau - \zeta}^t \|\tilde{v}(s)\|_{L^2}^2 ds \\ &\quad + 4 \int_{\tau - \zeta}^t \|z(\theta_{s+\zeta} \omega) - z(\theta_s \omega)\|_{L^2}^2 ds + 2 \int_{\tau - \zeta}^t \|u(s + \rho) - u(s)\|_{L^2}^2 ds. \end{aligned} \quad (17)$$

By (16)-(17), for $t \in [\tau, \tau + T]$ and $t \geq \tau - \zeta$, we obtain

$$\begin{aligned} \|\tilde{v}(t)\|_{L^2}^2 &\leq \|\tilde{v}(\tau - \zeta)\|_{L^2}^2 + c\rho \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2 + c \int_{\tau - \zeta}^t \|\tilde{v}(s)\|_{L^2}^2 ds \\ &\quad + c \int_{\tau - \zeta}^t \left(\|u^\rho(s + \zeta)\|_{L^p}^p + \|u(s)\|_{L^p}^p + 1 \right) ds + c \int_{\tau}^{\tau + 2\rho} \|u(s) - \phi\|_{L^2}^2 ds \\ &\quad + c \int_{\tau}^t \|u(s + \rho) - u(s)\|_{L^2}^2 ds + \int_{\tau}^t \|g(s + \zeta) - g(s)\|_{L^2}^2 ds + c \int_{\tau}^t p_8(\theta_s \omega) ds, \end{aligned} \quad (18)$$

where $p_8(\theta_s \omega) = p_7(\theta_s \omega) + \|z(\theta_s \omega)\|_{L^2}^2 + \|z(\omega)\|_{L^2}^2$. Therefore, it follows from (14) that for given $\eta > 0$, there exists $\rho_1 \in (0, 1]$ such that for all $\rho \leq \rho_1$, $\zeta \in [-\rho, 0]$, and $s \in [\tau, \tau + T]$,

$$\int_{\tau}^t p_8(\theta_s \omega) ds \leq c \int_{\tau}^t e^{\frac{\lambda}{2}s} r(\omega) ds \leq c\eta r(\omega). \quad (19)$$

Noting that $\lim_{\rho \rightarrow 0} \int_{\tau}^{\tau + 2\rho} \|u(s) - \phi\|_{L^2}^2 ds = 0$, we obtain that there exists $\rho_2 \leq \rho_1$ such that for all $\rho \leq \rho_2$,

$$\int_{\tau}^{\tau + 2\rho} \|u(s) - \phi\|_{L^2}^2 ds \leq \eta. \quad (20)$$

Since $u : [\tau, \tau + T + 1] \rightarrow L^2(\mathcal{O})$ is uniformly continuous, there exists $\rho_3 \leq \rho_2$ such that for all $\rho \leq \rho_3$ and $s \in [\tau, \tau + T]$,

$$\|u(s + \rho) - u(s)\|_{L^2} \leq \eta. \quad (21)$$

At the same time, noting that $g \in L_{loc}^2(\mathbb{R}, L^2(\mathcal{O}))$, we obtain

$$\lim_{\zeta \rightarrow 0} \int_{\tau}^{\tau + T} \|g(s + \zeta) - g(s)\|_{L^2}^2 ds = 0,$$

and hence that there exists $\rho_4 \leq \rho_3$ such that for all $\rho \leq \rho_4$, and $\zeta \in [-\rho, 0]$,

$$\int_{\tau}^{\tau + T} \|g(s + \zeta) - g(s)\|_{L^2}^2 ds \leq \eta. \quad (22)$$

By utilizing (8), we get for $\rho \in (0, 1]$

$$\int_{\tau - \zeta}^t \|u^\rho(s + \zeta)\|_{L^p}^p ds \leq c \left(\|\phi^\rho\|_{\mathcal{C}_2}^2 + \int_{\tau}^{\tau + T} \|g(s)\|_{L^2}^2 ds + r(\omega) + 1 \right). \quad (23)$$

Similarly, by (3) we have

$$\int_{\tau}^{\tau + T} \|u(s)\|_{L^p}^p ds \leq c \left(\|\phi\|_{L^2}^2 + \int_{\tau}^{\tau + T} \|g(s)\|_{L^2}^2 ds + r(\omega) + 1 \right). \quad (24)$$

It follows from (18)-(24) that for all $\rho \leq \rho_4$, $t \in [\tau, \tau + T]$, $t \geq \tau - \zeta$, and $\zeta \in [-\rho, 0]$,

$$\begin{aligned} \|\tilde{v}(t)\|_{L^2}^2 &\leq \|\tilde{v}(\tau - \zeta)\|_{L^2}^2 + c \int_{\tau - \zeta}^t \|\tilde{v}(s)\|_{L^2}^2 ds + c\eta \left(\|\phi^\rho\|_{\mathcal{C}_2}^2 + \|\phi\|_{L^2}^2 + r(\omega) + 1 \right) \\ &\quad + c\rho \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2. \end{aligned} \quad (25)$$

Now, the Gronwall's inequality yields that for all $\rho \leq \rho_4$, $t \in [\tau, \tau + T]$, $t \geq \tau - \zeta$, and $\zeta \in [-\rho, 0]$,

$$\|\tilde{v}(t)\|_{L^2}^2 \leq c \|\tilde{v}(\tau - \zeta)\|_{L^2}^2 + c\eta \left(\|\phi^\rho\|_{\mathcal{C}_2}^2 + \|\phi\|_{L^2}^2 + r(\omega) + 1 \right) + c\rho \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2. \quad (26)$$

Note that

$$\|\tilde{v}(\tau - \zeta)\|_{L^2}^2 \leq 4\|u^\rho(\tau) - \phi\|_{L^2}^2 + 4\|u(\tau - \zeta) - \phi\|_{L^2}^2 + 2\|z(\theta_\tau \omega)\|_{L^2}^2 + 2\|z(\theta_{\tau-\zeta} \omega)\|_{L^2}^2,$$

which along with (14) and the continuity of u shows that there exists $\rho_5 \leq \rho_4$ such that for all $\rho \leq \rho_5$,

$$\|\tilde{v}(\tau - \zeta)\|_{L^2}^2 \leq c\eta(r(\omega) + 1) + c \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2.$$

It follows from (26) that for all $\rho \leq \rho_5$, $t \in [\tau, \tau + T]$, $t \geq \tau - \zeta$, and $\zeta \in [-\rho, 0]$,

$$\|v^\rho(t + \zeta) - v(t)\|_{L^2}^2 \leq c\eta\left(\|\phi^\rho\|_{C_2}^2 + \|\phi\|_{L^2}^2 + r(\omega) + 1\right) + c \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2, \quad (27)$$

which implies that for all $\rho \leq \rho_5$, $t \in [\tau, \tau + T]$, $t \geq \tau - \zeta$, and $\zeta \in [-\rho, 0]$,

$$\|u^\rho(t + \zeta) - u(t)\|_{L^2}^2 \leq c\eta\left(\|\phi^\rho\|_{C_2}^2 + \|\phi\|_{L^2}^2 + r(\omega) + 1\right) + c \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2. \quad (28)$$

In the following, we consider the case $\tau \leq t \leq \tau - \zeta$. Let $\varsigma = t - \tau$, then $t = \varsigma + \tau$ and $0 \leq \varsigma \leq \rho$. In this case, we have

$$\|u^\rho(t + \zeta) - u(t)\|_{L^2}^2 \leq 2 \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2 + 2\|u(\varsigma + \tau) - \phi\|_{L^2}^2. \quad (29)$$

Combining $0 \leq \varsigma \leq \rho$ with the continuity of u at τ , we deduce that there exists $\rho_6 \leq \rho_5$ such that for all $\rho \leq \rho_6$ and $\tau \leq t \leq \tau - \zeta$,

$$\|u^\rho(t + \zeta) - u(t)\|_{L^2}^2 \leq c \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2 + c\eta,$$

which together with (28) means that for all $\rho \leq \rho_6$, $t \in [\tau, \tau + T]$, and $\zeta \in [-\rho, 0]$,

$$\|u^\rho(t + \zeta) - u(t)\|_{L^2}^2 \leq c\eta\left(\|\phi^\rho\|_{C_2}^2 + \|\phi\|_{L^2}^2 + r(\omega) + 1\right) + c \sup_{-\rho \leq \zeta \leq 0} \|\phi^\rho(\zeta) - \phi\|_{L^2}^2.$$

Therefore, the proof of Lemma 6 is completed. $\square$

We now present the uniform compactness of random attractors with respect to ρ , whose proof is similar to the proof of Lemma 7.2 in [31], and hence is omitted.

Lemma 7. Suppose that assumptions (A1)-(A6) hold. If $\rho_n \rightarrow 0$ and $u_n \in \mathcal{A}_{\rho_n}(\tau, \omega)$, then there exist a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ and $u \in L^2(\mathcal{O})$ satisfy

$$\lim_{m \rightarrow \infty} \sup_{-\rho_{n_m} \leq \zeta \leq 0} \|u_{n_m}(\zeta) - u\|_{L^2} = 0.$$

Finally, we provide the upper semicontinuity of random attractors as $\rho \rightarrow 0$.

Theorem 2. Suppose that assumptions (A1)-(A6) hold. Then for all $\tau \in \mathbb{R}$ and P -a.e. $\omega \in \Omega$,

$$\lim_{\rho \rightarrow 0} \text{dist}_H(\mathcal{A}_\rho(\tau, \omega), \mathcal{A}(\tau, \omega)) = 0, \quad (30)$$

where dist_H is the distance defined for any subset E of $C([-\rho_n, 0], X)$ and S of X by

$$\text{dist}_H(E, S) = \sup_{\phi \in E} \inf_{x \in S} \sup_{-\rho \leq s \leq 0} \|\phi(s) - x\|_X.$$

.

Proof. Let $\rho_n \rightarrow 0$, $\phi^{\rho_n} \in \mathcal{A}_{\rho_n}$, and $\phi \in L^2(\mathcal{O})$ satisfying $\sup_{-\rho_n \leq \zeta \leq 0} \|\phi^{\rho_n}(\zeta) - \phi\|_{L^2} \rightarrow 0$ as $n \rightarrow \infty$. It follows from Lemma 6 that for every $\tau \in \mathbb{R}$, $t \geq 0$, and P -a.e. $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} \sup_{-\rho_n \leq \zeta \leq 0} \|\Phi_{\rho_n}(t, \tau, \omega, \phi^{\rho_n})(\zeta) - \Phi_0(t, \tau, \omega, \phi)\|_{L^2} = 0,$$

which together with (7) and Lemma 7 shows that all conditions of Theorem 2.1 in [31] are satisfied, and hence (30) follows immediately. $\square$

6. Conclusions

In this paper, we successfully investigated the dynamics of solutions of a class of stochastic parabolic equations with delay and deterministic non-autonomous forcing. Firstly, we transformed the stochastic equations into deterministic versions with random coefficients by using Ornstein-Uhlenbeck transformation, and proved the existence of a continuous non-autonomous random dynamical system by exploiting the ideas from the theory of deterministic PDEs with polynomial nonlinearity and delay PDEs with Lipschitz nonlinearity. Then, we established the pullback asymptotical compactness of solutions as well as the existence of random attractors by deriving some uniform estimates of solutions with respect to delay and time-dependent forcing for the nonlinearity satisfying an arbitrary polynomial growth condition. Finally, we shown the upper semicontinuity of random attractors when the delay approaches zero by establishing the convergence of solutions.

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References

1. C.T. Anh, T.Q. Bao, Pullback attractors for a non-autonomous semi-linear degenerate parabolic equation, *Glasg. Math. J.*, **52** (2010), 537–554.
2. L. Arnold, *Random Dynamical Systems*, Berlin, Heidelberg: Springer, 1998.
3. A.V. Babin, M.I. Vishik, *Attractors of Evolution Equations*, Amsterdam: North-Holland, 1992.
4. T. Caraballo, G. Lukasiewicz, J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.*, **64** (2006), 484–498.
5. T. Caraballo, Nonlinear partial functional differential equations: Existence and stability, *J. Math. Anal. Appl.*, **262** (2001), 87–111.
6. V.V. Chepyzhov, M.I. Vishik, *Attractors for Equations of Mathematical Physics*, Providence, Rhode Island: American Mathematical society, 2003.
7. P.Y. Chen, X.H. Zhang, X.P. Zhang, Asymptotic behavior of non-autonomous fractional stochastic p-Laplacian equations with delay on R^n , *J. Dynam. Differential Equations*, **35** (2023), 3459–3485.
8. P.Y. Chen, X.P. Zhang, Upper semi-continuity of attractors for non-autonomous fractional stochastic parabolic equations with delay, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 4325–4357.
9. Z. Chen, W. Lin, D.D. Yang, X.D. Zuo, Random attractors and invariant measures for 3D stochastic globally modified Navier-Stokes equations with time-dependent delay and coefficient, *Discrete Contin. Dyn. Syst. Ser. S*, **16** (2023), 2721–2748.
10. H. Crauel, A. Debussche, F. Flandoli, Random attractors, *J. Dynam. Differential Equations*, **9** (1997), 307–341.
11. H. Crauel, F. Flandoli, Attractor for random dynamical systems, *Probab. Theory Related Fields*, **100** (1994), 365–393.
12. L. Dong, Random attractors for stochastic Navier-Stokes equation on a 2D rotating sphere with stable Lévy noise, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 5421–5448.
13. F. Flandoli, B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stoch. Stoch. Rep.*, **59** (1996), 21–45.

14. B. Gess, W. Liu, A. Schenke, Random attractors for locally monotone stochastic partial differential equations, *J. Differential Equations*, **269** (2020), 3414–3455.
15. B.L. Guo, Y.Q. Han, G.L. Zhou, Random attractor for the 2D stochastic nematic liquid crystals flows, *Commun. Pure Appl. Anal.*, **18** (2019), 2349–2376.
16. B.L. Guo, G.L. Zhou, Finite dimensionality of global attractor for the solutions to 3D viscous primitive equations of large-scale moist atmosphere, *Discrete Contin. Dyn. Syst. Ser. B*, **23** (2018), 4305–4327.
17. N.I. Karachalios, N.B. Zographopoulos, On the dynamics of a degenerate parabolic equation: global bifurcation of stationary states and convergence, *Calc. Var. Partial Differential Equations*, **25** (2006), 361–393.
18. N.I. Karachalios, N.B. Zographopoulos, Convergence towards attractors for a degenerate Ginzburg-Landau equation, *Z. Angew. Math. Phys.*, **56** (2005), 11–30.
19. S.Z. Li, S.J. Guo, Random attractors for stochastic semilinear degenerate parabolic equations with delay, *Phys. A*, **550** (2020), 124164.
20. S.Z. Li, S.J. Guo, Invariant measure and random attractors for stochastic differential equations with delay, *Qual. Theory Dyn. Syst.*, **21** (2022), Paper No: 40.
21. L. Xu, J.H. Huang, Q.Z. Ma, Random exponential attractor for stochastic non-autonomous suspension bridge equation with additive white noise, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022), 6323–6351.
22. Y. Li, Pullback random attractors of stochastic strongly damped wave equations with variable delays on unbounded domains, *AIMS Math.*, **6** (2021), 13634–13664.
23. G.D. Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge: Cambridge University Press, 1992.
24. G.R. Sell, Y. You, *Dynamics of Evolutionary Equations*, New York: Springer, 2002.
25. J. Simon, Compact sets in the space $L^p(0, T, B)$, *Ann. Mat. Pura Appl.*, **196** (1987), 65–96.
26. B.X. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (2012), 1544–1583.
27. B.X. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **34** (2014), 269–300.
28. B.X. Wang, Dynamics of fractional stochastic reaction-diffusion equations on unbounded domains driven by nonlinear noise, *J. Differential Equations*, **268** (2019), 1–59.
29. J.Y. Wang, Y.J. Wang, L. Yang, T. Caraballo, Random attractors for stochastic delay wave equations on R^n with linear memory and nonlinear damping, *Discrete Contin. Dyn. Syst. Ser. S*, **15** (2022), 3025–3057.
30. R.H. Wang, T. Caraballo, N.H. Tuan, Mean attractors and invariant measures of locally monotone and generally coercive SPDEs driven by superlinear noise, *J. Differential Equations*, **381** (2024), 209–259.
31. X.H. Wang, K.L. Lu, B.X. Wang, Random attractors for delay parabolic equations with additive noise and deterministic nonautonomous forcing, *SIAM J. Appl. Dyn. Syst.*, **14** (2015), 1018–1047.
32. J.H. Xu, T. Caraballo, J. Valero, Dynamics and Large Deviations for Fractional Stochastic Partial Differential Equations with Lévy Noise, *SIAM J. Math. Anal.*, **56** (2024), 1016–1067.
33. Q.G. Yang, X.F. Lin, C.B. Zeng, Random attractors for rough stochastic partial differential equations, *J. Differential Equations*, **371** (2023), 50–82.
34. Z. Chen, B.X. Wang, Limiting behavior of random attractors of stochastic supercritical wave equations driven by multiplicative noise, *Appl. Math. Optim.*, **88** (2023), 1–32.
35. G.L. Zhou, Random attractor for the 3D viscous primitive equations driven by fractional noises, *J. Differential Equations*, **266** (2019), 7569–7637.
36. G.L. Zhou, B.L. Guo, The global attractor for the 3D viscous primitive equations of large-scale moist atmosphere, *Commun. Math. Sci.*, **16** (2018), 2003–2032.

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