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LLM Assisted Mathematical Modeling: Homogeneous Laplace Equation in Cylinder with the Complete Electrode Model

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Posted Date: 31 March 2025

doi: 10.20944/preprints202503.2400.v1

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doi:10.20944/preprints202503.2400.v1

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LLM Assisted Mathematical Modeling: Homogeneous Laplace Equation in Cylinder with the Complete Electrode Model

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Abstract: This paper presents an analytical solution for the potential distribution in Electrical Impedance Tomography (EIT) under the complete electrode model, extending previous work on a 2D homogeneous disk to a more realistic 3D cylindrical geometry of finite height. By taking into account for the influence of electrode height, this approach offers an improved representation compared to the earlier 2D formulation. The mathematical modeling involved in deriving this solution was significantly facilitated by the use of a Large Language Model (LLM), DeepSeek, which aided in the tedious index manipulations inherent in solving the Laplace equation with appropriate boundary conditions in cylindrical coordinates using methods of separation variables. This work is organized into three main, presenting the main result and Conlusion/future works followed by an Appendix detailing the step-by-step mathematical derivation assisted by LLM Deepseek.

Keywords: finite length cylinder; complete electrode model; mathematical modeling; large language model assist

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1. Introduction

An analytical solution for the potential distribution in electrical impedance tomography (EIT) on a 2D homogeneous disk under the complete electrode model is presented by Demidenko [Demidenko], formulated as an infinite system of linear equations. The validity of this solution is supported by its agreement with a previously published elliptic integral solution for the shunt electrode model with two electrodes. The Dirichlet-to-Neumann map is derived to facilitate statistical estimation using nonlinear least squares. The proposed solution was validated through phantom experiments and applied to in vivo breast contact impedance estimation. Statistical hypothesis testing was performed to assess contact impedance characteristics, highlighting the potential of this method for rapid, real-time detection of poor surface contact in clinical environments.

Taking account for the influence of electrode height in the complete electrode model, the homogeneous Laplace equation was considered within a cylindrical geometry of finite height. This approach allows for a more realistic representation compared to 2D disk. The mathematical modeling involved in obtaining the solution to this problem facilitated by the use of a Large Language Model (LLM) DeepSeek, which assisted in navigating the nitty gritty of indices in solving the partial differential equations in this specific coordinate system and boundary conditions. Related with this work are for homogeneous ball with CEM in [Maulidi ea] and Point Electrode Model in [WeideltWeller]

In the following table, we provide a tabular comparison the results from Demidenko work on homogeneous Laplace equation on 2D disk with complete electrode model and current work.

The work organized into three parts, main result, future works/conclusion and an Appendix on step by step to obtain the main result.

2. Laplace Equation Within the Body

The potential distribution within the body is governed by the Laplace PDE:

$$\nabla \cdot (\sigma \nabla u) = 0.$$

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Demidenko	This work
An analytic solution of the potential distribution on a 2D homogeneous disk for EIT under the complete electrode model is expressed via an infinite system of linear equations.	YES
For the shunt electrode model with two electrodes, our solution coincides with the previously derived solution expressed via elliptic integral	Need verification
The Dirichlet-to-Neumann map is derived for statistical estimation via nonlinear least squares (NLeastSq).	DtN map NLeastSq format
The solution validated in phantom experiments and applied for breast contact impedance estimation in vivo.	no data
Statistical hypothesis testing is used to test whether the contact impedances are the same across electrodes or all equal zero.	no data
The solution can be especially useful for a rapid real-time test for bad surface contact in clinical setting.	no data

Notation

- σ: Spatial conductivity.
- u: Potential function.
- ∇: The gradient operator.

3. Boundary Conditions

3.1. Between Electrodes

On the boundary of the body between electrodes, there is no current flow:

$$\frac{\partial u}{dn} = 0, s \notin E_i.$$

Notation

- n: Normal vector on the surface of the body.
- s: Spatial coordinate vector.
- E_i: Nonoverlapping surface areas.

3.2. Complete Electrode Model

According to the complete electrode model,

$$u(s) + \zeta_i \sigma(s) \frac{\partial u}{dn} = U_i, s \in E_i$$

The supplied potential is constant over the surface of the electrode because it is made of a highly conductive metal material.

Notation

- ζ_i: Electrode contact (or surface) impedance.

 doi:10.20944/preprints202503.2400.v1

- U_i: Potentials applied at the electrodes.

3.3. Shunt Electrode Model

The shunt electrode model is a special case of the complete electrode model with $\zeta_i = 0, i = 1.2, ...L$ Thus, the boundary condition for this model takes the form

$$u(s) = U_i, s \in E_i.$$

4. The Homogeneous 3D Cylinder

4.1. Laplace Equation in Cylindrical Coordinates

The potential distribution on a homogeneous disk of radius R and conductivity with infinitesimal height is governed by the Laplace equation in cylindrical coordinates.

$$\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1}$$

Notation

- R: Radius
- r: Distance from the center
- θ: Angle
- z: axial

4.2. Solution Expressed in Fourier Series

The L nonoverlapping electrodes with half-width w (radians) and half height H are located on the cylinder surface of radius R at angle locations $\{\theta_i, i = 1, 2, ..., L\}$.

Full Solution for $u(r, \theta, z)$

$$\begin{aligned} u(r,\theta,z) &= u_0 \\ &+ \sum_{m=0}^{\infty} \frac{R}{m}\left(\frac{r}{R}\right)^m [a_m \cos(m\theta) + b_m \sin(m\theta)] \\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m\left(\frac{n\pi r}{2H}\right)}{I'_m\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right)[a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)], \end{aligned} \tag{2}$$

where coefficients a_m, b_m, a_{mn} and b_{mn} are found from the boundary condition

$$u(R,\theta,z)) + \zeta_i \sigma \frac{\partial u}{\partial r}|_{r=R} = U_i, i = 1, 2, ..., L; -H \leq z \leq H. \tag{3}$$

$$\frac{\partial u}{\partial r}|_{r=R} = f(\theta, z)$$

Boundary condition function for $f(\theta, z)$

$$f(\theta,z) = \begin{cases} \frac{1}{\zeta_i \sigma}(U_i - u_0 - u_N(R,\theta,z)), & \theta \in (\theta_i - w, \theta_i + w),\ i = 1,\ldots,L, \\ 0, & \text{otherwise.} \end{cases}$$

Coefficient Definitions

1. Azimuthal modes ($n = 0$, z-independent):

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\cos(m\theta)\,d\theta,\ b_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\sin(m\theta)\,d\theta,$$

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where $f_0(\theta)$ is the z-average of $f(\theta, z)$:

$$f_0(\theta) = \frac{1}{2H} \int_{-H}^{H} f(\theta, z)\, dz.$$

2. Axial modes ($n \geq 1$):

$$a_{m,n} = \frac{1}{\pi H} \int_0^{2\pi} \int_{-H}^{H} f(\theta, z) \cos(m\theta) \cos\left(\frac{n\pi z}{2H}\right) dz\, d\theta,$$

$$b_{m,n} = \frac{1}{\pi H} \int_0^{2\pi} \int_{-H}^{H} f(\theta, z) \sin(m\theta) \cos\left(\frac{n\pi z}{2H}\right) dz\, d\theta.$$

Key Notes

1. Constant u_0: Represents the background potential (e.g., determined by grounding $\sum_i U_i = 0$).

2. Modified Bessel Functions I_m: Govern the radial decay/growth of modes. For $r \to 0$, $I_m \sim r^m$, ensuring regularity.

Notation

- w: half-width (radians)
- a_n and b_n: Fourier coefficients
- u_0: Constant
- H: half-height (unit length)

4.3. Solution of an Infinitely Large System of Linear Equations

Alternatively, the expression for $u(r, \theta, z)$

$$\begin{aligned} u(r,\theta,z) &= u_0 \\ &+ \textstyle\sum_{m=0}^{\infty} \frac{R}{k} (\frac{r}{R})^k [a_m \cos(m\theta) + b_m \sin(m\theta)] \\ &+ \textstyle\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m(\frac{n\pi r}{2H})}{I'_m(\frac{n\pi R}{2H})} \cos(\frac{n\pi z}{2H}) [a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta)], \end{aligned}$$

with $a_{m,n}, b_{m,n}, u_0$ solved from

$$\begin{bmatrix} \delta_{mk'}\delta_{nn'} + \zeta_i \sigma \frac{n\pi}{2H} A_m B_n & 0 & C_m D_n \\ 0 & \delta_{mk'}\delta_{nn'} + \zeta_i \sigma \frac{n\pi}{2H} A'_m B_n & C'_m D_n \\ C_m D_n & C'_m D_n & 4wH \end{bmatrix} \begin{bmatrix} a_{m,n} \\ b_{m,n} \\ u_0 \end{bmatrix} = U_i \begin{bmatrix} C_m D_n \\ C'_m D_n \\ 4wH \end{bmatrix},$$

where:

$$A_m = \int_{\theta_i - w}^{\theta_i + w} \cos^2(m\theta)\, d\theta,\ A'_m = \int_{\theta_i - w}^{\theta_i + w} \sin^2(m\theta)\, d\theta,$$

$$B_n = \int_{-H}^{H} \cos^2(\frac{n\pi z}{2H}) dz,$$

$$C_m = \int_{\theta_i - w}^{\theta_i + w} \cos(m\theta)\, d\theta,\ C'_m = \int_{\theta_i - w}^{\theta_i + w} \sin(m\theta)\, d\theta,$$

$$D_n = \int_{-H}^{H} \cos(\frac{n\pi z}{2H}) dz.$$

HINT: The tensor notation seems plausible to be used for infinite representation of terms $[a_{mn} b_{mn} a_0]^T$, replacing vector terms in case of 2D disk in [Demidenko].

4.4. Equidistant-Equal-ζ Approximation

If an even number of electrodes (L) are equidistant (equally spaced) at $\theta_i = 2\pi i/L$, with equal ζ_i. This leads to the following equidistant-equal-ζ approximation:

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For L electrodes with spacing $\Delta\theta = \frac{2\pi}{L}$:
- Azimuthal modes m are restricted to $m = kL/2$ ($k \in \mathbb{Z}$) due to periodicity.
- Axial modes n are restricted to $n = pL/2$ ($p \in \mathbb{Z}$) due to $h_n = \frac{\sin(2nw)}{2n} \neq 0$.

Coefficient Matrix $\mathbf{M}$

The matrix $\mathbf{M}$ for the linear system $\mathbf{M}\mathbf{x} = \mathbf{c}$ (where $\mathbf{x} = [a_{m,n}, b_{m,n}, u_0]^T$) has entries:

$$M_{11}^{m,n} = \delta_{mm'}\delta_{nn'} + \zeta\sigma\frac{n\pi}{2H}\underbrace{\int_{-H}^{H}\cos^2(\frac{n\pi z}{2H})dz}_{=H}\underbrace{\int_{\theta_i-w}^{\theta_i+w}\cos^2(m\theta)d\theta}_{=\pi\delta_{m,kL/2}\cdot\text{sinc}(2mw)},$$

$$M_{22}^{m,n} = \delta_{mm'}\delta_{nn'} + \zeta\sigma\frac{n\pi}{2H}\underbrace{\int_{-H}^{H}\cos^2(\frac{n\pi z}{2H})dz}_{=H}\underbrace{\int_{\theta_i-w}^{\theta_i+w}\sin^2(m\theta)d\theta}_{=\pi\delta_{m,kL/2}\cdot\text{sinc}(2mw)}$$

$$M_{33} = L\cdot 2w\cdot 2H, \quad c_i = \sum_{i=1}^{L} U_i.$$

Matrix Structure

- Diagonal Dominance:

$\mathbf{M}$ is block-diagonal with blocks corresponding to modes $(m,n) = (kL/2, pL/2)$. Off-diagonal terms vanish due to orthogonality. - Simplified Entries:
For $m = kL/2$ and $n = pL/2$:

$$M_{11}^{m,n} = 1 + \zeta\sigma\frac{n\pi H}{2}\cdot\text{sinc}(2mw),$$

—

Key Observations

1. Mode Coupling:

Only modes $m = kL/2$, $n = pL/2$ contribute. Higher modes decay rapidly due to $(\frac{r}{R})^m$ and I_m.

2. Current-Voltage Relation:

The DtN map $\mathbf{I} = \sigma\mathbf{C}\mathbf{U}$ has entries:

$$C_{jk} = \frac{8R}{L}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{h_n^2}{1+\zeta\sigma\frac{n\pi H}{2}}\cos(m(\theta_j-\theta_k)).$$

—

Final Result

The coefficient matrix $\mathbf{M}$ is sparse and block-diagonal, with non-zero entries only for modes $(m,n) = (kL/2, pL/2)$. The solution $u(r,\theta,z)$ simplifies to:

$$u(r,\theta,z) = \sum_{k,p}\frac{4U\cdot h_{pL/2}}{L(1+\zeta\sigma\frac{pL\pi H}{4})}\left[\frac{R}{kL/2}(\frac{r}{R})^{kL/2}\cos(\frac{kL\theta}{2}) + \frac{2H}{pL\pi}\frac{I_{kL/2}(\frac{pL\pi r}{4H})}{I'_{kL/2}(\frac{pL\pi R}{4H})}\cos(\frac{pL\pi z}{4H})\right].$$

4.5. Dirichlet-to-Neumann Map

The current flowing through the jth electrode computed as the integral of the current density is

$$J(\theta) = \sigma\frac{\partial u(r,\theta,z)}{\partial r}|_{r=R}$$

over the electrode.

Namely,

$$I_j = R\int_{-H}^{H}\int_{\theta_j-w}^{\theta_j+w}J(\theta)d\theta = \sigma R\int_{-H}^{H}\int_{\theta_j-w}^{\theta_j+w}\left.\frac{\partial u}{\partial r}\right|_{r=R}d\theta dz.$$

 doi:10.20944/preprints202503.2400.v1

Substitute the radial derivative of u at $r = R$:

$$\begin{aligned} \left.\frac{\partial u}{\partial r}\right|_{r=R} &= \sum_{m=1}^{\infty} [a_m \cos(m\theta) + b_m \sin(m\theta)] \\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi z}{2H}\right)[a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)]. \end{aligned}$$

Thus:

$$\begin{aligned} I_j &= \sigma R \sum_{m=1}^{\infty} \left[a_m \cdot \mathcal{A}_m^{(j)} + b_m \cdot \mathcal{B}_m^{(j)}\right] & (4) \\ &+ \sigma R \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \left[a_{m,n} \cdot \mathcal{A}_{m,n}^{(j)} + b_{m,n} \cdot \mathcal{B}_{m,n}^{(j)}\right], & (5) \end{aligned}$$

where:

$$\begin{aligned} \mathcal{A}_m^{(j)} &= \int_{-H}^{H}\int_{\theta_j - w}^{\theta_j + w} \cos(m\theta) d\theta dz \\ &= 2H \cdot \frac{\sin(m(\theta_j + w)) - \sin(m(\theta_j - w))}{m}, \end{aligned}$$

$$\begin{aligned} \mathcal{B}_m^{(j)} &= \int_{-H}^{H}\int_{\theta_j - w}^{\theta_j + w} \sin(m\theta) d\theta dz \\ &= 2H \cdot \frac{-\cos(m(\theta_j + w)) + \cos(m(\theta_j - w))}{m}, \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{m,n}^{(j)} &= \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz \int_{\theta_j - w}^{\theta_j + w} \cos(m\theta) d\theta \\ &= \frac{4H}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cdot \frac{\sin(m(\theta_j + w)) - \sin(m(\theta_j - w))}{m}, \end{aligned}$$

$$\begin{aligned} \mathcal{B}_{m,n}^{(j)} &= \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz \int_{\theta_j - w}^{\theta_j + w} \sin(m\theta) d\theta \\ &= \frac{4H}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cdot \frac{-\cos(m(\theta_j + w)) + \cos(m(\theta_j - w))}{m}. \end{aligned}$$

Thus, we arrive at the DtN map, or the generalized Ohm's law of the cylinder, in the vector form as

$$\mathbf{I} = \sigma\, \mathbf{CU} \tag{6}$$

where $\mathbf{I} = (I_1, I_2, ..., I_L)'$ and $\mathbf{U} = (U_1, U_2, ..., U_L)'$ are columns vector of current and voltage measurements, and the elements of the $L \times L$ matrix $\mathbf{C}$ are given by

$$\mathbf{C}_{ij} = \frac{2HRw}{\pi}\delta_{jk} + \frac{4HR}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \frac{I_0\left(\frac{n\pi R}{2H}\right)}{I_0'\left(\frac{n\pi R}{2H}\right)} \cdot \mathrm{sinc}(nw)\cos(n(\theta_j - \theta_k)).$$

When electrodes are equidistant and have the same contact impedance, lead to an approximate DtN matrix with elements

 doi:10.20944/preprints202503.2400.v1

$$\mathbf{C}_{jk} = \frac{8R}{L} \sum_{p=1}^{\infty} \frac{\sin^2\left(\frac{pLw}{2}\right)}{pL/2\left(\frac{R}{pL/2} + \zeta\sigma\right)} \cos\left(\frac{pL(\theta_j - \theta_k)}{2}\right).$$

Notation

- $J(\theta)$: Current density.
- I_j: Current flowing through the jth electrode.
- $\mathbf{I}$: Column vector of current measurements.
- $\mathbf{U}$: Columns vector of voltage measurements.
- $\mathbf{C}$: $L \times L$ matrix.

4.6. Nonlinear Least Squares for contact impedance and conductivity

The nonlinear leastsquares to estimate contact impedance and conductivity from measurement is:

$$\sum_{i=1}^{16} |\mathbf{I}_i - \sigma \mathbf{C}(\sigma, \zeta_1, ..., \zeta_{16}) U_i|^2,$$

where the DtN map $\mathbf{C} = \mathbf{C}(\sigma, \zeta_1, ..., \zeta_{16})$ is defined above.

4.7. Computation of Voltage Drop Due to Contact Impedance

We assess the average drop by integrating the difference $(U_i - u(R, \theta, z))$ over the electrode $(\theta_i - w, \theta_i + w)$ as

$$\begin{aligned}
\Delta u_i &= U_i - u_0 \\
&- \frac{R}{w} \sum_{m=1}^{\infty} \frac{\sin(mw)}{m^2} [a_m \cos(m\theta_i) + b_m \sin(m\theta_i)] \\
&- \frac{4H}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \frac{I_0\left(\frac{n\pi R}{2H}\right)}{I_0'\left(\frac{n\pi R}{2H}\right)} a_{0,n} \\
&- \frac{4H}{w\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2 m} \frac{I_m\left(\frac{n\pi R}{2H}\right)}{I_m'\left(\frac{n\pi R}{2H}\right)} \sin(mw) [a_{m,n} \cos(m\theta_i) + b_{m,n} \sin(m\theta_i)].
\end{aligned}$$

NOTE: Need to resimplify

Besides ζ_i, one can report $100|u_i/U_i|$ as the percent voltage change due to contact impedance.

4.8. Special Case: 2 Electrodes with Shunt Model

Let the potential on the first electrode be U and on the opposite electrode be $-U$ with shunt electrode, i.e.$\zeta_i = 0$, the solution is expressed

$$\begin{aligned}
u(r, \theta, z) &= \sum_{m=1,3,5,...}^{\infty} \frac{4UR \sin(mw)}{\pi m^2} \left(\frac{r}{R}\right)^m \cos(m\theta) \\
&+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{4UH(-1)^{(n-1)/2}}{n^2 \pi^2} \frac{I_m\left(\frac{n\pi r}{2H}\right)}{I_m'\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right) \cos(m\theta),
\end{aligned}$$

5. Conclusions and Future Works

We derive solutions for homogeneous Laplace equation with Complete Electrode Model, to take into account effects of the finite length attached at the boundary. Also generated the Dirichlet to Neumann map which is important for diagnostics of the electrodes attachment and contact impedance estimation EIT for measurement.

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However, there are several details still to be working out. Such as the possibility for reduced model for Dirichlet-To-Neumann to obtain correction factor (Geometrical Factor) such that the Dirichlet-To-Neumann factor in 2D could be used/salvaged.

Appendix A

Appendix A.1 Homogeneous Laplace Equation with Neumann Condition

```
Q: I want to solve a homogeneous Laplace equations inside a cylinder with
outer radius R, and axially -H<z<H.

The boundary conditions at z=-H and z=H are zero Neumann's condition, while at the outer
boundary R, the boundary conditions are  non zero Neumann's  function of (\theta,z).

Assistant:
```

To solve the homogeneous Laplace equation inside a cylinder with the given boundary conditions, we'll follow these steps:

Problem Setup:

We seek a solution $u_N(r,\theta,z)$ to the Laplace equation in cylindrical coordinates (r,θ,z):

$$\nabla^2 u_N = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_N}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u_N}{\partial \theta^2} + \frac{\partial^2 u_N}{\partial z^2} = 0,$$

inside the cylinder defined by:

$$0 \le r \le R, 0 \le \theta \le 2\pi, -H \le z \le H.$$

Boundary Conditions:

1. Axial boundaries (top and bottom, $z = \pm H$):

$$\left.\frac{\partial u_N}{\partial z}\right|_{z=\pm H} = 0 \quad \text{(Neumann condition).}$$

2. Radial boundary (outer surface, $r = R$):

$$\left.\frac{\partial u_N}{\partial r}\right|_{r=R} = f(\theta,z) \quad \text{(Non-zero Neumann condition).}$$

Solution Approach:

We use separation of variables, assuming:

$$u_N(r,\theta,z) = P(r)\Theta(\theta)Z(z).$$

Substituting into the Laplace equation and dividing by u_N gives:

$$\frac{1}{rP}\frac{d}{dr}(r\frac{dP}{dr}) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0.$$

This separates into three ordinary differential equations (ODEs):

1. Angular part (Θ):

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0 \implies \Theta(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta),$$

where m is an integer ($m = 0,1,2,\ldots$) for periodicity in θ.

2. Axial part (Z):

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$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \implies Z(z) = C_k \cosh(kz) + D_k \sinh(kz),$$

where k is a separation constant. The Neumann condition at $z = \pm H$ implies:

$$\left.\frac{dZ}{dz}\right|_{z=\pm H} = 0 \implies kC_k \sinh(kH) \pm kD_k \cosh(kH) = 0.$$

For non-trivial solutions, we require $D_k = 0$ and $k \sinh(kH) = 0$. Since $\sinh(kH) = 0$ only if $k = 0$, the axial solution simplifies to:

$$Z(z) = C_0 \quad \text{(constant)}.$$

For $k \neq 0$, we must have $D_k = 0$ and $\sinh(kH) = 0$, which again implies $k = 0$. Thus, the only possibility is $k = 0$, meaning $Z(z)$ is constant.

However, this leads to a trivial solution unless we consider k as imaginary. Let $k = i\kappa$, then:

$$Z(z) = C_\kappa \cos(\kappa z) + D_\kappa \sin(\kappa z).$$

Applying the Neumann condition at $z = \pm H$:

$$\frac{dZ}{dz} = -\kappa C_\kappa \sin(\kappa z) + \kappa D_\kappa \cos(\kappa z),$$

$$\left.\frac{dZ}{dz}\right|_{z=H} = -\kappa C_\kappa \sin(\kappa H) + \kappa D_\kappa \cos(\kappa H) = 0,$$

$$\left.\frac{dZ}{dz}\right|_{z=-H} = -\kappa C_\kappa \sin(-\kappa H) + \kappa D_\kappa \cos(-\kappa H) = \kappa C_\kappa \sin(\kappa H) + \kappa D_\kappa \cos(\kappa H) = 0.$$

For non-trivial solutions, the determinant must vanish:

$$\begin{vmatrix} -\sin(\kappa H) & \cos(\kappa H) \\ \sin(\kappa H) & \cos(\kappa H) \end{vmatrix} = -\sin(\kappa H)\cos(\kappa H) - \sin(\kappa H)\cos(\kappa H) = -2\sin(\kappa H)\cos(\kappa H) = -\sin(2\kappa H) = 0.$$

Thus, $\sin(2\kappa H) = 0 \implies \kappa_n = \frac{n\pi}{2H}$, where $n = 0, 1, 2, \ldots$.

For $n = 0$ ($\kappa_0 = 0$):

$$\zeta_0(z) = C_0 \quad \text{(constant)}.$$

For $n \geq 1$:

$$\zeta_n(z) = C_n \cos(\frac{n\pi z}{2H}).$$

3. Radial part (P):

The radial equation becomes:

$$\frac{1}{r}\frac{d}{dr}(r\frac{dP}{dr}) - (\frac{m^2}{r^2} + \kappa_n^2)P = 0.$$

For $\kappa_n = 0$ ($n = 0$):

$$\frac{1}{r}\frac{d}{dr}(r\frac{dP}{dr}) - \frac{m^2}{r^2}P = 0,$$

which is the modified Bessel equation with solutions:

$$P(r) = E_m r^m + F_m r^{-m}.$$

Since r^{-m} is singular at $r = 0$, we set $F_m = 0$:

$$P(r) = E_m r^m.$$

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For $\kappa_n \neq 0$ ($n \geq 1$):

$$\frac{1}{r}\frac{d}{dr}(r\frac{dP}{dr}) - (\frac{m^2}{r^2} + (\frac{n\pi}{2H})^2)P = 0,$$

which is the modified Bessel equation with solutions:

$$P(r) = E_{m,n} I_m(\frac{n\pi r}{2H}) + F_{m,n} K_m(\frac{n\pi r}{2H}),$$

where I_m and K_m are modified Bessel functions. Since K_m is singular at $r = 0$, we set $F_{m,n} = 0$:

$$P(r) = E_{m,n} I_m(\frac{n\pi r}{2H}).$$

General Solution:

Combining all parts, the general solution is:

$$u_N(r,\theta,z) = \sum_{m=0}^{\infty} [E_m r^m (A_m \cos(m\theta) + B_m \sin(m\theta))] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} E_{m,n} I_m(\frac{n\pi r}{2H}) \cos(\frac{n\pi z}{2H})(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)).$$

Applying the Boundary Condition at $r = R$:

The Neumann condition at $r = R$ is:

$$\left.\frac{\partial u_N}{\partial r}\right|_{r=R} = f(\theta, z).$$

Substituting the general solution:

$$\sum_{m=0}^{\infty} mE_m R^{m-1}(A_m \cos(m\theta) + B_m \sin(m\theta)) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} E_{m,n} \frac{n\pi}{2H} I_m'(\frac{n\pi R}{2H}) \cos(\frac{n\pi z}{2H})(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)) = f(\theta, z).$$

This is a double Fourier series (in θ and z) for $f(\theta, z)$. The coefficients can be found by orthogonality:

1. For $n = 0$ (constant in z):

$$\sum_{m=0}^{\infty} mE_m R^{m-1}(A_m \cos(m\theta) + B_m \sin(m\theta)) = f_0(\theta),$$

where $f_0(\theta)$ is the θ-dependent part of $f(\theta, z)$ averaged over z:

$$f_0(\theta) = \frac{1}{2H}\int_{-H}^{H} f(\theta, z)\, dz.$$

The coefficients A_m, B_m, E_m are then determined by Fourier analysis in θ:

$$mE_m R^{m-1} A_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\cos(m\theta)\, d\theta,$$

$$mE_m R^{m-1} B_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\sin(m\theta)\, d\theta.$$

2. For $n \geq 1$:

The z-dependent terms involve $\cos(\frac{n\pi z}{2H})$. The coefficients $A_{m,n}, B_{m,n}, E_{m,n}$ are found by projecting $f(\theta, z)$ onto these modes:

$$E_{m,n}\frac{n\pi}{2H} I_m'(\frac{n\pi R}{2H}) A_{m,n} = \frac{1}{\pi H}\int_0^{2\pi}\int_{-H}^{H} f(\theta, z)\cos(m\theta)\cos(\frac{n\pi z}{2H})\, dz\, d\theta,$$

$$E_{m,n}\frac{n\pi}{2H} I_m'(\frac{n\pi R}{2H}) B_{m,n} = \frac{1}{\pi H}\int_0^{2\pi}\int_{-H}^{H} f(\theta, z)\sin(m\theta)\cos(\frac{n\pi z}{2H})\, dz\, d\theta.$$

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Final Solution:
The solution is:

$$u_N(r,\theta,z) = \sum_{m=0}^{\infty}\left(\frac{R}{m}\left(\frac{r}{R}\right)^m[a_m\cos(m\theta)+b_m\sin(m\theta)]\right) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{2H}{n\pi}\frac{I_m(\frac{n\pi r}{2H})}{I'_m(\frac{n\pi R}{2H})}\cos\left(\frac{n\pi z}{2H}\right)[a_{m,n}\cos(m\theta)+b_{m,n}\sin(m\theta)],$$

where:

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\cos(m\theta)\,d\theta, b_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\sin(m\theta)\,d\theta,$$

$$a_{m,n} = \frac{1}{\pi H}\int_0^{2\pi}\int_{-H}^{H} f(\theta,z)\cos(m\theta)\cos\left(\frac{n\pi z}{2H}\right)dz\,d\theta,$$

$$b_{m,n} = \frac{1}{\pi H}\int_0^{2\pi}\int_{-H}^{H} f(\theta,z)\sin(m\theta)\cos\left(\frac{n\pi z}{2H}\right)dz\,d\theta.$$

Key Observations:
1. The $n = 0$ mode corresponds to a solution independent of z.
2. For $n \geq 1$, the radial dependence involves modified Bessel functions I_m.
3. The boundary condition $f(\theta,z)$ determines the coefficients via Fourier analysis.

This gives the complete solution to the homogeneous Laplace equation with the given Neumann boundary conditions.

Appendix A.2 Complete Electrode Model

```
q: Hi, shall we continue ?
\begin{itemize}
\item Define: \[u(r,\theta,z) = u_0 + u_N(r,\theta,z), u_0 \quad {\text constant}\]
\item It is known that : \\
\begin{eqnarray*}
u_N(r, \theta, z) &=& \sum_{m=0}^\infty ( \frac{R}{m} ( \frac{r}{R})^m \left[ a_m
\cos(m\theta) + b_m \sin(m\theta) \right]) \\
&+& \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{2 H}{n \pi} \frac{I_m(\frac{n \pi r}{2 H})}
{I_m'( \frac{n \pi R}{2 H})} \cos(\frac{n \pi z}{2 H}) \left[ a_{m,n} \cos(m\theta)
+ b_{m,n} \sin(m\theta) \right],
\end{eqnarray*}
where:
\[
a_m = \frac{1}{\pi} \int_0^{2\pi} f_0(\theta) \cos(m\theta) \, d\theta, b_m = \frac{1}
{\pi} \int_0^{2\pi} f_0(\theta) \sin(m\theta) \, d\theta,
\]
\[
a_{m,n} = \frac{1}{\pi H} \int_0^{2\pi} \int_{-H}^H f(\theta, z) \cos(m\theta)
\cos(\frac{n \pi z}{2 H}) \, dz \, d\theta,
\]
\[
b_{m,n} = \frac{1}{\pi H} \int_0^{2\pi} \int_{-H}^H f(\theta, z) \sin(m\theta)
\cos(\frac{n \pi z}{2 H}) \, dz \, d\theta.
\]

\item Apply Boundary Conditions\\
At $ r = R $, the boundary condition becomes:
\begin{eqnarray}
u(R, \theta,z) + \zeta_i \sigma \frac{\partial u}{\partial r} \bigg|_{r=R} &=& U_i,
\theta_i\in (\theta_i-w,\theta_i+w)\quad i = 1, 2, \ldots, L. \\
```

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```
 \sigma \frac{\partial u}{\partial r} \bigg|_{r=R} &=& 0, else.
\end{eqnarray}

\begin{eqnarray*}
f(\theta,z) &=& \frac{1}{\zeta_i \sigma}(U_i - u(R,\theta,z)),\theta_i\in (\theta_i-w,
\theta_i+w),  i=1,\cdots,L\\
 &=& 0, else
    \end{eqnarray*}

The derivative with respect to $ r $ at $ r = R $ is:
$$
\frac{\partial u}{\partial r} \bigg|_{r=R} = \frac{\partial u_N}{\partial r} \bigg|_{r=R} .
$$

Substituting into the boundary condition:
$$
u(R, \theta, z) + \zeta_i \sigma \frac{\partial u_N}{\partial r} \bigg|_{r=R}  = U_i.
$$

\end{itemize}
```

`Assistant:` Let's proceed systematically by incorporating the new boundary conditions into our solution framework.

1. Full Solution Definition

The total solution is given by:

$$u(r, \theta, z) = u_0 + u_N(r, \theta, z),$$

where u_0 is a constant representing the background potential, and u_N is the Neumann solution derived earlier.

2. Boundary Conditions at $r = R$

The boundary conditions are piecewise-defined: - On electrodes ($\theta \in (\theta_i - w, \theta_i + w)$):

$$u(R, \theta, z) + \zeta_i \sigma \frac{\partial u}{\partial r}\bigg|_{r=R} = U_i, \ i = 1, 2, \ldots, L,$$

where ζ_i is a coupling coefficient, σ is conductivity, and U_i is the applied potential on the i-th electrode.
- Elsewhere (insulating regions):

$$\sigma \frac{\partial u}{\partial r}\bigg|_{r=R} = 0.$$

Since $\frac{\partial u}{\partial r} = \frac{\partial u_N}{\partial r}$, the boundary conditions simplify to:
- On electrodes:

$$u_0 + u_N(R, \theta, z) + \zeta_i \sigma \frac{\partial u_N}{\partial r}\bigg|_{r=R} = U_i.$$

- Elsewhere:

$$\frac{\partial u_N}{\partial r}\bigg|_{r=R} = 0.$$

3. Relating u_N and $f(\theta, z)$

From the earlier solution, the Neumann condition at $r = R$ is:

$$\frac{\partial u_N}{\partial r}\bigg|_{r=R} = f(\theta, z).$$

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Thus:
- On electrodes:

$$f(\theta, z) = \frac{1}{\zeta_i \sigma}(U_i - u_0 - u_N(R, \theta, z)).$$

- Elsewhere:

$$f(\theta, z) = 0.$$

4. Substituting u_N at $r = R$

The solution u_N evaluated at $r = R$ is:

$$u_N(R, \theta, z) = \sum_{m=0}^{\infty} \left(\frac{R}{m} [a_m \cos(m\theta) + b_m \sin(m\theta)] \right) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m(\frac{n\pi R}{2H})}{I'_m(\frac{n\pi R}{2H})} \cos\left(\frac{n\pi z}{2H}\right) [a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta)].$$

5. Self-Consistent Equations for Coefficients

The coefficients $a_m, b_m, a_{m,n}, b_{m,n}$ are defined via $f(\theta, z)$, which itself depends on $u_N(R, \theta, z)$. This creates a system of coupled equations:

For $n = 0$ modes:

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f_0(\theta) \cos(m\theta)\, d\theta, \; b_m = \frac{1}{\pi} \int_0^{2\pi} f_0(\theta) \sin(m\theta)\, d\theta,$$

where $f_0(\theta)$ is the z-average of $f(\theta, z)$:

$$f_0(\theta) = \frac{1}{2H} \int_{-H}^{H} f(\theta, z)\, dz.$$

For $n \geq 1$ modes:

$$a_{m,n} = \frac{1}{\pi H} \int_0^{2\pi} \int_{-H}^{H} f(\theta, z) \cos(m\theta) \cos\left(\frac{n\pi z}{2H}\right) dz\, d\theta,$$

$$b_{m,n} = \frac{1}{\pi H} \int_0^{2\pi} \int_{-H}^{H} f(\theta, z) \sin(m\theta) \cos\left(\frac{n\pi z}{2H}\right) dz\, d\theta.$$

6. Numerical Approach

The system is nonlinear due to the dependence of $f(\theta, z)$ on $u_N(R, \theta, z)$. A practical approach involves:

1. Initial guess: Start with $u_N = 0$ (i.e., $a_m = b_m = a_{m,n} = b_{m,n} = 0$).

2. Iterate:
- Compute $f(\theta, z)$ from the boundary conditions.
- Update the coefficients $a_m, b_m, a_{m,n}, b_{m,n}$ using the integral definitions.
- Recompute u_N and repeat until convergence.

7. Key Observations
- The constant u_0 represents the background potential and can be determined by imposing a reference condition (e.g., grounding one electrode).
- The coupling terms ζ_i account for electrode-specific effects (e.g., contact impedance).
- The solution naturally separates into azimuthal (m) and axial (n) modes, with the latter vanishing for z-independent problems.

Final Answer

The complete solution is:

$$u(r, \theta, z) = u_0 + u_N(r, \theta, z),$$

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where u_N is given by the series expansion above, and the coefficients are determined self-consistently from the boundary conditions. The system is solved iteratively due to the nonlinear coupling between u_N and $f(\theta, z)$.

Would you like to proceed with specific examples or numerical implementation details?

```
Q: I want an expression for  $u(r,\theta,z)$ please.
```

```
Assistant:
```

Here is the explicit expression for $u(r, \theta, z)$, incorporating all boundary conditions and the decomposition $u = u_0 + u_N$:

—

Full Solution for $u(r, \theta, z)$

$$u(r,\theta,z) = u_0 + \sum_{m=0}^{\infty} \frac{R}{m}\left(\frac{r}{R}\right)^m [a_m \cos(m\theta) + b_m \sin(m\theta)] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m\left(\frac{n\pi r}{2H}\right)}{I'_m\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right)[a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)],$$

Coefficient Definitions

1. Azimuthal modes ($n = 0$, z-independent):

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\cos(m\theta)\, d\theta,\ b_m = \frac{1}{\pi}\int_0^{2\pi} f_0(\theta)\sin(m\theta)\, d\theta,$$

where $f_0(\theta)$ is the z-average of $f(\theta, z)$:

$$f_0(\theta) = \frac{1}{2H}\int_{-H}^{H} f(\theta,z)\, dz.$$

2. Axial modes ($n \geq 1$):

$$a_{m,n} = \frac{1}{\pi H}\int_0^{2\pi}\int_{-H}^{H} f(\theta,z)\cos(m\theta)\cos\left(\frac{n\pi z}{2H}\right) dz\, d\theta,$$

$$b_{m,n} = \frac{1}{\pi H}\int_0^{2\pi}\int_{-H}^{H} f(\theta,z)\sin(m\theta)\cos\left(\frac{n\pi z}{2H}\right) dz\, d\theta.$$

Boundary Condition for $f(\theta, z)$

$$f(\theta,z) = \begin{cases} \frac{1}{\zeta_i\sigma}(U_i - u_0 - u_N(R,\theta,z)), & \theta \in (\theta_i - w, \theta_i + w),\ i = 1,\ldots,L, \\ 0, & \text{otherwise.} \end{cases}$$

Key Notes

1. Constant u_0: Represents the background potential (e.g., determined by grounding $\sum_i U_i = 0$).

2. Modified Bessel Functions I_m: Govern the radial decay/growth of modes. For $r \to 0$, $I_m \sim r^m$, ensuring regularity.

3. Convergence: The series must be truncated in practice ($m \leq M$, $n \leq N$) for numerical implementation.

4. Electrode Coupling: The term $\zeta_i\sigma$ models contact impedance at the i-th electrode.

—

Simplified Example (Single Electrode, z-Independent)

For a single electrode at $\theta \in (-\theta_0, \theta_0)$ and no z-dependence ($n = 0$):

$$u(r,\theta) = u_0 + \sum_{m=1}^{\infty} \frac{R}{m}\left(\frac{r}{R}\right)^m [a_m\cos(m\theta) + b_m\sin(m\theta)],$$

with $a_m = \frac{U - u_0}{\pi\zeta\sigma}\int_{-\theta_0}^{\theta_0}\cos(m\theta)\, d\theta$, $b_m = 0$ (symmetric case).

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—

Next Steps

1. Numerical Implementation:
Discretize θ and z, then solve for coefficients iteratively.

2. Validation:
Check convergence and physical behavior (e.g., $u \to u_0$ far from electrodes).

Would you like to refine any part of this expression or focus on a specific case?

Appendix A.3 Rearrangement Solutions in Series with Matrix Coefficient

```
Q: Shall we continue the works ?

From the full solution \( u(r, \theta, z) \) :
\begin{eqnarray*}
u(r, \theta, z) & = &u_0 + \sum_{m=0}^\infty \frac{R}{m} \left( \frac{r}{R} \right)^m
\left[ a_m \cos(m\theta) + b_m \sin(m\theta) \right] \\
& + & \sum_{m=0}^\infty \sum_{n=1}^\infty
 \frac{2 H}{n \pi} \\frac{I_m\left( \frac{n \pi r}{2 H} \right)}{I_m'\left( \frac{n \pi R}
 {2 H} \right)} \cos\left( \frac{n \pi z}{2 H} \right) \left[ a_{m,n} \cos(m\theta) +
 b_{m,n} \sin(m\theta) \right],
\end{eqnarray*}

Use algebraic manipulations, such that the coefficients $ a_{m,n} $ and $ b_{m,n} $
satisfy linear equations of the form :

M_{11} a_{m,n } + M_{12} b_{m,n} + M_{13}  u_0 - U_i  c_1 =0
M_{21} a_{m,n} + M_{22} b_{m,n}  + M_{23}  u_0 - U_i c_2 = 0
M_{31} a_{m,n}  + M_{32} b_{m,n} + M_{33} u_0 - U_i  c_3 = 0

that is of the form M [a_{m,n}  b_{m,n} u_0]^T = U_i*c^T.

Express explitely all of the unknown M_{11}, M_{12} , M_{13} ,  c_1, M_{21} , M_{22} ,
M_{23} , c_2, M_{31}, M_{32}, M_{33} ,  c_3.

Express u(r,\theta,z) in terms of matrix M and vector c.
```

Assistant: To derive the linear system $M\begin{bmatrix} a_{m,n} & b_{m,n} & u_0 \end{bmatrix}^T = U_i c^T$, we proceed as follows:

—

1. Boundary Condition at $r = R$
On the i-th electrode ($\theta \in (\theta_i - w, \theta_i + w)$):

$$u(R, \theta, z) + \zeta_i \sigma \frac{\partial u_N}{\partial r}\bigg|_{r=R} = U_i.$$

Substitute $u = u_0 + u_N$ and $\frac{\partial u_N}{\partial r}\big|_{r=R} = f(\theta, z)$:

$$u_0 + u_N(R, \theta, z) + \zeta_i \sigma f(\theta, z) = U_i.$$

From earlier, $f(\theta, z) = \frac{\partial u_N}{\partial r}\big|_{r=R}$, so:

$$u_0 + u_N(R,\theta,z) + \zeta_i \sigma \left(\sum_{m=0}^{\infty} \left[\frac{a_m}{R} \cos(m\theta) + \frac{b_m}{R} \sin(m\theta) \right] + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{n\pi}{2H} \cos\left(\frac{n\pi z}{2H}\right) [a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta)] \right) = U_i.$$

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—

2. Project onto Basis Functions

To extract $a_{m,n}, b_{m,n}, u_0$, multiply by orthogonal functions and integrate over θ and z:

Equation for $a_{m,n}$: Multiply by $\cos(k'\theta)\cos\left(\frac{n'\pi z}{2H}\right)$ and integrate over $\theta \in [0, 2\pi]$, $z \in [-H, H]$:

$$\int_{\theta_i-w}^{\theta_i+w}\int_{-H}^{H}\left(u_0 + u_N(R,\theta,z) + \zeta_i\sigma\frac{\partial u_N}{\partial r}|_{r=R}\right)\cos(m\theta)\cos\left(\frac{n\pi z}{2H}\right)dz\,d\theta = U_i\int_{\theta_i-w}^{\theta_i+w}\int_{-H}^{H}\cos(m\theta)\cos\left(\frac{n\pi z}{2H}\right)dz\,d\theta.$$

This yields:

$$M_{11}a_{m,n} + M_{12}b_{m,n} + M_{13}u_0 = U_i c_1,$$

where:

$$M_{11} = \delta_{mk'}\delta_{nn'} + \zeta_i\sigma\frac{n\pi}{2H}\int_{\theta_i-w}^{\theta_i+w}\cos^2(m\theta)\,d\theta\int_{-H}^{H}\cos^2\left(\frac{n\pi z}{2H}\right)dz,$$

$$M_{12} = 0,\ M_{13} = \int_{\theta_i-w}^{\theta_i+w}\cos(m\theta)\,d\theta\int_{-H}^{H}\cos\left(\frac{n\pi z}{2H}\right)dz,$$

$$c_1 = \int_{\theta_i-w}^{\theta_i+w}\cos(m\theta)\,d\theta\int_{-H}^{H}\cos\left(\frac{n\pi z}{2H}\right)dz.$$

Equation for $b_{m,n}$:

Multiply by $\sin(k'\theta)\cos\left(\frac{n'\pi z}{2H}\right)$:

$$M_{21} = 0,\ M_{22} = \delta_{mk'}\delta_{nn'} + \zeta_i\sigma\frac{n\pi}{2H}\int_{\theta_i-w}^{\theta_i+w}\sin^2(m\theta)\,d\theta\int_{-H}^{H}\cos^2\left(\frac{n\pi z}{2H}\right)dz,$$

$$M_{23} = \int_{\theta_i-w}^{\theta_i+w}\sin(m\theta)\,d\theta\int_{-H}^{H}\cos\left(\frac{n\pi z}{2H}\right)dz,$$

$$c_2 = \int_{\theta_i-w}^{\theta_i+w}\sin(m\theta)\,d\theta\int_{-H}^{H}\cos\left(\frac{n\pi z}{2H}\right)dz.$$

Equation for u_0:

Multiply by 1 and integrate:

$$M_{31} = \int_{\theta_i-w}^{\theta_i+w}\cos(m\theta)\,d\theta\int_{-H}^{H}\cos\left(\frac{n\pi z}{2H}\right)dz,\ M_{32} = \int_{\theta_i-w}^{\theta_i+w}\sin(m\theta)\,d\theta\int_{-H}^{H}\cos\left(\frac{n\pi z}{2H}\right)dz,$$

$$M_{33} = 2w\cdot 2H,\ c_3 = 2w\cdot 2H.$$

—

3. Explicit Matrix M and Vector c

The system $M\begin{bmatrix} a_{m,n} \\ b_{m,n} \\ u_0 \end{bmatrix} = U_i c$ is:

$$\begin{bmatrix} \delta_{mk'}\delta_{nn'} + \zeta_i\sigma\frac{n\pi}{2H}A_m B_n & 0 & C_m D_n \\ 0 & \delta_{mk'}\delta_{nn'} + \zeta_i\sigma\frac{n\pi}{2H}A'_m B_n & C'_m D_n \\ C_m D_n & C'_m D_n & 4wH \end{bmatrix}\begin{bmatrix} a_{m,n} \\ b_{m,n} \\ u_0 \end{bmatrix} = U_i\begin{bmatrix} C_m D_n \\ C'_m D_n \\ 4wH \end{bmatrix},$$

where:

$$A_m = \int_{\theta_i-w}^{\theta_i+w}\cos^2(m\theta)\,d\theta,\ A'_m = \int_{\theta_i-w}^{\theta_i+w}\sin^2(m\theta)\,d\theta,$$

$$B_n = \int_{-H}^{H}\cos^2\left(\frac{n\pi z}{2H}\right)dz,\ C_m = \int_{\theta_i-w}^{\theta_i+w}\cos(m\theta)\,d\theta,\ C'_m = \int_{\theta_i-w}^{\theta_i+w}\sin(m\theta)\,d\theta,$$

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$$D_n = \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz.$$

—

4. Final Expression for $u(r,\theta,z)$

$$u(r,\theta,z) = u_0 + \sum_{m=0}^{\infty} \frac{R}{k}\left(\frac{r}{R}\right)^k [a_m \cos(m\theta) + b_m \sin(m\theta)] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m\left(\frac{n\pi r}{2H}\right)}{I_m'\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right) [a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)],$$

with $a_{m,n}, b_{m,n}, u_0$ solved from $M\mathbf{x} = U_i c$.

—

Key Points

- Matrix M: Combines contributions from the Laplacian and boundary conditions.
- Vector c: Encodes the electrode geometry and applied potential U_i.
- Numerical Implementation: Truncate the series ($k \leq K, n \leq N$) and solve the linear system.

Would you like to proceed with specific electrode configurations or numerical examples?

Appendix A.4 Voltage Drop

```
Q: Let's continue, shall we ?
Let
\begin{eqnarray}
u(r, \theta, z)& =& u_0\\
&+& \sum_{m=0}^\infty \frac{R}{m} \left( \frac{r}{R} \right)^m \left[ a_m \cos(m\theta)
+ b_m \sin(m\theta) \right] \\
&+& \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{2 H}{n \pi} \frac{I_m( \frac{n \pi r}
{2 H})}{I_m'( \frac{n \pi R}{2 H})}
\cos( \frac{n \pi z}{2 H}) \left[ a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta) \right],
\end{eqnarray}
where coefficients $a_{mn}$ and $b_{mn}$ are to be found from the boundary condition
\begin{equation}\label{tag7}
u(R,\theta,z))+\zeta_i\sigma\frac{\partial u}{\partial r}|_{r=R}=U_i, i=1,2,...,L;
-H\leq z \leq H.
\end{equation}

Compute the average drop by integrating the difference
$(U_i - u(R,\theta,z))$ over the electrode $(\theta_i - w, \theta_i + w)\times(-H,H).$
```

`Assistant:` To compute the average drop ΔU_i over the i-th electrode, we integrate $U_i - u(R,\theta,z)$ over the electrode area $(\theta_i - w, \theta_i + w) \times (-H, H)$:

—

Average Drop Formula

$$\Delta U_i = \frac{1}{4wH} \int_{\theta_i - w}^{\theta_i + w} \int_{-H}^{H} [U_i - u(R,\theta,z)]\, dz\, d\theta.$$

Substitute $u(R,\theta,z) = u_0 + u_N(R,\theta,z)$:

$$\Delta U_i = U_i - u_0 - \frac{1}{4wH} \int_{\theta_i - w}^{\theta_i + w} \int_{-H}^{H} u_N(R,\theta,z)\, dz\, d\theta.$$

—

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Explicit Expression for ΔU_i
Substitute the series expansion for $u_N(R,\theta,z)$:

$$u_N(R,\theta,z) = \sum_{m=1}^{\infty} \frac{R}{m}[a_m \cos(m\theta) + b_m \sin(m\theta)] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m\left(\frac{n\pi R}{2H}\right)}{I'_m\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right)[a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)].$$

The average drop becomes:

$$\Delta U_i = U_i - u_0 - \frac{R}{w}\sum_{m=1}^{\infty} \frac{\sin(mw)}{m^2}[a_m \cos(m\theta_i) + b_m \sin(m\theta_i)] - \frac{4H}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \frac{I_0\left(\frac{n\pi R}{2H}\right)}{I'_0\left(\frac{n\pi R}{2H}\right)} a_{0,n} - \frac{4H}{w\pi^2}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2 m} \frac{I_m\left(\frac{n\pi R}{2H}\right)}{I'_m\left(\frac{n\pi R}{2H}\right)} \sin(mw)[a_{m,n}\cos(m\theta_i) + b_{m,n}\sin(m\theta_i)].$$

—

Matrix-Vector Form
Define the matrix $\mathbf{M}$ and vector $\mathbf{c}$ such that $\mathbf{Mx} = U_i\mathbf{c}$, where $\mathbf{x} = [a_{k,n}, b_{k,n}, u_0]^T$. The components are:

1. For $a_{k,n}$:

$$M_{11} = \delta_{kk'}\delta_{nn'} + \frac{\zeta_i \sigma n\pi}{2H}\int_{\theta_i - w}^{\theta_i + w} \cos^2(k\theta)\, d\theta \int_{-H}^{H} \cos^2\left(\frac{n\pi z}{2H}\right) dz, \; c_1 = \int_{\theta_i - w}^{\theta_i + w} \cos(k\theta)\, d\theta \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz.$$

2. For $b_{k,n}$:

$$M_{22} = \delta_{kk'}\delta_{nn'} + \frac{\zeta_i \sigma n\pi}{2H}\int_{\theta_i - w}^{\theta_i + w} \sin^2(k\theta)\, d\theta \int_{-H}^{H} \cos^2\left(\frac{n\pi z}{2H}\right) dz, \; c_2 = \int_{\theta_i - w}^{\theta_i + w} \sin(k\theta)\, d\theta \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz.$$

3. For u_0:

$$M_{33} = 4wH, \; c_3 = 4wH.$$

—

Final Form

$$\mathbf{M}\begin{bmatrix} a_{k,n} \\ b_{k,n} \\ u_0 \end{bmatrix} = U_i\mathbf{c},$$

where $\mathbf{M}$ encodes the system's geometry and material properties, and $\mathbf{c}$ depends on the electrode's angular/axial span. The solution $u(r,\theta,z)$ is fully determined once these coefficients are found.

Appendix A.5 Dirichlet to Neumann

```
Q: Let's continue, shall we ?

Let
\begin{eqnarray}
u(r, \theta, z)& =& u_0\\
&+& \sum_{m=0}^\infty \frac{R}{m} \left( \frac{r}{R} \right)^m \left[ a_m \cos(m\theta)
+ b_m \sin(m\theta) \right] \\
&+& \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{2 H}{n \pi} \frac{I_m( \frac{n \pi r}
{2 H})}{I_m'( \frac{n \pi R}{2 H})}
```

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```
\cos( \frac{n \pi z}{2 H}) \left[ a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta) \right],
\end{eqnarray}
where coefficients $a_{mn}$ and $b_{mn}$ are to be found from the boundary condition
\begin{equation}\label{tag7}
u(R,\theta,z))+\zeta_i\sigma\frac{\partial u}{\partial r}|_{r=R}=U_i, i=1,2,...,L;
-H\leq z \leq H.
\end{equation}

The current flowing through the jth electrode computed as the integral of the current
density is

\[J(\theta)=\sigma\frac{\partial u(r,\theta,z)}{\partial r}|_{r=R}\]

over the electrode.

Namely,

\[ I_j=R\int_{\theta_j-w}^{\theta_j+w}J(\theta)d\theta=\sigma~R\int_{\theta_j-w}^
{\theta_j+w}\frac{\partial u(r,\theta,z)}{\partial r}|_{r=R}d\theta
\]

Using general solution $u(r,\theta,z)$ above, the current $I_j$ flowing through electrode
 $j$ is expressed through basis coefficients.

Thus, we arrive at the DtN map, or the generalized Ohm's law of the cylinder surface, in the vector

\begin{equation}
\mathbf{I}=\sigma~\mathbf{C}\mathbf{U}
\end{equation}

where $\mathbf{I}=(I_1,I_2,...,I_{L})'$ and $\mathbf{U}=(U_1,U_2,...,U_{L})'$ are columns
vector of current and voltage measurements.
```

Assistant:

To express the current I_j through electrode j in terms of the coefficients $a_{m,n}$, $b_{m,n}$, and u_0, and derive the Dirichlet-to-Neumann (DtN) map $\mathbf{I} = \sigma\mathbf{C}\mathbf{U}$, we proceed as follows:

—

1. Current Expression

The current I_j is computed by integrating the radial current density over the j-th electrode:

$$I_j = \sigma R \int_{-H}^{H} \int_{\theta_j - w}^{\theta_j + w} \frac{\partial u}{\partial r}\bigg|_{r=R} d\theta dz.$$

Substitute the radial derivative of u at $r = R$:

$$\frac{\partial u}{\partial r}\bigg|_{r=R} = \sum_{m=1}^{\infty} [a_m \cos(m\theta) + b_m \sin(m\theta)] + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi z}{2H}\right) [a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta)].$$

Thus:

$$I_j = \sigma R \sum_{m=1}^{\infty} \left[a_m \cdot \mathcal{A}_m^{(j)} + b_m \cdot \mathcal{B}_m^{(j)}\right] + \sigma R \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[a_{m,n} \cdot \mathcal{A}_{m,n}^{(j)} + b_{m,n} \cdot \mathcal{B}_{m,n}^{(j)}\right],$$

where:

$$\mathcal{A}_m^{(j)} = \int_{-H}^{H} \int_{\theta_j - w}^{\theta_j + w} \cos(m\theta) d\theta dz = 2H \cdot \frac{\sin(m(\theta_j + w)) - \sin(m(\theta_j - w))}{m},$$

$$\mathcal{B}_m^{(j)} = \int_{-H}^{H} \int_{\theta_j - w}^{\theta_j + w} \sin(m\theta) d\theta dz = 2H \cdot \frac{-\cos(m(\theta_j + w)) + \cos(m(\theta_j - w))}{m},$$

$$\mathcal{A}_{m,n}^{(j)} = \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz \int_{\theta_j - w}^{\theta_j + w} \cos(m\theta) d\theta = \frac{4H}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cdot \frac{\sin(m(\theta_j + w)) - \sin(m(\theta_j - w))}{m},$$

$$\mathcal{B}_{m,n}^{(j)} = \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz \int_{\theta_j - w}^{\theta_j + w} \sin(m\theta) d\theta = \frac{4H}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cdot \frac{-\cos(m(\theta_j + w)) + \cos(m(\theta_j - w))}{m}.$$

—

2. DtN Map via Matrix **C**

The boundary condition $u(R, \theta, z) + \zeta_i \sigma \frac{\partial u}{\partial r}\big|_{r=R} = U_i$ links $a_{m,n}, b_{m,n}, u_0$ to U_i. Solving this system gives:

$$\begin{bmatrix} a_{m,n} \\ b_{m,n} \\ u_0 \end{bmatrix} = \mathbf{M}^{-1} \mathbf{c} U_i,$$

where **M** and **c** are defined by the integrals over electrode regions (see previous steps).

Substitute this into I_j:

$$I_j = \sigma R \sum_{k=1}^{L} \left[\sum_{m,n} \left(\mathcal{A}_{m,n}^{(j)} \cdot (\mathbf{M}^{-1}\mathbf{c})_{m,n,k} + \mathcal{B}_{m,n}^{(j)} \cdot (\mathbf{M}^{-1}\mathbf{c})_{m,n,k} \right) \right] U_k.$$

This yields the matrix form:

$$\mathbf{I} = \sigma \mathbf{C} \mathbf{U},$$

where the entries of **C** are:

$$C_{jk} = R \sum_{m,n} \left[\mathcal{A}_{m,n}^{(j)} \cdot (\mathbf{M}^{-1}\mathbf{c})_{m,n,k} + \mathcal{B}_{m,n}^{(j)} \cdot (\mathbf{M}^{-1}\mathbf{c})_{m,n,k} \right].$$

—

3. Explicit Matrix Entries

For each electrode pair (j, k):

$$C_{jk} = \frac{2HR}{\pi} \sum_{m=1}^{\infty} \frac{\sin(mw)}{m} \left[\cos(m\theta_j)\cos(m\theta_k) + \sin(m\theta_j)\sin(m\theta_k) \right] + \frac{4HR}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \frac{I_m\left(\frac{n\pi R}{2H}\right)}{I'_m\left(\frac{n\pi R}{2H}\right)}$$

$$\cdot \frac{\sin(mw)}{m} \left[\cos(m\theta_j)\cos(m\theta_k) + \sin(m\theta_j)\sin(m\theta_k) \right].$$

Simplified using orthogonality:

$$C_{jk} = \frac{2HRw}{\pi} \delta_{jk} + \frac{4HR}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \frac{I_0\left(\frac{n\pi R}{2H}\right)}{I'_0\left(\frac{n\pi R}{2H}\right)} \cdot \operatorname{sinc}(nw) \cos(n(\theta_j - \theta_k)).$$

—

Final Result

The DtN map is:

$$\boxed{\mathbf{I} = \sigma \mathbf{C} \mathbf{U}},$$

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where **C** is a symmetric matrix with entries dependent on the cylinder's geometry (R, H), electrode positions (θ_j), widths (w), and modified Bessel functions I_m.

Appendix A.6 Special Case: 2 Electrodes with Shunt

```
Let's continue, shall we ?

Let
\begin{eqnarray}
u(r, \theta, z)& =& u_0\\
&+& \sum_{m=0}^\infty \frac{R}{m} \left( \frac{r}{R} \right)^m \left[ a_m \cos(m\theta)
+ b_m \sin(m\theta) \right] \\
&+& \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{2 H}{n \pi} \frac{I_m( \frac{n \pi r}{2 H})}
{I_m'( \frac{n \pi R}{2 H})}
\cos( \frac{n \pi z}{2 H}) \left[ a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta) \right],
\end{eqnarray}
where coefficients $a_{mn}$ and $b_{mn}$ are to be found from the boundary condition
\begin{equation}\label{tag7}
u(R,\theta,z))+\zeta_i\sigma\frac{\partial u}{\partial r}|_{r=R}=U_i, i=1,2,...,L;
 -H\leq z \leq H.
\end{equation}

I need a special case for 2 electrodes, when the potential at the first electrode is
 $U$ while the opposite is $-U.$
```

Assistant

For a cylinder with two antisymmetric electrodes at potentials U and $-U$, the solution simplifies significantly due to symmetry. Here's the explicit expression for $u(r, \theta, z)$ and the resulting current-voltage relationship:

—

Simplified Solution for Two Antisymmetric Electrodes

1. Symmetry Assumptions:

- Electrodes are placed at $\theta = 0$ (potential U) and $\theta = \pi$ (potential $-U$), each spanning $\theta \in [-w, w]$. - The potential is antisymmetric: $u(r, \theta + \pi, z) = -u(r, \theta, z)$. - The solution is z-independent (electrodes span the full height $-H \leq z \leq H$).

2. Potential Distribution:

$$u(r, \theta) = \sum_{m\,\text{odd}} \frac{4UR}{\pi m(R + m\zeta\sigma)} (1 - \cos(mw)) \left(\frac{r}{R}\right)^m \sin(m\theta),$$

where only odd azimuthal modes ($m = 1, 3, 5, \ldots$) contribute.

3. Current Through Electrodes:

The total current I flowing through each electrode is:

$$I = \frac{8\sigma RU}{\pi} \sum_{m\,\text{odd}} \frac{(1 - \cos(mw))^2}{m(R + m\zeta\sigma)}.$$

4. DtN Map (Matrix Form):

For two electrodes, the current-voltage relationship is:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \sigma \mathbf{C} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

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where the conductance matrix $\mathbf{C}$ is:

$$\mathbf{C} = \frac{8R}{\pi} \sum_{m \text{ odd}} \frac{(1-\cos(mw))^2}{m(R+m\zeta\sigma)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

—

Key Steps

1. Boundary Conditions:
- On electrode 1 ($\theta \in [-w, w]$): $u(R,\theta) + \zeta\sigma \frac{\partial u}{\partial r}|_{r=R} = U$.
- On electrode 2 ($\theta \in [\pi - w, \pi + w]$): $u(R,\theta) + \zeta\sigma \frac{\partial u}{\partial r}|_{r=R} = -U$.

2. Fourier Series Expansion:
- Expand the boundary condition in sine series (odd symmetry).
- Project onto orthogonal modes to solve for coefficients b_m (only odd m survive).

3. Current Calculation:
- Integrate the radial current density $J(\theta) = \sigma \frac{\partial u}{\partial r}|_{r=R}$ over the electrode area.

—

Final Result

$$\boxed{\begin{aligned} u(r,\theta) &= \frac{4UR}{\pi} \sum_{m=1,3,5,\ldots} \frac{1-\cos(mw)}{m(R+m\zeta\sigma)} \left(\frac{r}{R}\right)^m \sin(m\theta), \\ I &= \frac{8\sigma RU}{\pi} \sum_{m=1,3,5,\ldots} \frac{(1-\cos(mw))^2}{m(R+m\zeta\sigma)}. \end{aligned}}$$

This represents the generalized Ohm's law for the cylinder, where the conductance depends on geometry (R, w), contact impedance (ζ), and material properties (σ).

Equidistant Electrodes

```
Let's continue, shall we ?

Given $L$ even number of electrodes with equidistant (equally spaced) at $\theta_i=2\pi i/L.$

Let
\begin{eqnarray}
u(r, \theta, z)& =& u_0\\
&+& \sum_{m=0}^\infty \frac{R}{m} \left( \frac{r}{R} \right)^m \left[ a_m \cos(m\theta)
+ b_m \sin(m\theta) \right] \\
&+& \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{2 H}{n \pi} \frac{I_m( \frac{n \pi r}{2 H})}
{I_m'( \frac{n \pi R}{2 H})}
\cos( \frac{n \pi z}{2 H}) \left[ a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta) \right],
\end{eqnarray}
where coefficients $a_{mn}$ and $b_{mn}$ are to be found from the boundary condition
\begin{equation}\label{tag7}
u(R,\theta,z))+\zeta_i\sigma\frac{\partial u}{\partial r}|_{r=R}=U_i, i=1,2,...,L;
-H\leq z \leq H.
\end{equation}
```

`Asistant:` To solve for the potential $u(r,\theta,z)$ with L equally spaced electrodes under the given boundary conditions, we proceed as follows:

—

1. General Solution Structure

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The potential is expressed as:

$$u(r,\theta,z) = u_0 + \sum_{m=0}^{\infty} \frac{R}{m}\left(\frac{r}{R}\right)^m [a_m \cos(m\theta) + b_m \sin(m\theta)] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m\left(\frac{n\pi r}{2H}\right)}{I'_m\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right)$$

$$[a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)].$$

Here
- u_0 is a constant background potential.
- a_m, b_m govern z-independent modes.
- $a_{m,n}, b_{m,n}$ govern z-dependent modes.
—
2. Boundary Conditions
At $r = R$, for each electrode $i = 1, 2, \ldots, L$:

$$u(R,\theta,z) + \zeta_i \sigma \frac{\partial u}{\partial r}\bigg|_{r=R} = U_i, \theta \in (\theta_i - w, \theta_i + w), -H \le z \le H.$$

Elsewhere (insulating regions):

$$\frac{\partial u}{\partial r}\bigg|_{r=R} = 0.$$

—
3. Coefficient Determination
Step 1:
Expand $u(R,\theta,z)$ and $\frac{\partial u}{\partial r}\big|_{r=R}$
Substitute $r = R$ into the general solution:

$$u(R,\theta,z) = u_0 + \sum_{m=0}^{\infty} \frac{R}{m}[a_m\cos(m\theta) + b_m\sin(m\theta)] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m\left(\frac{n\pi R}{2H}\right)}{I'_m\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right)[a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)],$$

$$\frac{\partial u}{\partial r}\bigg|_{r=R} = \sum_{m=1}^{\infty}[a_m\cos(m\theta) + b_m\sin(m\theta)] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{n\pi}{2H}\cos\left(\frac{n\pi z}{2H}\right)[a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)].$$

Step 2: Project Boundary Conditions onto Basis Functions
For each electrode i, multiply the boundary condition by $\cos(k\theta)\cos\left(\frac{l\pi z}{2H}\right)$ or $\sin(k\theta)\cos\left(\frac{l\pi z}{2H}\right)$ and integrate over $\theta \in (\theta_i - w, \theta_i + w)$, $z \in [-H, H]$. This yields a linear system for $a_{m,n}, b_{m,n}, u_0$.
—
4. Matrix Equation
The coefficients satisfy:

$$\mathbf{M}\begin{bmatrix} a_{m,n} \\ b_{m,n} \\ u_0 \end{bmatrix} = \mathbf{F},$$

where:
- $\mathbf{M}$ is a matrix encoding geometry and boundary conditions.
- $\mathbf{F}$ is a vector involving U_i, electrode positions θ_i, and integrals of basis functions over electrode regions.
Matrix Entries For each mode (m,n):

$$M_{11}^{m,n} = \delta_{mm'}\delta_{nn'} + \zeta_i \sigma \frac{n\pi}{2H}\int_{\theta_i - w}^{\theta_i + w} \cos^2(m\theta)d\theta \int_{-H}^{H} \cos^2\left(\frac{n\pi z}{2H}\right)dz,$$

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$$M_{12}^{m,n} = 0, M_{13} = \int_{\theta_i - w}^{\theta_i + w} \cos(m\theta) d\theta \int_{-H}^{H} \cos\left(\frac{n\pi z}{2H}\right) dz,$$

$$\mathbf{F}_i = U_i \int_{\theta_i - w}^{\theta_i + w} \int_{-H}^{H} \cos(m\theta) \cos\left(\frac{n\pi z}{2H}\right) dz d\theta.$$

—

5. Current-Voltage Relation (DtN Map)

The current I_j through electrode j is:

$$I_j = \sigma R \int_{\theta_j - w}^{\theta_j + w} \int_{-H}^{H} \frac{\partial u}{\partial r}\bigg|_{r=R} dz d\theta.$$

Substitute $\frac{\partial u}{\partial r}\big|_{r=R}$ and use the solved coefficients to express:

$$\mathbf{I} = \sigma \mathbf{C} \mathbf{U},$$

where $\mathbf{C}$ is a conductance matrix determined by the geometry and boundary conditions.

—

6. Symmetry Simplification for Equidistant Electrodes

For L electrodes at $\theta_i = \frac{2\pi i}{L}$:

- Only modes $m = kL/2$ (for integer k) contribute due to periodicity.
- The matrix $\mathbf{C}$ becomes circulant, simplifying computation.

—

Final Result

The potential $u(r, \theta, z)$ is fully determined by solving the linear system for $a_{m,n}, b_{m,n}, u_0$. The DtN map $\mathbf{I} = \sigma \mathbf{C} \mathbf{U}$ quantifies the current-voltage relationship, with:

$$\boxed{u(r, \theta, z) = u_0 + \sum_{m,n} [\text{Radial Terms}][\text{Angular and Axial Modes}], \mathbf{I} = \sigma \mathbf{C} \mathbf{U}.}$$

This framework applies to any L, with numerical implementation requiring truncation of the series and inversion of $\mathbf{M}$.

```
Q: Please revise the integral terms, as I want equidistant  $\theta_i=2\pi i/L,$ L is even.
```

```
Asistant
```

To address the case of L evenly spaced electrodes at $\theta_i = \frac{2\pi i}{L}$ $(i = 1, 2, \ldots, L)$, we exploit symmetry and periodicity to simplify the integrals and determine the coefficients a_m, b_m, u_0. Below is the revised formulation:

—

Revised Integral Terms for Equidistant Electrodes Assume L is even, and electrodes span $\theta \in [\theta_i - w, \theta_i + w]$, with uniform height $-H \leq z \leq H$. The solution $u(r, \theta)$ is z-independent, so we focus on the 2D case.

1. General Solution (z-independent):

$$u(r, \theta) = u_0 + \sum_{m=1}^{\infty} \frac{R}{m} \left(\frac{r}{R}\right)^m [a_m \cos(m\theta) + b_m \sin(m\theta)].$$

2. Boundary Conditions:

For each electrode i:

$$u(R, \theta) + \zeta\sigma \frac{\partial u}{\partial r}\bigg|_{r=R} = U_i, \theta \in [\theta_i - w, \theta_i + w].$$

Substitute $u(R,\theta)$ and $\frac{\partial u}{\partial r}\big|_{r=R}$:

$$u_0 + \sum_{m=1}^{\infty} \left[\frac{R}{m} + \zeta\sigma\right] [a_m \cos(m\theta) + b_m \sin(m\theta)] = U_i.$$

3. Project onto Fourier Modes:

For each mode k, multiply by $\cos(k\theta)$ or $\sin(k\theta)$, integrate over $\theta \in [\theta_i - w, \theta_i + w]$, and sum over all L electrodes.

Key Integrals:

1. Angular Integrals for Each Electrode i:

$$\int_{\theta_i - w}^{\theta_i + w} \cos(m\theta) d\theta = \frac{2}{m} \sin(mw) \cos(m\theta_i),$$

$$\int_{\theta_i - w}^{\theta_i + w} \sin(m\theta) d\theta = \frac{2}{m} \sin(mw) \sin(m\theta_i).$$

2. Sum Over Electrodes $i = 1, 2, \ldots, L$:

- Use orthogonality of $\cos(m\theta_i)$ and $\sin(m\theta_i)$:

$$\sum_{i=1}^{L} \cos(m\theta_i) = \begin{cases} L, & \text{if } m = kL/2 \ (k \in \mathbb{Z}), \\ 0, & \text{otherwise.} \end{cases}$$

Similarly for $\sin(m\theta_i)$.

4. Simplified Coefficients:

Only modes $m = kL/2$ (for integer k) survive due to symmetry. Define $m_p = pL/2$, $p \in \mathbb{Z}$.

- For a_{m_p}:

$$\left[\frac{R}{m_p} + \zeta\sigma\right] a_{m_p} = \frac{1}{L} \sum_{i=1}^{L} U_i \cos(m_p \theta_i).$$

- For b_{m_p}:

$$\left[\frac{R}{m_p} + \zeta\sigma\right] b_{m_p} = \frac{1}{L} \sum_{i=1}^{L} U_i \sin(m_p \theta_i).$$

- For u_0:

$$u_0 = \frac{1}{L} \sum_{i=1}^{L} U_i.$$

5. Final Solution:

$$\boxed{u(r,\theta) = \frac{1}{L} \sum_{i=1}^{L} U_i + \sum_{p=1}^{\infty} \frac{R}{m_p} \left(\frac{r}{R}\right)^{m_p} \left[a_{m_p} \cos(m_p \theta) + b_{m_p} \sin(m_p \theta)\right],}$$

where:

$$a_{m_p} = \frac{1}{\frac{R}{m_p} + \zeta\sigma} \cdot \frac{1}{L} \sum_{i=1}^{L} U_i \cos(m_p \theta_i),$$

$$b_{m_p} = \frac{1}{\frac{R}{m_p} + \zeta\sigma} \cdot \frac{1}{L} \sum_{i=1}^{L} U_i \sin(m_p \theta_i).$$

—

Key Observations:

1. Symmetry Reduction:

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Only modes $m = kL/2$ contribute, reducing the series to discrete harmonics aligned with electrode spacing.

2. Current-Voltage Relation:

The current I_j through electrode j is:

$$I_j = \sigma R \sum_{p=1}^{\infty} \frac{2\sin(m_p w)}{m_p} \left[a_{m_p} \cos(m_p \theta_j) + b_{m_p} \sin(m_p \theta_j) \right].$$

3. Matrix Form:

The Dirichlet-to-Neumann (DtN) map becomes diagonal in Fourier space:

$$\mathbf{I} = \sigma \mathbf{C}\mathbf{U}, C_{jk} = \frac{2R}{L} \sum_{p=1}^{\infty} \frac{\sin^2(m_p w)}{m_p \left(\frac{R}{m_p} + \zeta\sigma \right)} \cos(m_p(\theta_j - \theta_k)).$$

—

This framework leverages the periodicity and symmetry of equidistant electrodes to simplify computations, enabling efficient numerical implementation.

Appendix A.7 Equidistant and Uniform Contact

```
Q: Shall we continue ?

For the case  equidistant  $\theta_i=2\pi i/L,$ L is even. What will happen when
$\zeta_i = \zeta,$ a constant ?

    Assistant
```

For a cylinder with L equally spaced electrodes (positions $\theta_i = \frac{2\pi i}{L}$, L even) and uniform contact impedance $\zeta_i = \zeta$, the solution simplifies due to symmetry. Below is the explicit result:

—

Solution for L Equidistant Electrodes with Uniform Contact Impedance

The potential $u(r, \theta, z)$ and Dirichlet-to-Neumann (DtN) map are:

1. Potential Distribution

$$\boxed{u(r,\theta,z) = \sum_{p=1}^{\infty} \frac{4R\sin\left(\frac{pLw}{2}\right)}{pL\left(\frac{R}{pL/2} + \zeta\sigma\right)} \left(\frac{r}{R}\right)^{pL/2} \cos\left(\frac{pL\theta}{2}\right) + \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{8H(-1)^{(n-1)/2}}{n^2\pi^2} \frac{I_{pL/2}\left(\frac{n\pi r}{2H}\right)}{I'_{pL/2}\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right) \cos\left(\frac{pL\theta}{2}\right)}$$

2. Key Features

- Angular Modes: Only modes $m = pL/2$ ($p \in \mathbb{Z}^+$) survive due to periodicity and symmetry.
- Axial Modes: Only odd n contribute for antisymmetric z-boundary conditions.
- Coefficients:
- $a_{pL/2} = \frac{4\sin(pLw/2)}{pL\left(\frac{R}{pL/2}+\zeta\sigma\right)} \cdot \frac{1}{L}\sum_{i=1}^{L} U_i \cos(pL\theta_i/2)$,
- $b_{pL/2} = 0$ (symmetric electrodes cancel sine terms).

3. Current-Voltage Relation (DtN Map)

The current I_j through electrode j is:

$$\boxed{\mathbf{I} = \sigma\mathbf{C}\mathbf{U},\ C_{jk} = \frac{8R}{L} \sum_{p=1}^{\infty} \frac{\sin^2\left(\frac{pLw}{2}\right)}{pL/2\left(\frac{R}{pL/2} + \zeta\sigma\right)} \cos\left(\frac{pL(\theta_j - \theta_k)}{2}\right).}$$

—

Simplifications Due to Symmetry

1. Mode Truncation:

Only harmonics $m = pL/2$ survive, reducing the series to discrete multiples of $L/2$.

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2. Circulant Matrix:

The matrix $\mathbf{C}$ becomes circulant (diagonalizable by Fourier transform), with entries dependent only on $|j-k|$.

3. Uniform Contact Impedance:

The term $\frac{R}{pL/2} + \zeta\sigma$ modifies the decay rate of each mode, balancing radial resistance and contact impedance.

—

Physical Interpretation

- Lowest Mode ($p=1$): Dominates for wide electrodes ($w \gg 1/L$), decaying radially as $r^{L/2}$.
- Higher Modes ($p \geq 2$): Capture edge effects and fine potential variations.
- Axial Dependence: Governed by $I_{pL/2}$, modified Bessel functions that decay/grow with z

—

Steps for Numerical Implementation

1. Truncate Series: Retain $p \leq P$, $n \leq N$.
2. Compute Matrix $\mathbf{C}$: Exploit circulant symmetry for efficiency.
3. Solve $\mathbf{I} = \sigma\mathbf{CU}$: Invert $\mathbf{C}$ to find voltages $\mathbf{U}$ for given currents $\mathbf{I}$.

This framework leverages symmetry to simplify computations, enabling efficient modeling of multi-electrode systems with uniform contact impedance.

Appendix A.8 Errata

```
Q: Let's continue, shall we ?

I made a mistake in one of my previous query. Actually, my query is as follows.

Let
\begin{eqnarray}
u(r, \theta, z)& =& u_0\\
&+& \sum_{m=0}^\infty \frac{R}{m} \left( \frac{r}{R} \right)^m \left[ a_m \cos(m\theta) +
b_m \sin(m\theta) \right] \\
&+& \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{2 H}{n \pi} \frac{I_m( \frac{n \pi r}{2 H})}
{I_m'( \frac{n \pi R}{2 H})}
\cos( \frac{n \pi z}{2 H}) \left[ a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta) \right],
\end{eqnarray}
where coefficients $a_{mn}$ and $b_{mn}$ are to be found from the boundary condition
\begin{equation}\label{tag7}
u(R,\theta,z))+\zeta_i\sigma\frac{\partial u}{\partial r}|_{r=R}=U_i, i=1,2,...,L;
-H\leq z \leq H.
\end{equation}

I need a special case for 2 electrodes, where $\zeta_i=0, i=1,2$ and the potential at the first ele
```

Asistant: For two electrodes with potentials U and $-U$, and contact impedance $\zeta_i = 0$, the solution simplifies due to antisymmetry. Here's the explicit expression:

—

Special Case Solution (2 Electrodes, $\zeta_i = 0$)

$$\boxed{u(r,\theta,z) = \sum_{m=1,3,5,\ldots}^{\infty} \frac{4UR\sin(mw)}{\pi m^2}\left(\frac{r}{R}\right)^m \cos(m\theta) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{4UH(-1)^{(n-1)/2}}{n^2\pi^2} \frac{I_m\left(\frac{n\pi r}{2H}\right)}{I_m'\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right)\cos(m\theta),}$$

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Key Simplifications:
1. Azimuthal Symmetry (z-independent terms):
- Only odd azimuthal modes ($m = 1, 3, 5, \ldots$) contribute due to antisymmetric potentials. - Coefficients for $\cos(m\theta)$:

$$a_m = \frac{4U \sin(mw)}{\pi m^2}, \; b_m = 0.$$

2. Axial Dependence (z-dependent terms):
- Only odd axial modes ($n = 1, 3, 5, \ldots$) survive due to antisymmetry in z-integrated boundary conditions.
- Coefficients for $\cos\left(\frac{n\pi z}{2H}\right)$:

$$a_{m,n} = \frac{4UH(-1)^{(n-1)/2}}{n^2\pi^2} \frac{\delta_{m0}}{I_0'\left(\frac{n\pi R}{2H}\right)}, \; b_{m,n} = 0.$$

3. Background Potential:
- $u_0 = 0$, as the antisymmetric electrodes cancel the average potential.
Current-Voltage Relation (DtN Map):
The current I_j through electrode j is:

$$I_j = \frac{8\sigma RU}{\pi} \sum_{m=1,3,5,\ldots} \frac{\sin^2(mw)}{m^2} + \frac{16\sigma HU}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \frac{I_0\left(\frac{n\pi R}{2H}\right)}{I_0'\left(\frac{n\pi R}{2H}\right)}.$$

—

Interpretation:
- Radial Decay: The potential decays radially as $(r/R)^m$, dominated by the lowest odd mode ($m = 1$).
- Axial Modulation: Axial modes decay with z via I_m, modified Bessel functions.
- Electrode Width: Narrower electrodes ($w \to 0$) excite higher modes ($m \to \infty$).

This solution satisfies the mixed Dirichlet/Neumann boundary conditions and captures the antisymmetric potential distribution between the two electrodes.

Appendix B Equidistant and Equal Constant Value

```
Shall we continue ?

For the case  equidistant  $\theta_i=2\pi i/L,$ L is even. What will happens when
$\zeta_i = \zeta,$ a constant ?
Use the fact that: $h_n = \frac{\sin(2n w)}{2n}, if n =L/2, L, 3L/2, 2L, ...$; and
$h_n = 0,$ elsewhere.

Please study in details of the coefficient Matrix of the solution $u(r,\theta,z).$
```

Assistant

To analyze the coefficient matrix for the solution $u(r, \theta, z)$ with L equidistant electrodes ($\theta_i = \frac{2\pi i}{L}$, L even) and constant contact impedance ζ, we proceed as follows:

—

1. General Solution Structure

The potential $u(r, \theta, z)$ is:

$$u(r,\theta,z) = u_0 + \sum_{m=0}^{\infty} \frac{R}{m}\left(\frac{r}{R}\right)^m [a_m \cos(m\theta) + b_m \sin(m\theta)] + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{2H}{n\pi} \frac{I_m\left(\frac{n\pi r}{2H}\right)}{I_m'\left(\frac{n\pi R}{2H}\right)} \cos\left(\frac{n\pi z}{2H}\right) [a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)].$$

—

2. Symmetry and Mode Restriction

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For L electrodes with spacing $\Delta\theta = \frac{2\pi}{L}$:
- Azimuthal modes m are restricted to $m = kL/2$ ($k \in \mathbb{Z}$) due to periodicity.
- Axial modes n are restricted to $n = pL/2$ ($p \in \mathbb{Z}$) due to $h_n = \frac{\sin(2nw)}{2n} \neq 0$.

—

3. Boundary Conditions
At $r = R$, for each electrode i:

$$u(R,\theta,z) + \zeta\sigma \frac{\partial u}{\partial r}\bigg|_{r=R} = U_i, \quad \theta \in [\theta_i - w, \theta_i + w], \quad -H \leq z \leq H.$$

Substitute $u(R,\theta,z)$ and $\frac{\partial u}{\partial r}\big|_{r=R}$, then project onto orthogonal basis functions $\cos(m\theta)\cos\left(\frac{n\pi z}{2H}\right)$ and $\sin(m\theta)\cos\left(\frac{n\pi z}{2H}\right)$.

—

4. Coefficient Matrix **M**
The matrix **M** for the linear system $\mathbf{Mx} = \mathbf{F}$ (where $\mathbf{x} = [a_{m,n}, b_{m,n}, u_0]^T$) has entries:
For $a_{m,n}$:

$$M_{11}^{m,n} = \delta_{mm'}\delta_{nn'} + \zeta\sigma\frac{n\pi}{2H} \underbrace{\int_{-H}^{H} \cos^2\left(\frac{n\pi z}{2H}\right) dz}_{=H} \underbrace{\int_{\theta_i - w}^{\theta_i + w} \cos^2(m\theta) d\theta}_{=\pi\delta_{m,kL/2}\cdot \mathrm{sinc}(2mw)},$$

$$F_i^{m,n} = \frac{4U}{L} \cdot \frac{\sin(2nw)}{2n} \cdot \cos(m\theta_i).$$

For $b_{m,n}$:

$$M_{22}^{m,n} = \delta_{mm'}\delta_{nn'} + \zeta\sigma\frac{n\pi}{2H} \underbrace{\int_{-H}^{H} \cos^2\left(\frac{n\pi z}{2H}\right) dz}_{=H} \underbrace{\int_{\theta_i - w}^{\theta_i + w} \sin^2(m\theta) d\theta}_{=\pi\delta_{m,kL/2}\cdot \mathrm{sinc}(2mw)},$$

$$F_i^{m,n} = \frac{4U}{L} \cdot \frac{\sin(2nw)}{2n} \cdot \sin(m\theta_i).$$

For u_0:

$$M_{33} = L \cdot 2w \cdot 2H, \quad F_i = \sum_{i=1}^{L} U_i.$$

—

5. Matrix Structure
- Diagonal Dominance:
M is block-diagonal with blocks corresponding to modes $(m,n) = (kL/2, pL/2)$. Off-diagonal terms vanish due to orthogonality. - Simplified Entries:
For $m = kL/2$ and $n = pL/2$:

$$M_{11}^{m,n} = 1 + \zeta\sigma\frac{n\pi H}{2} \cdot \mathrm{sinc}(2mw),$$

$$F_i^{m,n} = \frac{4U}{L} \cdot h_n \cdot \cos(m\theta_i).$$

—

6. Key Observations
1. Mode Coupling:
Only modes $m = kL/2, n = pL/2$ contribute. Higher modes decay rapidly due to $\left(\frac{r}{R}\right)^m$ and I_m.
2. Current-Voltage Relation:

The DtN map $\mathbf{I} = \sigma\mathbf{CU}$ has entries:

$$C_{jk} = \frac{8R}{L}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{h_n^2}{1+\zeta\sigma\frac{n\pi H}{2}}\cos\big(m(\theta_j-\theta_k)\big).$$

3. Role of ζ:
Larger ζ suppresses higher modes ($n \to \infty$) due to the denominator $1+\zeta\sigma\frac{n\pi H}{2}$.

—

7. Final Result
The coefficient matrix $\mathbf{M}$ is sparse and block-diagonal, with non-zero entries only for modes $(m,n) = (kL/2, pL/2)$. The solution $u(r,\theta,z)$ simplifies to:

$$\boxed{u(r,\theta,z) = \sum_{k,p}\frac{4U\cdot h_{pL/2}}{L\left(1+\zeta\sigma\frac{pL\pi H}{4}\right)}\left[\frac{R}{kL/2}\left(\frac{r}{R}\right)^{kL/2}\cos\left(\frac{kL\theta}{2}\right)+\frac{2H}{pL\pi}\frac{I_{kL/2}\left(\frac{pL\pi r}{4H}\right)}{I'_{kL/2}\left(\frac{pL\pi R}{4H}\right)}\cos\left(\frac{pL\pi z}{4H}\right)\right].}$$

This framework enables efficient computation for systems with many electrodes by leveraging symmetry and sparsity.

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