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The Consistency of the Distribution Function Conditional Estimate and Application on the Consistency and Asymptotic Normality of the Conditional Hazard Function Estimate for High Dimensional Quasi-Associated Data

Hamza Daoudi [1,*], Zouaoui Chikr Elmezouar [2] and Fatimah Alshahrani [3]

[1] Department of electrotechnic, College of Science, University of Tahri Mohamed, Bechar, Algeria;
[2] Department of Mathematics, College of Science, King Khalid University, Abha, 61413, Saudi Arabia;
[3] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, Riyadh 11671, Saudi Arabia;
* Correspondence: Daoudiham63@gmail.com;

Abstract: The objective of this study is to examine a nonparametric estimate , using the kernel approach, of the conditional distribution function of a scalar response variable that is given a random variable whose values take place in a separable real Hilbert space. The observations will be dependent on one another in a quasi-associated fashion. The pointwise practically perfect consistencies with rates of this estimator are established by us under some broad conditions. The study's major objective is to investigate the convergence rate of the proposed estimator and its application in the convergence rate and asymptotic normality of the hazard function. The asymptotic normality of the developed estimator is established precisely. Simulation studies were conducted to investigate the behavior of the asymptotic property in the context of finite sample data.

Keywords: conditional distribution function; asymptotic normality, conditionalhazard function; quasi-associated; functional data

MSC: 62G05; 62F12; 62G20; 62M09

1. Introduction

Mathematics and statistical analysis techniques have been shown to be of considerable relevance in a variety of scientific sectors in recent years, including engineering, the economics, clinical medicine, and healthcare. In particular, the application of methods of mathematical and statistical analysis to engineering, the economics, healthcare, and clinical medicine, as well as demonstrating how these approaches may assist in such vital areas as comprehension, prediction, correlation, diagnosis, therapy, and data processing.

Functional data is the subject of this research. [3] excerpt: Statistics' functional data analysis (FDA) analyzes infinite-dimensional variables including curves, sets, and pictures. The "Big Data" revolution has spurred its rapid expansion over the past 20 years.

This may be demonstrated by researching the topic's past (for an example, see [1]). In [24], the topics of density and mode estimation for normed vector space data are discussed. In addition to that, he discussed the problem of excessive dimensionality in functional data and offered potential remedies. Nonparametric models were investigated for use in regression estimation in [22].

The treatment of functional data today typically involves contemporary theory. For example, the reference [18] presented the consistency rates of a variety of conditional distribution functionals, such as the regression function, conditional cumulative distribution, and conditional density, uniformly over a subset of the explanatory variable. These functionals include the conditional cumulative distribution

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and conditional density. The conditional cumulative distribution as well as the conditional density are both included in these functionals.

Additional examples include the conditional cumulative distribution and the conditional density. Uniformly in bandwidth (UIB) consistency was extended to the ergodic scenario, and the rates of consistency for different functional nonparametric models were investigated in [26]. The regression function was a part of these models, a conditional hazard function, conditional distribution, and conditional density.

In the field of statistical mathematics, in recent years, there has been a surge of curiosity on the statistical analysis of functional data. These numbers are used in econometrics, medicine, environmental science, and many other fields. In the statistical functional, [22] made the first attempt to estimate the conditional density function and its derivatives . In addition, they were the ones who were the first to do so in the scientific community.

These authors reached an extent of convergence in the case i.i.d. that was really near to being finished. Since this work was published, much more research has been done on estimating the conditional density and its derivatives, especially for computing the conditional mode. Since the publication of this work, the conditional mode has been the subject of a significant amount of research.

In point of fact, [19] demonstrated that a kernel estimator of the conditional mode will almost certainly converge to the true value. They did this by taking into account data that included a-mixing. The conditional density is maximized by the random variable that corresponds to this mode, which is defined as.

The point that nullifies the kernel density estimator derivative was used by [16,17] to estimate the conditional mode. This was done in order to determine the conditional mode. The outcomes were comparable using this approach.

The latter put more of an emphasis on the estimator's asymptotic normality, which was provided in both the iid and mixed circumstances respectively. Both scenarios included mixing. [27] was able to identify with what level of accuracy the terms that dominate the quadratic error that is produced by the kernel density estimator.

We suggest that the reader check out [28] for further information on the topic of the smoothing parameter that should be used in the process of estimating the conditional density in relation to the functional explanatory variable.

The concept of a quasi-association variable refers to a variable that exhibits some degree of association with another variable, Examples of research that processed data under both positive and negative dependent random variables are [34], [32], and [33]. [14] were the first people to offer the concept of quasi-association for the purpose of conducting an analysis of real-valued stochastic occurrences. This is a very striking illustration of the idea of weak reliance. It was used by [7] to real valued random fields, and it provides a unified technique for the study of families of positively dependent as well as negatively dependent random variable families.

According to our knowledge, the nonparametric estimation of quasi-associated random variables is addressed in a vanishingly small number of published papers. The study conducted by [13] focuses on a limit theorem for quasi associated Hilbertian random variables. Meanwhile, the research conducted by [5] explores asymptotic results for an M-estimator of the regression function for quasi-associated processes; and [36], who investigated both quasi-associated processes and asymptotic results. [9] investigated the asymptotic normality of this final estimator as part of their study, which focused on the single-index structure of the conditional hazard function.

To solve relative regression, [31] explored the nonparametric estimate for linked random variables. it was found to be significant by [10] and [10]. Both of these results were found to be significant. Both Daoudi and Mechab were responsible for conducting these investigations in their own separate ways.

It is important to keep in mind that the phenomena that our results are connected to the model's functional space in some manner, just as they are related to every previous asymptotic statistics functional nonparametric finding.

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In this research, we examine the conditional distribution function within the context of the quasi-associated situation, focusing on its asymptotic bias and dispersion. This study examined pointwise consistency for nonparametric conditional density function estimates.

This objective was successfully accomplished through the conduct of research on the rate of the almost-complete convergence (a.co.) for nonparametric estimates of the conditional distribution function.

The following is the structure of our paper: In the following section, we will describe our model. In the part 3, we present the hypotheses and remarks, along with a few notations, followed by a discussion of supplementary findings and supporting in paragraph 4. The Main Results are found in Section 5. The part 6 covers the application on consistency rates and the user is referring to the concepts of consistency and asymptotic normality in the context of estimating a conditional hazard function. he establishment of confidence bands for estimations in part 7. In Section 8, we will analyze and assess the behavior of our asymptotic normality results, on finite sample data. In conclusion, Auxiliary results and proofs are presented in Section 9.

2. Model and Estimator

To commence, we provide a precise delineation of quasi-association pertaining to random variables that possess values within a separable Hilbert space.

Consider a separable Hilbert space $(\mathcal{H}, < .,. >)$ furnished with an orthonormal basis. The sequence of elements is represented by $e_k, k \geq 1$. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of real random variables with values in $\mathcal{H}$. The statement is made that this sequence exhibits quasi-association in relation to the basis. The user defines e_k as a variable. The term "quasi-associated" is used to describe the sequence e_k if, for any positive integer d, the d-dimensional sequence $\{(< R_i, e_{j1} >, ..., < R_i, e_{jd} >), i \in \mathbb{N}\}$ is also quasi-associated.

In this analysis, we will examine a group of n quasi-associated random variables, which we will represent as $W_i = (R_i, S_i)_{1\leq i\leq n}$. The random variables mentioned exhibit the same distribution as the random variable $W = (R, S)$, which represents values in a separable real Hilbert space denoted as $\mathcal{H} \times \mathbf{R}$. This Hilbert space is equipped with an inner product denoted as $< .,. >$, which generates the norm. The user's text does not contain any meaningful information. The semi-metric d is considered, as given by. $\forall r, r' \in \mathcal{H}/d(x, x') = \|r - r'\|$. A point r in the $\mathcal{H}$ space, its fixed neighborhood $\mathcal{N}_r$, and a compact subset of $\mathbf{R}$ all have the notation $\mathcal{S}$ For each $r \in \mathcal{N}_r$, there is a S such that $R = r$. Using a sample of n dependent observations from random variables with the same distribution as $W := (R, S)$, the conditional distribution function $F^r(s)$ is estimated.

We present the $\widehat{F}^r$ estimator of F^r, a kernel type estimator, defined as:

$$\widehat{F}^r(s) = \frac{\sum_{i=1}^{n} K(h_H^{-1} d(r, R_i)) H(h_H^{-1}(s - S_i))}{K(h_H^{-1} d(r, R_i))} \quad \forall\, s \in \mathbb{R}$$

As n increases to infinity, the sequence of positive real integers $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) converges to zero.

Let K denote the kernel and H represent a particular distribution function.

3. Hypotheses and Notations

When there is no chance of misunderstanding, we shall designate any strictly positive generic constants along the actual paper by the notation l or/and l'. This will only happen when there is no risk of mistake. w in the process signifies a fixed point in $\mathcal{H}$, and $\mathcal{N}_w$ stands for a fixed neighborhood of w. Taking into account the fact that the random couple $\{(W_i, T_i), i \in \mathcal{N}\}$ is a process that is stationary. Let λ_k denote the covariance coefficient, which may be found by using the equation:

$$\lambda_k = sup_{s\geq k} \sum_{|i-j|\geq s} \lambda_{i,j}.$$

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Where

$$\lambda_{i,j} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |Cov(W_i^k, W_j^l)| + \sum_{k=1}^{\infty} |Cov(W_i^k, T_j)| + \sum_{l=1}^{\infty} |Cov(W_i, X_j^l)| + |Cov(T_i, T_j)|.$$

The k^{th} component of W_i , as represented by R_i^k, where $R_i^k :=< W_i, e^k >$ is the definition of Rk_i. Let's denote the ball as $B(w,h) := \{r' \in \mathcal{H}/d(w',w) < h\}$. This represents the ball with a center of w and a radius h, when $h > 0$.

For the purpose of establishing the virtually full convergence of the estimator of the $\widehat{F}^r(s)$, the following assumptions are required, and in order to have a functional explanation, we need to incorporate he following assumptions:

(P1) $\mathbb{P}(R \in B(r,h)) = \phi_r(h) > 0$ and the function $\phi_r(h)$ is a differentiable at 0.

(P2) The conditional distribution function $F^r(s)$ satisfies the Holder condition, that is: $\forall(r_1,r_2) \in \mathcal{N}_r \times \mathcal{N}_x, \forall(s_1,s_2) \in \mathcal{S}^2$

$$|F^{r_1}(s_1) - F^{r_2}(s_2)| \leq l\Big(d^{z_1}(r_1,r_2) + |s_1 - s_2|^{z_2}\Big) \quad ,z_1 > 0, \quad z_2 > 0.$$

$\mathcal{S}$ is a fixed compact subset of $\mathbb{R}$.

(P3) The kernel, denoted by H, is a differentiable function, and its inverse, denoted by H', is a positive, bounded, and Lipschitzian continuous function:

$$\int |t|^{z_2} H'(t)dt < \infty \qquad and \qquad \int H'^2(t)dt < \infty.$$

(P4) For K, we have the necessary conditions for it to be a bounded continuous Lipschitz function, which are:

$$l \|_{[0,1]}(.) < K(.) < l' \|_{[0,1]}(.)$$

$\|_{[0,1]}$ is referred to as an indicator function.

(P5) A quasi-association exists between the sequence of random pairings(R_i, S_i), where $i \in \mathcal{N}$, and the covariance coefficient $\lambda_k, k \in \mathcal{N}$, as long as the conditions are met.

$$\exists a > 0, \ \exists\, l > 0, \quad then \ \ \lambda_k \leq le^{-ak}$$

(P6) The joint distribution functions are defined so that they hold for each and every pair (i,j).

$$\Psi_{i,j}(h) = \mathbb{P}[(R_i, R_j) \in B(r,h) \times B(r,h))]$$

satisfy :

$$\sup_{i \neq j} \Psi_{i,j}(h) = O(\phi_r^2(h_k)) > 0.$$

(P7) The sequences of positive values that make up the bandwidths h_K and h_H fulfill the following equations when j is equal to 0 and 1:

$$\lim_{n\to\infty} \frac{\log^5(n)}{nh_H^j \phi_r(h_k)} = 0.$$

4. Remarks Concerning the Hypotheses

The attribute of concentration of the explanatory variable is denoted by the assumption (P1) in the context of tiny balls. The condition (P2) is applied to our model in order to regulate the regularity of the functional space. They are essential for calculating the bias component of the convergence rates and

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are needed to be present. Both the (P3) and (P4) hypotheses center their attention on the cumulative function H as well as the kernels $K^{'}H^{'}$, and $K^{'}$, respectively. In particular, this assumption is utilized to get rid of the word "bias" in the conclusion of asymptotic normalcy. The fifth assumption, denoted as P5, is widely regarded as a classical assumption in functional estimation within spaces of both finite and infinite dimensions. The hypothesis (P6) is a notation that refers to a structural condition that is applied to the quasi-associated data. In order for us to demonstrate the asymptotic normality of our model when it is exposed to quasi-association, it is required for us to have the assumption (P7). Following is a definition of the asymptotic behavior of the joint distribution of the pair (R_i, R_j).

5. Main Results

Theorem 1. *Under hypotheses $(P1)-(P7)$, we have:*

$$\widehat{F}^r(s) - F^r(s) = O(h_K^{b1} + h_K^{b2}) + O\left(\left(\frac{Log(n)}{n\phi_R(h_K)}\right)^{\frac{1}{2}}\right) \tag{1}$$

The following decomposition, together with the lemmas that are listed below, provide the foundation for the proof of Theorem 1:

$$\begin{aligned}\widehat{F}^r(s) - F^r(s) = \frac{1}{\widehat{F}_D^r}\left(\left(\widehat{F}_N^r(s) - \mathbb{E}\widehat{F}_N^r(s)\right) - \left(F^r(s) - \mathbb{E}\widehat{F}_N^r(s)\right)\right)\\ + \frac{F^r(s)}{\widehat{F}_D^r}\left(\widehat{F}_D^r - \mathbb{E}\widehat{F}_D^r\right)\end{aligned} \tag{2}$$

We can write:

$$\widehat{F}^r(s) = \frac{\widehat{F}_N^r(s)}{\widehat{F}_D^r}$$

Where:

$$\widehat{F}_N^r(s) = \frac{1}{n\mathbb{E}[K_1(r)]}\sum_{i=1}^{n} K_i(r)H_i(s)$$

and:

$$\widehat{F}_D(r) = \frac{1}{n\mathbb{E}[K_1(r)]}\sum_{i=1}^{n} K_i(r)$$

with :

$$K_i(r) = K(h_K^{-1}d(r, R_i)) \quad and \quad H_i(s) = H(h_H^{-1}(s - S_i))$$

Lemma 1. *Under hypotheses $(P1)$-$(P4)$ and $(P6)$:*

$$\frac{1}{\widehat{F}_D^r(s)}\left(\widehat{F}_N^r(s) - E\widehat{F}_N^r(s)\right) = O_{a.co}\left(\left(\frac{Log(n)}{n\phi_R(h_K)}\right)^{\frac{1}{2}}\right) \tag{3}$$

Corollary 1. *Under hypotheses $(P1)$-$(P4)$ and $(P6)$, we have:*

$$\sum_{i=1}^{\infty}\mathbb{P}\left(|\widehat{F}_D^r(s)| < 1/2\right) < \infty \tag{4}$$

Lemma 2. *Under hypotheses $(P1)$-$(P6)$, we have:*

$$\frac{1}{\widehat{F}_D^r(s)}\left(F^r(s) - E\widehat{F}_N^r(s)\right) = O(h_K^{b1} + h_K^{b2}) \tag{5}$$

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Lemma 3. *Under hypotheses* $(P1)$*-*$(P3)$ *and* $(P6)$*, we have:*

$$\widehat{F}_D^r(s) - E\widehat{F}_D^r(s) = O_{a.co}\left(\left(\frac{log(n)}{n\phi_r(h_K)}\right)^{\frac{1}{2}}\right) \tag{6}$$

6. Application on the Conditional Hazard Function Estimate

Using our most important result, The estimator of the conditional hazard function $\widehat{Z}^r(s)$ exhibits convergence behavior , it was examined in our analysis. This estimator is defined as follows:

$$\widehat{Z}^r(s) = \frac{\widehat{f}^r(s)}{1 - \widehat{F}^r(s)} \quad \forall\, s \in \mathbb{R}, \tag{7}$$

where $\widehat{f}^r(s)$ is an estimate of the conditional density given by

$$\widehat{f}^r(s) = \frac{h_H^{-1}\sum_{i=1}^{n} K(h_K^{-1}d(r,R_i))H'(h_H^{-1}(s - S_i))}{\sum_{i=1}^{n} K(h_K^{-1}d(r,R_i))} \quad \forall\, s \in \mathbb{R} \tag{8}$$

This is the derivative of H expressed as H'.

6.1. *Almost Complete Convergence with the Rate*

Theorem 2. *Assuming (P1)-(P7) and Theorem 1. of [23], we obtain:*

$$|\widehat{Z}^r(s) - Z^r(s)| = O\left(h_K^{b_1} + h_K^{b_2}\right) + O_{a.co}\left(\left(\frac{\log n}{nh_H\phi_r(h_K)}\right)^{\frac{1}{2}}\right). \tag{9}$$

The proof of Theorem 2 is the sum of this paper's proof and theorem's proof in [23]

6.2. *Asymptotic Normality*

Theorem 3. *Assuming (P1)-(P7) and Theorem 1. of [10] and [11], we obtain: for any* $r \in \mathcal{A}$

$$\sqrt{nh_H\phi(r,h_K)}\left(\widehat{Z}^r(s) - Z^r(s)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma_h^2(r)) \quad as\ n \quad \to \infty$$

where

$$\mathcal{A} = \{r \in \mathcal{H},\ f^r(s)(1 - F^r(s)) \neq 0\}$$

$$\sigma_h^2(r) = \frac{C_2 Z^r(s)}{C_1^2(1 - F(s,r))}\int H'^2(t)dt,$$

In

$$C_j = K(1) - \int_0^1 (K^j)'(s)\beta(r,s)ds, \quad for\ j = 1,2.$$

The symbol $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

7. Confidence Bands

One important aspect in statistical analysis is the establishment of confidence bands for estimations. These confidence bands provide a range of values within which we can be confident that the true value lies. By calculating and interpreting these confidence bands, we can gain a better understanding of the uncertainty associated with our estimations. The objective of this section is to

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utilize our asymptotic normality result (Theorem 3) to create confidence intervals for the actual value of $Z^r(s)$ for a specified curve with the format $< X, X' > = < x, x' >$. Estimation using nonparametric methods relies on the asymptotic variance, which is determined by a number of unknown functions. Regarding our situation, we have

$$\sigma_Z^2(r) = \frac{C_2 Z^r(s)}{C_1^2(1 - F^r(s))}$$

The variables $Z^r(s), F^r(s), C_1$, and C_2 are not known in advance and must be approximated during the practical implementation. It is possible to derive confidence bands even when $\sigma_Z^2(r)$ is functionally given. An estimate for the asymptotic standard deviation $\widehat{\sigma}_Z^2(r)$ may be produced using the estimators $\widehat{Z}^{(r)}(s)$, $\widehat{F}^{(r)}(s)$, $\widehat{C}_1$, and $\widehat{C}_2$ for $Z^{(r)}(s)$, $F^{(r)}(s)$, C_1, and C_2, respectively.

$$\widehat{\sigma}_Z^2(r) = \frac{\widehat{C}_2 \widehat{Z}^r(s)}{\widehat{C}_1^2(1 - \widehat{F}^r(s))}.$$

The constants C_1 and C_2 are estimated empirically in the following manner:

$$\widehat{C}_1 = \frac{1}{n\phi(r, h_K)} \sum_{i=1}^{n} K(h_K^{-1} d(r, R_i)), \quad \widehat{C}_2 = \frac{1}{n\phi(r, h_K)} \sum_{i=1}^{n} K^2(h_K^{-1} d(r, R_i))$$

where $\phi(r, h_K) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{|<r - R_i>| < h_K\}}$.

Now the asymptotic confidence band at asymptotic level $1 - \zeta$ for $Z^r(s)$ is given by

$$\left[\widehat{Z}^r(s) - u_{1-\frac{\zeta}{2}} \left(\frac{\widehat{\sigma}_Z^2(r)}{n\phi(r, h_K)}\right)^{1/2}, \ \widehat{Z}^r(s) + u_{1-\frac{\zeta}{2}} \left(\frac{\widehat{\sigma}_Z^2(r)}{n\phi(r, h_K)}\right)^{1/2}\right]$$

where $u_{1-\frac{\zeta}{2}}$ denotes the $1 - \frac{\zeta}{2}$ quantile of the standard normal distribution.

8. A simulation Study

In this part, we examine how our asymptotic normality conclusions behave across data from a limited sample.

Our primary purpose is to demonstrate how simple the conditional hazard function is to develop and to study the effect of dependency on this asymptotic characteristic.

We produce functional observations for this purpose by examining the following functional nonparametric model:

$$Z_i = r(W_i) + \epsilon_i \ \text{ for } i = 1, \ldots, n \tag{10}$$

Where $\epsilon_i \rightsquigarrow \mathcal{N}(0, .5)$. The linear process with quasi-associated variables is well known to satisfy the requirement (H7).
As a result, the functional regressor with quasi-associates shown below is constructed.

$$Z_i(t) = \sum_{j=i+1}^{i+m} \Gamma_j(t)$$

Where

$$\Gamma_j(t) = s_j t^2 + h_j * t + g_j \qquad t \in [0, 1]$$

$(s_j)_j \rightsquigarrow \mathcal{N}(0, \frac{1}{2})$ (resp. $(h_j)_j \rightsquigarrow \mathcal{N}(-1, \frac{1}{2})$ and $(g_j)_j \rightsquigarrow \mathcal{N}(1, \frac{1}{2})$).

 doi:10.20944/preprints202308.0307.v1

The Figure 1 shows the Z_i's curves discretized in the same 100-point grid in [0, 1] for three separate m=1 (independent case), and 10 values

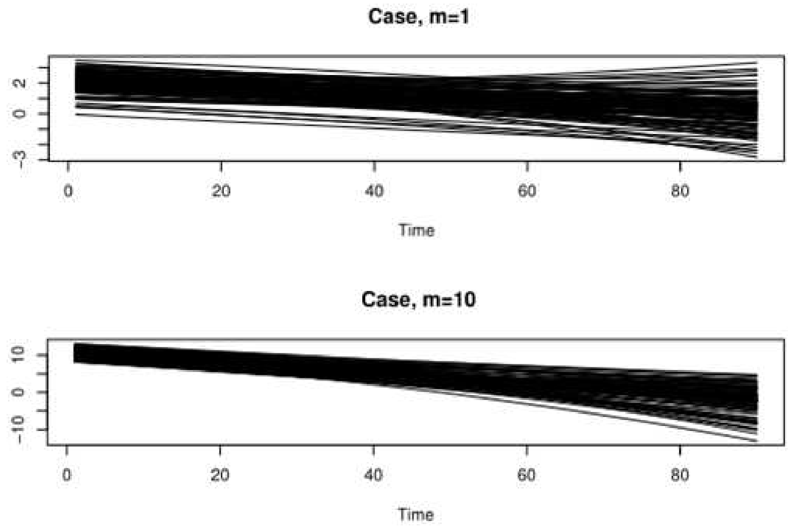

Figure 1. A sample of 200 curves.

Furthermore, the regression operator computes the scalar variable Z_i :

$$r(w) = 5 \int_0^1 \exp\{W(t)\}\,dt.$$

Given $W = w$, we may derive the theoretical conditional from the error distribution (ϵ_i). The distribution function of Z is supplied by e_i's distribution shifted by $r(w)$. As a result, the theoretical conditional hazard function is simple to compute. To demonstrate this function's asymptotic normality, we fix one curve, $w = W_0$, and $z = Z_0$, from the created data, then collect m independent n-samples of the same data and compute the quantity:

$$\sqrt{nh_H\phi(r, h_K)}\,\widehat{\sigma}_{h_K}^{-1}(r)\left(\widehat{Z}^r(s) - Z^r(s)\right)$$

where $\widehat{\sigma}_{h_K}$ is the previous section's standard deviation estimation.

The collected sample is next tested for normality. While the cross-validation approach determines , We utilized a quadratic kernel K represented as: the bandwidth parameters h_H and h_K dependent on the number of nearest neighbors and we have used quadratic kernel K expressed by

$$K(t) = \frac{3}{2}(1 - t^2)\mathbb{I}_{[0,1]}.$$

The Figure 2 depicts a Graph of the collected sample vs a typical normal distribution at varied m=1 and 10 values.

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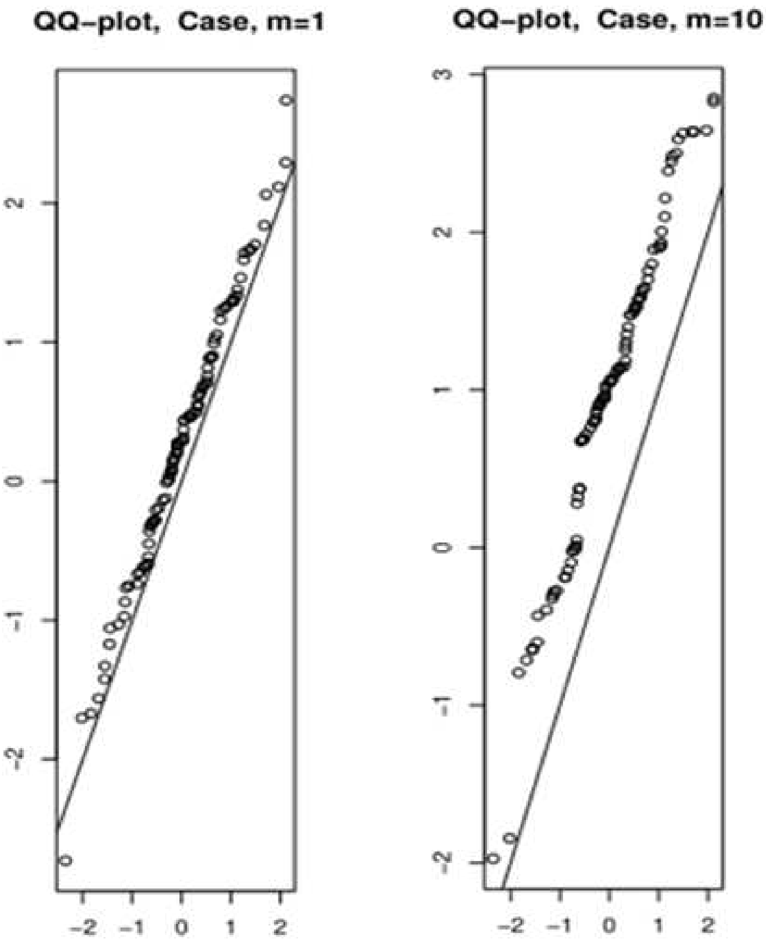

Figure 2. The QQ-plot of the obtained sample.

It is evident that our estimator is simple to use and behaves well in practice. Furthermore, the pace of asymptotic normality convergence is substantially related to data correlation. It diminishes in particular with respect to m values.

Table 1. Kolmogorov-Smirnov analysis.

m	1	10
P-value	0.88	0.43

9. Auxiliary Results and Proofs

Corollary 2. *19 (See, [13]). Let $(R_n)_{n\in\mathbb{N}}$ be a quasi-associated sequence of random variables with values in $\mathcal{H}$. Let $h \in BL(\mathcal{H}^{|I|}) \cap \mathbb{L}^\infty$ and $M \in BL(\mathcal{H}^{|J|}) \cap \mathbb{L}^\infty$, for some finite disjoint subsets $I, J \subset \mathbb{N}$. Then*

$$Cov(h(R_i, i \in I), M(R_j, j \in J)) \leq Lip(h)Lip(M) \sum_{i\in I}\sum_{j\in J}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left|Cov(R_i^k, R_j^l)\right|$$

where $(BL(\mathcal{H}^u; u > 0)$ is the set of bounded Lipschitz functions $h : \mathcal{H}^u \to \mathbb{R}$ and $\mathbb{L}^\infty$ is the set of bounded functions.

Corollary 3. *(See, [25]). Let $R_1, ..., R_n$ the real random variables such that $\mathbb{E}(R_j) = 0$ and $\mathbb{P}(| R_j |\leq M) = 1$ for all $j = 1, ..., n$ and some $M < \infty$, Let $\sigma_n^2 = Var(\sum_{i=1}^n \Delta_i)$.*

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Assume, furthermore, that there exist $K < \infty$ *and* $\beta > 0$ *such that, for all u-uplets* $(s_1, ..., s_u) \in \mathbb{N}^u$, $(t_1, ..., t_v) \in \mathbb{N}^v$ *with* $1 \leq s_1 \leq \cdots \leq s_u \leq t_1 \leq \cdots \leq t_v \leq n$.
The following inequality is fulfilled:

$$\mid cov(R_{s_1}...R_{s_u}, R_{t_1}...R_{t_v}) \mid \leq K^2 M^{u+v-2} v e^{-\beta(t_1 - s_u)}.$$

Then,

$$\mathbb{P}(\mid \sum_{j=1}^{n} R_j \mid > t) \leq \exp\{-\frac{t^2/2}{A_n + B_n^{1/3} t^{5/2}}\}$$

for some

$$A_n \leq \sigma_n^2$$

and

$$B_n = (\frac{16nK^2}{9An(1-e^-\beta) \vee 1})\frac{2(K \vee M)}{1-e^-\beta}.$$

The proof of Lemma 1

We put:

$$\Delta_i = \frac{1}{nh_H E[K_1]}\chi(R_i, S_i), \qquad 1 \leq i \leq n. \tag{11}$$

where :

$$\psi(R_i, S_i) = K(h_K^{-1} d(r, R_i))H(s - S_i) - E[K_1 H_1] \quad 1 \leq i \leq n, \tag{12}$$

$R_i \in \mathcal{H}$, $S_i \in \mathbb{R}$: $E(\Delta_i) = 0$ and,

$$|\widehat{F}_N^r(s) - E\widehat{F}_N^r(s)| = \sum_{i=1}^{n} \Delta_i \tag{13}$$

Moreover, we can write :

$$||\psi||_\infty \leq 2C||K||_\infty ||H||_\infty \tag{14}$$

and :

$$Lip(\psi) \leq C\Big(h_K^{-1}||H||_\infty Lip(K) + h_H^{-1}||K||_\infty Lip(H)\Big). \tag{15}$$

In order to use the lemma that was developed by [25], we need to do an evaluation of the variance term $Var(\sum_{i=1}^{n} \Delta_i)$ as well as the covariance term $Cov\Big(\prod_{i=1}^{u} \Delta_{s_i}, \prod_{j=1}^{v} \Delta_{t_j}\Big)$. Both of these terms are denoted by the following for all $(s_1, ..., s_u) \in \mathcal{N}_u$, $\quad (t_1, ..., t_v) \in \mathcal{N}_v$ with $1 \leq s_1 \leq ... \leq s_u \leq t_1 \leq ... \leq t_v \leq n$.
We explore the following cases with respect to the covariance term: if $t_1 = s_u$. Utilizing the Reality that

$$E[|K_1 H_1|] = O_{a.co}\Big(\phi_r(h_K)\Big)$$

and

$$E[|K_1|] = O_{a.co}\Big(\phi_r(h_K)\Big)$$

we have:

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$$\begin{aligned}\Big|Cov\Big(\prod_{i=1}^{u}\Delta_{s_i},\prod_{j=1}^{v}\Delta_{t_j}\Big)\Big| &\leq \left(\frac{C}{nE[K_1]}\right)^{u+v} E_\psi|R_1,S_1|^{u+v} \\ &\leq \left(\frac{C||K||_\infty||H||_\infty}{nE[K_1]}\right)^{u+v} E|K_1H_1| \\ &\leq \phi_r(h_K)\left(\frac{C}{n\phi_r(h_K)}\right)^{u+v} \end{aligned} \quad (16)$$

If it $t_1 > s_u$, Under $(P5)$, the quasi-association yields:

$$\begin{aligned}\Big|Cov\Big(\prod_{i=1}^{u}\Delta_{s_i},\prod_{j=1}^{v}\Delta_{t_j}\Big)\Big| &\leq \left(\frac{h_K^{-1}Lip(K)+h_H^{-1}Lip(H)}{nE[K_1]}\right)^2 \times \left(\frac{C}{nE[K_1]}\right)^{u+v-2} \sum_{i=1}^{u}\sum_{j=1}^{v}\lambda_{s_i,\,t_j} \\ &\leq (h_K^{-1}Lip(K)+h_H^{-1}Lip(H))^2\left(\frac{C}{nE[K_1]}\right)^{u+v} v\lambda_{t_1-s_u} \\ &\leq (h_K^{-1}Lip(K)+h_H^{-1}Lip(H))^2\left(\frac{C}{\phi_r(h_K)}\right)^{u+v} ve^{-\alpha(t_1-s_u)} \end{aligned} \quad (17)$$

On the other hand, if we take into account $(P6)$, we have:

$$\begin{aligned}\Big|Cov\Big(\prod_{i=1}^{u}\Delta_{s_i},\prod_{j=1}^{v}\Delta_{t_j}\Big)\Big| &\leq \left(\frac{C||K||_\infty||H||_\infty}{nE[K_1]}\right)^{u+v-2}\Big(E|\Delta_{s_u},\Delta_{t_1}|+E|\Delta_{s_u}|E|\Delta_{t_1}|\Big) \\ &\leq \left(\frac{C||K||_\infty||H||_\infty}{nE[K_1]}\right)^{u+v-2}\left(\frac{C}{nE[K_1]}\right) \\ &\times h_H\Big(sup_{i\neq j}\mathbb{P}\Big((R_i,R_j)\in B(r,h_K)\times B_\theta(r,h_K) \\ &+\mathbb{P}(R_1\in B(r,h_K))^2\Big)\Big) \\ &\leq \left(\frac{C}{h_H\phi_r(h_K)}\right)^{u+v}(\phi_r(h_K))^2 \end{aligned} \quad (18)$$

In addition, by taking a $\gamma - power$ of (10), and a $(1-\gamma) - power$ of (11), we are able to derive an upper-bound of the tree terms as follows, for: $1 \leq s_1 \leq ...s_u \leq t_1 \leq .... \leq t_v \leq n$,

$$\Big|Cov\Big(\prod_{i=1}^{u}\Delta_{s_i},\prod_{j=1}^{v}\Delta_{t_j}\Big)\Big| \leq h_H\phi_r(h_K)\left(\frac{C}{nh_H\phi_r(h_K)}\right)^{u+v} \quad (19)$$

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Second, we entered the following information for the $Var(\sum_{i=1}^{n}\Delta_i)$ variance term for all $1 \leq i \leq n$:

$$\begin{aligned}\left|Var\left(\prod_{i=1}^{u}\Delta_{s_i},\prod_{j=1}^{v}\Delta_{t_j}\right)\right| &= \left(\frac{1}{nE[K_1]}\right)^2\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(K_iH_i,K_jH_j)\\ &= \underbrace{\left(\frac{1}{nE[K_1]}\right)^2 Var(K_1H_1)}_{T_1}\\ &+ \underbrace{\left(\frac{1}{nE[K_1]}\right)^2\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}Cov(K_iH_i,K_jH_j)}_{T_2}\end{aligned} \tag{20}$$

For the first term T_1, we have :

$$Var(K_1H_1) = E(K_1^2H_1^2) - (E(K_1H_1))^2 \tag{21}$$

Then,

$$\mathbb{E}[K_1^2H_1^2] = E[K_1^2E[H_1^2/X_1]] \tag{22}$$

As a result, considering (P2) and (P3) and integrating over the real component y gives us the following:

$$\mathbb{E}\left[H_1^2 \mid R_1\right] = O\left(h_H\right) \tag{23}$$

As for all $j \geq 1$:

$$\mathbb{E}[K_1^j] = O\left(\phi_x(h_K)\right).$$

Then,

$$\mathbb{E}[K_1^2H_1^2] = O(\phi_x(h_K)).$$

It follows that:

$$\left(\frac{1}{n(\mathbb{E}[K_1])^2}\right)^2 Var(K_1H_1) = O(n\phi_x(h_K)). \tag{24}$$

Regarding the covariance term in $(ref26)$, the following decomposition will be utilized.

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}Cov(K_iH_i,K_jH_j) &= \underbrace{\sum_{i=1}^{n}\sum_{j=1,0<|i-j|\leq m_n}^{n}Cov(K_iH_i,K_jH_j)}_{I}\\ &+ \underbrace{\sum_{i=1}^{n}\sum_{j=1,|i-j|>m_n}^{n}Cov(K_iH_i,K_jH_j)}_{II}\end{aligned}$$

where (m_n) is an infinite series of positive integers as n tends to infinity. The $(P1)$-$(P3)$ and $(P6)$ Presuppositions

we obtain, if $i \neq j$:

$$\begin{aligned} I &\leq nm_n\left(\max_{i\neq j}\left|\mathbb{E}\left(K_iH_iK_jH_j\right)\right| + \left(\mathbb{E}\left(K_1H_1\right)\right)^2\right)\\ &\leq Cnm_n\left(\phi_r^2\left(h_K\right) + \left(\phi_r\left(h_K\right)\right)^2\right)\\ &\leq \mathrm{Cnm_n}\left(\phi_r^2\left(h_K\right)\right).\end{aligned} \tag{25}$$

H and K are both bounded and Lipschitz kernels, we obtain:

$$\begin{aligned} II &\leq \left(h_K^{-1}Lip(K)+h_H^{-1}Lip(H)\right)^2\sum_{i=1}^{u}\sum_{j=1_{|i-j|>m_n}}^{v}\lambda_{i,j} \\ &\leq C\left(h_K^{-1}Lip(K)+h_H^{-1}Lip(H)\right)^2\sum_{i=1}^{u}\sum_{j=1_{|i-j|>m_n}}^{v}\lambda_{i,j} \\ &\leq Cn\left(h_K^{-1}Lip(K)+h_H^{-1}Lip(H)\right)^2\lambda_{m_n} \\ &\leq Cn\left(h_K^{-1}Lip(K)+h_H^{-1}Lip(H)\right)^2 e^{-\alpha m_n}. \end{aligned} \tag{26}$$

Then, by (20) and (21), we get

$$\sum_{j=1,i\neq j}^{n} Cov(K_iH_i,K_jH_j)\leq C\left(nm_n(h_H^2\phi_r^2(h_K))+n(h_K^{-1}Lip(K)+h_H^{-1}Lip(H))^2e^{-\alpha m_n}\right)$$

By choosing :

$$m_n=\log\left(\frac{\left(h_K^{-1}\,\mathrm{Lip}(K)+h_H^{-1}\,\mathrm{Lip}(H)\right)^2}{\alpha\phi_r^2\,(h_K)}\right)$$

We get :

$$\frac{1}{\phi_r(h_K)}\sum_{j=1,i\neq j}^{n}Cov(K_iH_i,K_jH_j)\xrightarrow[n\to+\infty]{}0. \tag{27}$$

Finally, we get this result by integrating the previous three results (13), (18), and (22).

$$Var\Big(\sum_{i=1}^{n}\Delta_i\Big)=O\left(\frac{1}{n\phi_r(h_K)}\right)$$

Therefore, the criteria of the lemma are satisfied by the variables Δ_i, where $i=1,...,n$.

$$K_n=\frac{C}{n\sqrt{\phi_r(h_K)}},\quad M_n=\frac{C}{n\phi_r(h_K)},\qquad and\quad A_n=Var\Big(\sum_{i=1}^{n}\Delta_i\Big)$$

Thus,

$$\begin{aligned} \mathbb{P}\left(\left|\widehat{F}_N^r(s)-\mathbb{E}\widehat{F}_N^r(s)\right|>\eta\frac{\log n}{n\phi_r\,(h_K)}\right) &= \mathbb{P}\left(\left|\sum_{i=1}^{n}\Delta_i\right|>\eta\frac{\log n}{n\phi_r\,(h_K)}\right) \\ &\leq \exp\left\{-\frac{\eta^2\log n}{n\phi_r\,(h_K)\,\mathrm{Var}\,(\sum_{i=1}^{n}\Delta_i)+\frac{\log^{5/6}n}{(n\phi_r(h_K))^{(7/6)}}}\right\} \\ &\leq \exp\left\{-\frac{\eta^2\log n}{C+\frac{(\log n)^{5/6}}{n\phi_r(h_K)^{(7/6)}}}\right\} \\ &\leq C'\exp\left\{-C\eta^2\log n\right\} \end{aligned} \tag{28}$$

by $(P7)$. Last but not least, with an appropriate selection of η, Borel-Cantelli's lemma makes it possible to conclude the proof of Lemma 1.

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The proof of Corollary 1

We have:

$$\left\{\left|F_D^R\right| \leq \frac{1}{2}\right\} \subseteq \left\{\left|F_D^R - 1\right| > \frac{1}{2}\right\} \tag{29}$$

Therefor,

$$\begin{aligned}\mathbb{P}\left\{\left|F_D^R\right| \leq \frac{1}{2}\right\} \quad &\leq \mathbb{P}\left\{\left|F_D^R - 1\right| > \frac{1}{2}\right\} \\ &\leq \mathbb{P}\left\{\left|F_D^X - EF_D^X\right| > \frac{1}{2}\right\}\end{aligned} \tag{30}$$

For $E(F_D^R) = 1$, we apply the result of Lemma 1 we show that :

$$\mathbb{P}\left\{\left|F_D^R\right| \leq \frac{1}{2}\right\} < \infty \tag{31}$$

The proof of the Corollary 1 has finished.

The proof of Lemma 2

We have :

$$\begin{aligned}\mathbb{E}\widehat{F}_N^R(S) - F^R(S) &= \frac{1}{n\mathbb{E}K_1}\sum_{i=1}^{n}\mathbb{E}K_iH_i(y) - F^R(S) \\ &= \frac{1}{\mathbb{E}K_1}\left[\mathbb{E}K_1H_1\left(\frac{s-S_i}{h_H}\right) - F^R(S)\right] \\ &= \frac{1}{\mathbb{E}K_1}\mathbb{E}\left(K_1\left[\mathbb{E}\left(H_1\left(h_H^{-1}(s-S_i)\right)/R\right) - F^R(S)\right]\right)\end{aligned} \tag{32}$$

We derive the following by making use of the stationarity of the data, the conditioning by the variable that is doing the explaining, and the customary change in the variable $t = \frac{s-u}{h_H}$:

$$\begin{aligned}\mathbb{E}\left(H_1\left(h_H^{-1}(s-S_i)\right)/R\right) &= \int_R H\left(\frac{s-u}{h_H}\right)f^R(u)du \\ &= \int_R H^{(1)}(t)F^R(s-h_Ht)\,dt\end{aligned} \tag{33}$$

and we deduce ,

$$\begin{aligned}\left|\mathbb{E}\left(H_1\left(h_H^{-1}(s-S_i)\right)/R\right) - F^R(s)\right| &= \left|\int_R H^{(1)}(t)F^R(s-h_Ht)\,dt - F^R(s)\right| \\ &\leq \int_R H^{(1)}(t)\left|F^R(s-h_Ht) - F^R(s)\right|dt\end{aligned}$$

So, by (H2) we get

$$\left|\mathbb{E}\left(H_1\left(h_H^{-1}(s-S_i)\right)/R\right) - F^R(s)\right| \leq A_R\int_R H^{(1)}\left(h_K^{b_1} + |t|^{b_2}h_H^{b_2}\right)dt$$

This inequality holds true everywhere in y and after substituting in (29) and simplifying the expression $(\mathbb{E}K_1)$ we get that:

$$\mathbb{E}\widehat{F}_N^R(s) - F^R(s) \leq A_R\left(h_K^{b_1}\int_R H^{(1)}(t)dt + h_H^{b_2}\int_R |t|^{b_2}H^{(1)}(t)dt\right)$$

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In conclusion, the evidence of Lemma 2 is provided by Hypothesis (P4) and Corollary 1.

Substituting for $\psi(.,.)$ in Lemma 1's **proof of Lemma 3** yields a simple application of Lemma 3.

$$\psi(R_i) = K(h_K^{-1} d(r, R_i)) - E[K_1], \quad \forall \quad R_i \in \mathcal{H}. \tag{34}$$

The proof of **Theorem 1** has finished. □

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References

1. Araujo, A.; Giné, E. *The Central Limit Theorem for Real and Banach Valued Random Variables*; Wiley Series in Probability and Mathematical Statistics; JohnWiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia, 1980; p. xiv+233.
2. Aït Saidi, A.; Ferraty, F.; Kassa, R. Cross-validated estimation inthe single functional index model. *Statistics* **2008**, *42*, 475–494.
3. Aneiros, G.; Cao, R.; Fraiman, R.; Genest, C.; Vieu, P. Recent advances in functional data analysis and high-dimensional statistics. . *Multivar. Anal* **2019**, *170* , 3—9.
4. Attaoui, S.; Laksaci, A.; Ould-Saïd, E. A note on the conditional densityestimate in the single functional index model. *Statist. Probab.* **2011**, *81*, 45–53, .
5. Attaoui, S.; Laksaci, A.; Ould-Saïd, E. Asymptotic Results for an Mestimatorof the Regression Function for Quasi-Associated Processes. FunctionalStatistics and Applications. *Contributions to Statistics* **2015**, *10*.1007/978-3-319-22476-3-1.
6. Attaoui, S.; Ling, N. Asymptotic results of a nonparametric conditionalcumulative distribution estimator in the single functional indexmodeling for time series data with applications. *Metrika* **2016**, 79:485511 DOI 10.1007/s00184-015-0564-6.
7. Bulinski, A.; Suquet, C. Normal approximation for quasi-associatedrandom fields. *Statist. Probab. Lett* **2001**, *54* , 215–226.
8. Bulinski, A.; Shabanovich, E. Asymptotical behaviour of some functionalsof positively and negatively dependent random fields. (in Russian). *Fundam. Appl. Math***1998**, *4*, 479–492, .
9. Daoudi, H.; Mechab, B.; Chikr Elmezouar, Z. Asymptotic normality ofa conditional hazard function estimate in the single index for quasi-associateddata, *Communications in Statistics - Theory and Methods*, **2020** *49*:3, 513–530.

10. Daoudi, H.; Mechab, B., Asymptotic Normality of the Kernel Estimate of Conditional Distribution Function for the quasi-associated data. *Pakistan Journal of Statistics and Operation Research* **2019**, *15*(4), 999–1015.
11. Daoudi, H.; Mechab, B.; Benaissa, S..; Rabhi. A. Asymp-totic normality of the nonparametric conditional density function esti-mate with functional variables for the quasi-associated data, *Interna-tional Journal of Statistics and Economics* **2019**, *20*(3), 94–106.
12. Doob, J.L. *StochasticProcesses*, John Wiley and Sons, New York, 1953.
13. Douge, L. Théorèmes limites pour des variables quasi-associées hilbertiennes. *Ann. I.S.U.P*, **2010**, *54*, 51–60.
14. Doukhan, P. Louhichi,S., A new weak dependence condition andapplications to moment inequalities. *Stoch. Proc. Appl***1999**, *84*, 313–342, .
15. Esary, J. D. Proschan,F., Walkup,D.,.Association of random variables with applications. *Ann. Math. Statist***1967**, *38*, 1466–1476.
16. Ezzahrioui, M.; Ould-Saïd, E. Asymptotic normality of nonparametric estimators of the conditional mode for functional data. *Technical report* **2005**, *249*.
17. Ezzahrioui, M.; Ould-Saïd, E. On the asymptotic properties of a nonparametric estimator of the conditional mode for functional dependent data. *Preprint* **2008**, *277*.
18. Ferraty, F.; Laksaci, A.; Tadj, A.; Vieu, P. Rate of uniform consistency for nonparametric estimates with functional variables. *J. Statist. Plann. Inference* **2010**, *140*, 335—352.
19. Ferraty, F.; Peuch, A.; Vieu, P. Modèle à indice fonctionnel simple. *C. R. A. S.Mathématiques Paris* **2003**, *336*, 1025–1028, .
20. Ferraty, F.; Rabhi, A.; Vieu, P. Estimation nonparametric de lafonction de hasard avec variable explicative fonctionnelle. *Rev. Roumaine Math. Pures Appl* **2008**, *53*, 1–18.
21. Ferraty, F.; Mas, A.; Vieu, P. Advances in nonparametric regression for functional variables, Aust. and New Zeal. *J. of Statist* **2007**, *49*, 1–20.
22. Ferraty, F.; Vieu. P. *Nonparametric functional data analysis: theoryand practice* , Springer Series in Statistics, New York, USA; 2006.
23. Bouaker, I.; Belguerna, A.; Daoudi, H. The consistency of the kernel estimation of the Function conditional density for associated Data in high-dimensional statistics, Journal of science and arts, *22*:2, 247–256, **2022**.
24. Gasser, T.; Hall, P.; Presnell, B. Nonparametric estimation of the mode of a distribution of random curves. J. R. Stat. Soc. Ser. B Stat. Methodol. **1998**, *60*, 681—691.
25. Kallabis, R.S.; Neumann, M.H. An exponential inequality under weakdependence. *Bernoulli* **2006**, *12*, 333–350, .
26. Kara-Zaitri, L.; Laksaci, A.; Rachdi, M.; Vieu, P.,Uniform in bandwidth consistency for various kernel estimators involving functional data. *J. Nonparametr. Stat* **2017**, *29*, 85-–107.
27. Laksaci, A. Convergence en moyenne quadratique de l'estimateur à noyau de la densité conditionnelle avec variable explicative fonctionnelle. *Ann. I.S.U.P***2007**, *51*, 69–80.
28. Laksaci, A.; Maref, F. Estimation non paramétrique de quantiles conditionnels pour des variables fonctionnelles spatialement dépendantes. *C. R. Math. Acad. Sci. Paris***2009**, *347*, 1075–1080.
29. Laksaci, A.; Mechab, B. Estimation non-paramétrique de la fonctionde hasard avec variable explicative fonctionnelle : cas des données spatiales. *Rev. Roumaine Math. Pures Appl* **2010**, *55* , 35–51.
30. Laksaci,A.; Mechab, B. Conditional hazard estimate for functionalrandom fields. *Journal of Statistical Theory and Practice* **2014**, *8*, 192–200.
31. Laksaci, A.; Mechab, W. Nonparametric relative regression for associatedrandom variables. *Metron* **2016**, *10*.1007/s40300-016-0084-9.
32. Matula, P. A note on the almost sure convergence of sums of negatively dependent random variables. *Statist. Probab. Lett***1992**, *15*, 209–213.
33. Roussas, G. G. *Positive and negative dependence with some statistical applications In : Asymptotics, Nonparametrics and Time Series* (S. Ghosh,Ed). Marcell Dekker, Inc., New York, 1999; pp. 757–788.
34. Newman, C.M., Asymptotic independence and limit theorems for positivelyand negatively dependent random variables : In Inequalities in Statisticsand Probability, *IMS Lect. Notes-Monographs Series* **1984**, *5*, 127–140, .
35. Quintela-del-RÃo, A. Hazard function given of functional variable: Nonparametricestimation under strong mixing conditions. *J. Nonparametr. Stat***2008**, *20*, 413–430.

 doi:10.20944/preprints202308.0307.v1

36. Tabti, H.; Ait Saïdi, A. Estimation and simulation of conditionalhazard function in the quasi-associated framework when the observationsare linked via a functional single-index structure, *Commu. Stat. Theory and Methods* **2017**, *47*, 816–838.